Modelling of Bedload Sediment Transport for Weak and Strong Regimes

C. Escalante, E. D. Fernández-Nieto, T. Morales de Luna, and G. Narbona-Reina

Abstract A two-layer shallow water type model is proposed to describe bedload sediment transport for strong and weak interactions between the fluid and the sediment. The key point falls into the definition of the friction law between the two layers, which is a generalization of those introduced in Fernández-Nieto et al. (https://doi.org/10.1051/m2an/2016018). Moreover, we prove formally that the two-layer model converges to a Saint-Venant-Exner system (SVE) including gravitational effects when the ratio between the hydrodynamic and morphodynamic time scales is small. The SVE with gravitational effects is a degenerated nonlinear parabolic system, whose numerical approximation can be very expensive from a computational point of view, see for example T. Morales de Luna et al. (https://doi.org/10.1007/s10915-010-9447-1). In this work, gravitational effects are introduced into the two-layer system without any parabolic term, so the proposed model may be a advantageous solution to solve bedload sediment transport problems.

Keywords Bedload · Saint Venant Exner · Sediment friction law · Two-layer model

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1 Introduction

Our goal is to obtain a general model for bedload sediment transport that is valid in any regime, for strong and weak interactions between the fluid and sediment.

In most models a weak interaction between the sediment and the fluid is assumed. In this case Saint-Venant-Exner models [7] are usually considered (SVE in what follows). For the case of high bedload transport rate, two-layer shallow water type model are considered instead, see for example [18, 20, 21]. In this work we focus into the definition of a two-layer shallow water type model that can be applied in both situations.

For the case of uniform flows the thickness of the moving sediment layer can be predicted, because erosion and deposition rates are equal in those situations. This is a general hypothesis that is assumed when modeling weak bedload transport. The usual approach is to consider a coupled system consisting of a Shallow Water system for the hydrodynamical part combined with a morphodynamical part given by the so-called Exner equation. The whole system is known as Saint Venant Exner system [7]. Exner equation depends on the definition of the solid transport discharge. Different classical definitions can be found for the solid transport discharge, for instance the ones given by Meyer-Peter and Müller [14], Van Rijn's [23], Einstein [5], Nielsen [17], Fernández-Luque and Van Beek [8], Ashida and Michiue [1], Engelund and Fredsoe [6], Kalinske [12], Charru [4], etc. A generalization of these classical models was introduced in [10] where the morphodynamical component is deduced from a Reynolds equation and includes gravitational effects in the sediment layer. Classical models do not take into account in general such gravitational effects because in their derivation the hypothesis of nearly horizontal sediment bed is used (see for example [13]).

In general, classical definitions for solid transport discharge can be written as follows,

$$\frac{q_b}{Q} = \operatorname{sgn}(\tau)\, \frac{k_1}{(1-\varphi)}\, \theta^{m_1}\, (\theta - k_2\,\theta_c)_+^{m_2} \left(\sqrt{\theta} - k_3\sqrt{\theta_c}\right)_+^{m_3}, \tag{1}$$

where Q represents the characteristic discharge, $Q = d_s\sqrt{g(1/r-1)d_s}$, $r = \rho_1/\rho_2$ is the density ratio, ρ_1 being the fluid density and ρ_2 the density of the sediment particles; d_s the mean diameter of the sediment particles, and φ is the averaged porosity. The coefficients k_l and m_l, $l = 1, 2, 3$, are positive constants that depend on the model. We usually find $m_2 = 0$ or $m_3 = 0$, for example, Meyer-Peter and Müller model takes $m_3 = 0$ and Ashida and Michiue's model uses $m_2 = 0$.

The Shields stress, θ, is defined as the ratio between the agitating and the stabilizing forces, $\theta = |\tau| d_s^2/(g(\rho_2 - \rho_1)d_s^3)$, τ being the shear stress at the bottom. For example, for Manning's law, we have $\tau = \rho_1 g h_1 n^2 u_1|u_1|/h_1^{4/3}$. Where h_1 and u_1 are the thickness and the velocity of the fluid layer, respectively, and n is the Manning coefficient.

Finally, θ_c is the critical Shields stress. The positive part, $(\cdot)_+$, in the definition implies that the solid transport discharge is null if $\theta \leq k\theta_c$ (with $k = k_2$ when $m_2 > 0$ and $k = \sqrt{k_3}$ when $m_3 > 0$). If the velocity of the fluid is zero, $u_1 = 0$, we have $\theta = 0 < k\theta_c$, and for any model that can be written under the structure (1) we obtain that $q_b = 0$, which means that there is no movement of the sediment layer. This is even true when the sediment layer interface is not horizontal which is a consequence of the fact that classical models do not take into account gravitational effects.

In order to introduce gravitational effects in classical models, Fowler et al. proposed in [11] (see also [16]) a modification of the Meyer-Peter and Müller formula that consists in replacing θ by θ_{eff}, where:

$$\theta_{\text{eff}} = |\text{sgn}(u_1)\theta - \vartheta\,\partial_x(b + h_2)|\,, \tag{2}$$

with

$$\vartheta = \frac{\theta_c}{\tan\delta}, \tag{3}$$

δ being the angle of repose of the sediment particles. The sediment surface is defined by $z = b + h_2$, where h_2 is the thickness of the sediment layer and b the topography function or bedrock layer. Then, θ_{eff} is defined in terms of the gradient of sediment surface.

In [10], a multi-scale analysis is performed taking into account that the velocity of the sediment layer is smaller than the one of the fluid layer. This leads to a shallow water type system for the fluid layer and a lubrication Reynolds equation for the sediment one. The model includes gravitational effects and the authors deduce that it can also be seen as a modification of classical models: θ is replaced by the proposed values $\theta_{\text{eff}}^{(L)}$ or $\theta_{\text{eff}}^{(Q)}$, depending on whether the friction law between the fluid and the sediment is linear or quadratic. In the case when h_2 is of order of d_s/ϑ, for a linear friction law, the definition of the effective shear stress proposed in [10] can be written as follows:

$$\theta_{\text{eff}}^{(L)} = \left|\text{sgn}(u_1)\theta - \vartheta\,\partial_x(b + h_2) - \vartheta\,\frac{\rho_1}{\rho_2 - \rho_1}\partial_x(b + h_1 + h_2)\right|. \tag{4}$$

Let us remark that if the water free surface is horizontal, the definition of $\theta_{\text{eff}}^{(L)}$ coincides with θ_{eff} (2), proposed by Fowler et al. in [11]. Otherwise, the main difference is that this definition for the effective shear stress takes into account not only the gradient of the sediment surface but also the gradient of the water free surface.

For the case of a quadratic friction law, although the definition is a combination of the same components, it is rather different. In this case we can write the effective

Shields parameter proposed in [10] as follows:

$$\theta_{\text{eff}}^{(Q)} = \left| \text{sgn}(u_1)\sqrt{\theta} - \sqrt{\frac{\vartheta\rho_1}{\rho_2 - \rho_1}|\partial_x(\frac{\rho_1}{\rho_2}h_1 + h_2 + b)|}\ \text{sgn}\left(\partial_x(\frac{\rho_1}{\rho_2}h_1 + h_2 + b)\right)\right|^2 . \quad (5)$$

In the case of submerged bedload sediment transport, the drag term is defined by a quadratic friction law. Thus, we should consider an effective Shields stress given by $\theta_{\text{eff}}^{(Q)}$. Nevertheless, in the bibliography θ_{eff} (2) is usually considered, regardless the fact that θ_{eff} is an approximation of $\theta_{\text{eff}}^{(L)}$ which is deduced from a linear friction law. In any case, considering the definitions θ_{eff} (2), $\theta_{\text{eff}}^{(L)}$ (4), or $\theta_{\text{eff}}^{(Q)}$ (5), means that the corresponding SVE system with gravitational effects is a parabolic degenerated partial differential system with non linear diffusion. Moreover, the system cannot be written as combination of a hyperbolic part plus a diffusion term.

Let us remark that in the literature a linearized version can be found, where gravitational effects are included by considering a classical SVE model with an additional viscous term, see for example [15, 22] and references therein. The drawback of this approach is that the diffusive term should not be present in stationary situations, for instance when the velocity is not high enough and sediment slopes are under the one given by the repose angle. In such situations it is necessary to include some external criteria that controls whether the diffusion term is applied or not. This is not the case in definitions (4) or (5) where the effective Shields stress is automatically limited by the effect of the Coulomb friction angle.

In this work we propose a two-layer shallow water model for bedload transport. The model converges to a generalization of SVE model with gravitational effects for low transport regimes while being valid for higher transport regimes as well. Moreover, it has the advantage that the inclusion of gravitational effects does not imply to approximate any non-linear parabolic degenerated term, as for the case of SVE model with gravitational effects.

In the next section we propose the new two-layer Shallow Water model for bedload transport. We also show the formal convergence to the SVE model and the associated energy balance.

2 Proposed Model

We consider a domain with two immiscible layers corresponding to water (upper layer) and sediment (lower layer). The sediment layer is in turn decomposed into a moving layer of thickness h_m and a sediment layer that does not move of thickness h_f, adjacent to the fixed bottom. These thicknesses are not fixed because there is an exchange of sediment material between the layers. Particles are eroded from the lower sediment layer and come into motion in the upper sediment layer. Conversely, particles from the upper layer are deposited into the lower sediment layer and stop moving.

We propose a 2D shallow water model is obtained by averaging on the vertical direction the Navier-Stokes equations and taking into account suitable boundary conditions. In particular, at the free surface we impose kinematic boundary conditions and vanishing pressure; at the bottom a Coulomb friction law is considered. The friction between water and sediment is introduced through the term F at the water/sediment interface and the mass transference term in the internal sediment interface is denoted by T. The general notation for the water layer corresponds to the subindex 1 and for the sediment layer to the subindex 2. Thus, the water of layer has a thickness h_1 and moves with horizontal velocity u_1. The thickness of the total sediment layer is denoted by $h_2 = h_f + h_m$, and the moving sediment layer h_m flows with velocity u_m. The fixed bottom or bedrock is denoted by b. See Fig. 1 for a sketch of the domain.

Note that the velocity of the sediment layer is defined as $u_2 = u_m$ in the moving layer and $u_2 = 0$ in the static layer. We assume an hydrostatic pressure regime.

Then we propose the following two-layer shallow water model:

$$\partial_t h_1 + \nabla \cdot (h_1 u_1) = 0 \tag{6a}$$

$$\partial_t (h_1 u_1) + \nabla \cdot (h_1 u_1 \otimes u_1) + g h_1 \nabla_x (b + h_1 + h_2) = -F \tag{6b}$$

$$\partial_t h_2 + \nabla \cdot (h_m u_m) = 0 \tag{6c}$$

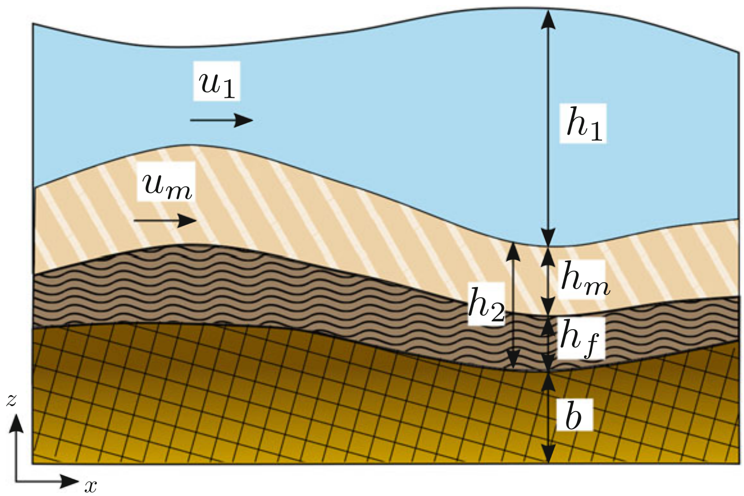

Fig. 1 Sketch of the domain for the fluid-sediment problem

$$\partial_t(h_m u_m) + \nabla \cdot (h_m u_m \otimes u_m) + g h_m \nabla_x (b + r h_1 + h_2) = rF + \frac{1}{2} u_m T - (1-r) g h_m \operatorname{sgn}(u_m) \tan\delta \tag{6d}$$

$$\partial_t h_f = -T \tag{6e}$$

where $r = \rho_1/\rho_2$ is the ratio between the densities of the water, ρ_1, and the sediment particles, ρ_2. δ is the internal Coulomb friction angle. In the next lines we give the closures for the friction term F and the mass transference T.

Following [10] we consider two types of friction laws at the interface: linear and quadratic. The friction term for the linear friction law is defined as

$$F_L = C_L(u_1 - u_m) \quad \text{with} \quad C_L = g\left(\frac{1}{r} - 1\right) \frac{h_1 h_m}{\vartheta(h_1 + h_m)\sqrt{(\frac{1}{r} - 1) g d_s}} \tag{7}$$

and for the quadratic friction law,

$$F_Q = C_Q(u_1 - u_m)|u_1 - u_m| \quad \text{with} \quad C_Q = \frac{1}{\beta} \frac{h_1 h_m}{\vartheta(h_1 + h_m)}, \tag{8}$$

d_s being the mean diameter of the sediment particles. ϑ is defined by Eq. (3). This definition of ϑ complies with the analysis of Seminara et al. [19], who concluded that the drag coefficient is proportional to $\tan(\delta)/\theta_c$.

Remark that the calibration coefficient β has units of length so that C_Q is non-dimensional. In [10], $\beta = d_s$ was assumed for the bedload in low transport situations. In our case, given that we deal with a complete bilayer system for any regime, this value is not always valid. In bedload framework, we can establish from experimental observations that the region of particles moving at this level is at most 10–20 particle-diameter in height [3].

So we may assume that the thickness of the bed load layer is $h_m = k\, d_s$ with $k \in [0, k_{\max}]$ ($k_{\max} = 10$ or 20). So that, when $h_m \le k_{\max} d_s$ we are in a bedload low rate regime and it makes sense to consider the friction coefficient as in [10], that is, of the order of d_s. Conversely, when $h_m > k_{\max} d_s$ we are in an intense bedload regime and then we must turn to a more appropriate friction coefficient. Thus, to be consistent with our previous work, we propose to take:

$$\beta = \begin{cases} h_m & \text{if } h_m > k_{\max} d_s \\ d_s & \text{if } h_m \le k_{\max} d_s \end{cases}$$

Another possibility would be to define $\beta = k_{\max} d_s$ when $h_m \le k_{\max} d_s$. The coefficient $k_{\max}$ can be then considered as a calibration constant for the friction law.

The mass transference between the moving and the static sediment layers T is defined in terms of the difference between the erosion rate, $\dot{z}_e$, and the deposition rate $\dot{z}_d$. There exists in the literature different forms to close the definition of the erosion and deposition rates, all of them depending on calibration parameters (see for example [4]). For instance the following definitions are given in [9]:

$$T = \dot{z}_e - \dot{z}_d \quad \text{with} \quad \dot{z}_e = K_e(\theta_e - \theta_c)_+ \frac{\sqrt{g(1/r-1)d_s}}{1-\varphi}, \quad \dot{z}_d = K_d h_m \frac{\sqrt{g(1/r-1)d_s}}{d_s}.$$

The coefficients K_e and K_d are erosion and deposition constants, respectively, φ is the porosity. For the case of nearly flat sediment bed, $\theta_e = \theta$ is usually set. This corresponds to the Bagnold's relation (see [2]). Nevertheless, in order to take into account the gradient of the sediment bed θ_e must be defined in terms of the effective Shields stress (see [10]). Then we define θ_e in terms of the friction law between the fluid and the sediment layers: for a linear friction law it is given by (4) and Eq. (5) gives its value for a quadratic friction law.

2.1 *Convergence to the Classical SVE System for Weak Regimes*

In this subsection we show formally the convergence of system (6) to the Saint-Venant-Exner model presented in [10]. This model is also obtained from an asymptotic approximation of the Navier-Stokes equations but following a different derivation for water and sediment under the hypothesis of large morphodynamic time scale. In particular, it has the following advantages: it preserves the mass conservation, the velocity (and hence, the discharge) of the bedload layer is explicitly deduced, and it has a dissipative energy balance.

The model introduced in [10] reads as follows:

$$\begin{cases} \partial_t h_1 + \nabla_x \cdot q_1 = 0, \\ \partial_t q_1 + \nabla_x \cdot \big(h_1(u_1 \otimes u_1)\big) + g h_1 \nabla_x (b + h_2 + h_1) = -\dfrac{g h_m}{r} \mathcal{P}, \\ \partial_t h_2 + \nabla_x \cdot \Big(h_m\, v_b \sqrt{(1/r-1) g d_s}\Big) = 0, \\ \partial_t h_f = -T. \end{cases} \tag{9}$$

with

$$\mathcal{P} = \nabla_x (r h_1 + h_2 + b) + (1-r)\operatorname{sgn}(u_2) \tan\delta. \tag{10}$$

The definition of the non-dimensional sediment velocity v_b depends on the friction law. When a linear friction law is considered, it reads:

$$v_b^{(LF)} = \frac{u_1}{\sqrt{(1/r-1)gd_s}} - \frac{\vartheta}{1-r}\mathcal{P}, \tag{11}$$

where

$$\operatorname{sgn}(u_2) = \operatorname{sgn}\left(\frac{u_1}{\sqrt{(1/r-1)gd_s}} - \frac{\vartheta}{1-r}\nabla_x(rh_1+h_2+b)\right).$$

For a quadratic friction law:

$$v_b^{(QF)} = \frac{u_1}{\sqrt{(1/r-1)gd_s}} - \left(\frac{\vartheta}{1-r}\right)^{1/2}|\mathcal{P}|^{1/2}\operatorname{sgn}(\mathcal{P}), \tag{12}$$

where $\operatorname{sgn}(u_2) = \operatorname{sgn}(\Psi)$ and

$$\Psi = \frac{u_1}{\sqrt{(1/r-1)gd_s}} - \left|\frac{\vartheta}{1-r}\nabla_x(rh_1+h_2+b)\right|^{1/2}\operatorname{sgn}\left(\frac{\vartheta}{1-r}\nabla_x(rh_1+h_2+b)\right).$$

The convergence is obtained when we assume the adequate asymptotic regime in terms of the time scales. As it is well known, for the weak bedload transport problem, the morphodynamic time is much larger than the hydrodynamic time, which makes the pressure effects much more important than the convective ones. As a consequence, the behavior of the sediment layer is just defined by the solid mass equation (Exner equation), omitting a momentum equation. This large morphodynamic time turns into an assumption of a smaller velocity for the lower layer. In order to fall into the low bedload transport regime we must also assume that the thickness of the bottom layer is smaller, because it represents the layer of moving sediment. Thus, we suppose:

$$u_m = \varepsilon_u\tilde{u}_m; \quad h_m = \varepsilon_h\tilde{h}_m; \quad T = \varepsilon_u\tilde{T}.$$

with ε_h and ε_u small parameters. Now we take these values into the momentum conservation equation for the lower layer (6d):

$$\begin{aligned}\partial_t(\varepsilon_h\varepsilon_u\tilde{h}_m\tilde{u}_m) + \nabla\cdot(\varepsilon_h\varepsilon_u^2\tilde{h}_m\tilde{u}_m\otimes\tilde{u}_m) + g\varepsilon_h\tilde{h}_m\nabla_x(b+rh_1+h_2)\\ = r\tilde{F} + \varepsilon_u^2\frac{1}{2}\tilde{u}_m\tilde{T} - (1-r)g\varepsilon_h\tilde{h}_m\operatorname{sgn}(\tilde{u}_m)\tan\delta\end{aligned}$$

Then, if we neglect second order terms in $(\varepsilon_h,\ \varepsilon_u)$, we get

$$g\varepsilon_h\tilde{h}_m\nabla_x(b+rh_1+h_2) = r\tilde{F} - (1-r)g\varepsilon_h\tilde{h}_m\operatorname{sgn}(\tilde{u}_m)\tan\delta.$$

Returning to dimension variables, this equation reads:

$$rF = gh_m \nabla_x(b + rh_1 + h_2) + (1 - r)gh_m \mathrm{sgn}(u_m)\tan\delta = gh_m\mathcal{P},$$

where the last equality follows from the definition of $\mathcal{P}$, (10). Thus the expression of the friction term is

$$rF = gh_m\mathcal{P}; \tag{13}$$

which coincides with the friction term in the momentum equation of layer 1, r.h.s. of (9).

Now, from this equation and using the expressions of F, for linear (7) and quadratic (8) laws, we have to compute the value of u_m to check that it fits with (11) and (12) respectively.

- Linear friction law:

$$\begin{aligned}\tilde{F} &= g\Big(\frac{1}{r} - 1\Big)\frac{1}{\vartheta\sqrt{(\frac{1}{r} - 1)gd_s}}\frac{\varepsilon_h\tilde{h}_m}{1 + \varepsilon_h\frac{\tilde{h}_m}{h_1}}(u_1 - \varepsilon_u\tilde{u}_m)\\ &= g\Big(\frac{1}{r} - 1\Big)\frac{\varepsilon_h\tilde{h}_m}{\vartheta\sqrt{(\frac{1}{r} - 1)gd_s}}(u_1 - \varepsilon_u\tilde{u}_m) + O(\varepsilon_h^2)\end{aligned}$$

  where in the last equality we have used that $\dfrac{1}{1 + \varepsilon_h\frac{\tilde{h}_m}{h_1}} = 1 - \varepsilon_h\dfrac{\tilde{h}_m}{h_1} + O(\varepsilon_h^2)$.

  So turning to the dimension variables and neglecting second order terms, Eq. (13) reads:

$$rg\Big(\frac{1}{r} - 1\Big)\frac{h_m}{\vartheta\sqrt{(\frac{1}{r} - 1)gd_s}}(u_1 - u_m) = gh_m\mathcal{P}.$$

  From where we directly obtain that $u_m = v_b^{(LF)}\sqrt{(\frac{1}{r} - 1)gd_s}$.

- Quadratic friction law:

  Note that in this case β reduces to d_s and then

$$\tilde{F} = \frac{\varepsilon_h h_1\tilde{h}_m}{\vartheta d_s(h_1 + \varepsilon_h\tilde{h}_m)}(u_1 - \varepsilon_u\tilde{u}_m)|u_1 - \varepsilon_u\tilde{u}_m| = \frac{\varepsilon_h\tilde{h}_m}{\vartheta d_s}(u_1 - \varepsilon_u\tilde{u}_m)|u_1 - \varepsilon_u\tilde{u}_m| + O(\varepsilon_h^2).$$

  Following the same reasoning as above, Eq. (13) reads:

$$r\frac{h_m}{\vartheta d_s}(u_1 - u_m)|u_1 - u_m| = gh_m\mathcal{P}.$$

From where we obtain that

$$r\frac{1}{\vartheta d_s}(u_1 - u_m)^2 = g\mathcal{P}\,\text{sgn}(\mathcal{P}) \quad \text{and then} \quad u_m = v_b^{(QF)}\sqrt{\Big(\frac{1}{r} - 1\Big)gd_s}.$$

2.2 *Energy Balance*

The proposed model has an exact dissipative energy balance, which is an easy consequence of two-layer shallow water systems. We obtain the following result.

Theorem 1 *System* (6) *admits a dissipative energy balance that reads:*

$$\partial_t\left(rh_1\frac{|u_1|^2}{2} + h_m\frac{|u_m|^2}{2} + \frac{1}{2}g(rh_1^2 + h_2^2) + g\,rh_1h_2 + gb(rh_1 + h_2)\right)$$
$$+\nabla\cdot\left(rh_1u_1\frac{|u_1|^2}{2} + h_mu_m\frac{|u_m|^2}{2} + g\,rh_1u_1(h_1 + h_2 + b) + gh_mu_m(rh_1 + h_2 + b)\right)$$
$$\leq -r(u_1 - u_m)F - (1 - r)gh_m|u_m|\tan\delta;$$

where the friction term F is given by (7) *or* (8)*.*

The proof of the previous result is straightforward and for the sake of brevity we omit it.

Notice that classical SVE model does not verify in general a dissipative energy balance. In [10] a modification of a classical SVE models by including gravitational effects has been proposed that allows to verify this property. Nevertheless the proposed model in this work presents several advantages: it preserves the mass conservation, it accounts with a direct energy balance and it has a better structure to be solved from a numerical point of view. Moreover it can be applied for both regimes, weak and strong bedload transport without any a priori prescription. Numerical approximation and tests will be presented in a forthcoming paper.

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