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# OPERADORES DE VOLTERRA-COMPOSICIÓN: TEORÍA ESPECTRAL Y CICLICIDAD

(COMPOSITION VOLTERRA OPERATORS: SPECTRAL THEORY AND CYCLICITY)

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Alfonso Montes Rodríguez

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porque me hizo comprender que debía ser matemático.

# Contents

# Resumen en Castellano

Para hacer más accesibles sus contenidos, se incluye en esta memoria de investigación el presente capítulo en castellano. En este capítulo se da cuenta de los principales resultados, enunciándolos en el orden en que aparecen y respetando su numeración en el texto.

## R-1 Resumen del Capítulo 2.

El este capítulo se definen los operadores sobre los que versa la presente memoria de investigación y se establecen sus propiedades básicas. Dada una aplicación medible $\varphi$ del intervalo [0,1] en sí mismo, se define el operador de tipo Volterra $V_\varphi$ como

$$(V_\varphi f)(x) = \int_0^{\varphi(x)} f(t)\,dt \qquad \text{con } f \in L^p[0,1], \quad 1 \leqslant p \leqslant \infty.$$

Obsérvese que cuando el símbolo $\varphi$ es la identidad, se obtiene el clásico operador de Volterra, $V$. Estos operadores son compactos en todos los espacios $L^p[0,1]$, para $1 \leqslant p \leqslant \infty$ [5, p. 44], ya que podemos factorizarlos como $V_\varphi = C_\varphi V$, donde $V$ es compacto y $C_\varphi$ es acotado sobre el rango de $V$, que está contenido en $L^\infty[0,1]$.

Los trabajos previos sobre estos operadores son casi inexistentes, ya que se reducen a tan sólo tres referencias: Lyubic [30], que se limita a enunciarlos y preguntar por su quasi-nilpotencia, Whitley [56] y Tong [54]. Estos autores encontraron la caracterización de la quasi-nilpotencia de los operadores de tipo Volterra. Aquí presentamos una prueba más corta, la cual se deduce de una caracterización más general que abarca a todos los operadores integrales con núcleo positivo en el espacio $L^2([0,1]^2)$. Las pruebas se basan en la teoría de operadores nucleares y Hilbert-Schmidt, y en particular en un resultado de Lidskii [45]. El teorema principal es el siguiente.

**Teorema. 2.1.1** *Sea $K \in L^2([0,1]^2)$ no negativo. Entonces el operador integral con núcleo $K$ es quasi-nilpotente si y sólo si*

$$K(t_1,t_2)K(t_2,t_3)\cdots K(t_{n-1},t_n)K(t_n,t_1)$$

*se anula e.c.t.* $[0,1]^n$ *para cada* $n \geqslant 2$.

Utilizando que los operadores de tipo Volterra se pueden escribir como operadores integrales con núcleo

$$K_\varphi(x,t) = \begin{cases} 1, & \text{if } t \leqslant \varphi(x); \\ 0, & \text{if } t > \varphi(x), \end{cases}$$

se obtiene la siguiente caracterización.

**Corolario. 2.1.2** *Sea* $\varphi$ *una aplicación medible de* $[0,1]$ *en sí mismo. Entonces* $V_\varphi$ *es quasi-nilpotente si y sólo si* $\varphi(x) \leqslant x$ *e.c.t.*

El resultado anterior junto con el Teorema 2.2.8 son la causa de que los símbolos para los que es más natural estudiar el espectro sean los que verifican la condición $\varphi(x) \geqslant x$.

Otro resultado básico que presentamos es la fórmula para el operador adjunto a un operador de tipo Volterra cuando el símbolo $\varphi$ es creciente. Si definimos

$$\varphi_{-1}(x) = \begin{cases} \sup\{y : \varphi(y) < x\}, & \text{if } x > \varphi(0); \\ 0, & \text{otherwise}, \end{cases}$$

se tiene que

$$(V_\varphi^\star f)(x) = \int_{\varphi_{-1}(x)}^1 f(t)\,dt \qquad \text{con } f \in L^p[0,1], \quad 1 \leqslant p < \infty.$$

En este punto debemos recalcar que la involución isométrica $(Uf)(x) = f(1-x)$ jugará un papel fundamental en el trabajo, ya que $UV_\varphi^\star U = V_{\widetilde{\varphi}}$, donde $\widetilde{\varphi}(x) = 1 - \varphi_{-1}(1-x)$. Como consecuencia:

$$\sigma(V_{\widetilde{\varphi}}) = \sigma(V_\varphi).$$

Otra propiedad general de operadores es la inyectividad. En particular, es útil conocer el núcleo de los operadores con los que se trabaja. Para enunciar el resultado recordemos que el rango esencial de una aplicación medible $\varphi$ en $[0,1]$ es

$$\operatorname{ess}(\varphi) = \big\{y \in \mathbb{R} \text{ tales que } \mu\{t : |y - \varphi(t)| < \varepsilon\} > 0 \text{ para todo } \varepsilon > 0\big\}.$$

Se tiene:

**Proposición. 2.1.3** *Sea* $\varphi$ *una aplicación medible de* $[0,1]$ *en sí mismo. Entonces el* $\ker V_\varphi$ *es trivial si y sólo si el rango esencial de* $\varphi$ *es* $[0,1]$. *Es más,* $\ker V_\varphi$ *es de dimensión finita si y sólo si* $\ker V_\varphi \neq \{0\}$.

El siguiente resultado atañe a la acotación de las normas de los operadores de tipo Volterra. En algunas situaciones, la norma se computa exactamente en el espacio $L^2[0,1]$.

**Proposición. 2.1.5** *Sean $\varphi$ y $\psi$ transformaciones medibles de $[0,1]$ en sí mismo. Entonces*

(i) $\|V_\varphi - V_\psi\|_p \leqslant \left\| |\varphi - \psi|^{p-1} \right\|_1^{1/p} \leqslant 1$ *para* $1 \leqslant p < \infty$.

(ii) $\|V_\varphi - V_\psi\|_\infty = \|\varphi - \psi\|_\infty$ *para* $p = \infty$.

Tomando $\psi \equiv 0$ en la proposición anterior, obtenemos el siguiente resultado.

**Corolario. 2.1.6** *Sea $\varphi$ una aplicación medible de $[0,1]$ en sí mismo. Entonces $\|V_\varphi\|_p \leqslant \left\|\varphi^{p-1}\right\|_1^{1/p}$ para $1 \leqslant p < \infty$ y $\|V_\varphi\|_\infty = \|\varphi\|_\infty$ para $p = \infty$. En particular, $V_\varphi$ siempre es una contracción.*

Usando un método bien conocido para el cálculo de normas en espacios de Hilbert, [15, p. 300], se puede computar la del operador de tipo Volterra con símbolo $\varphi_\alpha(x) = x^\alpha$.

**Ejemplo. 2.1.7** *Sea $\varphi_\alpha(x) = x^\alpha$ con $0 < \alpha < \infty$. Entonces $\|V_{\varphi_\alpha}\|_2$ es igual a raíz cuadrada del mayor zero de*

$$J_{-(1+\alpha)^{-1}}\left(2(1+\alpha)^{-1}\alpha^{1/2}\lambda^{-1/2}\right),$$

*donde $J_{-(1+\alpha)^{-1}}$ es la función de Bessel de tipo uno y orden $-(1+\alpha)^{-1}$.*

Los siguientes dos resultados son ejemplos particulares de símbolos para los que es posible calcular las autofunciones y los autovalores de los correspondientes operadores de tipo Volterra.

**Teorema. 2.1.10** *Sea $\varphi(x) = x^\alpha$ con $0 < \alpha < 1$. Entonces los autovalores de $V_\varphi$ son simples y $\sigma(V_\varphi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$. Es más, para cada $n \geqslant 0$, la autofunción correspondiente a $(1-\alpha)\alpha^n$ es $f_n(x) = x^{\alpha/(1-\alpha)}p_n(\ln x)$, donde*

$$p_n(x) = x^n + \sum_{j=1}^{n} \frac{n!(1-\alpha)^j \alpha^{(j^2-j)/2}}{(n-j)!} \left(\prod_{l=1}^{j} \frac{1}{1-\alpha^l}\right) x^{n-j}.$$

*En particular, las autofunciones $\{f_n\}_{n\geqslant 0}$ tienen span lineal denso en $L^2[0,1]$.*

El segundo ejemplo de cálculo concreto es el siguiente.

**Teorema. 2.1.11** *Sea* $\psi(x) = 1-(1-x)^{1/\alpha}$ *con* $0 < \alpha < 1$. *Entonces los autovalores de* $V_\psi$ *son simples y* $\sigma(V_\psi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$. *Es más, para cada* $n \geqslant 0$, *la autofunción de* $V_\psi$ *correspondiente a* $(1-\alpha)\alpha^n$, *es*

$$f_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(1-\alpha)^k\alpha^{nk}}{(\alpha^{-1}-1)\cdots(\alpha^{-k}-1)}(1-x)^{\frac{\alpha^{-k-1}-\alpha^{-1}}{\alpha^{-1}-1}}.$$

*En particular, las autofunciones de* $V_\psi$ *no tienen span lineal denso en* $L^2[0,1]$.

Un primer resultado respecto a la ciclicidad de los operadores de tipo Volterra, es el siguiente.

**Proposición. 2.1.13** *Sea* $\varphi(x) = x^\alpha$ *con* $\alpha > 0$. *Entonces la función* $\phi(x) = x^\beta$ *con* $\beta > -1/p$ *es cíclica para* $V_\varphi$ *acting on* $L^p[0,1]$, $1 \leqslant p < \infty$, *si y sólo si* $0 < \alpha \leqslant 1$.

La demostración del resultado anterior es una aplicación del clásico Teorema de Müntz-Szász.

Ahora prestaremos atención a la acotación del espectro y en particular al radio espectral. Por el Teorema de Kreĭn-Rutman sabemos que los operadores compactos y positivos tienen siempre un autovalor simple en el valor del radio espectral, véase [26] o [31, Theorem 4.1.4]. Este resultado nos permite obtener el teorema siguiente.

**Teorema. 2.2.1** *Sea* $K$ *un núcleo no negativo en* $L^2([0,1]^2)$. *Si el radio espectral del correspondiente operador integral con núcleo es positivo, tenemos que este mismo valor es un autovalor del operador, al que corresponde una única autofunción no negativa.*

Estableciendo una relación de acotación entre los operadores de tipo Volterra aquí definidos y los operadores integrales con núcleo, obtenemos la siguiente mejora del Teorema 2.2.1.

**Teorema. 2.2.2** *Sea* $K$ *un núcleo no negativo en* $L^2([0,1]^2)$ *y sea positivo el radio espectral del operador integral asociado. Si hay una aplicación continua* $\psi$ *de* $[0,1]$ *en sí mismo, de manera que* $\psi(x) > x$ *cuando* $0 < x < 1$ *y* $K(x,t) > 0$ *e.c.t. cuando* $0 < t \leqslant \psi(x) \leqslant 1$, *entonces la autofunción suministrada en el Theorem* 2.2.1 *es estrictamente positiva e.c.t. Es más, no hay más autofunciones generalizadas de signo definido, diferentes la correspondiente al radio espectral.*

Como consecuencia tenemos:

**Corolario. 2.2.3** *Sea* $\varphi$ *una aplicación medible de* $[0,1]$ *en sí mismo con* $\mu\{x : \varphi(x) > x\} > 0$. *Entonces* $r(V_\varphi) > 0$ *es un autovalor al que corresponde una autofunción positiva. Es más, si* $\varphi$ *es continua y* $\varphi(x) > x$ *para* $0 < x < 1$, *entonces la autofunción* *correspondiente al radio espectral es estrictamente positiva e.c.t. y no hay más autofunciones generalizadas de signo definido.*

El siguiente resultado aporta una cota inferior óptima para el radio espectral de los operadores de tipo Volterra.

**Proposición. 2.2.4** *Sea $\varphi$ una aplicación creciente de $[0,1]$ en sí mismo. Entonces,*

$$r(V_\varphi) \geqslant \|(\varphi(x)-x)^+\|_\infty.$$

Respecto a las autofunciones de los operadores de tipo Volterra, podemos enunciar los dos siguientes resultados que atañen a su multiplicidad y su soporte.

**Lema. 2.2.5** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo, tal que $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$. Si $f$ es una función de $L^2[0,1]$ tal que $V_\varphi f = \lambda f$ para algún $\lambda \neq 0$, entonces $f$ no puede ser ortogonal a las funciones constantes. Es más, si $0 < \alpha \leqslant 1$ es un punto fijo de $\varphi$ y $\max_{[0,\alpha]} \varphi \leqslant \alpha$, entonces o bien $f(\alpha) \neq 0$ ó $f$ se anula en $[0,\alpha]$.*

Un corolario directo del resultado anterior es:

**Corolario. 2.2.6** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo, tal que $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$. Entonces los autovalores no nulos de $V_\varphi$ son simples.*

El siguiente resultado establece bajo ciertas condiciones naturales, que las autofunciones de los operadores de tipo Volterra están soportadas en todo el intervalo $[0,1]$.

**Proposición. 2.2.7** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo tal que $\varphi(x) > x$ para $0 < x < 1$ y no constante en ningún subintervalo. Entonces el soporte de todas las autofunciones de $V_\varphi$ asociadas a autovalores no nulos es $[0,1]$.*

El siguiente resultado descompone el espectro de un operador de tipo Volterra en la unión de los espectros de varios operadores de tipo Volterra con símbolos más simples. Dada una aplicación medible $\varphi$ de $[0,1]$ en sí mismo, consideramos el conjunto

$$S_\varphi = \{y \in [0,1] : \max_{[0,y]} \varphi \leqslant y\}$$

y escribimos su frontera como $\partial S_\varphi$. Entonces tenemos el siguiente resultado.

**Teorema. 2.2.8** *Sea $\varphi$ una aplicación medible de $[0,1]$ en sí mismo, tal que*

$$\mu(\partial S_\varphi) = 0.$$

*Entonces*

$$\sigma(V_\varphi) = \bigcup_{j\in J} \sigma(V_\varphi^{j,j}),$$

*donde los operadores* $V_\varphi^{j,j}$ *están definidos como en (2.2.1).*

Como último tema dentro del primer capítulo, consideramos las posibles simetrías o propiedades geométricas y de distribución de los autovalores de los operadores de tipo Volterra. En particular se tiene,

**Teorema. 2.2.11** *Sea* $\varphi$ *una aplicación creciente de* $[0,1]$ *en sí mismo. Entonces todos los autovalores de* $V_\varphi$ *son reales y positivos. Es más,*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) \leqslant \mu(\{x \in [0,1] : \varphi(x) > x\}) \leqslant 1.$$

Las técnicas para obtener estos resultados se basan en la aproximación de operadores compactos por matrices de entradas reales especialmente elegidas. En particular se utilizan las propiedades descritas en [32] para matrices con menores positivos. De estos métodos se deduce automáticamente que,

**Teorema. 2.2.15** *Si* $\varphi$ *es una aplicación medible de* $[0,1]$ *en sí mismo, entonces el espectro de* $V_\varphi$ *es simétrico con respecto al eje real.*

## R-2 Resumen del Capítulo 3.

Este capítulo está dedicado a un estudio más profundo del espectro de los operadores de tipo Volterra. En concreto, se construye una función analítica a partir de los operadores de tipo Volterra con símbolos ligeramente regulares, y se establece que los inversos de los ceros de estas funciones analíticas son los autovalores de los operadores correspondientes.

Haciendo uso de los teoremas clásicos para funciones enteras que relacionan crecimiento y distribución de ceros obtendremos, entre otras cosas, varias caracterizaciones de la finitud del espectro de nuestros operadores y los exponentes de sumación de las sucesiones de autovalores. Para terminar el capítulo, estudiamos la transmisión de la analiticidad del símbolo $\varphi$ a las autofunciones del correspondiente operador.

En este capítulo se trabaja sobre los símbolos del conjunto siguiente,

$$\Lambda = \{\varphi : [0,1] \to [0,1] \text{ continuas y tales que } \varphi(x) \geqslant x \text{ para } 0 \leqslant x \leqslant 1\},$$

que consideramos dotado de la topología que hereda del espacio de Banach $\mathcal{C}[0,1]$.

Ahora, para cada $\varphi$ en $\Lambda$, consideremos el operador acotado

$$(W_\varphi f)(x) = \int_{\varphi(x)}^1 f(t)\,dt, \qquad f \in L^2[0,1].$$

Si $\varphi(x) = x$, simplemente escribimos $W_\varphi = W$. Definimos la función $\mathcal{F} : \Lambda \times [0,1] \times \mathbb{C} \to \mathbb{C}$ que a cada terna $(\varphi, x, z)$, le asigna

$$\mathcal{F}^\varphi(x,z) = \mathcal{F}_x^\varphi(z) = \sum_{n=0}^{\infty} (-1)^n a_n^\varphi(x) z^n, \tag{R-2.1}$$

donde $a_0^\varphi(x) = 1$ y $a_n^\varphi(x) = (WW_\varphi^{n-1}1)(x)$ para cada $n \geqslant 1$.

En esta sección del capítulo, los resultados son esencialmente técnicos y se centran en probar que la función $\mathcal{F}$ está bien definida, es analítica y se dan ciertas propiedades de monotonía sobre sus coeficientes y sobre su función módulo máximo.

El siguiente resultado es fundamental para establecer la relación entre los ceros de $\mathcal{F}^\varphi$ y los autovalores de $V_\varphi$.

**Proposición. 3.1.2** *La función $\mathcal{F}$ está bien definida, es diferenciable con respecto a $x$, holomorfa con respecto a $z$ y tanto $(\varphi, x) \mapsto \mathcal{F}^\varphi(x, \cdot)$ como $(\varphi, x) \mapsto \frac{\partial \mathcal{F}^\varphi}{\partial x}(x, \cdot)$ son aplicaciones continuas de $\Lambda \times [0,1]$ en $\mathcal{H}(\mathbb{C})$. Es más,*

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = z\mathcal{F}^\varphi(\varphi(x), z),$$
$$\mathcal{F}^\varphi(1,z) = 1.$$

*Además, tenemos la siguiente representación en serie de Taylor*

$$\frac{\partial \mathcal{F}^{\varphi}}{\partial x}(x,z) = \sum_{n=1}^{\infty}(-1)^n b_n(x) z^n,$$

*donde* $b_n(x) = (V_{\psi}^{n-1}1)(1-x)$ *con* $\psi(x) = 1-\varphi(1-x)$.

El siguiente resultado de existencia y unicidad de soluciones de un cierto tipo de ecuación diferencial, es el puente entre la función $\mathcal{F}^{\varphi}$ y el operador $V_{\varphi}$.

**Proposición. 3.1.5** *Sean* $a \leqslant \alpha \leqslant b$ *y sea* $\varphi$ *una aplicación continua de* $[a,b]$ *en sí mismo, tal que*

$$\begin{cases} x \leqslant \varphi(x) \leqslant \alpha, & \text{si } a \leqslant x \leqslant \alpha; \\ \alpha \leqslant \varphi(x) \leqslant x, & \text{si } \alpha \leqslant x \leqslant b. \end{cases}$$

*Sean también* $T$ *un operador acotado en un espacio real o complejo de Banach* $\mathcal{B}$, $x_0$ *en* $\mathcal{B}$ *y* $G$ *en* $\mathcal{C}([a,b],\mathcal{B})$. *Entonces, el problema de Cauchy*

$$\begin{cases} H'(x) = TH(\varphi(x)) + G(x), \\ H(\alpha) = x_0 \end{cases}$$

*tiene una única solución* $H : [a,b] \to \mathcal{B}$, *que pertenece al espacio* $\mathcal{C}^1([a,b],\mathcal{B})$.

Ahora pasamos a establecer de forma explícita la relación entre $\mathcal{F}^{\varphi}$ y el espectro de $V_{\varphi}$.

**Teorema. 3.1.7** *Sea* $\varphi$ *una aplicación continua de* $[0,1]$ *en sí mismo, tal que* $\varphi(x) \geqslant x$ *para* $0 \leqslant x \leqslant 1$. *Entonces,* $\lambda \neq 0$ *es un cero de orden* $k$ *de* $\mathcal{F}_0^{\varphi}$ *si y sólo si* $\lambda^{-1}$ *es un autovalor de multiplicidad algebraica* $k$ *de* $V_{\varphi}$. *Más aún, en tal caso, una base del espacio* $\ker(V_{\varphi} - \lambda^{-1}I)^k$ *es*

$$g_j(x) = \left.\frac{\partial^{j+1}\mathcal{F}^{\varphi}}{\partial x \partial z^j}(x,z)\right|_{z=\lambda}, \qquad \text{para} \quad 0 \leqslant j \leqslant k-1.$$

Gracias a este resultado se pueden probar entre otros resultados, que si el símbolo $\varphi$ satisface que $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$, entonces $V_{\varphi}$ no tiene autovalores negativos. Más aún, este último resultado nos permite aplicar la teoría de distribución de ceros de funciones enteras a la localización de nuestros autovalores y mediante algunos lemas técnicos de acotación, podemos concluir lo siguiente.

**Corolario. 3.1.13** *Sea* $\varphi$ *una aplicación continua de* $[0,1]$ *en sí mismo, tal que* $\varphi(x) \geqslant x$ *para* $0 \leqslant x \leqslant 1$. *Si el conjunto de los puntos para los que* $\varphi(x) = x$ *tiene medida de Lebesgue cero, entonces* $\mathcal{F}_x^{\varphi}$ *es de tipo exponencial* $0$ *para* $0 \leqslant x \leqslant 1$.

A continuación, establecemos varias equivalencias con la finitud del espectro de $V_\varphi$. Es de destacar que la condición $\max_{[0,x]} \varphi \leqslant x$ para cada punto fijo $x$ de $\varphi$, es una condición menos restrictiva que la de creciente. En lo sucesivo, dada una aplicación $\varphi$ de $[0,1]$ en sí mismo, denotamos por $\varphi_0$ a la identidad y escribimos $\varphi_n = \varphi \circ \varphi_{n-1}$ para cada $n$ entero positivo.

**Teorema. 3.1.15** *Sea $\varphi$ una aplicación de $[0,1]$ en sí mismo, tal que $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$, satisfaciendo $\sup\{x : \varphi(x) > x\} = 1$ y $\max_{[0,y]} \varphi \leqslant y$ para cada punto fijo $y$ de $\varphi$. Entonces las siguientes afirmaciones son equivalentes:*

(i) *El espectro $\sigma(V_\varphi)$ es finito.*

(ii) *Existe un entero positivo $n$ tal que $\varphi_n(x) \equiv 1$.*

(iii) *La aplicación $\varphi \equiv 1$ en un entorno de 1 y $\varphi(x) > x$ para $0 \leqslant x < 1$.*

(iv) *Para algún entero positivo $n$, el operador $V_\varphi^n$ es de rango finito.*

(v) *Si $P$ es la proyección sobre las funciones constantes, entonces $P - V_\varphi$ es nilpotente.*

Como corolario se obtiene la siguiente aplicación a ciertos símbolos ligeramente más adecuados.

**Corolario. 3.1.16** *Sea $\varphi$ un aplicación continua de $[0,1]$ en sí mismo, tal que $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$ y satisfaciendo que $\max_{[0,y]} \varphi \leqslant y$ para cada punto fijo $y$ de $\varphi$. Entonces las siguientes afirmaciones son equivalentes:*

(i) *El espectro $\sigma(V_\varphi)$ es finito.*

(ii) *O bien $\varphi$ es la identidad, o bien existen $0 < \beta < \alpha = \sup\{x : \varphi(x) > x\}$ de manera que*

$$\begin{cases} \varphi(x) > x, & si\ 0 \leqslant x < \beta; \\ \varphi(x) = \alpha, & si\ \beta \leqslant x \leqslant \alpha; \\ \varphi(x) = x, & si\ \alpha < x \leqslant 1. \end{cases}$$

Por supuesto, si el símbolo $\varphi$ es continuo y permanece estrictamente sobre la diagonal principal en todo el intervalo $[0,1]$, i.e. $\varphi(x) > x$ para $0 < x < 1$, con $\varphi(0) = 0$ ó satisface que $\varphi^{-1}(1)$ no contiene ningún intervalo de la forma $[1-\varepsilon, 1]$ para algún $\varepsilon > 0$, entonces el espectro $V_\varphi$ es infinito.

Como resultado de la asociación entre funciones analíticas y operadores, podemos estimar, y en algunos casos calcular, las sumas de los espectros de nuestros operadores y sus exponentes de convergencia. Desde ahora, $s(\varphi) = s(\{\lambda_n(\varphi)\})$ denotará el exponente de convergencia de la sucesión de autovalores $\{\lambda_n\}$. Entre otros resultados, obtenemos la siguiente fórmula exacta para símbolos cuyos autovalores suman absolutamente.

**Teorema. 3.1.26** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo, con $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$ y asumamos que la sucesión $\{\lambda_n(\varphi)\}$ de autovalores de $V_\varphi$ es absolutamente sumable. Entonces,*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) = \mu(\{x \in [0,1] : \varphi(x) > x\}).$$

Obtenemos además la siguiente fórmula exacta para símbolos crecientes.

**Corolario. 3.1.27** *Sea $\varphi$ una aplicación continua y creciente de $[0,1]$ en sí mismo, con $\varphi(x) \geqslant x$ para $0 \leqslant x \leqslant 1$ . Entonces*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) = \mu(\{x \in [0,1] : \varphi(x) > x\}).$$

Los siguientes resultados tratan la transmisión de la analiticidad entre el símbolo $\varphi$ y las autofunciones del correspondiente operador $V_\varphi$.

Una herramienta fundamental en esta parte de la memoria es la fórmula de Fao de Bruno [52, Chapter 3], para la derivada enésima de una composición de funciones.

**Lema. 3.2.1** *Sean $f$ y $g$ funciones en $\mathcal{C}^n[u,v]$. Entonces para cada $u \leqslant x \leqslant v$, tenemos*

$$(g \circ f)^{(n)}(x) = n! \sum_{k_1+\cdots+nk_n=n} \frac{g^{(k_1+\cdots+k_n)}(f(x))}{k_1!\cdots k_n!(1!)^{k_1}\cdots(n!)^{k_n}} (f'(x))^{k_1}\cdots(f^{(n)}(x))^{k_n}.$$

Como consecuencia se obtiene la formula exacta para la siguiente suma.

**Lema. 3.2.2** *Para todo $c$ en $\mathbb{C}$ y todo $n$ natural, tenemos*

$$\sum_{k_1+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} c^{k_1+\cdots+k_n} = c(c+1)^{n-1}.$$

El principal resultado de esta sección es el siguiente.

**Teorema. 3.2.6** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo, tal que $\varphi(x) > x$ para $0 < x < 1$, y sea $f$ una autofunción generalizada de $V_\varphi$ asociada a un autovalor no nulo. Si $\varphi$ es analítica en $[\alpha,1]$ para algún $0 < \alpha < 1$ y $\varphi'(1) \neq 1$, entonces $f$ es analítica en $[\alpha,1]$. Lo mismo es cierto si $\alpha = 0$, supuesto que $\varphi(0) > 0$.*

Para completar este resultado, se aporta un contraejemplo que muestra que las condiciones impuestas sobre la derivada de $\varphi$ en el punto 1 y sobre $\varphi$ en el punto 0, no se pueden eliminar. El ejemplo es la función $\varphi(x) = (2-x)^{-1}$, que satisface todas las condiciones del Teorema anterior para $\alpha = 0$, excepto que $\varphi'(1) = 1$.

# R-3 Resumen del Capítulo 4.

Como se verá más adelante, los operadores de tipo Volterra para los que es interesante estudiar la superciclicidad, son tan sólo los quasi-nilpotentes con símbolos continuos y estrictamente crecientes. Por ello, en este capítulo, que es de carácter puramente técnico, buscamos estimaciones de las normas de las potencias de $V_\varphi$ y de las normas de las órbitas $\{V_\varphi^n f\}_{n\geqslant 0}$, cuando $V_\varphi$ es supercíclico. Los dos primeros resultados relevantes de este capítulo son los siguientes.

**Corolario. 4.1.4** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$ y $\varphi(1) = 1$. Entonces, si $\varphi$ es diferenciable en $0$ y en $1$,y $\varphi'(0) = 0$, tenemos que para todo $1 \leqslant p \leqslant \infty$,*

$$\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = \frac{1}{\sqrt{\varphi'(1)}}.$$

*Si $\varphi$ es diferenciable en $0$ y $\varphi'(1) = \infty$, entonces*

$$\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = \sqrt{\varphi'(0)}.$$

**Corolario. 4.1.5** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$, $\varphi(1) = 1$ y $\varphi$ es diferenciable en $0$ y en $1$. Entonces, el conjunto de las funciones $f$ de $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, para las que se satisface*

$$\lim_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} = \phi(\varphi'(0), 1/\varphi'(1))$$

*es denso en $L^p[0,1]$, donde $\phi(u,v)$ está definida por*

$$\phi(u,v) = \begin{cases} \exp\left(\dfrac{\ln u \ln v}{2\ln(uv)}\right), & \textit{si } u > 0, v > 0 \textit{ y } (u,v) \neq (1,1); \\ \sqrt{|u-v|}, & \textit{si } u = 0 \textit{ ó } v = 0; \\ 1, & \textit{si } (u,v) = (1,1), \end{cases}$$

Dentro de este capítulo se dedica una sección a las acotaciones superiores de las órbitas de $V_\varphi$. El principal resultado en este sentido es el siguiente.

**Lema. 4.2.1** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$, $\varphi(1) = 1$ y*

$$\delta_1^+ = \delta_1^+(\varphi) = \overline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)}.$$

*Si además $f$ en $L^p[0,1]$, $1 \leqslant p < \infty$, satisface que $\inf \operatorname{supp}(f) > 0$, tenemos que,*

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{\delta_1^+}.$$

*En particular, si $\varphi$ es diferenciable en 1, tenemos*

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{1/\varphi'(1)}.$$

Otra sección de este capítulo estudia las preimágenes iteradas del operador $V_\varphi$. El teorema principal de esta sección es el siguiente.

**Teorema. 4.3.1** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$ y $\varphi(1) = 1$. Además, supongamos que $\varphi$ es analítica en 0 y $\varphi'(0) > 0$. Entonces, para todo $b > 1/\varphi'(0)$, el conjunto*

$$F_b = \left\{ f \in V_\varphi^\infty(\mathcal{C}_0[0,1]) \text{ tales que } \overline{\lim_{n\to\infty}} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{b} \right\}$$

*es una variedad lineal densa en $\mathcal{C}_0[0,1]$, que satisface $V_\varphi(F_b) = F_b$ y $V_\varphi^{-1}(F_b) = F_b$.*

La última sección del capítulo está dedicada a las acotaciones inferiores de las órbitas de $V_\varphi$. El principal resultado de esta sección es el siguiente.

**Corolario. 4.4.2** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(1) = 1$ y $\varphi(x) < x$ para $0 < x < 1$. Si además tenemos que $\varphi$ es analítica en 1 y diferenciable en 0 con $\varphi'(0) = 0$, entonces, para cada función no nula $f$ en $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, se tiene que*

$$\lim_{n\to\infty} \|V_\varphi f\|_p^{1/n^2} = \frac{1}{\sqrt{\varphi'(1)}}.$$

## R-4 Resumen del Capítulo 5.

En esta sección se estudia la ciclicidad de los operadores de tipo Volterra. Recordemos que un operador se dice cíclico si la envolvente lineal de alguna de sus órbitas es densa en el espacio entero. En el caso de que sea suficiente la envolvente proyectiva de alguna de sus órbitas para alcanzar la densidad en el espacio, el operador se llama supercíclico. Una de las nociones más fuertes de ciclicidad es la hiperciclicidad, que consiste en la existencia de una órbita que es densa en el espacio por sí misma.

En la primera sección del capítulo se encuentran operadores de tipo Volterra que son cíclicos y cuyos símbolos permanecen bajo el de la identidad. Esto significa que el operador de Volterra no es el caso límite, como sí lo es para el espectro. El principal resultado de la primera sección es el siguiente.

**Corolario. 5.1.3** *Sea $\varphi$ una aplicación continua de $[0,1]$ en sí mismo, tal que $\varphi(x) > x$ para $0 < x < 1$, $\varphi$ es diferenciable en $0$ y en $1$ con $1 < \varphi'(0) \leqslant \infty$ y $\varphi'(1) < 1$. Entonces la función constante $1$ es cíclica para $V_\psi$, donde $\psi(x) = 1 - \varphi(1-x)$, si y sólo si la envolvente lineal de las autofunciones de $V_\varphi$ es denso en $L^2[0,1]$.*

En la siguiente sección, que está dedicada a la Teoría General de operadores, se emplea el concepto de núcleo generalizado para probar la existencia de operadores hipercíclicos.

Recordemos que el núcleo generalizado de un operador $T$ es el espacio

$$\ker^\star T = \bigcup_{n=1}^{\infty} \ker T^n.$$

Se tiene

**Corolario. 5.2.3** *Sea $T$ un operador acotado en un $\mathcal{F}$-espacio separable $X$, de manera que $\ker^\star T$ es denso en $X$ y $T(\ker T^{n+1})$ es denso en $\ker T^n$ para cada entero positivo $n$. Entonces $I + T$ es hypercyclic.*

El primer resultado general de superciclicidad es el siguiente.

**Proposición. 5.2.13** *Sean $X$ un $\mathcal{F}$-espacio separable y $T$ un operador acotado en él. Si el núcleo generalizado de $T$ es denso y $T$ tiene rango denso, entonces $T$ es supercíclico.*

El siguiente resultado establece la existencia de gran cantidad de operadores hipercíclicos, con adjunto hipercíclico.

**Corolario. 5.2.17** *Sea $\mathcal{Q}$ el conjunto de los operadores compactos y quasi-nilpotentes sobre un espacio de Hilbert separable e infinito dimensional. Se tiene que el subconjunto de los operadores $T$ en $I+\mathcal{Q}$, tales que $T$ y $T^{\star}$ son hipercíclicos es un $G_{\delta}$ denso en $I+\mathcal{Q}$.*

En la siguiente sección del capítulo pasamos a estudiar la superciclicidad de los operadores de tipo Volterra y la hiperciclicidad de los operadores de la forma $I+V_{\varphi}$, tanto en los espacios $L^p[0,1]$, $1 \leqslant p < \infty$, como en el espacio $\mathcal{C}_0[0,1]$ dotado con la norma del supremo. Utilizando los resultados anteriores, probamos que para todo símbolo $\varphi$ continuo y estrictamente creciente, tal que $\varphi(x) < x$ para $0 < x \leqslant 1$ (nótese que $\varphi(1) < 1$), él correspondiente operador $V_{\varphi}$ es supercíclico y el operador $I+V_{\varphi}$ es hipercíclico. Esto aporta nuevos ejemplos de operadores hipercíclicos y quasi-nilpotentes, que vienen a unirse a los ya existentes de Hilden y Wallen [21] por una parte, y de Salas [49] por otra. Además, se prueba que existen símbolos continuos y estrictamente crecientes, con $\varphi(x) < x$ para $0 < x < 1$, de manera que tanto $V_{\varphi}$ como $V_{\varphi}^{\star}$ son supercíclicos y los operadores $I+V_{\varphi}$ e $I+V_{\varphi}^{\star}$ son hipercíclicos. Para terminar se demuestra que para los símbolos $\varphi$ continuos y estrictamente crecientes, con $\varphi(x) < x$ para $0 < x < 1$, $\varphi(1) = 1$ y analíticos en 0 y en 1, se tiene que si $\varphi'(0)\varphi'(1) > 1$, entonces $V_{\varphi}$ es supercíclico, y si $\varphi'(0)\varphi'(1) < 1$, entonces $V_{\varphi}$ no es cíclico. Debemos señalar que en algunos enunciados no se especifican los espacios en los que los operadores cumplen una cierta propiedad cíclica. Esto se debe a que, gracias a las características especiales de los operadores de tipo Volterra, sus propiedades de ciclicidad no dependen del espacio $L^p[0,1]$, $1 \leqslant p < \infty$ o $C_0[0,1]$ en el que actúan.

**Teorema. 5.3.10** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x \leqslant 1$. Entonces $V_{\varphi}$ es supercíclico y $I+V_{\varphi}$ es hipercíclico.*

**Corolario. 5.3.12** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$ y $\varphi(1) = 1$. Si $\varphi$ es analítica en 0 y diferenciable en 1 con $\varphi'(0)\varphi'(1) > 1$, entonces $V_{\varphi}$ es supercíclico.*

El primer resultado de no-ciclicidad que presentamos es el siguiente.

**Corolario. 5.3.14** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$ y $\varphi(1) = 1$. Entonces, si además asumimos que $\varphi$ es analítica en 1 y diferenciable en 0 con $\varphi'(0)\varphi'(1) < 1$, tenemos que $V_{\varphi}$ no es cíclico.*

Como consecuencia de los dos resultados anteriores, tenemos

**Corolario. 5.3.15** *Sea $\varphi$ una aplicación continua y estrictamente creciente de $[0,1]$ en sí mismo, tal que $\varphi(x) < x$ para $0 < x < 1$ y $\varphi(1) = 1$. Asumamos también que $\varphi$ es analítica en 0 y en 1. Se tine que*

(i) *Si $\varphi'(0)\varphi'(1) > 1$, entonces $V_\varphi$ es supercíclico.*

(ii) *Si $\varphi'(0)\varphi'(1) < 1$, entonces $V_\varphi$ no es cíclico.*

La siguiente definición es necesaria para estudiar los conjuntos de operadores, supercíclicos e hipercíclicos, cuyos adjuntos también lo son. Consideremos el conjunto

$$\Omega = \{\varphi \in \mathcal{C}_0[0,1] \text{ tales que } 0 \leqslant \varphi(x) \leqslant x \quad \text{para} \quad 0 \leqslant x \leqslant 1 \quad \text{y } \varphi \text{ es creciente}\}$$

dotado de la métrica

$$d(\varphi,\psi) = \sum_{n=1}^{\infty} 2^{-n} \max_{[3^{-n},1-3^{-n}]} |\varphi - \psi|.$$

Consideremos también el subconjunto de $\Omega$

$$\Omega_0 = \{\varphi \in \Omega : \varphi(x) < x \text{ for } 0 < x < 1,\ \varphi(1) = 1 \ \text{ y } \ \varphi \text{ es estrictamente creciente}\}.$$

Entonces tenemos el siguiente resultado.

**Teorema. 5.3.16** *El conjunto de símbolos $\varphi$ en $\Omega_0$ para los que $V_\varphi$ y $V_\varphi^\star$ son supercíclicos y tanto $I + V_\varphi$ como $I + V_\varphi^\star$ son hipercíclicos es un $G_\delta$ denso en $\Omega_0$.*

# Introduction

An operator is nothing else than a continuous linear transformation on a vector space endowed with a norm, and the branch of mathematics that treats operators is called Operator Theory. As in any other branch of mathematics, general results usually come from the study of particular examples. The part of Operator Theory that provides such study of examples is known as Concrete Operator Theory but many times it has been written and said that there is a lack of objects in the theory that limits its development. In this work we present and develop the spectral and cyclic theory of a family of operators that we call composition Volterra operators. Namely, given a Lebesgue measurable self-map $\varphi$ of the interval $[0, 1]$, the composition Volterra operator is defined as

$$(V_\varphi f)(x) = \int_0^{\varphi(x)} f(t)\, dt, \qquad f \in L^p[0,1],\ 1 \leqslant p \leqslant \infty.$$

Results listed here provide understanding on how the behavior of these operators depends on the geometry or the qualitative and quantitative properties of the symbol $\varphi$. Therefore, composition Volterra operators might be useful to produce ad hoc instances of operators for particular situations in the theory. To be precise, for a class of natural symbols $\varphi$, finiteness of the spectrum is characterized. An interesting instance of finite spectrum occurs when it reduces to the singleton $\{0\}$. Operators with such property are known as quasi-nilpotent operators and they are characterized in the family of composition Volterra operators. When the spectrum is infinite, a formula for the convergence exponent of eigenvalues is provided. We also treat the symmetry and the positivity of the spectrum as well as the analyticity of the eigenfunctions. The text is illustrated by some examples of symbols $\varphi$ to which the theory can be applied and, in particular, norms, eigenvalues and eigenfunctions are computed explicitly. Finally, we show that the cyclic behavior of $V_\varphi$ is essentially determined by the behavior of the symbol $\varphi$ at 0 and at 1. In particular, this leads to new examples of quasi-nilpotent supercyclic operators, which parallel previous ones of Héctor Salas on weighted shifts.

However, before we go into the technical part of the text, in these introductory lines we would like to expose roughly the origins of Operator Theory and some of the main difficulties that one encounters while studding it. We emphasize by means of examples how far of being parallel are the finite and the infinite dimensional settings.

Operator Theory could very well be considered as a natural extension of the study of finite matrices. As a mater of fact, operators were first treated in finite dimensional spaces, where a huge effort was made by many mathematicians to construct a corpus of results, known nowadays as linear algebra. Such accumulation of knowledge lets us almost clearly understand the way in which a matrix transforms a finite dimensional vector space. Among others, Jordan Canonical Forms and Schur triangular representations, are tools aimed to exhibit the main properties of finite dimensional linear transformations and to classify them up to isomorphisms.

For the sake of the exposition, we sacrifice the chronological order. Indeed, although Jordan Canonical Forms are triangular matrices, surprisingly enough, it was Marie Ennemond Camille Jordan (1838-1922) who first stated his result, and a little later, Issai Schur (1875-1941) found triangular representations of matrices. The latter representations are more intuitive and in some sense they could even be considered as an intermediate step to reach Jordan's result, which is more involved and complete. Jordan's work on Group Theory [22], *Traité des substitutions et des équations algebraique*, was published in 1870, when Schur was only 5 years old. There, along with many other results that founded Group Theory, appears the Jordan Canonical Form, not over the complex numbers, but over a finite field. It approximately says that *each square matrix is similar to a block-diagonal matrix which is triangular.* Some works in the same direction were first developed by Weierstrass, but it seems clear that Jordan was not aware of these. It was not until 1909, that Schur [51] found his result, *each square matrix is similar to a triangular one.* A triangular matrix at first glance displays its eigenvalues, the diagonal elements, and their algebraic multiplicities, the number of times that an eigenvalue is repeated, and as we next see, it is more elementary to compute than the Jordan Canonical Form. In order to show the existence of the triangular representation we follow a constructive method. Since we work on the field of complex numbers, on finite dimensional spaces the unit ball is compact and matrices are continuous transformations, we can ensure the existence of a preferred direction in the space under the action of the matrix; i.e. there are a vector $u_1$, called eigenvector, and a complex number $\lambda_1$, called eigenvalue, such that $Au_1 = \lambda_1 u_1$. If we restrict the transformation $A$ to the orthogonal complement of $u_1$ in $V$, in symbols $V \ominus u_1$, by the same argument as before, there exist an eigenvector $u_2$ and an eigenvalue $\lambda_2$ in $V \ominus u_1$. Of course, although in general $u_2$ is not an eigenvector of $A$ acting on $V$, at most, $Au_2$ is a linear combination of $u_1$ and $u_2$. Iterating this process, we clearly end up with an orthogonal basis $\{u_1, \ldots, u_n\}$ of $V$, in which $A$ has an upper triangular matrix. Moreover, that an $n$-square matrix $A$ be similar to a triangular matrix, is equivalent to the existence of a maximal chain of invariant subspaces of $A$. That is, a chain of subspaces $\{0\} = M_0 \subset \cdots \subset M_n = V$ with $\dim(M_k) = k$ and such that $A(M_k) \subset M_k$ for $0 \leqslant k \leqslant n$. The collection of all invariant subspaces of $A$ is usually denoted by $\mathrm{Lat}(A)$. Observe that the same orthonormal basis could have been obtained by taking orthogonal differences in the chain $\{0\} = M_0 \subset \cdots \subset M_n = V$. This reformulation provides a different approach, that will be useful below, and points out the interest of knowing the lattice of invariant subspaces of a matrix. These subspaces provide a good understanding of the transformation and let us construct nice representations for the matrix. Jordan's Theorem provides more information than Schur's, and it naturally appears from a detailed study of a particular example; the finite dimensional version of the linear application known as backward shift, $B$ say. This is the transformation that sends the first element of the basis to the null vector and every other element of the basis is sent to its predecessor: $Be_0 = 0$ and $Be_k = e_{k-1}$ for each $k > 0$, where obviously, $\{e_k : k = 0, 1, \ldots, n\}$ is a basis of the $n$-dimensional vector space. The only eigenvalue of $B$ is 0, thus its algebraic multiplicity is $n$, and the only eigenvector associated to 0 is $e_0$. The geometric multiplicity of an eigenvalue is the dimension of the space of its eigenvectors. Therefore, in the case of $B$, the geometric multiplicity of 0 is 1. The main idea of the Jordan Canonical Form follows analyzing the orbits of $B$. Indeed, $B$ is the classical example of a nilpotent linear transformation. These are matrices $M$ for which there exists a natural number $p$, called index of nilpotency, such that $M^p = 0$ but $M^{p-1} \neq 0$. Obviously, for each nilpotent transformation of index $p$, one can ensure the existence of at least one non-null vector, $x$ say, such that $M^{p-1}x \neq 0$. In fact, geometrically one can think of a nilpotent matrix as a set of holes, or vectors in its kernel, towards which the different invariant subspaces are shrunk while iterating the application. This rough idea points out the possibility of writing a nilpotent linear transformation, in a canonical form, as the sum of several backward shifts, each of them acting on an invariant subspace. Since at the end of the day, all this goes about invariant subspaces, it is interesting and necessary to introduce here a particulary nice kind of invariant subspaces; the so called reducing subspaces, that consist on invariant subspaces with invariant orthogonal complement. The reducing subspaces of a linear application reduce, or decompose, the application into a direct sum of two applications acting on the respective subspaces. The following result cited from [17] belongs to the folklore of the theory and leads, via a finite induction process, to the hinted canonical form for nilpotent matrices.

**Theorem.** *If $A$ is a nilpotent linear transformation of index $q$ on a finite-dimensional vector space $V$, and $x_0$ is a vector for which $A^{q-1}x_0 \neq 0$, then $x_0, Ax_0, \ldots, A^{q-1}x_0$ are linearly independent. If $H$ is the subspace spanned by these vectors, then there exists a subspace $K$ such that $H \oplus K$ is the whole space and such that the pair $(H, K)$ reduces $A$.*

Obviously, since $A$ was nilpotent on $V$, it is so on the reducing subspace $K$, and a finite induction finishes the construction of the canonical form for nilpotent matrices.

Once the nilpotent matrices are understood, the same ideas may be adapted to obtain the Jordan Canonical Form. The adaptation consists on observing that if a matrix $A$ has the eigenvalue $\lambda$ with algebraic multiplicity $k$, then $\ker(A - \lambda I)$ is not trivial. Therefore, there is a subspace of dimension $k$ in which the matrix given by $A - \lambda I$ is nilpotent of order at most $k$. Such space is called the generalized eigenspace of $A$ associated to the eigenvalue $\lambda$, and vectors in it are generalized eigenvectors. Denote by $A_\lambda$ to the restriction of $A$ to the generalized eigenspace associated to $\lambda$. We can write $A_\lambda = (A_\lambda - \lambda I) + \lambda I$, where the matrix in brackets is nilpotent of order at most $k$ and therefore, using the above result, we can construct a basis of the generalized eigenspace such that $A_\lambda - \lambda I$ coincides with the direct sum of a number of backward shifts, $\widetilde{B}_\lambda$ say. The number of backward shifts added in $\widetilde{B}_\lambda$ is precisely the geometric multiplicity of $\lambda$, or what is the same, the dimension of $\ker(A - \lambda I)$. As a result of this, we have just ensured the existence of a basis in each generalized eigenspace such that, for each eigenvalue $\lambda$, $A_\lambda = \widetilde{B}_\lambda + \lambda I$.

$$A_\lambda \approx \begin{pmatrix} \lambda & 1 & & & & & & \\ & \ddots & \ddots & & & & \text{\Large 0} & \\ & & \lambda & 1 & & & & \\ & & & \lambda & 0 & & & \\ & & & & \lambda & 1 & & \\ & & & & & \lambda & 1 & \\ & \text{\Large 0} & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{pmatrix}$$

Now it is clear that, similarly to the nilpotent case, each generalized eigenspace is shrunk upon its associated eigenspace, or what is the same, each generalized eigenvector becomes an eigenvector under suitable iterations of the matrix. The latter behavior will help in the last needed step. The only remaining task is to prove the linear independence of the generalized eigenspaces but, as we show next, it reduces to the linear independence of the eigenspaces, which we prove first. Indeed, let $v$ be an eigenvector corresponding to an eigenvalue $\lambda$, which is linear combination of linearly independent eigenvectors $v_i$ associated to eigenvalues $\lambda_i$ all different from $\lambda$, where $i$ runs over the needed index set.

$$v = \sum_i \alpha_i v_i.$$

Applying now the linear transformation on both sides we get,

$$\lambda v = \sum_i \alpha_i \lambda_i v_i.$$

Using the first equality to substitute $v$ in the second we have

$$\lambda \sum_i \alpha_i v_i = \sum_i \alpha_i \lambda_i v_i,$$

or what is the same

$$\sum_i \alpha_i (\lambda_i - \lambda) v_i = 0,$$

a contradiction. Assume now that a non-nilpotent linear application $A$ has eigenvalues $\lambda_i$ with corresponding algebraic multiplicities $k_i$, where again $i$ runs over the needed index set. Assume also that there is a null non-trivial linear combination of generalized eigenvectors $x_i$, corresponding to different eigenvalues. That is,

$$0 = \sum_i \alpha_i x_i.$$

If $k$ is the largest among the algebraic multiplicities, then we apply $A^k$ to the last equality to get

$$0 = \sum_i \alpha_i \lambda_i^{k-k_i+1} A^{k_i-1} x_i.$$

From what has been discussed, we know that $A^{k_i-1}x_i$ must be an eigenvector for each $i$, contradicting their linear independence. Observe here that since we have let $A$ be a non-quasinilpotent transformation, the tautology $0 = 0$ is not possible in the last display.

As a conclusion of the discussion about nilpotent operators and the construction of the Jordan Canonical Form we can say that this representation provides a complete description of the lattice of invariant subspaces of each matrix. If we turn our attention to the infinite dimensional setting, it appears that this is one of the most important problems in Operator Theory. *Does each operator acting on the separable Hilbert space have a non-trivial invariant subspace?* The first natural attempt to answer such question is to look for an eigenvector for each operator, but, unfortunately, we can provide a very easy counterexample. Namely, on the space $\ell^2(\mathbb{N})$ of squared modulus summable sequences we define the so called forward shift $S$ acting on a basis $\{e_n\}$ by $S(e_n) = e_{n+1}$. This operator is clearly an isometry of $\ell^2(\mathbb{N})$ without eigenvectors, what by the way shows that the unit ball of the separable infinite dimensional Hilbert space is not compact. As a consequence of this we also get that the constructive proof provided for the Schur triangular representation does not work anymore. Does the result still hold? The answer is no, not each operator has a triangular matrix representation on the separable infinite dimensional Hilbert space. In order to provide a counterexample we move to the space of functions supported on $[0, 1]$ with square integrable modulus, $L^2[0, 1]$. Now, consider one of the oldest operators, the classical Volterra operator $V$ defined as,

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad \text{for each } f \in L^2[0,1],$$

and recall the equivalent formulation of Schur's result in terms of invariant subspaces. It is clear, that the existence of a triangular matrix representation of the Volterra operator should imply the existence of a chain in $\mathrm{Lat}(V)$ with one-dimensional jumps from one space to the next. The lattice of $V$ has been computed through various methods, for instances see [39] and references therein, and it is,

$$\mathrm{Lat}(V) = \{L^2[a, 1] : \ a \in [0, 1]\}.$$

Since there are no one-dimensional jumps between its spaces, our claim is proved. Even more, since the Volterra operator is compact, or what is the same, it is limit of finite matrices, this counterexample makes somehow fruitless the task of finding the class of operators to which Schur's result extends. The best general result in this direction is due to Halmos, and although it is a bit disappointing, in view of the impossibility to provide triangular representations for general compact operators, it is rather sharp. *Every operator acting on a separable Hilbert space has a matrix finite by columns indexed on* $\mathbb{N}$ [15] . Here the expression *finite by columns* means that each column has finitely many non-zero elements.

Particulary nice representations that will be useful below are those which are as close as possible to be triangular. These are called triangular plus one, meaning that the last non-zero element of each column is its first sub-diagonal element.

Concerning Jordan's Theorem, the lack of eigenvectors for many operators makes impossible to find a satisfactory extension to the infinite dimensional setting. Nonetheless, things work when we keep our operators 'close' to finite dimensional ones, or we impose very restrictive conditions. In fact, there are some good substitutes for particular classes of operators. For instance, functional calculus may be constructed for self-adjoint and normal operators, and compact self-adjoint operators might be represented as diagonal matrices.

Other differences between finite and infinite dimensional operators are due to the existence of cyclic vectors. An operator $A$ is called cyclic if there exists a cyclic vector $x$, such that

$$\mathrm{span}\{A^n x : \ n = 0, 1, 2, \dots\}$$

is dense in the space. If instead of the linear span of the orbit, it suffices to take its projective span, in symbols,

$$\{\lambda A^n x : \ \lambda \in \mathbb{C} \text{ and } n = 0, 1, 2, \dots\}, \tag{0.4.1}$$

then $A$ is said to be supercyclic and $x$ is a supercyclic vector. The strongest form of cyclicity is called hypercyclicity, and it occurs when there is a hypercyclic vector $x$ such that the orbit

$$\{A^n x : \ n = 0, 1, 2, \dots\} \tag{0.4.2}$$

is itself dense in the space. Observe that both in (0.4.1) and (0.4.2), finitely many values of $n$ can be omitted without any loss. Rolewicz [42] provided the first examples of hypercyclic operators on a Hilbert space, precisely on $\ell^2$. Such examples are scalar multiples of the backward shift $\lambda B$, with $|\lambda| > 1$. By the way, this also provides examples of supercyclic operators, but it was not noticed at that time. Such an extremal behavior for a linear transformation of the space is difficult to imagine *a priory*. Indeed, in a private communication, Rolewicz explained that he found his examples while trying to establish the impossibility of hypercyclicity on separable infinite dimensional Hilbert spaces. The last two forms of cyclicity, supercyclicity and hypercyclicity, are characteristic of the infinite dimensional setting, being impossible on finite dimensional spaces. Namely, the existence of at least an eigenvector and the Angle Criterion, introduced by A. Montes-Rodríguez and H. Salas [34], will be enough to accomplish the proof of the assertion. First observe that hypercyclic operators are all supercyclic and that each scalar multiple of a supercyclic operator is itself supercyclic. Therefore, without loose of generality we can assume the existence of a supercyclic operator $A$, with norm $1/2$, and supercyclic normalized vector $x$ on a finite dimensional vector space. Now, let $\lambda$ be an eigenvalue of the adjoint matrix $A^\star$ with corresponding normalized eigenvector $y$. Then, we have

$$\langle A^n x, y\rangle = \langle x, A^{n\star} y\rangle = \langle x, A^{\star n} y\rangle = \overline{\lambda}^n \langle x, y\rangle.$$

Since $|\lambda| \leqslant 1/2$, the above display has a geometrical meaning. The orbit

$$\{A^n x : \ n = 1, 2, \dots\}$$

is bounded away from a cone around $y$, which is a contradiction. Notice that even if $\rangle x, y\langle = 0$, the prove still works.

# Chapter 1

# Preliminaries

This expository chapter presents the spaces on which our results are established, to expose some general properties of compact operators and to state several classical and remarkable results that are used in the following chapters. We do not try to give an exhaustive description of the respective theories, but just to enhance the self-containment of the present work.

## 1.1 Function spaces

The only measure used all along the work is the Lebesgue measure. For instance, measurable functions on the interval $[0,1]$ are always measurable with respect to the Lebesgue measure, but it will not be mentioned in order to make the statements more readable. Indeed, for $1 \leqslant p < \infty$, we denote by $L^p[0,1]$ to the vector space of complex valued measurable functions $f$ supported on $[0,1]$ such that

$$\int_0^1 |f(t)|^p \, dt$$

is finite. For $p = \infty$, the space $L^\infty[0,1]$ is the one of Lebesgue essentially bounded, complex measurable functions on $[0,1]$. These vector spaces are known to be Banach when endowed with the corresponding norms

$$\|f\|_p^p = \int_0^1 |f(t)|^p \, dt \qquad \text{for } 1 \leqslant p < \infty,$$

and

$$\|f\|_\infty = \text{essensup}_{[0,1]}|f| \qquad \text{for } p = \infty.$$

This parametric family of spaces form a decreasing chain with respect to contention and some of their more remarkable properties are the following: The only Hilbert space among them is $L^2[0,1]$ and if as usual, we say that $1 \leqslant p < \infty$ and $q$ are conjugated when $1/p + 1/q = 1$, then the dual space of $L^p[0,1]$, $1 \leqslant p < \infty$, is isometrically isomorphic to $L^q[0,1]$. The 1-$\infty$ pairing is the most singular in the family, since $L^\infty[0,1]$ is not separable and its dual is bigger than $L^1[0,1]$. The non-separability of $L^\infty[0,1]$ makes senseless to look for a cyclic operator acting on it. Hence, in the last chapter, it will be substituted by the space of continuous functions that vanish at zero, $\mathcal{C}_0[0,1]$.

As an historical note, we may emphasize that soon after Lebesgue defined his integral at the beginning of the XX$^{\text{th}}$ century, first F. Riesz [40] and E. Fisher [7] (1907) defined the space $L^2[0,1]$ and later F. Riesz [41] (1910) found the $L^p[0,1]$ spaces.

## 1.2 The spectrum of a compact operator

If $T$ is a bounded operator on a Banach space $\mathcal{B}$, its *spectrum*, denoted by $\sigma(T)$, is the set of complex numbers such that $T - \lambda I$ is not invertible and it is always a non-empty compact set. The *eigenvalues* of $T$ are those complex numbers $\lambda$ for which $\ker(T-\lambda I) = \{f \in \mathcal{B} \,:\, (T-\lambda I)f = 0\}$ is not the null space and they clearly belong to $\sigma(T)$. The dimension of $\ker(T-\lambda I)$ is called the *geometric multiplicity* of $\lambda$ and each non-zero element in $\ker(T-\lambda I)$ is called an *eigenvector* (or an *eigenfunction*) of $T$ corresponding to the eigenvalue $\lambda$.

A compact operator on a reflexive Banach space is an operator that takes the unit ball of the space to a pre-compact set. In Hilbert spaces, compact operators coincide with the norm closure of finite rank operators. The set of compact operators is a closed ideal of the Banach algebra of bounded linear operators on $\mathcal{B}$. It is well known that for a compact operator $T$, such as our $V_\varphi$, acting on an infinite dimensional Banach space, the spectrum consists of either a finite set of eigenvalues joint with $\{0\}$ or a sequence $\{\lambda_n(T)\}_{n\geqslant 0}$ of isolated eigenvalues that converges to zero together with $\{0\}$, since $\sigma(T)$ is closed. In any case, the non-zero eigenvalues are of finite geometric multiplicity. A detailed study of these facts is in [5].

The resolvent $R(T) = (T-\lambda I)^{-1}$ of a compact operator $T$ is an operator-valued analytic function that only has poles at the non-zero eigenvalues, that is, the non-zero eigenvalues are always *normal* [12]. The order of the pole of $R(T)$ at a non-zero eigenvalue is called *algebraic multiplicity.* The algebraic multiplicity is always greater than or equal to the geometric multiplicity. The sequence $\{\lambda_n(T)\}$ of eigenvalues of a compact operator $T$ is arranged in decreasing order of moduli and each non-zero element in the sequence is repeated as many times as warranted by algebraic multiplicity. We also assume that if the spectrum of $T$ is finite, then $\lambda_n(T) = 0$ for $n$ greater than the sum of the algebraic multiplicities of all non-zero eigenvalues of $T$. When dealing with composition Volterra operators we write $\lambda_n(\varphi)$ instead of $\{\lambda_n(V_\varphi)\}$. Recall that the *spectral radius* of a bounded operator is

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}. \tag{1.2.1}$$

The limit always exists and is $\max\{|z| : z \in \sigma(T)\}$.

## 1.3 Traces of operators

The matrix trace of an $n$-square matrix is defined as the sum of its diagonal coefficients. Indeed, this value is an invariant of the matrix, that is, it remains the same for the whole similarity orbit of each given matrix. Moreover, the matrix trace equals the spectral trace, or what is the same, the sum of its eigenvalues. Following the same lane as we did in the introduction, a big effort was put to determine how far remains good for operators this property of matrices. The first candidates are compact operators, but once more, things are not so easy. For instance, the operator defined on the Hilbert sequence space $\ell^2(\mathbb{N})$ by the diagonal matrix with diagonal coefficients $\{1/n\}_{n\geqslant 0}$, is compact, but obviously, its 'trace' is infinite. In relation with traces of operators emerges the crucial concepts of nuclear operator and Hilbert-Schmidt operator. An operator $A$ on a Hilbert space is said to be Hilbert-Schmidt if the sum $\sum_{n=0}^{\infty} \|Ae_n\|^2$ converges for each orthonormal basis $\{e_n\}$, and it is said to be nuclear if the sum $\mathrm{tr}(A) = \sum_{n=0}^{\infty}\langle Ae_n, e_n\rangle$ converges for each orthonormal basis $\{e_n\}$. We denote the sets of nuclear and Hilbert-Schmidt operators by $\mathcal{N}$ and $\mathcal{H}$-$\mathcal{S}$ respectively. In any case, the latter sums can be shown to be independent of the basis, and the value $\mathrm{tr}(A)$ is usually called matrix trace of the operator $A$. Indeed, let $\{u_n\}_{n\geqslant 0}$ and $\{e_n\}_{n\geqslant 0}$ be orthonormal bases of a Hilbert space. Then,

$$\begin{aligned}\sum_{n=0}^{\infty}\langle Ae_n, e_n\rangle &= \sum_{n=0}^{\infty}\left\langle A\left(\sum_{i=0}^{\infty}\langle e_n, u_i\rangle u_i\right), \sum_{j=0}^{\infty}\langle e_n, u_j\rangle u_j\right\rangle \\ &= \sum_{n=0}^{\infty}\left\langle \sum_{i=0}^{\infty}\langle e_n, u_i\rangle Au_i, \sum_{j=0}^{\infty}\langle e_n, u_j\rangle u_j\right\rangle \\ &= \sum_{i,j=0}^{\infty}\langle Au_i, u_j\rangle \sum_{n=0}^{\infty}\langle e_n, u_i\rangle\langle u_j, e_n\rangle \\ &= \sum_{i,j=0}^{\infty}\langle Au_i, u_j\rangle \langle u_j, u_i\rangle \\ &= \sum_{i=0}^{\infty}\langle Au_i, u_i\rangle\end{aligned}$$

A nice introduction to these topics may be found in [6, §18]. Among the main properties of Hilbert-Schmidt operators outstands that they constitute a sub-ideal of the ideal of compact operators. Moreover, an operator is nuclear if and only of it is the product of two Hilbert-Schmidt operators. Both nuclear and Hilbert-Schmidt operators are complete Banach operator ideals when endowed with the appropriate norms. In the case of nuclear operators, the matrix trace is also absolutely summable and it is related to the spectral trace of the operator. The spectral trace of an operator is the sum of its eigenvalues repeated according to their algebraic multiplicity. Among the first calculated matrix traces we find those of integral operators with kernel. For each function $K$ in $L^2([0,1]^2)$, the integral operator with kernel $K$ is

$$(J_K f)(x) = \int_0^1 K(x,t)f(t)\,dt \qquad \text{for each} \quad f \in L^2[0,1].$$

Applying Fubini's Theorem, it follows from the last definition that the iterate $J_K^n$ of an integral operator with kernel $K$, is again an integral operator $J_{K^n}$ with kernel

$$K^n(x,t) = \int_{[0,1]^{n-1}} K(x,t_1)K(t_1,t_2)\cdots K(t_{n-2},t_{n-1})K(t_{n-1},t)\,dt_1,\cdots dt_{n-1} \quad (1.3.1)$$

for each $n \geqslant 2$. Integral operators with kernel are known to be Hilbert-Schmidt operators, see for instance [16, pp. 18–19], what implies that $J_K^n$ are nuclear operators for each kernel $K$ in $L^2([0,1]^2)$ and each $n \geqslant 2$.

It was in 1909 that J. Mercer stated his classical Theorem, see for instance [57], from which automatically outcomes the following.

**Mercer's Theorem 1.3.1.** *Let $J_K$ be an integral operator with Hermitian positive semi-definite kernel $K$ in $L^2([0,1]^2)$. Then,*

$$\operatorname{tr}(J_K) = \int_0^1 K(x,x)\,dx.$$

The first results relating traces and spectra of nuclear operators were stated with the extra condition of positivity. The full theorem is due to Lidskii.

**Lidskii's Theorem 1.3.2.** *If the operator $A$ is nuclear, then its matrix trace coincides with its spectral trace:*

$$\sum_{j=1}^{\infty}\langle Ae_j, e_j\rangle = \sum_i \lambda_i(A),$$

*where $\{e_n\}_{n\geqslant 1}$ is an arbitrary orthonormal basis in $H$ and $\lambda_i(A)$ are the eigenvalues of the operator $A$.*

Lidskii's Theorem is one of the tops of a lot of works involving variational estimates of the spectrum of a compact operator and that is spotlighted by results of Hilbert, H. Weyl, Horn, Grothendieck and some others, see [45].

In light of Lidskii's Theorem, when dealing with nuclear operators we will not distinguish matrix and spectral traces, that will be denoted the same, tr.

Now, since we could not find an appropriate reference for it, we prove an explicit formula for the trace of a product of integral operators with kernel, that belongs to the folklore of the theory. As we have said, such products are nuclear operators, what will be essential in the proofs. Before we get to the general version of the result, we need some lemmas.

**Lemma 1.3.3.** *Let $A$ be a nuclear operator acting on the infinite dimensional separable Hilbert space $H$ and let $P_n$ be an increasing sequence of orthogonal projections such that $\bigcup P_n(H)$ is dense in $H$. Then*

$$\operatorname{tr}(A) = \lim_{n\to\infty} \operatorname{tr}(P_n A P_n).$$

*Proof.* It is standard that putting together the orthonormal bases obtained from the spaces $P_n(H) \ominus P_{n-1}(H)$, each of them denoted by $\{e_1^n, \dots, e_{m_n}^n\}$, we end up with a basis of the whole space $H$. Now, since $A$ is nuclear, the following limit exist and is finite.

$$\lim_{n\to\infty} \operatorname{tr}(P_n A P_n) = \lim_{n\to\infty} \sum_{l=1}^{n}\sum_{k=1}^{m_n}\langle Ae_k^l, e_k^l\rangle = \sum_{l=1}^{\infty}\sum_{k=1}^{m_n}\langle Ae_k^l, e_k^l\rangle = \operatorname{tr}(A).$$

□

Now we prove a formula for the trace of a nuclear integral operator with continuous kernel.

**Lemma 1.3.4.** *Let $K$ be in $\mathcal{C}([0,1]^2)$ such that the associated integral operator $J_K$ is nuclear. Then*

$$tr(J_K) = \int_0^1 K(x,x)\,dx.$$

*Proof.* In order to apply Lemma 1.3.3, for each natural entire $n \geqslant 0$ we take a regular partition of the unit interval of diameter $2^{-n}$. Each of the segments generated by the $n^{\text{th}}$ partition will be denoted by $I_k^n$, $1 \leqslant k \leqslant 2^n$, and $e_k^n$ will stand for the normalized characteristic function $2^{n/2}\chi_k^n$ of the segment $I_k^n$, where again $k$ runs from 1 to $2^n$. Now, for each $n$, consider the space $\text{span}\{e_1^n, \dots, e_{2^n}^n\}$ and its associated orthogonal projection $P_n$. As it is well known, $\text{span}\{e_k^n : n \geqslant 0 \text{ and } k = 1, \dots, 2^n\}$ is dense in $L^2[0,1]$. Therefore we can apply Lemma 1.3.3.

$$\begin{aligned}
\text{tr}(J_K) &= \lim_{n\to\infty} \text{tr}(P_n J_K P_n) \\
&= \lim_{n\to\infty} \sum_{k=1}^{2^n} \langle J_K e_k^n, e_k^n \rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^{2^n} \int_0^1 \int_0^1 K(x,t) e_k^n(t)\,dt\, e_k^n(x)\,dx \\
&= \lim_{n\to\infty} 2^n \sum_{k=1}^{2^n} \int_{I_k^n \times I_k^n} K(x,t)\,dt\,dx.
\end{aligned}$$

Since $K$ in continuous on the compact set $[0,1]^2$, it is uniformly continuous. Therefore, for each natural $n$ and each $1 \leqslant k \leqslant 2^n$, we find at least a point $(x_k^n, t_k^n)$ in $I_k^n \times I_k^n$ such that

$$\begin{aligned}
\lim_{n\to\infty} 2^n \sum_{k=1}^{2^n} \int_{I_k^n \times I_k^n} K(x,t)\,dt\,dx &= \lim_{n\to\infty} 2^n \sum_{k=1}^{2^n} 2^{-2n} K(x_k^n, t_k^n) \\
&= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{2^n} K(x_k^n, t_k^n).
\end{aligned}$$

To finish, we know that by uniform continuity of the kernel $K$, it is possible to choose points $r_k^n$ in each $I_k^n$ such that for a given $\varepsilon > 0$ and each $n$ big enough, we have that

$|K(x_k^n, t_k^n) - K(r_k^n, r_k^n)| < \varepsilon$. Putting all together we can compute de limit above.

$$\begin{aligned}\operatorname{tr}(J_K) &= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{2^n} K(x_k^n, t_k^n) \\ &= \lim_{n\to\infty} \left( 2^{-n} \sum_{k=1}^{2^n} K(r_k^n, r_k^n) + 2^{-n} \sum_{k=1}^{2^n} K(x_k^n, t_k^n) - K(r_k^n, r_k^n) \right) \\ &= \int_0^1 K(x,x)\,dx,\end{aligned}$$

where in the last equality we have used the definition of the Riemann integral. □

A straightforward consequence of the last Lemma is,

**Corollary 1.3.5.** *Given $K_1, \ldots, K_n$ in $\mathcal{C}([0,1]^2)$, $n \geqslant 2$, let $J_K$ be the integral operator with kernel $K$, defined by $J_K \equiv J_{K_1} \cdots J_{K_n}$. Then*

$$\operatorname{tr}(J_K) = \int_{[0,1]^n} K_1(x_1, x_2) K_2(x_2, x_3) \cdots K_{n-1}(x_{n-1}, x_n) K_n(x_n, x_1)\, dx_1 \cdots dx_n. \tag{1.3.2}$$

*Proof.* Using formula (1.3.1), we get that

$$K(x,t) = \int_{[0,1]^{n-1}} K_1(x, x_2) K_2(x_2, x_3) \cdots K_{n-1}(x_{n-1}, x_n) K_n(x_n, t)\, dx_2 \cdots dx_n,$$

is a continuous function in $\mathcal{C}[0,1]^2$. Moreover, since an operator is nuclear if and only if it is the product of two Hilbert-Schmidt operators, we know that $J_K$ is nuclear. It just rests to use Lemma 1.3.4. □

We need just one more lemma to prove the general formula for traces of integral operators with kernel.

**Lemma 1.3.6.** *For each $n \geqslant 2$, the functional that maps the $n$-tuple $(K_1, \ldots, K_n)$ from $\left(L^2[0,1]^2\right)^n$ to the trace of $J_{K_1} \cdots J_{K_n}$, is a bounded $n$-linear functional.*

*Proof.* This proof consist in recalling some known facts. Observe that since $n \geqslant 2$, the product operator $J_{K_1} \cdots J_{K_n}$ is always a nuclear operator. Now, the following three mappings are bounded when the respective spaces are endowed with the norms that makes them complete. Namely, the trace norm in $\mathcal{N}$ and the Hilbert-Schmidt norm in $\mathcal{H}$-$\mathcal{S}$. This can be found in [6] and partially in [16]. First, the mapping that takes a function $K$ in $L^2[0,1]^2$ to the associated Hilbert-Schmidt operator $J_K$ in $\mathcal{H}$-$\mathcal{S}$ is a linear isometry. Second, the mapping that takes an $n$-tuple $(T_1, \ldots, T_n)$ in $(\mathcal{H}\text{-}\mathcal{S})^n$ to its product $T_1 \cdots T_n$ in $\mathcal{N}$ is an $n$-linear contraction. Finally, the mapping denoted by tr, that takes each operator in $\mathcal{N}$ to its trace in $\mathbb{C}$, is a continuous linear functional. □

We are ready to state and prove the formula for the trace of a product of integral operators with kernels in $L^2([0,1]^2)$.

**Theorem 1.3.7.** *Given $K_1,\ldots,K_n$ in $L^2([0,1]^2)$, $n \geqslant 2$, let $J_K$ be the integral operator with kernel $K$, defined by $J_K \equiv J_{K_1}\cdots J_{K_n}$. Then*

$$\operatorname{tr}(J_K) = \int_{[0,1]^n} K_1(x_1,x_2)K_2(x_2,x_3)\cdots K_{n-1}(x_{n-1},x_n)K_n(x_n,x_1)\,dx_1\cdots dx_n. \tag{1.3.3}$$

*Proof.* First, parallel arguments to those of formula (1.3.1) suffices to see that $J_K$ is well defined as an integral operator with kernel

$$K(x,t) = \int_{[0,1]^{n-1}} K_1(x,x_2)K_2(x_2,x_3)\cdots K_{n-1}(x_{n-1},x_n)K_n(x_n,t)\,dx_2\cdots dx_n.$$

Moreover, the kernel $K$ evaluated on the diagonal elements of $[0,1]^2$ is absolutely integrable, that is, $K(x,x)$ is in $L^1[0,1]$. To see this, it suffices to show that given $M$ and $N$ in $L^2[0,1]$, we have that

$$\int_0^1 M(x,s)N(s,t)\,ds$$

belongs to $L^2[0,1]^2$. Indeed,

$$\begin{aligned}
\left\| \int_0^1 M(x,s)N(s,t)\,ds \right\|_2^2 &= \int_0^1\int_0^1 \left| \int_0^1 M(x,s)N(s,t)\,ds \right|^2 dx\,dt \\
&\leqslant \int_0^1\int_0^1\int_0^1 |M(x,s)|^2\,ds \int_0^1 |N(s,t)|^2\,ds\,dx\,dt \\
&= \int_{[0,1]^2} |M(x,s)|^2\,ds\,dx \int_{[0,1]^2} |N(s,t)|^2\,ds\,dt \\
&= \|M\|_2^2\|N\|_2^2.
\end{aligned}$$

Therefore, a finite induction is enough to prove that $K(x,t)$ is in $L^2[0,1]^2$, and Hölder's inequality lays that $K(x,x)$ is in $L^1[0,1]$. Thus we have that (1.3.3) defines a bounded linear functional on $\left(L^2[0,1]^2\right)^n$, that may be written

$$\operatorname{tr}(J_K) = \int_0^1 K(x,x)\,dx.$$

By Corollary 1.3.5, the latter functional coincides with the trace functional on the space $\left(\mathcal{C}[0,1]^2\right)^n$, which is dense in $\left(L^2[0,1]^2\right)^n$. Therefore, by Lemma 1.3.6 we have that formula (1.3.3) is the only possible extension of the trace functional to $\left(L^2[0,1]^2\right)^n$. $\square$

## 1.4 Zeros of analytic functions

In order to obtain several properties of the spectrum of composition Volterra operators, we establish a relation between the zeros of an entire function and the named spectrum. Once this is done, the more we can say about the growth, distribution or geometry of the zeros of the entire function, the more we will know about the spectrum of our operators. For this reason, we use several classical results that deal with the relation between the growth of an entire function and the distribution of its zeros, all of them cited in the first chapter of the book by B. Ja. Levin [29].

It is known that given an entire function $f(z)$, the growth rate of

$$M(f,r) = \max_{|z|=r} |f(z)|$$

exceeds the growth of all polynomials. Therefore, functions of the kind

$$e^{r^k} \qquad \text{with } k > 0$$

are used in order to have an scale of growth. Then, if an entire function satisfies asymptotically the inequality

$$M(f,r) < e^{r^k}$$

for a positive constant $k$, then we say that $f$ is of finite order $\rho(f)$. The most extended definition of the order of an entire function $f$ is

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\ln\ln M(f,r)}{\ln r}.$$

Among the entire functions with the same order $\rho$, we discriminate their growth rate by using the finer quantity,

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\ln M(f,r)}{r^\rho}.$$

The value $\tau(f)$ is called type of the entire function $f$. If $\tau = 0$ the function is said to be of minimal type, if $0 < \tau < \infty$ of normal type, and $\tau = \infty$ of maximal type. If the order $\rho$ of an entire function is one, then its type is called exponential type.

One of the main aims of the study of entire functions is to understand the relation between the order of growth of a given entire function and the distribution of its zeros. The first milestone of such study is the Weierstrass representation of entire functions as infinite products. Indeed, given $n$ arbitrary points of the complex plane $\mathbb{C}$, it is easy to produce a monic polynomial of degree $n$, as a product of $n$ monomials, with zeros at the prescribed points. Following this idea, for each moduli increasing sequence of points $\{a_n\}$ in $\mathbb{C}$, all different from zero and accumulating just at infinity, we can construct the infinite product

$$\Pi(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}\,;\, p_n\right),$$

in which

$$G(u\,;\,p) = (1-u)e^{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}}, \qquad G(u\,;\,0) = 1-u$$

and $\{p_n\}$ is an arbitrary sequence of distinct natural numbers. For instance $p_n = n$, $n \geqslant 1$.

Since these products depend on the choice of $\{p_n\}$ they are not unique, but at least they can be shown to define entire functions that just vanish at $\{a_n\}$. Nonetheless, the infinite product representation above may be improved just by imposing to the sequence of zeros $\{a_n\}$ that the sum

$$\sum \frac{1}{|a_n|^s} \tag{1.4.1}$$

converges with the help of the positive exponent $s$. In such a case, if we let $p$ be the smallest integer such that $p+1$ can replace $s$ in the display above, then the so called canonical product

$$\Pi(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}\,;\, p\right),$$

is well defined as an entire function, and $p$ is call the genus of the canonical product. Notice that each sequence of zeros $\{a_n\}$ satisfying the summability condition (1.4.1) produces a unique genus and therefore a unique canonical product.

It has appeared at last a clear relation between the growth of a entire function and the 'density' of its zeros. The product representation shows the importance of measuring the 'density' of a sequence of points $\{a_n\}$ on $\mathbb{C}$, all different from zero, with no finite limit points. One of the measures of such density is the convergence exponent of the sequence $\{a_n\}$, in symbols $s(\{a_n\})$, and is defined as the infimum of $c > 0$ for which

$$\sum_{n=0}^{\infty} \frac{1}{|a_n|^c}$$

is finite. Observe that $s(\{a_n\})$ might be both zero or infinity. We must precise here that when measuring the convergence exponent of a sequence of points converging to zero, with no risk of confusion, we use the same definition but replacing the sum of the inverses by the sum of the elements in the sequence.

The following theorem is due to Borel [29, p. 30] and it provides control on the order of a canonical product, means the convergence exponent of its sequence of zeros.

**Borel's Theorem 1.4.1.** *The order $\rho$ of a canonical product*

$$\Pi(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}\,;\,p\right)$$

*does not exceeds the convergence exponent $s$ of the sequence $\{a_n\}$.*

It is a standard result that both the type and the order of the product of two entire functions are those of the factor with larger ones. The latter fact along with Borel's Theorem lays one of the most classical results in the theory of entire functions,

**Hadamard's Theorem 1.4.2.** *The entire function $f(z)$ of finite order $\rho$ can be represented in the form*

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\omega} G\left(\frac{z}{a_n}\,;\,p\right) \qquad \omega \leqslant \infty,$$

*where $a_n$ are the nonzero roots of $f(z)$, $p \leqslant \rho$, $P(z)$ is a polynomial whose degree $q$ does not exceed $\rho$, and $m$ is the multiplicity of the zero at the origin.*

Now we look for results providing control on the amount of zeros that a holomorphic function can have in a circle or radius $r$.

**Jensen's Theorem 1.4.3.** *Let $f(z)$ be holomorphic in a circle of radius $r$ with center at the origin, and $f(0) \neq 0$. Them*

$$\int_0^r \frac{n_f(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|\,d\theta - \ln|f(0)|$$

*where $n_f(t)$ is the number of zeros of $f(z)$ in the circle $|z| < t$.*

Jensen's Theorem provides control on the growth of the zero-counting function of a holomorphic function. The following Lemma is, in words of Levin, *an important estimate for the number of zeros of $f(z)$ in a circle.* [29, p. 15].

**Jensen's Lemma 1.4.4.** *If $f(z)$ is holomorphic in the circle*

$$|z| \leqslant er$$

*and $|f(0)| = 1$, then*

$$n_f(r) \leqslant \ln M_f(er).$$

As a straightforward consequence of Jensen's Lemma, we have

**Theorem 1.4.5.** *The convergence exponent of the zeros of an arbitrary entire function does not exceed its order.*

The only tricky step of the proof is to realize that one may consider functions with $f(0) = 1$, just replacing $f$ by

$$\tilde{f}(z) = n! z^{-n} \frac{f(z)}{f^{(n)}(0)},$$

where $n$ is the order of the possible zero at zero. Notice that $\tilde{f}$ keeps the same order and convergence exponent for its zeros.

**Remark.** A direct consequence of Borel's Theorem and Theorem 1.4.5 is that *for canonical products the convergence exponent of the zeros is equal to the order of the function.*

The maximum principle states that a function that is holomorphic in a domain and continuous on its closure, attains the maximum value of its modulus at a boundary point. This extremely useful result was extended by E. Phragmén and E. Lindelöf to holomorphic functions with controlled growth at some discontinuity points at the boundary of the domain, and even to domains with infinity at its boundary. Next theorem is a consequence of the Phragmén-Lindelöf Theorem, [29, Theorem 22, p. 50].

**Theorem 1.4.6.** *Let the function $f(z)$ be holomorphic inside an angle of opening $\pi/\alpha$ and continuous on the boundary. Assume that on the sides of the angle*

$$|f(z)| \leqslant M$$

*and that the order $\rho$ of the function $f(z)$ is less than $\alpha$. Then*

$$|f(z)| \leqslant M$$

*throughout the angle.*

Now, we are ready to begin the study of composition Volterra operators.

# Chapter 2

# Basic Theory of composition Volterra operators

For each Lebesgue measurable self-map $\varphi$ of the unit interval $[0,1]$, the *composition Volterra operator* on $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, is defined as

$$(V_\varphi f)(x) = \int_0^{\varphi(x)} f(t)\,dt.$$

If $\varphi$ is the identity map, the operator $V_\varphi$ becomes the classical Volterra operator, which is simply denoted by $V$. There are several ways to see that $V_\varphi$ acts compactly on $L^p[0,1]$ for $1 \leqslant p \leqslant \infty$. For instance, let $C_\varphi$ denote the operator that to each function $f$ assigns the function $f \circ \varphi$. Clearly, $V_\varphi = C_\varphi V$ and, although in general $C_\varphi$ may be unbounded on $L^p[0,1]$, $1 \leqslant p < \infty$, it is always bounded from $L^\infty[0,1]$ into itself. Since $V$ from $L^p[0,1]$ into $L^\infty[0,1]$ is compact, see [5, p. 44], it follows that $V_\varphi$ acting on $L^p[0,1]$ is compact.

The literature on composition Volterra operators is very scarce. Indeed, it reduces to a few references, see the works by Whitley [56] and Tong [54], and the note by Lyubic [30]. One of the reasons seems to be the lack of a satisfactory formula for the iterates of these operators. As will be seen along this work, this problem may be overcome in a suitable way.

In Section 2.1, we provide the most basic facts about composition Volterra operators. In particular, we present a shorter proof of the fact that $V_\varphi$ is quasi-nilpotent if and only if $\varphi(x) \leqslant x$ a.e., that was independently proved in [56] and [54]. Thus the most natural set of symbols, whose spectrum is other than the zero point, are those for which $\varphi(x) > x$. The quasi-nilpotency criterion will be derived from a characterization of the quasi-nilpotency of Volterra kernel operators in terms of their kernels, which is of an independent interest. The proof is based on Lidskii's Theorem about traces and the relation between the traces of the powers and the spectrum of a nuclear operator. We also present some elementary results about the formula for the adjoint, norms and the distance to quasi-nilpotent operators. In Subsection 2.1.5, to motivate our study of the spectra of composition Volterra operators, we provide some examples of symbols $\varphi$ in which we can exhibit explicitly the eigenvalues and eigenfunctions of $V_\varphi$.

In Section 2.2, by means of Kreĭn-Rutman's Theorem we show that if $\varphi(x) > x$ on a set of positive Lebesgue measure, then the spectral radius is a positive eigenvalue and corresponding to it there is a non-negative eigenfunction. Sharp upper and lower bounds are provided for the spectral radius. As a straightforward consequence, we get a much simpler characterization of quasinilpotent composition Volterra operators when the symbol $\varphi$ is increasing. Using elements of the theory of totally positive matrices, we show that if $\varphi$ is increasing then all eigenvalues of $V_\varphi$ are real and non-negative. For general symbols it is always symmetric with respect to the real line.

## 2.1 Characterization of Quasi-nilpotency and basic formulas

In this section, we prove the most basic facts about the spectrum of composition Volterra operators. We also present the formula for the adjoint, estimates of the norm and of the distance of $V_\varphi$ to quasi-nilpotent operators.

### 2.1.1 Quasi-nilpotent integral operators with positive kernel

An operator is quasi-nilpotent if $\sigma(T) = \{0\}$, or equivalently, the spectral radius $r(T) = 0$. Independently, Whitley [56] and Tong [54] proved that a composition Volterra operator is quasi-nilpotent if and only if the set $\{x \in [0,1] : \varphi(x) > x\}$ has zero Lebesgue measure. This theorem can be derived from our next result that characterizes quasi-nilpotent integral operators with kernel. For each function $K$ in $L^2([0,1]^2)$, the integral operator with kernel $K$ is

$$(J_K f)(x) = \int_0^1 K(x,t) f(t)\, dt \qquad \text{for each} \quad f \in L^2[0,1].$$

The operator $J_K$ is always a Hilbert–Schmidt operator, see [16, pp. 18–19], that is, $\sum_{n=0}^{\infty} \|J_K e_n\|^2$ is finite for each orthonormal basis $\{e_n\}$ of $L^2[0,1]$. Composition Volterra operators are integral operators with kernel. Indeed, if $\varphi$ is a measurable self-map of $[0,1]$, then $V_\varphi = J_{K_\varphi}$ where

$$K_\varphi(x,t) = \begin{cases} 1, & \text{if } t \leqslant \varphi(x); \\ 0, & \text{if } t > \varphi(x). \end{cases}$$

Thus, in particular, $V_\varphi$ is always Hilbert-Schmidt. Observe that $K_\varphi$ is non-negative. Recall that a bounded operator on $L^2[0,1]$ is said to be *nuclear* if $\sum_{n=0}^{\infty}\langle Te_n, e_n\rangle$ converges for each orthonormal basis $\{e_n\}$. For nuclear operators the latter sum does not depend on the orthonormal basis $\{e_n\}$ and is called the *trace* of $T$ which is denoted by $\operatorname{tr} T$, see 1.3 or [45]. Tong [54] also provided a characterization of quasi-nilpotent integral operators in terms of the existence of certain measurable sets. The following theorem provides a simpler characterization only in terms of the kernel.

**Theorem 2.1.1.** *Let $K \in L^2([0,1]^2)$ be non-negative. Then the integral operator with kernel $K$ is quasi-nilpotent if and only if*

$$K(t_1,t_2)K(t_2,t_3)\cdots K(t_{n-1},t_n)K(t_n,t_1) \tag{2.1.1}$$

*vanishes a.e. on $[0,1]^n$ for each $n \geqslant 2$.*

*Proof.* Since the corresponding integral operator $J_K$ is Hilbert-Schmidt, we find that $J_K^n$ is nuclear for each $n \geqslant 2$. As a consequence of Theorem 1.3.7 we have,

$$\operatorname{tr} J_K^n = \int_{[0,1]^n} K(t_1,t_2)K(t_2,t_3)\cdots K(t_{n-1},t_n)K(t_n,t_1)\,dt_1,\cdots dt_n \text{ for each } n \geqslant 2.$$

Since $K$ is non-negative, it follows that $\operatorname{tr} J_K^n = 0$ if and only if the integrand in the above display vanishes a.e. on $[0,1]^n$. By Lidskii's Theorem, see 1.3.2 or [45, p. 331], for instance,

$$\operatorname{tr} J_K^n = \sum_{m=0}^{\infty} (\lambda_m(J_K))^n, \qquad \text{for each} \quad n \geqslant 2.$$

If there is $n \geqslant 2$ such that the function in (2.1.1) does not vanish a.e. on $[0,1]$, then $\operatorname{tr} J_K^n \neq 0$ and, therefore, the above display implies that $J_K$ must have non-zero eigenvalues, or what is the same, $J_K$ is not quasi-nilpotent.

Conversely, if for each $n \geqslant 2$ the function in (2.1.1) vanish a.e., then $\operatorname{tr} J_K^n = 0$ for each $n \geqslant 2$. It is well known that the eigenvalues of a nuclear operator on a Hilbert space are uniquely determined by the traces of the powers of the operator. Thus $\sigma(J_K^2) = \{0\}$ and, therefore, the spectrum $\sigma(J_K) = \{0\}$. □

In what follows, the Lebesgue measure is denoted by $\mu$.

**Corollary 2.1.2.** *Let $\varphi$ be a measurable self-map of $[0,1]$. Then $V_\varphi$ is quasi-nilpotent if and only if $\varphi(x) \leqslant x$ a.e.*

*Proof.* First, assume that $\varphi(x) \leqslant x$ a.e. on [0,1]. Without loss of generality we may assume that $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$. In such a case, the function in (2.1.1) for $K = K_\varphi$ vanishes at each point in $[0,1]^n$ with pairwise different components and therefore vanishes a.e. By Theorem 2.1.1, $V_\varphi$ is quasi-nilpotent.

Conversely, assume that $\varphi(x) > x$ holds on a set of positive measure. Take $\varepsilon > 0$ such that $A = \{x \in (0,1) : \varphi(x) > x + \varepsilon\}$ has positive measure and let $x_0$ be a density point of $A$. It is easy to see that $(x_0, x_0)$ is a density point of $\{(t_1,t_2) \in [0,1]^2 : K_\varphi(t_1,t_2)K_\varphi(t_2,t_1) = 1\}$. Hence, upon applying Theorem 2.1.1, for $K = K_\varphi$ and $n = 2$, we see that $V_\varphi$ is not quasi-nilpotent. □

**Remark.** Prof. P. Ahern provided a proof of Corollary 2.1.2 which is different of those in Whitley [56], Tong [54] and the one above. The proof for the sufficiency provided by Prof. P. Ahern, is essentially the same as in Whitley [56], but his proof of the necessity consists on an elegant reduction to Gronwall's inequality. Namely, assume the conditions of Corollary 2.1.2 and let $\lambda$ be a non-null complex number and $f$ be a non-trivial function in $L^2[0,1]$ satisfying the equation

$$(V_\varphi f)(x) = \lambda f(x).$$

Then we have,

$$|f(x)| = \frac{1}{|\lambda|}\left|\int_0^{\varphi(x)} f(t)\,dt\right| \leqslant \frac{1}{|\lambda|}\int_0^x |f(t)|\,dt. \tag{2.1.2}$$

Now let the function $g$ be defined as

$$g(x) = \int_0^x |f(t)|\,dt$$

and observe that from (2.1.2), we deduce

$$g'(x) - \frac{1}{|\lambda|}g(x) \leqslant 0.$$

Let us now define the function $h(x) = e^{-x/|\lambda|}g(x)$, which satisfies: $h(0) = g(0) = 0$, $h(x) \geqslant 0$ and $h'(x) \leqslant 0$. Therefore $h \equiv 0$, hence $g \equiv 0$ and $f \equiv 0$, what is a contradiction.

Corollary 2.1.2 along with the decomposition theorem 2.2.8 is the reason why the symbols satisfying $\varphi(x) \geqslant x$ are the most interesting ones in connection with the spectrum.

### 2.1.2 The adjoint of $V_\varphi$ and the kernels of $V_\varphi$ and $V_\varphi^\star$

If $\varphi$ is increasing, the adjoint of $V_\varphi$ is unitarily similar to a composition Volterra operator. Indeed, for an increasing self-map $\varphi$ of $[0,1]$, we may define

$$\varphi_{-1}(x) = \begin{cases} \sup\{y : \varphi(y) < x\}, & \text{if } x > \varphi(0); \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\varphi_{-1}$ is also increasing and for $f$ in $L^p[0,1]$, $1 \leqslant p < \infty$, and $g$ in $L^q[0,1]$, where $1/p + 1/q = 1$, we have

$$\int_0^1 (V_\varphi f)(x) g(x)\, dx = \int_0^1 \int_0^{\varphi(x)} f(t) g(x)\, dt\, dx = \int_0^1 \int_{\varphi_{-1}(t)}^1 f(t) g(x)\, dx\, dt.$$

Hence, the adjoint of $V_\varphi$ is

$$(V_\varphi^\star f)(x) = \int_{\varphi_{-1}(x)}^1 f(t)\, dt. \tag{2.1.3}$$

Now, consider the involutive isometry defined by $(Uf)(x) = f(1-x)$. Then $UV_\varphi^\star U = V_{\widetilde{\varphi}}$, where $\widetilde{\varphi}(x) = 1 - \varphi_{-1}(1-x)$. Thus

$$\sigma(V_{\widetilde{\varphi}}) = \sigma(V_\varphi).$$

Note also that since the set of real valued functions is invariant with respect to any composition Volterra operator, formula (2.1.3) for the Banach space adjoint of $V_\varphi$ gives for $p = 2$ the Hilbert space adjoint as well.

**Remark 1.** Although for continuous increasing $\varphi$ the function $\widetilde{\varphi}$ may fail to be continuous, for a continuous strictly increasing $\varphi$ with $\varphi(0) = 0$ and $\varphi(1) = 1$, the map $\varphi_{-1}$ is the inverse of $\varphi$ and $\widetilde{\varphi}$ is again continuous and strictly increasing with $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(1) = 1$.

**Remark 2.** For decreasing $\varphi$ it is possible to define $\varphi_{-1}(x) = \sup\{y : \varphi(y) > x\}$ if $x < \varphi(0)$ and 0 otherwise. Now, the adjoint is a composition Volterra operator

$$(V_\varphi^\star f)(x) = (V_{\varphi_{-1}} f)(x) = \int_0^{\varphi_{-1}(x)} f(t)\, dt.$$

Next proposition characterizes when $\ker V_\varphi$ is trivial. Recall that the *essential range* of a measurable self-map $\varphi$ of $[0,1]$ is

$$\operatorname{ess}(\varphi) = \big\{ y \in \mathbb{R} \text{ such that } \mu\{t : |y - \varphi(t)| < \varepsilon\} > 0 \text{ for each } \varepsilon > 0 \big\}.$$

**Proposition 2.1.3.** *Let $\varphi$ be a measurable self-map of $[0,1]$. Then $\ker V_\varphi$ is the null space if and only if the essential range of $\varphi$ is $[0,1]$. Furthermore, $\ker V_\varphi$ is infinite dimensional if and only if $\ker V_\varphi \neq \{0\}$.*

*Proof.* It is clear that $f$ belongs to $\ker V_\varphi$ if and only if

$$F(x) = \int_0^x f(t)dt$$

vanishes on $\mathrm{ess}\,(\varphi)$. If $\mathrm{ess}\,(\varphi) = [0,1]$, it follows that $F$ vanishes on the whole interval $[0,1]$ and, therefore, so does $f$. Conversely, assume that $\mathrm{ess}\,(\varphi) \neq [0,1]$. Since $\mathrm{ess}\,(\varphi)$ is closed, the complement $[0,1]\backslash \mathrm{ess}\,(\varphi)$ contains an interval $(a,b)$. It is straightforward to see that if $f$ belongs to $L^p[0,1]$, then $\mathrm{supp}\,(f)$ is contained in $[a,b]$ and

$$\int_a^b f(t)\,dt = 0,$$

then $V_\varphi f$ is the null function. It follows that $\ker V_\varphi$ is infinite dimensional as soon as $\mathrm{ess}\,(\varphi)$ is not the whole interval $[0,1]$. The result is proved. □

**Corollary 2.1.4.** *Let $\varphi$ be a continuous self-map of $[0,1]$. Then $\ker V_\varphi$ is the null space if and only if $\varphi$ is onto.*

### 2.1.3 Norms

The norm of composition Volterra operators can be easily estimated. In many situations the norm in $L^2[0,1]$ can be computed exactly. First, we prove

**Proposition 2.1.5.** *Let $\varphi$ and $\psi$ be measurable self-maps of $[0,1]$. Then*

(i) $\|V_\varphi - V_\psi\|_p \leqslant \big\||\varphi-\psi|^{p-1}\big\|_1^{1/p} \leqslant 1$ *for* $1 \leqslant p < \infty$.

(ii) $\|V_\varphi - V_\psi\|_\infty = \|\varphi - \psi\|_\infty$ *for* $p = \infty$.

*Proof.* For $p=1$ the formula is trivial. For $f$ in $L^p[0,1]$, $1 < p < \infty$, we set $1/q = 1 - 1/p$ and apply Hölder's inequality:

$$\|\,(V_\varphi - V_\psi)\, f\|_p^p = \int_0^1 \left|\int_{\psi(x)}^{\varphi(x)} f(t)\,dt\right|^p dx \leqslant \int_0^1 \left|\int_{\psi(x)}^{\varphi(x)} |f(t)|^p\,dt\right| \left|\int_{\psi(x)}^{\varphi(x)} 1\,dt\right|^{p/q} dx.$$

Hence,

$$\|\,(V_\varphi - V_\psi)\, f\|_p^p \leqslant \|f\|_p^p \int_0^1 |\varphi(x)-\psi(x)|^{p-1} dx = \|f\|_p^p \big\||\varphi-\psi|^{p-1}\big\|_1.$$

For $f$ in $L^\infty[0,1]$, we have

$$\|\,(V_\varphi - V_\psi)\, f\|_\infty = \sup_{0\leqslant x\leqslant 1} \left|\int_{\psi(x)}^{\varphi(x)} f(t)\,dt\right| \leqslant \|f\|_\infty \|\varphi-\psi\|_\infty.$$

On the other hand, for the constant function 1, the last quantity above is attained. The result is proved. □

Taking $\psi = 0$ in Proposition 2.1.5, we obtain,

**Corollary 2.1.6.** *Let $\varphi$ be a measurable self-map of $[0,1]$. Then $\|V_\varphi\|_p \leqslant \left\|\varphi^{p-1}\right\|_1^{1/p}$ for $1 \leqslant p < \infty$ and $\|V_\varphi\|_\infty = \|\varphi\|_\infty$ for $p = \infty$. In particular, $V_\varphi$ is always a contraction.*

For $p = 2$, one may proceed as for the Volterra operator, see [15, p. 300]. Indeed, $V_\varphi^\star V_\varphi$ is compact, self-adjoint and positive. In particular, its eigenvalues are non-negative and the norm of $V_\varphi$ coincides with the square root of the greatest eigenvalue of $V_\varphi^\star V_\varphi$. It is also possible to consider $V_\varphi V_\varphi^\star$ that shares with $V_\varphi^\star V_\varphi$ its eigenvalues. The latter makes simpler the computations in some of the examples below.

Suppose that $\varphi$ in $\mathcal{C}^2[0,1]$ is strictly increasing with $\varphi(0) = 0$ and $\varphi(1) = 1$. To find the eigenvalues, one has to solve the integral equation

$$(V_\varphi V_\varphi^\star f)(x) = \int_0^{\varphi(x)} \int_{\varphi^{-1}(t)}^1 f(s)\,ds\,dt = \lambda f(x).$$

Differentiating twice the above display, one arrives to a second order differential equation for the eigenfunctions

$$\lambda\varphi'(x)f''(x) - \lambda\varphi''(x)f'(x) + (\varphi'(x))^2 f(x) = 0, \tag{2.1.4}$$

which is to be solved with the boundary conditions $f(0) = 0$ and $f'(1) = 0$. If one considers $V_\varphi^\star V_\varphi$, it is enough to replace $\varphi$ by $\varphi_{-1}$ in the above equation. The boundary conditions are then $f(1) = 0$ and $f'(0) = 0$.

The norm of the Volterra operator, which is $2/\pi$, is a special case of the example below.

**Example 2.1.7.** *Assume that $\varphi_\alpha(x) = x^\alpha$ with $0 < \alpha < \infty$. Then $\|V_{\varphi_\alpha}\|_2$ is equal to the square root of the greatest positive zero of*

$$J_{-(1+\alpha)^{-1}}\left(2(1+\alpha)^{-1}\alpha^{1/2}\lambda^{-1/2}\right),$$

*where $J_{-(1+\alpha)^{-1}}$ is the Bessel function of the first kind and of order $-(1+\alpha)^{-1}$.*

*Proof.* In the present case, equation (2.1.4) becomes

$$\lambda\alpha x^{\alpha-1}f''(x) + \lambda(\alpha-\alpha^2)x^{\alpha-2}f'(x) + \alpha^2 x^{2\alpha-2}f(x) = 0. \tag{2.1.5}$$

Upon setting

$$\begin{aligned} g(t) &= ((\alpha+1)\sqrt{\lambda/\alpha}\,t/2)^{-\alpha/(1+\alpha)} f(((\alpha+1)\sqrt{\lambda/\alpha}\,t/2)^{2/(1+\alpha)}),\\ t &= 2\sqrt{\alpha/\lambda}(\alpha+1)^{-1}x^{(\alpha+1)/2}\\ \text{and}\quad \nu &= \alpha/(\alpha+1), \end{aligned}$$

we obtain, after some elementary computations, that $g$ satisfies Bessel's differential equation

$$t^2 g''(t) + t g'(t) + \left(t^2 - \nu^2\right) g(t) = 0.$$

Since $0 < \nu < 1$, a linear independent system of solutions of the above equation is $\{J_\nu(t), J_{-\nu}(t)\}$, where $J_\nu$ denotes the Bessel function of the first kind and of order $\nu$, see [55, pp. 38–44], for instance. Therefore, the general solution of (2.1.5) is $c_0 f_{\lambda,0} + c_1 f_{\lambda,1}$, where $c_0$ and $c_1$ are constants and

$$f_{\lambda,j}(x) = x^{\alpha/2} J_{(-1)^j \alpha/(\alpha+1)} \left( 2\sqrt{\frac{\alpha}{\lambda}} \frac{x^{(\alpha+1)/2}}{\alpha+1} \right), \qquad j = 0, 1.$$

While $f_{\lambda,0}$ vanishes at 0, the function $f_{\lambda,1}$ does not, here $f_{\lambda,1}(0)$ is defined as the limit at 0. Thus the boundary condition $f(0) = 0$ implies that the eigenfunctions are $f_\lambda = f_{\lambda,0}$. Using that the Bessel functions satisfy $xJ'_\nu(x) + \nu J_\nu(x) = xJ_{\nu-1}(x)$, see [55, p. 45], it is elementary to check that

$$f'_\lambda(x) = \sqrt{\alpha/\lambda}\, x^{\alpha-1/2} J_{-(\alpha+1)^{-1}} \left( 2(\alpha+1)^{-1} \sqrt{\alpha/\lambda}\, x^{(\alpha+1)/2} \right).$$

Thus imposing the boundary condition $f'(1) = 0$, we obtain the eigenvalue equation and the desired result follows. □

**Remark.** Parallel arguments apply to a strictly decreasing self-map $\varphi$ of $[0,1]$. For instance, if $\varphi$ belongs to $\mathcal{C}^2[0,1]$ with $\varphi(0) = 1$ and $\varphi(1) = 0$, the differential equation for the eigenvalues is (2.1.4) with the plus sign in the last term replaced by the minus sign. In the computation of the norms, it is convenient to keep in mind that $\|V_{\varphi_{-1}}\|_2 = \|V_\varphi\|_2$ for decreasing $\varphi$ and $\|V_\varphi\|_2 = \|V_{1-\varphi_{-1}(1-x)}\|_2$ for increasing $\varphi$.

### 2.1.4 The distance to the quasi-nilpotent operators

Let $\mathcal{Q}_p$ denote the class of quasi-nilpotent operators in the class of bounded operators on $L^p[0,1]$, $1 \leqslant p \leqslant \infty$. The distance from $T$ to the quasi-nilpotent operators is

$$\operatorname{dist}_p(T, \mathcal{Q}_p) = \inf_{Q \in \mathcal{Q}_p} \|T - Q\|_p.$$

Now, Herrero [19] proved that $\operatorname{dist}_2(T, \mathcal{Q}_2) \leqslant r(T)/2$ for any compact operator $T$. Since for a composition Volterra operator, the spectral radius does not depend on the underlying space, $r = r(T) \leqslant \|V_\varphi\|_p$, $1 \leqslant p \leqslant \infty$, from Corollary 2.1.6, we have the upper estimate

$$\operatorname{dist}_2(V_\varphi, \mathcal{Q}_2) \leqslant \frac{1}{2} \inf_{1 \leqslant p \leqslant \infty} \|V_\varphi\|_p \leqslant \frac{1}{2} \inf_{1 < p < \infty} \|\varphi^{p-1}\|_1^{1/p}.$$

Below there is another estimate in terms of the part of the graph of $\varphi$ which is over the graph of the identity. As usual $\phi^+ = \max\{\phi, 0\}$ and $\phi^- = \min\{\phi, 0\}$.

**Proposition 2.1.8.** *Let $\varphi$ be a measurable self-map of $[0,1]$. Then, for $1 \leqslant p < \infty$, we have*

$$\operatorname{dist}_p(V_\varphi, \mathcal{Q}_p) \leqslant \left\| \left( (\varphi(x) - x)^+ \right)^{p-1} \right\|_1^{1/p} \leqslant p^{-1/p}$$

*and* $$\operatorname{dist}_\infty(V_\varphi, \mathcal{Q}_\infty) \leqslant \left\| (\varphi(x) - x)^+ \right\|_\infty \leqslant 1.$$

*Proof.* Since $\phi(x) = (\varphi(x) - x)^- \leqslant 0$, it follows that $\psi(x) = \phi(x) + x \leqslant x$. By Corollary 2.1.2, the operator $V_\psi$ is quasi-nilpotent. Therefore,

$$\operatorname{dist}_p(V_\varphi, \mathcal{Q}_p) \leqslant \|V_\varphi - V_\psi\|_p.$$

Assume now that $1 \leqslant p < \infty$. By Proposition 2.1.5 (i), we have

$$\|V_\varphi - V_\psi\|_p \leqslant \left\| |\varphi - \psi|^{p-1} \right\|_1^{1/p} = \left\| \left( (\varphi(x) - x)^+ \right)^{p-1} \right\|_1^{1/p} \leqslant p^{-1/p},$$

where the last inequality follows from the elementary estimate $(\varphi(x) - x)^+ \leqslant 1 - x$. For $p = \infty$, the result follows by applying Proposition 2.1.5 (ii). □

### 2.1.5 The eigenfunctions of $V_\varphi$. Examples

Before going into a deeper study of the spectrum of $V_\varphi$, we present some examples of symbols for which we can provide the eigenvalues and the eigenfunctions exactly.

We start by introducing the concept of generalized eigenvector. Let $T$ be a bounded operator acting on a Banach space $\mathcal{B}$. Recall that $\lambda$ is a normal eigenvalue of algebraic multiplicity $k$ of $T$ if and only if $\mathcal{B}$ is the direct sum of $T$-invariant subspaces $\mathcal{B}_0^\lambda$ and $\mathcal{B}_1^\lambda$ such that $\dim \mathcal{B}_0^\lambda = k$, the restriction of $(T - \lambda I)$ to $\mathcal{B}_0^\lambda$ is nilpotent and the restriction of $(T - \lambda I)$ to $\mathcal{B}_1^\lambda$ is invertible. The spaces $\mathcal{B}_0^\lambda$ and $\mathcal{B}_1^\lambda$ are uniquely determined by $T$ and $\lambda$. Indeed,

$$\mathcal{B}_0^\lambda = \bigcup_{m=1}^{\infty} \ker (T - \lambda I)^m = \ker (T - \lambda I)^k \quad \text{and} \quad \mathcal{B}_1^\lambda = \bigcap_{m=1}^{\infty} (T - \lambda I)^m (\mathcal{B}) = (T - \lambda I)^k (\mathcal{B}).$$

The elements in $\mathcal{B}_0^\lambda$ are called *generalized eigenvectors* corresponding to the normal eigenvalue $\lambda$. From the Jordan decomposition theorem, it follows that if $\lambda$ is a normal eigenvalue of $T$ of algebraic multiplicity $k$ and geometric multiplicity 1, then there is $f \in \mathcal{B}$ for which $(T - \lambda I)^k f = 0$ and $(T - \lambda I)^{k-1} f \neq 0$ and for such $f$ the space of generalized eigenvectors corresponding to $\lambda$ is

$$\operatorname{span}\{f, (T - \lambda I) f, \ldots, (T - \lambda I)^{k-1} f\}.$$

**Remark.** It is clear that if $f$ in $L^1[0,1]$ is a generalized eigenfunction of $V_\varphi$, with $\varphi$ a measurable self-map of $[0,1]$, then $f$ is automatically in $L^\infty[0,1]$ and, therefore, in all the spaces $L^p[0,1]$, $1 \leqslant p \leqslant \infty$. Thus, the spectrum $\sigma(V_\varphi)$ as well as the generalized eigenspaces corresponding to non-zero eigenvalues do not depend on the underlying space. Consequently, all the statements concerning the spectra will be done in the friendly confines of the Hilbert space $L^2[0,1]$. Observe also that if $\varphi$ is a self-map of $[0,1]$ of class $\mathcal{C}^k$ for some non-negative integer $k$, then the non-zero eigenvalues and the corresponding generalized eigenvectors of $V_\varphi$ acting on $L^2[0,1]$ coincide with those of $V_\varphi$ acting on $\mathcal{C}^k[0,1]$.

The next lemma provides a useful tool that guarantees we will have found all the eigenvalues and eigenfunctions in our examples.

**Lemma 2.1.9.** *Let $T$ be a compact operator acting on an infinite dimensional Banach space. If a sequence of generalized eigenvectors of $T$ corresponding to non-zero eigenvalues has dense span, then this span contains all generalized eigenvectors of $T$ that correspond to non-zero eigenvalues.*

*Proof.* Let $\{\lambda_k\}$ be the finite or infinite sequence of all non-zero eigenvalues of $T$ and $m_k$ be the multiplicity of $\lambda_k$. Here we assume that $\lambda_k \neq \lambda_n$ for $k \neq n$. Recall that Fredholm's Alternative Theorem [5, 44] asserts that dim ker $(T-\lambda_k)^{m_k} =$ dim ker $(T^\star - \lambda_k)^{m_k}$ for each $k$, $\langle f, g\rangle = 0$ for $f$ in ker $(T-\lambda_k)^{m_k}$ and $g$ in ker $(T^\star - \lambda_n)^{m_n}$ with $k \neq n$ and the functionals in ker $(T^\star - \lambda_k)^{m_k}$, separate points of ker $(T-\lambda_k)^{m_k}$.

Let $L$ denote the dense span of a sequence of eigenvectors of $T$ corresponding to certain non-zero eigenvalues of $T$. Suppose that there is a non-negative integer $k$ such that ker $(T-\lambda_k)^{m_k}$ is not contained in $L$. Consequently, there is $y \neq 0$ in ker $(T^\star - \lambda_k)^{m_k}$ that vanishes on ker $(T-\lambda_k)^{m_k} \cap L$ and, of course, on ker $(T-\lambda_n)^{m_n} \cap L$ for all $n \neq k$. Since

$$L = \bigoplus_n \left(\ker (T-\lambda_n)^{m_n} \cap L\right),$$

we find that $g$ vanishes on $L$, which contradicts the density of $L$. □

In the following result we compute eigenfunctions and eigenvalues of $V_\varphi$ for $\varphi(x) = x^\alpha$, $0 < \alpha < 1$.

**Theorem 2.1.10.** *Assume that $\varphi(x) = x^\alpha$ with $0 < \alpha < 1$. Then the eigenvalues of $V_\varphi$ have algebraic multiplicity 1 and $\sigma(V_\varphi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$. Furthermore, for each $n \geqslant 0$, the eigenfunction corresponding to $(1-\alpha)\alpha^n$ is $f_n(x) = x^{\alpha/(1-\alpha)} p_n(\ln x)$,* *where*

$$p_n(x) = x^n + \sum_{j=1}^{n} \frac{n!(1-\alpha)^j \alpha^{(j^2-j)/2}}{(n-j)!} \left( \prod_{l=1}^{j} \frac{1}{1-\alpha^l} \right) x^{n-j}.$$

*Moreover, the eigenfunctions* $\{f_n\}_{n\geqslant 0}$ *span* $L^2[0,1]$.

*Proof.* If we denote $a_{n,j}$ the $j$-th coefficient of $p_n$, then $a_{n,n} = 1$ for every $n$ and $a_{n,j}$ satisfy

$$a_{n,j}(\alpha^j - \alpha^n) = (1-\alpha)\alpha^{n-1}(j+1)a_{n,j+1} \quad \text{for} \quad 0 \leqslant j \leqslant n-1.$$

From the above display, one sees that

$$p_n(\alpha t) = \alpha^n p_n(t) + (1-\alpha)\alpha^{n-1} p_n'(t). \tag{2.1.6}$$

Using (2.1.6) in the second equality below, we have

$$\begin{aligned} \varphi'(x) f_n(\varphi(x)) =& \alpha x^{(2\alpha-1)/(1-\alpha)} p_n(\alpha \ln x) \\ =& \alpha x^{(2\alpha-1)/(1-\alpha)} \left( \alpha^n p_n(\ln x) + (1-\alpha)\alpha^{n-1} p_n'(\ln x) \right) \\ =& (1-\alpha)\alpha^n f_n'(x). \end{aligned}$$

Set $f_n(0) = 0$. Since $f_n$ is absolutely continuous on $[0,1]$, we may integrate in the above display from 0 to $x$ to obtain $V_\varphi f_n = (1-\alpha)\alpha^n f_n$, which means that each $f_n$ is an eigenfunction of $V_\varphi$ corresponding to the eigenvalue $(1-\alpha)\alpha^n$.

Now we prove that span$\{f_n : n \geqslant 0\}$ is dense. Since the operator of multiplication by $x^{\alpha/(1-\alpha)}$ has dense range, it is enough to prove that $\{p_n(\ln x)\}_{n\geqslant 0}$ spans $L^2[0,1]$. The change of variables $t = -\ln x$ shows that this is equivalent to the fact that $\{p_n(-t)e^{-t}\}_{n\geqslant 0}$ spans $L^2[0,\infty)$, which follows by a standard argument.

Finally, Lemma 2.1.9 shows that span$\{f_n : n \geqslant 0\}$ coincides with the span of all generalized eigenfunctions of $V_\varphi$ corresponding to non-zero eigenvalues. Hence, it follows that $\sigma(V_\varphi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$ and the eigenvalues $(1-\alpha)\alpha^n$ have algebraic multiplicity 1. The result is proved. □

**Theorem 2.1.11.** *Let* $\psi(x) = 1 - (1-x)^{1/\alpha}$ *with* $0 < \alpha < 1$. *Then the eigenvalues of* $V_\psi$ *have algebraic multiplicity* 1 *and* $\sigma(V_\psi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$. *Furthermore, for each* $n \geqslant 0$, *the eigenfunction of* $V_\psi$ *corresponding to* $(1-\alpha)\alpha^n$, *is*

$$f_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (1-\alpha)^k \alpha^{nk}}{(\alpha^{-1}-1)\cdots(\alpha^{-k}-1)} (1-x)^{\frac{\alpha^{-k-1}-\alpha^{-1}}{\alpha^{-1}-1}}. \tag{2.1.7}$$

*In particular, the eigenfunctions of* $V_\psi$ *do not span* $L^2[0,1]$.

*Proof.* Since $V_\psi$ is similar to $V_\varphi^\star$, where $\varphi(x) = x^\alpha$, by Theorem 2.1.10, we find that

$$\sigma(V_\psi) = \sigma(V_\varphi^\star) = \sigma(V_\varphi) = \{(1-\alpha)\alpha^n\}_{n\geqslant 0} \cup \{0\}$$

and the eigenfunctions have algebraic multiplicity 1. To obtain the eigenfunctions we begin with $\alpha = 1/m$, where $m$ is an integer $m \geqslant 2$. Set

$$F(x) = \int_0^x f(t)\,dt$$

and suppose that $F$ is analytic on a neighborhood of 1. Then

$$F(x) = \sum_{n=0}^{\infty} a_n(1-x)^n.$$

Thus if $\lambda$ is an eigenvalue, then $F(\psi(x)) = \lambda F'(x)$ implies that

$$\sum_{n=0}^{\infty} a_n(1-x)^{nm} = -\lambda \sum_{n=1}^{\infty} n a_n (1-x)^{n-1} = -\lambda \sum_{n=0}^{\infty} (n+1) a_{n+1}(1-x)^n.$$

Taking $a_0 = 1$, one finds that (2.1.7) is true for $\alpha = 1/m$. For $\alpha \neq 1/m$, it consists on a computation to check directly that the function $f_n$ also satisfies the eigenfunction equation.

To prove that the eigenfunctions do not span $L^2[0,1]$ observe that

$$\overline{\text{span}}\,\{f_n : n \geqslant 0\} \subset \overline{\text{span}}\,\left\{(1-x)^{\frac{\alpha^{-k-1}-\alpha^{-1}}{\alpha^{-1}-1}} : k \geqslant 0\right\}.$$

Since the sum of the inverses of the exponents in the monomials above is finite, by the Müntz-Szász Theorem, see [43], it follows that the right-hand side above is different from $L^2[0,1]$. The proof is complete. □

**Remark.** Observe that in Theorem 3.2, or in Theorem 3.3, the 'trace' of $V_\varphi$ equals to 1, independently of $\alpha$, which will be just a trivial application of Theorem 3.1.26.

It is worth noting that $V_\varphi$ acting on $L^2[0,1]$ can be self-adjoint, which holds if and only if $\{(x,t) \in [0,1]^2 : t \leqslant \varphi(x)\}$ is symmetric with respect to $y = x$ up to a set of plane Lebesgue measure zero. For instance, it follows that for $\varphi$ increasing, the operator $V_\varphi$ is self-adjoint just only in two cases: when $\varphi \equiv 0$, in which case $V_\varphi$ is the zero operator, and when $\varphi \equiv 1$, in which case $V_\varphi$ is the orthogonal projection onto the space of constant functions.

The class of decreasing $\varphi$'s for which $V_\varphi$ is self-adjoint is much richer. This class contains, for instance, the strictly decreasing and onto self-maps $\varphi$ of $[0,1]$ such that

$\varphi = \varphi_{-1}$. The simplest example is $\varphi(x) = 1 - x$. We shall compute explicitly the eigenvalues and eigenfunctions of $V_\varphi$. Unlike the previous examples, there are infinitely many negative eigenvalues.

**Example 2.1.12.** *Assume that* $\varphi(x) = 1 - x$ *for* $0 \leqslant x \leqslant 1$. *Then* $\sigma(V_\varphi) = \{2(-1)^n/(\pi(2n+1))\}_{n\geqslant 0} \cup \{0\}$. *Furthermore, the eigenvalues have algebraic multiplicity* 1 *and the corresponding sequence of eigenfunctions is* $\{\cos((2n+1)\pi x/2)\}_{n\geqslant 0}$.

*Proof.* The eigenvalue equation is easily checked. On the other hand, in [15, pb. 188], Halmos proved that the sequence $\{\sqrt{2}\cos((2n+1)\pi x/2)\}_{n\geqslant 0}$ is an orthonormal basis in $L^2[0,1]$. Thus, the eigenvalues have algebraic multiplicity 1 and there are no other eigenvalues. □

Observe that the norm of $V_\varphi$ in example above coincides with that of the Volterra operator, see Example 2.1.7.

### 2.1.6 Cyclicity

Now, we will see examples of cyclic composition Volterra operators. Recall that an operator $T$ on a Banach space $\mathcal{B}$ is *cyclic* if there is $f$ in $\mathcal{B}$ such that span $\{T^n f : n \geqslant 0\}$ is dense in $\mathcal{B}$.

**Proposition 2.1.13.** *Assume that* $\varphi(x) = x^\alpha$ *with* $\alpha > 0$. *Then* $\phi(x) = x^\beta$ *with* $\beta > -1/p$ *is cyclic for* $V_\varphi$ *acting on* $L^p[0,1]$, $1 \leqslant p < \infty$, *if and only if* $0 < \alpha \leqslant 1$.

*Proof.* An elementary computation shows that

$$(V_\varphi^n \phi)(x) = c x^{\beta\alpha^n + \frac{\alpha-\alpha^{n+1}}{1-\alpha}}, \qquad \text{for each} \quad n \geqslant 0,$$

where $c \neq 0$ depends only on $n$, $\alpha$ and $\beta$. Thus, the result follows from the Müntz-Szász Theorem. □

Observe that the statement of Proposition 2.1.13 above is still true if for $p = \infty$ we consider the space $\mathcal{C}_0[0,1]$ of continuous functions on $[0,1]$ vanishing at 0, endowed with the supremum norm.

From Proposition 2.1.13 the constant function is not cyclic for $V_\varphi$ when $\varphi(x) = x^\alpha$ with $\alpha > 1$. In contrast, the cyclicity of the constant function 1 for $V_\varphi$ is also possible when $\varphi(x) < x$ for $0 < x < 1$. Indeed, in Section 5.1, it will be shown that the eigenfunctions of $V_\varphi$ span $L^2[0,1]$ if and only if 1 is a cyclic vector for $V_\psi$, where $\psi(x) = 1 - \varphi(1-x)$, see Theorem 5.1.1. Thus as corollaries of Theorems 2.1.10 and 2.1.11 and Proposition 2.1.13, we have

**Corollary 2.1.14.** *Assume that* $\alpha > 1$. *Then the constant function* $1$ *is cyclic for* $V_\psi$, *where* $\psi(x) = 1 - (1-x)^{1/\alpha}$ *and is not cyclic for* $V_\varphi$, *where* $\varphi(x) = x^\alpha$.

## 2.2 Fundamental facts on the eigenfunctions of $V_\varphi$

In this section, we begin the study of the eigenfunctions of $V_\varphi$ when $\varphi(x) > x$ on a set of positive measure.

### 2.2.1 The spectral radius as an eigenvalue

Kreĭn-Rutman's Theorem asserts that if a compact operator preserves the cone of positive functions, then the spectral radius is an eigenvalue to which corresponds a non-negative eigenfunction, see [26] or [31, Theorem 4.1.4]. Integral operators with kernel is one of the most natural settings where Kreĭn-Rutman's Theorem applies. As its immediate corollary, we have

**Theorem 2.2.1.** *Assume that* $K$ *in* $L^2([0,1]^2)$ *is non-negative. If the spectral radius* $r(J_K)$ *is positive, then it is an eigenvalue of* $J_K$ *for which there is a non-negative eigenfunction* $f$ *in* $L^2[0,1]$.

The next result asserts that under suitable hypotheses the eigenfunction furnished by Theorem 2.2.1 is strictly positive and all generalized eigenfunctions corresponding to any other eigenvalue change their sign.

**Theorem 2.2.2.** *Assume that* $K$ *in* $L^2([0,1]^2)$ *is non-negative and the spectral radius* $r = r(J_K) > 0$. *Assume also that there is a continuous self-map* $\psi$ *of* $[0,1]$ *with* $\psi(x) > x$ *for* $0 < x < 1$ *and* $K(x,t) > 0$ *a.e. whenever* $0 < t \leqslant \psi(x) \leqslant 1$. *Then the eigenfunction provided by Theorem* 2.2.1 *is strictly positive a.e. Furthermore, there are no non-negative non-zero generalized eigenfunctions corresponding to an eigenvalue of* $J_K$ *different from the spectral radius.*

*Proof.* Without loss of generality, we may assume that $\psi$ is strictly increasing and $\psi(0) = 0$. Let $f$ be the eigenfunction provided by Theorem 2.2.1. Since $f$ is different from zero, we see that $\alpha = \min \operatorname{supp}(f) < 1$. Since $K(x,t) > 0$ for $t \leqslant \psi(x)$, using Fubini's Theorem we have

$$f(x) = \frac{1}{r}(J_K f)(x) = \frac{1}{r}\int_0^1 K(x,t) f(t)\,dt > 0 \text{ a.e.}, \qquad \text{whenever } \psi_{-1}(\alpha) \leqslant x \leqslant 1.$$

Thus $\psi_{-1}(\alpha) \geqslant \alpha$ and, therefore, $\alpha < 1$ is a fixed point of $\psi$. Hence, $\alpha = 0$ and the above display implies that $f(x) > 0$ a.e.

To prove the second statement of the theorem, first observe that under the involutive isometry $(Uf)(x) = f(1-x)$ the operator $J_K^\star$ is similar to $J_{\widetilde{K}}$, where $\widetilde{K}(t,x) = K(1-t,1-x)$. In particular, $J_K$ and $J_{\widetilde{K}}$ have the same spectral radius and $\widetilde{K}$ as well as $\widetilde{\psi}(x) = 1-\psi_{-1}(1-x)$ satisfy the same hypotheses as $K$ and $\psi$. Therefore, according to what is already proved, there is a positive a.e. function $h$ in $L^2[0,1]$ such that $J_{\widetilde{K}}h = rh$. Thus $J_K^\star f = rf$, where $f(x) = h(1-x) > 0$ a.e. Suppose now that a non-negative generalized eigenfunction $g$ in $L^2[0,1]$ corresponds to an eigenvalue $\lambda \neq r$ of $J_K$. Then $(J_K - \lambda I)^n g = 0$ for some positive integer $n$. Therefore,

$$\begin{aligned}
0 &= (r-\lambda)^{-n}\langle (J_K - rI + (r-\lambda)I)^n g, f\rangle \\
&= \langle g, f\rangle + \sum_{k=1}^{n} \frac{(r-\lambda)^{-k} n!}{k!(n-k)!}\langle (J_K - rI)^k g, f\rangle \\
&= \langle g, f\rangle + \sum_{k=1}^{n} \frac{(r-\lambda)^{-k} n!}{k!(n-k)!}\langle g, (J_K - rI)^{\star k} f\rangle.
\end{aligned}$$

Since $J_K^\star f = rf$, we have $\langle g, f\rangle = 0$. Since $f(x) > 0$ a.e. and $g(x) \geqslant 0$, it follows that $g$ is the null function, which is a contradiction. The proof is complete. □

The next corollary follows applying the previous theorem to $J_{K_\varphi}$ and Corollary 2.1.2.

**Corollary 2.2.3.** *Let $\varphi$ be a measurable self-map of $[0,1]$ with $\mu\{x : \varphi(x) > x\} > 0$. Then $r(V_\varphi) > 0$ is an eigenvalue for which there is a non-negative eigenfunction. Furthermore, if $\varphi$ is continuous and $\varphi(x) > x$ for $0 < x < 1$, then the eigenfunction corresponding to the spectral radius is strictly positive and there are no non-negative non-zero generalized generalized eigenfunctions corresponding to an eigenvalue different from the spectral radius.*

### 2.2.2 Estimates on the spectral radius

The next proposition provides a lower estimate on the spectral radius of $V_\varphi$.

**Proposition 2.2.4.** *Let $\varphi$ be an increasing self-map of $[0,1]$. Then,*

$$r(V_\varphi) \geqslant \|(\varphi(x) - x)^+\|_\infty.$$

*Proof.* If $\varphi(x) \leqslant x$, then by Corollary 2.1.2, we have $r(V_\varphi) = 0$ and there is nothing to prove. Thus assume that there is $0 \leqslant x_0 < 1$ for which $\varphi(x_0) > x_0$. Then set $\phi(x) = \varphi(x_0)\chi_{[x_0,1]}(x)$ and observe that $\varphi(x) \geqslant \phi(x)$ for $0 \leqslant x \leqslant 1$. Therefore, for positive $f$, we have $V_\varphi^n f \geqslant V_\phi^n f \geqslant 0$ for each positive integer $n$. Since, by Kreĭn-Rutman's Theorem, the spectral radius is determined by the positive functions, we see that $r(V_\varphi) \geqslant r(V_\phi)$. As $\varphi(x_0) - x_0$ is an eigenvalue of the rank one operator $V_\phi$, we find that $r(V_\phi) = \varphi(x_0) - x_0$ and the result follows by just taking supremum of $\varphi(x) - x$ on $\{x \in [0,1] : \varphi(x) > x\}$. □

### 2.2.3 The dimension of $\ker(V_\varphi - \lambda I)$

**Lemma 2.2.5.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Assume also that $f$ in $L^2[0,1]$ satisfies that $V_\varphi f = \lambda f$ for some $\lambda \neq 0$. Then $f$ cannot be orthogonal to the constant functions. Furthermore, if $0 < \alpha \leqslant 1$ is a fixed point of $\varphi$ and $\max_{[0,\alpha]} \varphi \leqslant \alpha$, then either $f(\alpha) \neq 0$ or $f$ vanishes on $[0,\alpha]$.*

*Proof.* The first statement of the lemma follows easily from the second one by just taking $\alpha = 1$. Indeed, if $f$ is orthogonal to the constant functions, then

$$f(1) = \frac{1}{\lambda} \int_0^1 f(t)\, dt = 0$$

and, therefore, $f$ is the null function.

To show the second statement of the lemma, observe that $f$ must be continuous. Now, suppose that

$$f(\alpha) = \frac{1}{\lambda} \int_0^\alpha f(t)\, dt = 0.$$

Let $\beta$ be the minimum of $t \in [0,\alpha]$ for which $f$ vanishes on $[t,\alpha]$. If $\beta = 0$, there is nothing to prove. Thus we may assume that $0 < \beta \leqslant \alpha$.

*Case* $\varphi(\beta) = \beta$. Since $f(\beta) = 0$ and $f$ does not vanish on $[\beta - \varepsilon, \beta]$ for $0 < \varepsilon < \beta$, there is a strictly increasing sequence $\{\beta_n\}$ in $(0,\beta)$ such that $\beta_n$ tend to $\beta$ as $n$ tends to $\infty$ and $|f(\beta_n)| = \max_{[\beta_n,\beta]} |f| \neq 0$. Since $\varphi(\beta) = \beta$ and $f(\beta) = 0$, we have

$$\int_0^\beta f(t)\, dt = \lambda f(\beta) = 0.$$

Hence,

$$|\lambda||f(\beta_n)| = \left| \int_0^{\varphi(\beta_n)} f(t)\, dt \right| = \left| \int_{\varphi(\beta_n)}^\beta f(t)\, dt \right| \leqslant |\beta - \varphi(\beta_n))| f(\beta_n)|.$$

Thus, $|\lambda| \leqslant |\beta - \varphi(\beta_n)|$ for each positive integer $n$. Since $\varphi(\beta_n)$ tends to $\varphi(\beta) = \beta$, we have $\lambda = 0$, which is a contradiction.

*Case* $\beta < \varphi(\beta)$. Since $\varphi$ is continuous and $\varphi([0,\alpha]) \subseteq [0,\alpha]$, there is $\delta > 0$ such that $\beta \leqslant \varphi(x) \leqslant \alpha$ for $\beta - \delta \leqslant x \leqslant \beta$. Since $f$ vanishes on $[\beta,\alpha]$, we have

$$f(x) = \frac{1}{\lambda} \int_0^{\varphi(x)} f(t)\, dt = \frac{1}{\lambda} \int_0^{\varphi(\beta)} f(t)\, dt = f(\beta) = 0, \qquad \text{for } x \in [\beta - \delta, \beta],$$

which contradicts the minimality of $\beta$. The result is proved.

□

**Corollary 2.2.6.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Then each non-zero eigenvalue of $V_\varphi$ has geometric multiplicity 1.*

*Proof.* Let $f$ and $g$ be linearly independent eigenfunctions corresponding to a non-zero eigenvalue. Then, by Lemma 2.2.5, we have $f(1) \neq 0$ and $g(1) \neq 0$. Thus $h(x) = g(1)f(x) - f(1)g(x)$ is an eigenfunction with $h(1) = 0$ that corresponds to the same eigenvalue. Thus $h$ is null by Lemma 2.2.5, which is a contradiction. □

**Remark.** It is worth mentioning that for increasing self-maps $\varphi$ of $[0,1]$ we have

$$V_\varphi + V^\star_{\varphi_{-1}} = P,$$

where $P$ is the orthogonal projection on the space of constant functions. For instance, it can be used to prove the first statement of Lemma 2.2.5 for increasing self-maps $\varphi$ of $[0,1]$ with $\varphi(x) \geqslant x$.

In what follows, for any self-map $\varphi$ of $[0,1]$ we denote by $\varphi_0$ the identity map and $\varphi_n = \varphi \circ \varphi_{n-1}$ for each positive integer $n$.

**Proposition 2.2.7.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and non-constant on any subinterval. Then the support of each eigenfunction of $V_\varphi$ corresponding to a non-zero eigenvalue is $[0,1]$.*

*Proof.* Let $f$ be different from zero such that $V_\varphi f = \lambda f$ for a complex number $\lambda \neq 0$. Suppose that $f$ vanishes on $[a,b]$ with $0 \leqslant a < b \leqslant 1$. The eigenvalue equation implies that

$$F(x) = \int_0^x f(t)\,dt$$

vanishes on $\varphi([a,b])$ and, therefore, so does $f = F'$, since the interval $\varphi([a,b])$ is non-trivial. Upon iterating this argument, we find that $f$ vanishes on $\varphi_n([a,b])$, which contains the interval $[\min\{\varphi_n(a), \varphi_n(b)\}, \max\{\varphi_n(a), \varphi_n(b)\}]$ for every positive $n$. In particular, for each positive integer $n$ we have

$$\int_0^{\varphi_n(b)} f(t)\,dt = \lambda f(\varphi_{n-1}(b)) = 0.$$

Since $\varphi(x) > x$ for $0 < x < 1$, the sequence $\{\varphi_n(b)\}$ tends to 1. Thus the display above implies that $f$ is orthogonal to the constant functions, which contradicts Lemma 2.2.5 and the result follows. □

**Remark.** Using Proposition 2.2.7, it is not difficult to produce an example in which $0 \leqslant \alpha < 1$ is a fixed point of $\varphi$ and $\max_{[0,\alpha]} \varphi$ is not contained in $[0,\alpha]$, then the conclusion in Lemma 2.2.5 is not true.

We shall express $\sigma(V_\varphi)$ in terms of the spectra of simpler composition Volterra operators defined on subintervals of $[0,1]$. For a measurable self-map $\varphi$ of $[0,1]$, we can consider

$$S_\varphi = \{y \in [0,1] : \max_{[0,y]} \varphi \leqslant y\}$$

and denote its boundary by $\partial S_\varphi$. If $\partial S_\varphi$ has zero Lebesgue measure, then it induces an orthogonal decomposition of $L^2[0,1]$ in the obvious way. Indeed, since $\partial S_\varphi$ is closed, we have

$$[0,1] \setminus \partial S_\varphi = \bigcup_{j\in J} I_j,$$

where $I_j$ are pairwise disjoint open intervals and $J$ is countable. We clearly have

$$L^2[0,1] = \bigoplus_{j\in J} L^2(I_j).$$

Upon writing $I_j = (a_j, b_j)$, $j \in J$, we see that $\{a_j : j \in J\}$ as well as $\{b_j : j \in J\}$ are subsets of $\partial S_\varphi$. The key point here is that $V_\varphi$ has a block lower triangular matrix with respect to the decomposition above whenever $J$ is ordered in the obvious way, that is, $i < j$ if $b_i \leqslant a_j$. Indeed, let $P_j$ be the orthogonal projections that correspond to the above orthogonal decomposition and let $V_\varphi^{i,j} = P_i V_\varphi P_j$, then it is easy to show that $V_\varphi^{i,j}$ is equal to zero whenever $i < j$. To compute the spectrum of $V_\varphi$, it is enough to compute the spectrum of each $V_\varphi^{j,j}$. We have

**Theorem 2.2.8.** *Let $\varphi$ be a measurable self-map of $[0,1]$ with $\mu(\partial S_\varphi) = 0$. Then*

$$\sigma(V_\varphi) = \bigcup_{j\in J} \sigma(V_\varphi^{j,j}). \tag{2.2.1}$$

*Proof.* First, observe that the operators involved are compact and thus zero is in both sides of (2.2.1). Now, suppose that $J$ is finite. In this case, the left-right inclusion is well-known, see [15, Problem 72], for instance. The right-left inclusion is elementary and follows by induction because all the elements in the spectra are eigenvalues.

Finally, suppose that $J$ is infinite. Then for each positive integer, consider

$$\varphi_n(x) = \begin{cases} \varphi(x), & \text{if } x \in I_j \text{ and } b_j - a_j > 1/n; \\ 0, & \text{otherwise.} \end{cases}$$

Note that this time the subscript $n$ does not mean the sequence of iterates. Clearly, $V_{\varphi_n}$ has a finite number of non-zero blocks with respect to the orthogonal decomposition induced by $\varphi$. In addition, the non-zero diagonal blocks of $V_{\varphi_n}$ coincide with the corresponding ones of $V_\varphi$. Since $\|V_{\varphi_n} - V_\varphi\|$ tends to zero as $n$ tends to $\infty$ and $V_\varphi$ has totally disconnected spectrum, we find that $\{\sigma(V_{\varphi_n})\}$ converges to the spectrum of $V_\varphi$ in the Hausdorff metric, see [37, Theorem 3]. The proof is complete. □

The next proposition follows easily from the fact that $\|V_\varphi^{j,j}\| \leqslant b_j - a_j$ for each $j \in J$ and we omit its proof.

**Proposition 2.2.9.** *Let $\varphi$ be a measurable self-map of $[0,1]$ with $\mu(\partial S_\varphi) = 0$. Then for each non-zero eigenvalue $\lambda$ of $V_\varphi$ the set $\{j \in J : \lambda \in \sigma(V_\varphi^{j,j})\}$ is finite.*

Thus, for each non-zero eigenvalue $\lambda$ of $V_\varphi$, we may consider

$$j(\lambda) = \max\{j \in J : \lambda \in \sigma(V_\varphi^{j,j})\}. \tag{2.2.2}$$

The next corollary improves Lemma 2.2.5.

**Corollary 2.2.10.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and $\max_{[0,y]} \varphi \leqslant y$ for a fixed point $y$ of $\varphi$ with $\mu(\partial S_\varphi) = 0$. If $f$ is an eigenfunction of $V_\varphi$ corresponding to an eigenvalue $\lambda \neq 0$, then $\operatorname{supp}(f) \subset [a_{j(\lambda)}, 1]$ and $f(y) \neq 0$ whenever $a_{j(\lambda)} < y$.*

*Proof.* Since $\lambda$ belongs to $\sigma(V_\varphi^{j(\lambda),j(\lambda)})$, by Theorem 2.2.8 and (2.2.2), there is a corresponding non-zero eigenfunction $f$ of $V_\varphi$ whose support is contained in $[a_{j(\lambda)}, 1]$. In addition, Lemma 2.2.5 ensures that $f(b_{j(\lambda)}) \neq 0$. Let $\widetilde{\varphi}$ be the restriction of $\varphi$ to $[b_{j(\lambda)}, 1]$. Theorem 2.2.8, along with (2.2.2) shows that $\lambda$ is not $\sigma(V_{\widetilde{\varphi}})$. Thus $V_{\widetilde{\varphi}} - \lambda$ is invertible. Now, a straightforward computation shows that

$$g(x) = \begin{cases} 0, & \text{if } 0 \leqslant x \leqslant a_{j(\lambda)}; \\ f(x), & \text{if } a_{j(\lambda)} < x \leqslant b_{j(\lambda)}; \\ -\lambda f(b_{j(\lambda)})\big((V_{\widetilde{\varphi}} - \lambda)^{-1}(1)\big)(x), & \text{if } b_{j(\lambda)} < x \leqslant 1 \end{cases}$$

is an eigenfunction of $V_\varphi$ corresponding to the eigenvalue $\lambda$. Since the hypotheses of Corollary 2.2.6 are satisfied, we find that $\lambda$ has geometric multiplicity 1 and therefore $g = cf$ for some constant $c$. Since $f(b_{j(\lambda)}) = g(b_{j(\lambda)}) \neq 0$, we have $f = g$, which proves the first statement. The second one is just an application of Lemma 2.2.5. □

### 2.2.4 Positivity of the eigenvalues

In this section we show that eigenvalues of $V_\varphi$ are non-negative provided $\varphi$ is increasing. We also estimate the sum of eigenvalues.

**Theorem 2.2.11.** *Let $\varphi$ be an increasing self-map of $[0,1]$. Then all eigenvalues of $V_\varphi$ are real and non-negative. Furthermore,*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) \leqslant \mu(\{x \in [0,1] : \varphi(x) > x\}) \leqslant 1. \tag{2.2.3}$$

The key to prove Theorem 2.2.11 is the concept of totally positive matrix. Given an $n$-square matrix $(a_{j,k})$, we will be considering $m$-square submatrices $(a_{j_p,k_q})$, where $1 \leqslant j_1 < \cdots < j_m \leqslant n$ and $1 \leqslant k_1 < \cdots < k_m \leqslant n$. A square matrix with real entries is *totally positive* when all its square submatrices have non-negative determinant. The following theorem, whose proof can be found in [32], furnishes the main spectral property of totally positive matrices.

**Theorem P.** *The eigenvalues of a totally positive matrix are real and non-negative.*

We need to consider matrices with 0–1 entries, which are totally positive. Let $\mathcal{M}$ denote the set of $n$-square matrices $(a_{j,k})$ defined by

$$a_{j,k} = \begin{cases} 0, & \text{if } j \leqslant m_k; \\ 1, & \text{otherwise,} \end{cases} \qquad \text{for each choice } 0 \leqslant m_1 \leqslant \cdots \leqslant m_n \leqslant n.$$

It is easy to check that $\det A \geqslant 0$ for $A$ in $\mathcal{M}$. Indeed, if $m_n = n$, then the last column of $A$ is zero and, therefore, $\det A = 0$. If there is $k$ such that $m_k = m_{k+1}$, then the $k$-th and $(k+1)$-th columns of $A$ coincide and $\det A = 0$ again. Finally, if $0 = m_1 < \cdots < m_n < n$, then $A$ is lower triangular with 1 on each entry of the main diagonal and $\det A = 1$. Consequently, since each square submatrix of a matrix in $\mathcal{M}$ is clearly in $\mathcal{M}$ again, each matrix in $\mathcal{M}$ is totally positive. Thus as a consequence of Theorem P, we have

**Proposition 2.2.12.** *The eigenvalues of each matrix in $\mathcal{M}$ are real and non-negative.*

Now, we can prove Theorem 2.2.11.

*Proof of Theorem 2.2.11.* Let $[r]$ denote the integer part of the real number $r$. For each positive integer $n$, consider the self-map of $[0,1]$ defined by

$$\varphi_n(x) = \frac{[n\varphi([nx]/n)]}{n},$$

where again the subscript $n$ does not mean iterate. Clearly, $\|\varphi - \varphi_n\|_1$ tends to 0 as $n$ tends to $\infty$ and, therefore, by Proposition 2.1.5,

$$\|V_\varphi - V_{\varphi_n}\| \to 0 \quad \text{as } n \to \infty. \tag{2.2.4}$$

Now, for $1 \leqslant j \leqslant n$, consider the characteristic functions $\chi_{j,n} = \chi_{((j-1)/n, j/n)}$. Since $\varphi_n$ is constant on each interval $((j-1)/n, j/n)$ and takes values in $\{j/n : 0 \leqslant j \leqslant n\}$, the range of $V_{\varphi_n}$ is contained in span$\{\chi_{j,n} : 1 \leqslant j \leqslant n\}$. Thus, non-zero eigenvalues and the corresponding generalized eigenfunctions of $V_{\varphi_n}$ are those of itself acting on its reducing subspace span$\{\chi_{j,n} : 1 \leqslant j \leqslant n\}$. Let $A_n = (a^n_{j,k})$ be the corresponding matrix with respect to the basis $\{\chi_{j,n} : 1 \leqslant j \leqslant n\}$. Using that $\varphi$ is increasing, one can check that

$$na^n_{j,k} = \begin{cases} 0, & \text{if } \varphi((j-1)/n) < k/n; \\ 1, & \text{otherwise.} \end{cases} \tag{2.2.5}$$

Thus $nA_n$ is in $\mathcal{M}$. By Lemma 2.2.12, the eigenvalues of $nA_n$ and, therefore, of $A_n$ are real and non-negative. Since $\sigma(V_{\varphi_n}) = \sigma(A_n) \cup \{0\}$, it follows from (2.2.4) that all eigenvalues of $V_\varphi$ are also real and non-negative.

It remains to prove (2.2.3). From (2.2.5), we have $na_{j,j} = 1$ if and only if

$$\varphi((j-1)/n) \geqslant j/n,$$

which implies that $\varphi(x) > x$ for $(j-1)/n < x < j/n$. Thus, the trace of $nA_n$ does not exceed the number of $j$'s in $\{1, \dots, n\}$ for which $\varphi(x) > x$ for $(j-1)/n < x < j/n$. It follows that

$$\operatorname{tr} V_{\varphi_n} = \operatorname{tr} A_n = \frac{1}{n} \operatorname{tr}(nA_n) \leqslant \mu(\{x \in [0,1] : \varphi(x) > x\}).$$

Since the eigenvalues of $V_{\varphi_n}$ are all real and non-negative, for each positive integer $k$ we have

$$\sum_{m=0}^{k} \lambda_m(\varphi_n) \leqslant \operatorname{tr} V_{\varphi_n} \leqslant \mu(\{x \in [0,1] : \varphi(x) > x\}), \qquad \text{for} \quad 1 \leqslant k < n.$$

Now, for fixed $k$, the map that to each operator $T$ assigns $\sum_{m=0}^{k} \lambda_m(T)$ is operator norm continuous on the space of compact operators. Thus we can make $n$ tend to $\infty$ in the above display to obtain that

$$\sum_{m=0}^{k} \lambda_m(\varphi) \leqslant \mu(\{x \in [0,1] : \varphi(x) > x\}), \qquad \text{for each positive integer } k.$$

Passing to the limit as $k$ tends to $\infty$, we obtain the inequality (2.2.3). □

From the proof of Theorem 2.2.11 we have

**Example 2.2.13.** *Consider the piecewise constant function* $\varphi_n(x) = ([nx]+1)/n$. *Then range of* $V_{\varphi_n}$ *is span*$\{\chi_{j,n} : 1 \leqslant j \leqslant n\}$*, as in the proof of Theorem* 2.2.11. *In* *this case, the only non-zero eigenvalue of $V_{\varphi_n}$ is $1/n$ and the Jordan form of $A_n$ is just one block. It follows that eigenvalues of a composition Volterra operator can have any algebraic multiplicity.*

**Remark.** Observe that Proposition 2.2.4 and Theorem 2.2.11 provide upper and lower sharp bounds for the spectral radius and the 'trace' of composition Volterra operators with increasing symbols. Indeed, if $\varphi = b\chi_{[a,1]}$, for $0 \leqslant a < b \leqslant 1$, the spectrum of $V_\varphi$ is a one-point set and the inequalities in Proposition 2.2.4 and Theorem 2.2.11 become equalities. Moreover, in Theorem 3.1.26 we will see that if $\varphi$ is continuous and $\varphi(x) \geqslant x$ for each $0 \leqslant x \leqslant 1$, then the upper bound found for the 'trace' in Theorem 2.2.11 is always achieved. Also, from Proposition 2.2.4 and Theorem 2.2.11, one immediately deduces the characterization of quasi-nilpotent of $V_\varphi$ for increasing symbols.

We close by observing that the spectrum of each $V_\varphi$ is symmetric with respect to the real axis.

**Proposition 2.2.14.** *Let $T$ be a bounded linear operator on $L^2[0,1]$ such that the space $L^2([0,1],\mathbb{R})$ is invariant under $T$. Then the spectrum of $T$ is symmetric with respect to the real axis.*

*Proof.* Since $L^2([0,1],\mathbb{R})$ is invariant under $T$, we have that $\overline{Tf} = T\overline{f}$ for each $f$ in $L^2[0,1]$. It follows that the Hilbert space adjoint $T^\star$ of $T$ coincides with the Banach space adjoint $T^*$, being the adjoint with respect to the dual pairing

$$(f,g) = \int_0^1 f(t)g(t)\,dt.$$

The equalities $\sigma(T^*) = \sigma(T)$ and $\sigma(T^\star) = \overline{\sigma(T)}$, which are true for any bounded linear operator on $L^2[0,1]$, imply that $\sigma(T^\star) = \overline{\sigma(T)}$. ☐

**Corollary 2.2.15.** *Let $\varphi$ be a measurable self-map of $[0,1]$. Then the spectrum of $V_\varphi$ is symmetric with respect to the real axis.*

# Chapter 3

# Spectrum of $V_\varphi$ and analyticity of its eigenfunctions

In Section 3.1, we go into a deeper analysis of the spectrum of $V_\varphi$ for those symbols satisfying $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. As usual in the Fredholm determinants Theory, specially appropriate when dealing with integral operators, there is an entire function $\mathcal{F}^\varphi$ associated to $V_\varphi$. Solving a differential equation, we prove not only that the inverses of the zeros of $\mathcal{F}^\varphi$ correspond to the eigenvalues of $V_\varphi$, but also that their multiplicities coincide. Once this is done, analyzing the growth of $\mathcal{F}^\varphi$, we provide several characterizations of the finiteness of the spectrum of $V_\varphi$. Under suitable hypotheses, the exponent of convergence of the sequence of eigenvalues of $V_\varphi$ is computed. For increasing, continuous symbols with the graph over the main diagonal we show that the sequence of eigenvalues is absolutely summable, which reflects that $V_\varphi$ behaves like a nuclear operator, although it is not. In such cases, we provide an explicit formula for the 'trace' of $V_\varphi$.

In Section 3.2, we turn our attention to the analyticity of the eigenfunctions of $V_\varphi$ being $\varphi$ analytic. The analyticity of the symbol $\varphi$ is inherited by the eigenfunctions provided that $\varphi(x) > x$ for $0 \leqslant x < 1$ and $\varphi'(1) < 1$. Here, the Fao de Bruno formula for the derivative of compositions will play a key role. If $\varphi'(1) = 1$, then we can provide examples in which the eigenfunctions are non-analytic although $\varphi$ is.

## 3.1 Spectral properties of $V_\varphi$

The main result in this section is that the eigenvalues of $V_\varphi$ are the inverses of the zeros of an entire function. Furthermore, the multiplicity of each zero of the entire function is the same as the algebraic multiplicity of the corresponding eigenvalue. Using this, we shall characterize the finiteness of the spectrum of $V_\varphi$. In view of the characterization of quasi-nilpotent composition Volterra operators and Theorem 2.2.8 about decomposition of the spectrum, we will focus on the following set of symbols

$$\Lambda = \{\varphi : [0,1] \to [0,1] \text{ continuous and such that } \varphi(x) \geqslant x \text{ for } 0 \leqslant x \leqslant 1\},$$

that we consider endowed with the topology inherited from the Banach space $\mathcal{C}[0,1]$.

### 3.1.1 The map $\mathcal{F}$

For each $\varphi$ in $\Lambda$, consider the bounded operator

$$(W_\varphi f)(x) = \int_{\varphi(x)}^{1} f(t)\, dt, \qquad f \in L^2[0,1].$$

For $\varphi(x) = x$, we just write $W_\varphi = W$. Now, we may define $\mathcal{F} : \Lambda \times [0,1] \times \mathbb{C} \to \mathbb{C}$ that to each element $(\varphi, x, z)$ assigns

$$\mathcal{F}^\varphi(x,z) = \mathcal{F}^\varphi_x(z) = \sum_{n=0}^{\infty} (-1)^n a_n^\varphi(x) z^n, \tag{3.1.1}$$

where $a_0^\varphi(x) = 1$ and $a_n^\varphi(x) = (WW_\varphi^{n-1}1)(x)$ for each $n \geqslant 1$.

In order to prove that $\mathcal{F}$ is well defined, some properties of $a_n^\varphi$ are needed. All the properties listed in proposition below follow immediately from the definition of $a_n^\varphi$ and, thus, their proofs are omitted.

**Proposition 3.1.1.** *For each $\varphi$ in $\Lambda$, the functions $a_n^\varphi$ satisfy the following properties.*

(i) $a_0^\varphi(1) = 1$ *and* $a_n^\varphi(1) = 0$ *for* $n \geqslant 1$.

(ii) *For each $n \geqslant 0$ the function $a_n^\varphi$ belongs to $\mathcal{C}^1[0,1]$ and*

$$(a_0^\varphi)' = 0 \quad \text{and} \quad (a_n^\varphi)'(x) = -a_{n-1}(\varphi(x)) = -(W_\varphi^{n-1}1)(x) \quad \text{for } n \geqslant 1.$$

(iii) *For each $n \geqslant 0$, the map $\varphi \mapsto a_n^\varphi$ is continuous from $\Lambda$ into $\mathcal{C}^1[0,1]$.*

(iv) *For each $n \geqslant 0$ and $0 \leqslant x \leqslant 1$, we have*

$$0 \leqslant a_n^\varphi(x) \leqslant \frac{(1-x)^n}{n!} \quad \text{and} \quad 0 \leqslant -(a_{n+1}^\varphi)'(x) \leqslant \frac{(1-x)^n}{n!}.$$

(v) *For each $n \geqslant 0$, the function $a_n^\varphi$ is decreasing.*

(vi) *If $\psi$ is in $\Lambda$ and $\varphi(x) \leqslant \psi(x)$ for $0 \leqslant x \leqslant 1$, then $a_n^\varphi(x) \geqslant a_n^\psi(x)$ for $n \geqslant 0$ and $0 \leqslant x \leqslant 1$.*

In what follows, $\mathcal{H}(\mathbb{C})$ stands for the space of entire functions endowed with the topology of uniform convergence on compact sets. As it was recalled in 1.4, for $F$ in $\mathcal{H}(\mathbb{C})$ the maximum modulus function

$$M(F,R) = \max_{|z|=R} |F(z)|, \qquad 0 \leqslant R < \infty,$$

is well defined and increasing.

The next proposition records some of the fundamental properties of $\mathcal{F}$. The differential equation satisfied by $\mathcal{F}^\varphi$ already suggests that $\mathcal{F}^\varphi$ is intimately related to the eigenvalue equation for $V_\varphi$.

**Proposition 3.1.2.** *The function $\mathcal{F}$ is well defined, differentiable with respect to $x$, holomorphic with respect to $z$ and $(\varphi, x) \mapsto \mathcal{F}^\varphi(x,\cdot)$ as well as $(\varphi,x) \mapsto \frac{\partial \mathcal{F}^\varphi}{\partial x}(x,\cdot)$ are continuous mappings from $\Lambda \times [0,1]$ into $\mathcal{H}(\mathbb{C})$. Furthermore,*

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = z\mathcal{F}^\varphi(\varphi(x),z), \tag{3.1.2}$$

$$\mathcal{F}^\varphi(1,z) = 1. \tag{3.1.3}$$

*In addition, we have the Taylor series representation*

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = \sum_{n=1}^{\infty} (-1)^n b_n(x) z^n, \tag{3.1.4}$$

*where $b_n(x) = (V_\psi^{n-1}1)(1-x)$ with $\psi(x) = 1 - \varphi(1-x)$.*

*Proof.* Clearly, Proposition 3.1.1 (iv) implies uniform convergence and uniform boundedness of the sums of the series in (3.1.1) and the series

$$\sum_{n=0}^{\infty} (-1)^n (a_n^\varphi)'(x) z^n$$

on $\Lambda \times [0,1] \times D_R$ for each $R > 0$, where $D_R = \{z \in \mathbb{C} : |z| \leqslant R\}$. Therefore, $\mathcal{F}$ is well defined, differentiable with respect to $x$, holomorphic with respect to $z$ and $(\varphi,x) \mapsto \mathcal{F}^\varphi(x,\cdot)$ as well as

$$(\varphi,x) \mapsto \frac{\partial \mathcal{F}^\varphi}{\partial x}(x,\cdot)$$

are continuous mappings from $\Lambda \times [0,1]$ into $\mathcal{H}(\mathbb{C})$. Since $(a_0^\varphi)' = 0$, it is clear that

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = \sum_{n=1}^{\infty} (-1)^n (a_n^\varphi)'(x) z^n. \tag{3.1.5}$$

Hence, by Proposition 3.1.1 (ii), we have

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = \sum_{n=1}^{\infty} (-1)^{n-1} a_{n-1}^\varphi(\varphi(x)) z^n = z \sum_{n=0}^{\infty} (-1)^n a_n^\varphi(\varphi(x)) z^n = z\mathcal{F}^\varphi(\varphi(x),z),$$

which is equality (3.1.2). From Proposition 3.1.1 (i), we also have the initial value (3.1.3).

Finally, consider the involutive isometry $(Uf)(x) = f(1-x)$ and compare (3.1.4) with (3.1.5). It follows from Proposition 3.1.1 (ii) that

$$b_n(x) = -(a_n^\varphi)'(x) = (W_\varphi^{n-1}1)(x).$$

On the other hand, $W_\varphi = UV_\psi U$. Therefore,

$$b_n(x) = (UV_\psi^{n-1}U1) = (V_\psi^{n-1}1)(1-x)$$

and (3.1.4) is also proved. The proof finished. □

The next proposition provides some basic monotonic properties of $\mathcal{F}$.

**Proposition 3.1.3.** *Let $\varphi$ be in $\Lambda$ and $c > 0$. We have,*

(a) *If $0 \leqslant x \leqslant y \leqslant 1$, then $1 \leqslant \mathcal{F}^\varphi(y,-c) \leqslant \mathcal{F}^\varphi(x,-c) \leqslant e^{c(1-x)}$.*

(b) *If $|z| \leqslant c$, then $|\mathcal{F}^\varphi(x,z)| \leqslant \mathcal{F}^\varphi(x,-c)$ for $0 \leqslant x \leqslant 1$.*

(c) *If $\psi$ is in $\Lambda$ with $\varphi(x) \leqslant \psi(x)$ for $0 \leqslant x \leqslant 1$, then $\mathcal{F}^\varphi(x,-c) \geqslant \mathcal{F}^\psi(x,-c)$ for $0 \leqslant x \leqslant 1$.*

*Proof.* The inequality $\mathcal{F}^\varphi(x,-c) \leqslant e^{c(1-x)}$ follows from (3.1.1) and Proposition 3.1.1 (iv). The rest of the inequalities in (a) and (b) follow from (3.1.1) and Proposition 3.1.1 (v). Finally, (c) also follows from (3.1.1) and Proposition 3.1.1 (i) and (vi). □

Some monotonic properties of $\mathcal{F}$ extend to the maximum modulus.

**Corollary 3.1.4.** *Let $\varphi$ be in $\Lambda$ and $R > 0$. Then the function $M(\mathcal{F}_x^\varphi, R)$ is decreasing with respect to $x$ and*

$$M(\mathcal{F}_x^\varphi, R) = \mathcal{F}^\varphi(x,-R) \leqslant e^{R(1-x)}, \qquad \text{for } \ 0 \leqslant x \leqslant 1.$$

*In addition, for $\psi$ in $\Lambda$ with $\varphi(x) \leqslant \psi(x)$ for $0 \leqslant x \leqslant 1$, we have*

$$M(\mathcal{F}_x^\varphi, R) \geqslant M(\mathcal{F}_x^\psi, R), \qquad \text{for } 0 \leqslant x \leqslant 1.$$

*Proof.* The first statement follows from Proposition 3.1.3 (a) and (b). The second statement follows from Proposition 3.1.3 (b) and (c). □

### 3.1.2 Existence and uniqueness of solution of a differential equation

We need an existence and uniqueness theorem of solution of certain differential equations. For a Banach space $\mathcal{B}$, the space $\mathcal{C}([a,b],\mathcal{B})$ is the one of continuous functions from $[a,b]$ into $\mathcal{B}$ endowed with the supremum norm.

**Proposition 3.1.5.** *Assume that $a \leqslant \alpha \leqslant b$ and let $\varphi$ be a continuous self-map of $[a,b]$ such that*

$$\begin{cases} x \leqslant \varphi(x) \leqslant \alpha, & \text{if } a \leqslant x \leqslant \alpha; \\ \alpha \leqslant \varphi(x) \leqslant x, & \text{if } \alpha \leqslant x \leqslant b. \end{cases}$$

*Assume also that $T$ is a bounded operator on a (real or complex) Banach space $\mathcal{B}$, $x_0$ belongs to $\mathcal{B}$ and $G$ belongs to $\mathcal{C}([a,b],\mathcal{B})$. Then the Cauchy problem*

$$\begin{cases} H'(x) = TH(\varphi(x)) + G(x), \\ H(\alpha) = x_0 \end{cases} \tag{3.1.6}$$

*has a unique solution $H : [a,b] \to \mathcal{B}$, which belongs to $\mathcal{C}^1([a,b],\mathcal{B})$.*

*Proof.* Consider the bounded operator $Q$ acting on $\mathcal{C}([a,b],\mathcal{B})$ defined by

$$(Qf)(x) = \int_x^\alpha Tf(\varphi(t))\,dt.$$

One easily checks that

$$\|Q^n\| \leqslant \frac{c^n\|T\|^n}{n!}, \qquad \text{where } c = \max\{\alpha - a, b - \alpha\}.$$

Thus $Q$ is quasi-nilpotent and, therefore, $I + Q$ is invertible. Upon integrating the equation in (3.1.6), we see that Cauchy problem (3.1.6) is equivalent to

$$H + QH = R, \qquad \text{where } \ R(x) = x_0 - \int_x^\alpha G(t)\,dt.$$

Thus $H = (I+Q)^{-1}R$ is the unique solution of (3.1.6). It is also obvious that $H$ is in $\mathcal{C}^1([a,b],\mathcal{B})$. □

As an immediate application of Proposition 3.1.5, we have the following lemma, which will be used later.

**Lemma 3.1.6.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and $\max_{[0,\alpha]} \varphi \leqslant \alpha$ for a fixed point $\alpha$ of $\varphi$. If $\mathcal{F}^\varphi(\alpha,\lambda) = 0$ for some $\lambda$ in $\mathbb{C}$, then $\mathcal{F}^\varphi(x,\lambda) = 0$ for $0 \leqslant x \leqslant \alpha$.*

*Proof.* Consider the functions $f, g : [0,1] \to \mathbb{C}$ defined by

$$f(x) = \mathcal{F}^\varphi(x,\lambda) \quad \text{and} \quad g(x) = \begin{cases} \mathcal{F}^\varphi(x,\lambda), & \text{if } \alpha \leqslant x \leqslant 1; \\ 0, & \text{if } 0 \leqslant x < \alpha. \end{cases}$$

Clearly, by Proposition 3.1.2, $f$ as well as $g$ are solutions of the Cauchy problem

$$\begin{cases} H'(x) = \lambda H(\varphi(x)), \\ H(1) = 1. \end{cases}$$

Therefore, by Proposition 3.1.5, we find $f = g$ and the result follows. □

### 3.1.3 The zeros of $\mathcal{F}_0^\varphi$ and the eigenvalues of $V_\varphi$

Theorem below not only states that the eigenvalues of $V_\varphi$ are the inverses of the zeros of $\mathcal{F}_0^\varphi$, but also there is a correspondence between the multiplicity of the zeros and the algebraic multiplicity of the eigenvalues.

**Theorem 3.1.7.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Then $\lambda \neq 0$ is a zero of order $k$ of $\mathcal{F}_0^\varphi$ if and only if $\lambda^{-1}$ is an eigenvalue of algebraic multiplicity $k$ of $V_\varphi$. Furthermore, in such a case, a basis for $\ker (V_\varphi - \lambda^{-1} I)^k$ is formed by*

$$g_j(x) = \frac{\partial^{j+1}\mathcal{F}^\varphi}{\partial x \partial z^j}(x,z)\bigg|_{z=\lambda}, \qquad \textit{for} \quad 0 \leqslant j \leqslant k-1.$$

*Proof.* First, suppose that $\mathcal{F}_0^\varphi(\lambda) \neq 0$ with $\lambda \neq 0$ and there is a non-zero $f$ in $L^2[0,1]$ such that $V_\varphi f = \lambda^{-1} f$. By Lemma 2.2.5, we may assume that

$$\int_0^1 f(t)\,dt = 1.$$

In particular,

$$F(x) = \int_0^x f(t)\,dt$$

belongs to $\mathcal{C}^1[0,1]$ with $F(0) = 0$ and $F(1) = 1$ and is a solution of the Cauchy problem

$$\begin{cases} H'(x) = \lambda H(\varphi(x)), \\ H(1) = 1. \end{cases}$$

On the other hand, from (3.1.2) and (3.1.3), we know that $\mathcal{F}^\varphi(\cdot,\lambda)$ is also a solution of the above Cauchy problem. Thus, by Proposition 3.1.5, we have $F = \mathcal{F}^\varphi(\cdot,\lambda)$ and, therefore,

$$0 \neq \mathcal{F}_0^\varphi(\lambda) = \mathcal{F}^\varphi(0,\lambda) = F(0) = 0,$$

a contradiction. Hence $\lambda^{-1}$ is not an eigenvalue of $V_\varphi$.

Conversely, suppose that $\lambda \neq 0$ is a zero of order $k \geqslant 1$ of $\mathcal{F}_0^\varphi$ and consider

$$G_j(x) = \left.\frac{\partial^j \mathcal{F}^\varphi(x,z)}{\partial z^j}\right|_{z=\lambda}, \qquad \text{for} \;\; j \geqslant 0.$$

Clearly, $G_j$ are in $\mathcal{C}^1[0,1]$ and, therefore, $g_j(x) = G_j'(x)$ are in $\mathcal{C}[0,1]$ for $j \geqslant 0$. Since $\lambda$ is a zero of order $k$ of $\mathcal{F}_0^\varphi$, we have

$$G_j(0) = 0, \quad \text{for } 0 \leqslant j \leqslant k-1 \qquad \text{and} \qquad G_k(0) \neq 0. \tag{3.1.7}$$

In particular, we have

$$G_j(x) = \int_0^x g_j(t)\,dt \qquad \text{for} \;\; 0 \leqslant j \leqslant k-1. \tag{3.1.8}$$

Upon differentiating successively with respect to $z$ on both sides of (3.1.2), and taking into account that $\mathcal{F}^\varphi(1,z) = 1$, we obtain

$$G_0'(x) = \lambda G_0(\varphi(x)) \quad \text{and} \quad G_j'(x) = \lambda G_j(\varphi(x)) + jG_{j-1}(\varphi(x)), \qquad \text{for} \;\; j \geqslant 1. \tag{3.1.9}$$

The last display along with (3.1.8) implies that

$$(I - \lambda V_\varphi)g_0 = 0 \quad \text{and} \quad (I - \lambda V_\varphi)g_j = jV_\varphi g_{j-1}, \qquad \text{for} \;\; 1 \leqslant j \leqslant k-1. \tag{3.1.10}$$

Now, an induction argument along with (3.1.10) shows that

$$(I - \lambda V_\varphi)g_j = \sum_{m=0}^{j-1} (-1)^{m+j-1} \frac{j!}{m!\lambda^{j-m}} g_m, \qquad \text{for} \;\; 1 \leqslant j \leqslant k-1. \tag{3.1.11}$$

Since, by Corollary 2.2.6, $\dim\ker(I/\lambda - V_\varphi) \leqslant 1$, it follows from (3.1.10) that $\ker(I/\lambda - V_\varphi)$ is one-dimensional and is spanned by $g_0$. In addition, from (3.1.11), it follows, for $1 \leqslant j \leqslant k$, that $\ker(I/\lambda - V_\varphi)^j$ has dimension $j$ and is spanned by $g_0, \ldots, g_{j-1}$. Thus $\lambda^{-1}$ is an eigenvalue of algebraic multiplicity at least $k$. To show that it is precisely $k$, it suffices to prove that $g_{k-1}$ is not in the range of $(I/\lambda - V_\varphi)$. But if $g_{k-1}$ is in the range of $(I/\lambda - V_\varphi)$, then by (3.1.10), there is $g$ in $L^2[0,1]$ such that $(I - \lambda V_\varphi)g = kV_\varphi g_{k-1}$. Therefore,

$$H_1(x) = G(x) - G(1)G_0(x), \quad \text{where} \;\; G(x) = \int_0^x g(t)\,dt$$

is a solution of the Cauchy problem

$$\begin{cases} H'(x) = \lambda H(\varphi(x)) + kG_{k-1}(x), \\ H(1) = 0. \end{cases}$$

Since $G_k$ is also a solution of the above Cauchy problem, Proposition 3.1.5 implies that $H_1 = G_k$. Hence

$$G_k(0) = H_1(0) = G(0) - G(1)G_0(0) = 0,$$

which contradicts (3.1.7). Thus $\lambda^{-1}$ has algebraic multiplicity $k$. The proof is complete. □

**Corollary 3.1.8.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Then $V_\varphi$ has no real negative eigenvalues.*

*Proof.* By Proposition 3.1.3, we have $\mathcal{F}_0^\varphi(-t) \geqslant 1$ for each $t \geqslant 0$. Thus $\mathcal{F}_0^\varphi$ has no negative zeros and the result follows from Theorem 3.1.7. □

**Remark.** Example 2.1.12 shows that the hypothesis $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ in Corollary 3.1.8 cannot be omitted.

### 3.1.4 Finiteness of $\sigma(V_\varphi)$

Next we proceed to characterize when $\sigma(V_\varphi)$ is finite. In order to do this, we need to estimate the growth of the entire function $\mathcal{F}_0^\varphi$. We begin with the following lemma.

**Lemma 3.1.9.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. If $0 \leqslant \alpha < \beta \leqslant 1$, then, for each $R > 0$, we have*

$$(1 + (\beta - \alpha)R)\mathcal{F}^\varphi(\eta, -R) \leqslant \mathcal{F}^\varphi(\alpha, -R) \leqslant (1 + (\beta - \alpha)R)\mathcal{F}^\varphi(\gamma, -R), \tag{3.1.12}$$

*where*

$$\gamma = \min\{\beta, \min_{[\alpha,\beta]} \varphi\} \quad \text{and} \quad \eta = \max_{[\alpha,\beta]} \varphi. \tag{3.1.13}$$

*Proof.* Using (3.1.2) in the second equality below, we have

$$\begin{aligned} \mathcal{F}^\varphi(\alpha, -R) &= \mathcal{F}^\varphi(\beta, -R) - \int_\alpha^\beta \frac{\partial \mathcal{F}^\varphi}{\partial x}(x, -R)\, dx \\ &= \mathcal{F}^\varphi(\beta, -R) + R\int_\alpha^\beta \mathcal{F}^\varphi(\varphi(x), -R)\, dx. \end{aligned}$$

Since $\mathcal{F}^\varphi(x, -R)$ is decreasing with respect to $x$, see Proposition 3.1.3 (a), and $\gamma \leqslant \varphi(x) \leqslant \eta$ for $\alpha \leqslant x \leqslant \beta$, we find that

$$\mathcal{F}^\varphi(\eta, -R) \leqslant \mathcal{F}^\varphi(\varphi(x), -R) \leqslant \mathcal{F}^\varphi(\gamma, -R),$$

which along with the above display implies (3.1.12). The result is proved. □

The next two lemmas establish that under suitable hypotheses the function $\mathcal{F}^\varphi(\cdot, -R)$ cannot grow arbitrarily.

**Lemma 3.1.10.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Assume also that $\varphi(x) > x$ for $u \leqslant x \leqslant v$ for certain $0 \leqslant u < v \leqslant 1$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$, with $\gamma_n \geqslant 0$, such that*

$$\sum_{n\in\mathbb{Z}} \gamma_n = v - u$$

*and*

$$\mathcal{F}^\varphi(u, -R) \leqslant \mathcal{F}^\varphi(v, -R) \prod_{n=-\infty}^{+\infty} (1 + \gamma_n R), \qquad \text{for each } R > 0. \tag{3.1.14}$$

*Proof.* Since $\varphi$ is continuous and $\varphi(x) > x$ for $u < x < v$, we may choose a bilateral sequence $\{\alpha_n\}$ satisfying $u \leqslant \alpha_n \leqslant \alpha_{n+1} \leqslant v$ for each $n$ in $\mathbb{Z}$ with $\alpha_n$ tending to $u$ as $n$ tends to $-\infty$ and $\alpha_n$ tending to $v$ as $n$ tends to $+\infty$, and in such a way that $\varphi(x) \geqslant \alpha_{n+1}$ for $\alpha_n \leqslant x \leqslant \alpha_{n+1}$. Setting $\gamma_n = \alpha_{n+1} - \alpha_n \geqslant 0$, we have

$$\sum_{n\in\mathbb{Z}} \gamma_n = v - u.$$

Moreover, Lemma 3.1.9 implies that

$$\mathcal{F}^\varphi(\alpha_n, -R) \leqslant \mathcal{F}^\varphi(\alpha_{n+1}, -R)(1 + \gamma_n R), \qquad \text{for each } n \in \mathbb{Z} \text{ and each } R > 0.$$

Therefore, the required inequality (3.1.14) follows by iterating inequality above and that $\mathcal{F}^\varphi(x, z)$ is continuous with respect to $x$. The result is proved. □

**Lemma 3.1.11.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Assume also that $\varphi(x) > x$ for $u < x < v$ for certain $0 \leqslant u < v \leqslant 1$. If either $\varphi(u) = u$ or $\varphi(x) < v$ for $u < x < v$, then there exists a sequence $\{\gamma_n\}_{n\geqslant 0}$, with $\gamma_n > 0$, such that*

$$\sum_{n=0}^{\infty} \gamma_n \leqslant v - u$$

*and*

$$\mathcal{F}^\varphi(u, -R) \geqslant \mathcal{F}^\varphi(v, -R) \prod_{n=0}^{\infty} (1 + \gamma_n R), \qquad \text{for each } R > 0. \tag{3.1.15}$$

*Proof.* If $\varphi(u) = u$, we take a strictly decreasing sequence $\{a_n\}_{n\geqslant 0}$ with $u < a_n < v$ and satisfying that $\{a_n\}$ tends to $u$ as $n$ tends to $\infty$ and $\varphi(a_{n+1}) < a_n$ for each non-negative integer $n$. If $\varphi(x) < v$ for $u < x < v$, we take a strictly increasing sequence

$\{a_n\}$ with $u < a_n < v$ satisfying that $\{a_n\}$ tends to $v$ as $n$ tends to $\infty$ and such that $\varphi(a_n) < a_{n+1}$ for each non-negative integer $n$. Thus, in any case there is a sequence of pairwise disjoint intervals $(\alpha_n, \delta_n)$ contained in $(u, v)$ such that $\varphi(\alpha_n) < \delta_n$ for each non-negative integer $n$. Now, we take $\beta_n$ in $(\alpha_n, \delta_n)$ such that $\max_{[\alpha_n,\beta_n]} \varphi \leqslant \delta_n$ and set $\gamma_n = \beta_n - \alpha_n$. Clearly, $\gamma_n > 0$ and since the $(\alpha_n, \delta_n)$ are pairwise disjoint and contained in $(u, v)$, we also have

$$\sum_{n=0}^{\infty} \gamma_n \leqslant v - u.$$

Hence Lemma 3.1.9 implies that

$$\mathcal{F}^\varphi(\alpha_n, -R) \geqslant \mathcal{F}^\varphi(\delta_n, -R)(1 + \gamma_n R), \qquad \text{for each } n \geqslant 0 \text{ and each } R > 0.$$

Since $\mathcal{F}^\varphi(x, -R)$ is decreasing with respect to $x$, see Proposition 3.1.3 (a), and $(\alpha_n, \delta_n)$ are disjoint, we have

$$\frac{\mathcal{F}^\varphi(u, -R)}{\mathcal{F}^\varphi(v, -R)} \geqslant \prod_{n=0}^{\infty} \frac{\mathcal{F}^\varphi(\alpha_n, -R)}{\mathcal{F}^\varphi(\delta_n, -R)} \geqslant \prod_{n=0}^{\infty} (1 + \gamma_n R),$$

which proves (3.1.15). The result is proved. □

Recall from 1.4 that the *exponential type* of an entire function $F$ is

$$\tau(F) = \varlimsup_{R\to\infty} \frac{\ln M(F, R)}{R}.$$

For a measurable self-map $\varphi$ of $[0, 1]$ we set $E(\varphi) = \{x \in [0, 1] \ : \ \varphi(x) = x\}$ and $\tau_x = \mu(E(\varphi) \cap [x, 1])$ where, as usual, $\mu$ is the Lebesgue measure. The next proposition shows that $\tau_x$ is precisely the exponential type of $\mathcal{F}^\varphi_x$.

In the reminder we make use of the widely extended Landau's asymptotic notation to compare functions. Given $f$ and $g$, functions defined on the real line, satisfying that there exists $M > 0$ such that for all large enough $x$ we have

$$|f(x)| \leqslant M|g(x)|,$$

we say that $f$ is big-$O$ of $g$ as $x$ goes to $+\infty$, in symbols

$$f(x) = O(g(x)), \qquad \text{as } x \to +\infty.$$

If on the other hand, for each $M > 0$ and all large enough $x$ we have

$$|f(x)| \leqslant M|g(x)|,$$

then we say that $f$ is small-$o$ of $g$ as $x$ goes to $+\infty$, in symbols

$$f(x) = o(g(x)), \qquad \text{as } x \to +\infty.$$

**Proposition 3.1.12.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Then for each $0 \leqslant x \leqslant 1$, there exists a sequence $\{\gamma_n\}_{n\geqslant 0}$, with $\gamma_n \geqslant 0$, such that*

$$\sum_{n=0}^{\infty} \gamma_n = 1 - x - \tau_x$$

*and*

$$e^{\tau_x R} \leqslant M(\mathcal{F}_x^{\varphi}, R) \leqslant e^{\tau_x R} \prod_{n=0}^{\infty} (1 + \gamma_n R), \qquad \textit{for each } R > 0. \tag{3.1.16}$$

*In particular, $\tau(\mathcal{F}_x^{\varphi}) = \tau_x$ for $0 \leqslant x \leqslant 1$.*

*Proof.* For $x = 1$ the result is trivial. Thus suppose that $0 \leqslant x < 1$ and $R > 0$. Propositions 3.1.2 and 3.1.3 imply that $\mathcal{F}^{\varphi}(\cdot, -R)$ is continuously differentiable, decreasing and $\mathcal{F}^{\varphi}(t, -R) \geqslant \mathcal{F}^{\varphi}(1, -R) = 1$ for $0 \leqslant t \leqslant 1$. Thus we may consider $f(t) = \ln(\mathcal{F}^{\varphi}(t, -R))$ which is clearly non-negative, decreasing and $f(1) = 0$. Therefore, by (3.1.2), we have

$$f(x) = -\int_x^1 f'(t)\,dt = R \int_x^1 \frac{\mathcal{F}^{\varphi}(\varphi(t), -R)}{\mathcal{F}^{\varphi}(t, -R)}\,dt.$$

Since $\mathcal{F}^{\varphi}(\varphi(t), -R) \leqslant \mathcal{F}^{\varphi}(t, -R)$ for $\varphi(t) \geqslant t$ and $\varphi(t) = t$ for $t$ in $E(\varphi)$, and the last set is closed, we find that $[x, 1] \setminus E(\varphi)$ is a union of disjoint open intervals $(u_j, v_j)$, $j \geqslant 1$, where there are possibly finitely many non-empty $(u_j, v_j)$. Therefore, we may split the last integral in the above display so that

$$R \int_{E(\varphi)\cap[x,1]} dt \leqslant f(x) = R \int_{E(\varphi)\cap[x,1]} dt + \sum_{j=1}^{\infty} (f(u_j) - f(v_j)),$$

where the last series may have finitely many terms different from zero. Hence,

$$\tau_x R \leqslant f(x) = \tau_x R + \sum_{j=1}^{\infty} \ln \frac{\mathcal{F}^{\varphi}(u_j, -R)}{\mathcal{F}^{\varphi}(v_j, -R)}. \tag{3.1.17}$$

Since $\varphi(t) > t$ for $u_j < t < v_j$ for each $j \geqslant 1$, we may apply Lemma 3.1.10 and, therefore, there is a sequence $\{\gamma_{j,k} : j \geqslant 1 \text{ and } k \in \mathbb{Z}\}$ of non-negative real numbers, such that

$$\sum_{k=-\infty}^{\infty} \gamma_{j,k} = v_j - u_j \qquad \text{and} \qquad \frac{\mathcal{F}^{\varphi}(u_j, -R)}{\mathcal{F}^{\varphi}(v_j, -R)} \leqslant \prod_{k=-\infty}^{\infty} (1 + \gamma_{j,k} R), \qquad \text{for each } j \geqslant 1. \tag{3.1.18}$$

Upon rearranging $\{\gamma_{j,k}\}$ in just one sequence $\{\gamma_n\}_{n\geqslant 0}$, we see that

$$\sum_{n=0}^{\infty} \gamma_n = 1 - x - \tau_x.$$

It also follows from (3.1.17) and (3.1.18) that

$$e^{\tau_x R} \leqslant e^{f(x)} = \mathcal{F}^{\varphi}(x,-R) \leqslant e^{\tau_x R}\prod_{n=0}^{\infty}(1+\gamma_n R),$$

from which (3.1.16) follows.

Finally, for each $\varepsilon > 0$, we take a positive integer $m$ such that $\sum_{k=m}^{\infty}\gamma_k < \varepsilon$. Then

$$\ln \prod_{k=m}^{\infty}(1+\gamma_k R) = \sum_{k=m}^{\infty}\ln(1+\gamma_k R) \leqslant R\sum_{k=m}^{\infty}\gamma_k < R\varepsilon.$$

The above inequality along with (3.1.16) shows that

$$e^{\tau_x R} \leqslant M(\mathcal{F}_x^{\varphi}, R) = O(R^m e^{(\tau_x+\varepsilon)R}), \quad \text{as } R \to \infty.$$

Therefore, $\tau_x \leqslant \tau(\mathcal{F}_x^{\varphi}) \leqslant \tau_x + \varepsilon$. Since $\varepsilon$ was arbitrary, we conclude $\tau(\mathcal{F}_x^{\varphi}) = \tau_x$. The result is proved □

The most important case is the one for which $\mu(E(\varphi)) = 0$. We have

**Corollary 3.1.13.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Assume also that the set of points for which $\varphi(x) = x$ has zero Lebesgue measure. Then $\mathcal{F}_x^{\varphi}$ is of exponential type $0$ for each $0 \leqslant x \leqslant 1$.*

As mentioned in 1.4, the *order* of an entire function $f$ is defined as

$$\rho(f) = \overline{\lim_{R\to\infty}} \frac{\ln\ln M(f,R)}{\ln R}.$$

It should be noted that if $0 < \tau(f) < \infty$, then $\rho(f) = 1$ and if $\tau(f) = 0$, then $\rho(f) \leqslant 1$. It is well known that if $F$ in $\mathcal{H}(\mathbb{C})$ has finitely many zeros, then $F(z) = p(z)e^{g(z)}$, where $p$ is a polynomial and $g \in \mathcal{H}(\mathbb{C})$. We need a more precise result.

**Lemma 3.1.14.** *Let $F$ be an entire function with finitely many zeros and of finite exponential type. Then $F(z) = p(z)e^{az}$, where $p$ is a polynomial and $|a| = \tau(F)$.*

*Proof.* Since $F$ is of finite exponential type, we have $\rho(F) \leqslant 1$. Thus Hadamard's Theorem, see 1.4.2 or [29, p. 24], implies that

$$F(z) = z^m e^{bz+c}\prod_{k=1}^{n}\left(1-\frac{z}{\lambda_k}\right)e^{z/\lambda_k},$$

where $m \geqslant 0$, $b$ and $c$ belongs to $\mathbb{C}$ and $\lambda_k$, $1 \leqslant k \leqslant n$, are the zeros of $F$ repeated according to their multiplicities. Therefore, $F(z) = p(z)e^{az}$, where $p$ is a polynomial and $a$ belongs to $\mathbb{C}$. It is also clear that $|a| = \tau(F)$. □

The next theorem provides several equivalent conditions for $\sigma(V_\varphi)$ to be finite. Observe that the assumption $\max_{[0,x]}\varphi \leqslant x$ for each fixed point $x$ of $\varphi$ is a weaker assumption than increasing. Recall that $\varphi_n$ denotes the $n$-th iterate of $\varphi$.

**Theorem 3.1.15.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ satisfying $\sup\{x : \varphi(x) > x\} = 1$ and $\max_{[0,y]}\varphi \leqslant y$ for each fixed point $y$ of $\varphi$. Then the following are equivalent.*

(i) *The spectrum $\sigma(V_\varphi)$ is finite.*

(ii) *There exists a positive integer $n$ such that $\varphi_n(x) = 1$ for each $0 \leqslant x \leqslant 1$.*

(iii) *The map $\varphi$ is identically 1 on a neighborhood of 1 and $\varphi(x) > x$ for $0 \leqslant x < 1$.*

(iv) *For some positive integer $n$, the operator $V_\varphi^n$ has finite rank.*

(v) *If $P$ is the orthogonal projection onto the constant functions, then $P - V_\varphi$ is nilpotent.*

*Proof.* It is elementary to verify that (ii) is equivalent to (iii).

Now we shall prove that (ii) is equivalent to (v). First, it is easy to see that $(P - V_\varphi)^n$ is the integral operator $J_{K_n}$ with kernel

$$K_n(y_0, y_n) = \int_{[0,1]^{n-1}} \chi_n(y_0, y_1, \ldots, y_{n-1}, y_n)\, dy_1 \cdots dy_{n-1},$$

where $\chi_n$ is the characteristic function of

$$S_n = \{y \in [0,1]^{n+1} \text{ such that } y_j \geqslant \varphi(y_{j-1}), \quad \text{for } 1 \leqslant j \leqslant n+1\}.$$

Clearly, $(P - V_\varphi)^n = 0$ if and only if $K_n = 0$ a.e. By Fubini's Theorem, this is equivalent to the fact that $S_n$ has zero Lebesgue measure. Now, it is easy to check that $S_n$ has zero Lebesgue measure if and only if $\varphi_n$ is identically 1. For instance, if $\varphi_n(x) = 1$ for $0 \leqslant x \leqslant 1$, then $S_n$ is contained in $\{y \in [0,1]^{n+1}$ such that $y_n = 1\}$. Thus, (ii) is equivalent to (v).

Since $P$ is a rank one operator, we have that (v) implies (iv). Obviously (iv) implies (i).

Thus the proof will be finished once we have shown that (i) implies (ii). To this end, suppose that $\sigma(V_\varphi)$ is finite and $\varphi$ does not satisfy (ii). We will distinguish two cases.

*Case 1. There is $0 \leqslant w < 1$ such that $\varphi(w) = w$.* The hypotheses on $\varphi$ allow us to choose fixed points $u$ and $v$ of $\varphi$ such that $\varphi(x) > x$ for $u < x < v$. Proposition 3.1.12

implies that $\mathcal{F}_u^\varphi$ and $\mathcal{F}_v^\varphi$ have the same exponential type

$$\tau = \mu(E(\varphi) \cap [v, 1]).$$

The hypotheses imply that $\varphi(x) \leqslant u$ if $x \leqslant u$ and $\varphi(x) \leqslant v$ if $x \leqslant v$. Therefore from Lemma 3.1.6 and Theorem 3.1.7, it follows that if $\lambda \neq 0$ is a zero of $\mathcal{F}_u^\varphi \mathcal{F}_v^\varphi$, then $\lambda^{-1}$ is an eigenvalue of $V_\varphi$. Since $V_\varphi$ has finitely many eigenvalues, we find that $\mathcal{F}_u^\varphi$ and $\mathcal{F}_v^\varphi$ have finitely many zeros. Consequently, by Lemma 3.1.14, there are polynomials $p$ and $q$ and $a$ and $b$ in $\mathbb{C}$ such that $|a| = |b| = \tau$ and $\mathcal{F}_u^\varphi = p(z)e^{az}$ and $\mathcal{F}_v^\varphi = q(z)e^{bz}$. In addition, by Proposition 3.1.3, we have

$$M(\mathcal{F}_u^\varphi, R) = \mathcal{F}_u^\varphi(-R) \quad \text{and} \quad M(\mathcal{F}_v^\varphi, R) = \mathcal{F}_v^\varphi(-R), \quad \text{for } R > 0.$$

Therefore $a = b = -\tau$. Hence $\mathcal{F}_u^\varphi(z) = p(z)e^{-\tau z}$ and $\mathcal{F}_v^\varphi(z) = q(z)e^{-\tau z}$. Thus there exists a positive integer $m$ such that

$$\mathcal{F}^\varphi(u, -R) = O(R^m \mathcal{F}^\varphi(v, -R)), \qquad \text{as } R \to \infty. \tag{3.1.19}$$

On the other hand, since $\varphi(x) > x$ for $u < x < v$ and $\varphi(u) = u$, by Lemma 3.1.11, there is a sequence of positive numbers $\{\gamma_k\}_{k \geqslant 0}$ such that

$$\mathcal{F}^\varphi(u, -R) \geqslant \mathcal{F}^\varphi(v, -R) \prod_{k=0}^{\infty} (1 + \gamma_k R),$$

which contradicts (3.1.19).

*Case 2.* $\varphi(x) > x$ *for* $0 \leqslant x < 1$. By Corollary 3.1.13 and Theorem 3.1.7, we find that $\mathcal{F}_0^\varphi$ has zero exponential type and finitely many zeros. Therefore, by Lemma 3.1.14, we have that $\mathcal{F}_0^\varphi$ is a polynomial. In particular, there is a positive integer $n$ such that $M(\mathcal{F}_0^\varphi, R) = O(R^n)$ as $R$ tends to $\infty$ and thus, by Proposition 3.1.3, we have $M(\mathcal{F}_x^\varphi, R) = O(R^n)$ for each $0 \leqslant x \leqslant 1$. Hence, $\mathcal{F}_x^\varphi$ is a polynomial of degree at most $n$ for each $0 \leqslant x \leqslant 1$ and, therefore, $a_m^\varphi = 0$ for $m \geqslant n + 1$. Upon applying successively Proposition 3.1.1 (ii), we see that $a_{n-j+1}(\varphi_j(x)) = 0$ for $1 \leqslant j \leqslant n$ and $0 \leqslant x \leqslant 1$. Since $\varphi_n$ is not identically 1, we find that $\varphi_n([0, 1])$ is equal to $[\alpha, 1]$ for some $0 \leqslant \alpha < 1$. We conclude that all $a_n^\varphi$ for $n \geqslant 1$ vanish identically on $[\alpha, 1]$. Thus, according to (3.1.1), we find that $\mathcal{F}^\varphi(x, z)$ is identically 1 on $[\alpha, 1]$ and $z$ in $\mathbb{C}$, which contradicts equation (3.1.2). The proof is complete. □

**Remark.** If $\varphi$ is a continuous self-map of $[0, 1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and $\alpha = \sup\{x : \varphi(x) > x\} < 1$, then the spectra of $V_\varphi$ acting on $L^2[0, 1]$ and $V_\varphi$ acting on $L^2[0, \alpha]$ coincide. Moreover, each generalized eigenfunction of $V_\varphi$ acting on $L^2[0, \alpha]$ is the restriction of a generalized eigenfunction of $V_\varphi$ acting on $L^2[0, 1]$. Therefore, from Theorem 3.1.15, we have

**Corollary 3.1.16.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and $\max_{[0,y]} \varphi \leqslant y$ for each fixed point $y$ of $\varphi$. Then the following are equivalent.*

(i) *The spectrum of $V_\varphi$ is finite.*

(ii) *Either $\varphi$ is the identity or there exist $0 < \beta < \alpha = \sup\{x : \varphi(x) > x\}$ such that*

$$\begin{cases} \varphi(x) > x, & \text{if } 0 \leqslant x < \beta; \\ \varphi(x) = \alpha, & \text{if } \beta \leqslant x \leqslant \alpha; \\ \varphi(x) = x, & \text{if } \alpha < x \leqslant 1. \end{cases}$$

The following corollary is an immediate consequence of Theorem 3.1.15.

**Corollary 3.1.17.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$. If $\varphi(0) = 0$ or $\varphi^{-1}(1)$ does not contain an interval of the form $[1-\varepsilon, 1]$ for some $\varepsilon > 0$, then the spectrum of $V_\varphi$ is infinite.*

### 3.1.5 Exponent of convergence of the eigenvalues of $V_\varphi$

Recall that the sequence of non-zero eigenvalues of $V_\varphi$ is denoted by $\{\lambda_n(\varphi)\}$, arranged in decreasing order of moduli and such that each eigenvalue appears as many times as its multiplicity indicates. Recall also from 1.4 that the *convergence exponent* $s(\{\lambda_n\})$ of a sequence $\{\lambda_n\}_{n\geqslant 0}$ is the infimum of $c > 0$ for which

$$\sum_{n=0}^{\infty} |\lambda_n|^c$$

is finite. For the sake of brevity we write $s(\varphi) = s(\{\lambda_n(\varphi)\})$.

Let $\varphi$ be a continuous increasing self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and $\varphi(y) < \varphi(x)$ whenever $y < x$ and $\varphi(y) < 1$, which means that $\varphi$ is strictly increasing on $\varphi_{-1}([0,1])$, where $\varphi_{-1}$ is as in subsection 2.1.2. In the present case, $\varphi_{-1}$ is clearly continuous on $[0,1]$. Thus we can extend the notation $\varphi_n$ for the iterates of $\varphi$ in the obvious way to all the integers $\mathbb{Z}$. Observe that if $\varphi(0) = 0$ and $\varphi^{-1}(1) = \{1\}$, then $\varphi$ is invertible and $\varphi_{-1}$ is just the inverse of $\varphi$.

The next proposition records some of the properties of the iterates $\varphi_n$, which are very easy to check. Throughout the remainder of this subsection, we denote by $\{\gamma_n(\varphi, c)\}$ the sequence of differences of $\{\varphi_n(c)\}$, that is, $\gamma_n(\varphi, c) = \varphi_n(c) - \varphi_{n-1}(c)$.

**Proposition 3.1.18.** *Let $\varphi$ be a continuous increasing self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and $\varphi(y) < \varphi(x)$ whenever $y < x$ and $\varphi(y) < 1$. Assume also that $0 < c < 1$. Then*

(i) *For each $n$ in $\mathbb{Z}$, the interval $\varphi([\varphi_{n-1}(c), \varphi_n(c)])$ is contained $[\varphi_n(c), \varphi_{n+1}(c)]$.*

(ii) *The sequence $\{\varphi_n(c)\}_{n\in\mathbb{Z}}$ is increasing.*

(iii)

$$\lim_{n\to-\infty} \varphi_n(c) = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \varphi_n(c) = 1.$$

(iv) *If $\varphi(0) = 0$, then $\varphi(\varphi_n(c)) = \varphi_{n+1}(c)$ for each $n$ in $\mathbb{Z}$. If in addition, $\varphi(x) = 1$ only for $x = 1$, then $\{\varphi_n(c)\}_{n\in\mathbb{Z}}$ is strictly increasing.*

(v) *The sequence of differences $\{\gamma_n(\varphi, c)\}_{n\in\mathbb{Z}}$ satisfies*

$$\sum_{n=-\infty}^{\infty} \gamma_n(\varphi, c) = 1.$$

We observe that $\varphi_n(c) > 0$ for each $n \in \mathbb{Z}$ if and only if $\varphi(0) = 0$. Also, $\varphi_n(c) < 1$ for each $n \in \mathbb{Z}$ if and only if $\varphi(x) = 1$ only for $x = 1$.

**Proposition 3.1.19.** *Let $\varphi$ be a continuous self-map of $[0, 1]$ with $\varphi(x) > x$ for $0 < x < 1$ and $\varphi(y) < \varphi(x)$ whenever $y < x$ and $\varphi(y) < 1$. Then for each $0 < c < 1$ and $R > 0$, we have*

$$\prod_{n=-\infty}^{\infty} (1 + \gamma_{2n}(\varphi, c)R) \leqslant \mathcal{F}^\varphi(0, -R) \leqslant \prod_{n=-\infty}^{\infty} (1 + \gamma_n(\varphi, c)R). \tag{3.1.20}$$

*Proof.* We set $\gamma_n = \gamma_n(\varphi, c)$. Upon taking $[\alpha, \beta] = [\varphi_n(c), \varphi_{n+1}(c)]$ in Lemma 3.1.9, we have

$$(1+\gamma_{n+1}R)\mathcal{F}^\varphi(\varphi_{n+2}(c),-R) \leqslant \mathcal{F}^\varphi(\varphi_n(c),-R) \leqslant (1+\gamma_{n+1}R)\mathcal{F}^\varphi(\varphi_{n+1}(c),-R). \tag{3.1.21}$$

Since $\mathcal{F}^\varphi(x, -R)$ is continuous, decreasing with respect to $x$ and $\mathcal{F}^\varphi(1, -R) = 1$, see Propositions 3.1.2 and 3.1.3, the second inequality in (3.1.20) follows by applying the second one in (3.1.21) for each $n$ in $\mathbb{Z}$ and taking limits. Similarly, the first inequality in (3.1.20) follows from the first one in (3.1.21), but considering only odd integers. The result is proved. □

The following lemma must be known to the experts. Since we have not found a precise reference, we include a proof.

**Lemma 3.1.20.** *Let $F$ be an entire function of exponential type 0. Let $\{\lambda_n\}_{n\geqslant 0}$ be the sequence of non-null zeros of $F$ repeated accordingly to their multiplicity. Then the exponent of convergence $s(\{\lambda_n^{-1}\})$ is equal to the order of $F$.*

*Proof.* First, for any entire function we have that $s(\{\lambda_n^{-1}\}) \leqslant \rho(F)$, see 1.4.5 or [29, Theorem 2, p. 18]. If $\rho(F) < 1$, the statement is just that of Theorem 1 in [29, p. 31] or the remark after 1.4.5.

Now, proceeding by contradiction, assume that $s(\{\lambda_n^{-1}\}) < \rho(F) = 1$, then by Hadamard's Theorem [29, p. 26] we have $F(z) = z^m e^{az+b} G(z)$, where $a$ and $b$ are complex numbers, $m$ is a non-negative integer and

$$G(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{\lambda_n}\right).$$

Moreover, by Borel's Theorem, see 1.4.1 or [29, p. 30], $\rho(G) = s(G) = s(\{\lambda_n^{-1}\}) < 1$. Hence $a \neq 0$, otherwise $\rho(F) = \rho(G) < 1$. But then one checks that $\tau(F) = |a| \neq 0$, which is a contradiction. ☐

The following Theorem provides estimates on the exponent of convergence of the eigenvalues of $V_\varphi$.

**Theorem 3.1.21.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$. Set*

$$\rho_+ = \rho_+(\varphi) = \max\left\{ \overline{\lim_{x\to 0}} \frac{\ln(\varphi(x) - x)}{\ln x},\ \overline{\lim_{x\to 1}} \frac{\ln(\varphi(x) - x)}{\ln(1-x)} \right\}$$

*and*

$$\rho_- = \rho_-(\varphi) = \max\left\{ \underline{\lim_{x\to 0}} \frac{\ln(\varphi(x) - x)}{\ln x},\ \underline{\lim_{x\to 1}} \frac{\ln(\varphi(x) - x)}{\ln(1-x)} \right\}.$$

*Then we have*

$$\frac{\rho_- - 1}{\rho_-} \leqslant s(\varphi) \leqslant \frac{\rho_+ - 1}{\rho_+}. \tag{3.1.22}$$

*Proof.* We begin by observing that $1 \leqslant \rho_- \leqslant \rho_+ \leqslant \infty$. By Theorem 3.1.7, we know that $s(\varphi)$ is equal to the exponent of convergence of the inverse of the zeros of $\mathcal{F}_0^\varphi$. Since by Corollary 3.1.13, we have $\tau(\mathcal{F}_0^\varphi) = 0$, Lemma 3.1.20 implies $0 \leqslant s(\varphi) \leqslant 1$. Therefore, if $\rho_+ = \infty$, the right inequality in (3.1.22) is trivially satisfied. If $\rho_+ < \infty$, then we can take $r > 0$ such that $(r+1)/r > \rho_+$. Clearly, we can choose a continuous, strictly increasing self-map $\psi$ of $[0,1]$ with $\psi(0) = 0$ and $\psi(x) > x$ for $0 < x < 1$ satisfying

$$\psi(x) \leqslant \varphi(x) \qquad \text{for } 0 \leqslant x \leqslant 1,$$

$$\psi(x) = \frac{x}{(1 - x^{1/r})^r} \qquad \text{on a neighborhood of } 0,$$

and

$$\psi(x) = 1 - \frac{1-x}{(1 + (1-x)^{1/r})^r} \qquad \text{on a neighborhood of } 1.$$

Now, take $0 < c < 1$ and set $\gamma_n = \gamma_n(\psi, c) = \psi_{n+1}(c) - \psi_n(c)$ for each $n$ in $\mathbb{Z}$. Plainly, for large enough $a > 0$, we have

$$\psi\left((1+a)^{-r}\right) = a^{-r} \quad \text{and} \quad \psi\left(1 - a^{-r}\right) = 1 - (a+1)^{-r}.$$

Hence, there is a positive integer $n_0$ and real numbers $a_+$ and $a_-$ such that $\psi_n(c) = 1 - (n + a_+)^{-r}$ for $n \geqslant n_0$ and $\psi_n(c) = (|n| + a_-)^{-r}$ for $n \leqslant -n_0$. Therefore,

$$\gamma_n \sim r|n|^{-r-1} \qquad \text{as } |n| \to \infty.$$

On the other hand, by Corollary 3.1.4 and Proposition 3.1.19, for each $R > 0$, we have

$$M(\mathcal{F}_0^\varphi, R) \leqslant M(\mathcal{F}_0^\psi, R) \leqslant \prod_{n=-\infty}^{\infty} (1 + \gamma_n R) = M(G, R)$$

where

$$G(z) = \prod_{n=-\infty}^{\infty} (1 + \gamma_n z).$$

By Lemma 3.1.20, we have $s(\varphi) = \rho(\mathcal{F}_0^\varphi) \leqslant \rho(G) = s(G)$. Finally, since the asymptotic formula for $\gamma_n$ shows that $s(G) = (r+1)^{-1}$, we find that

$$s(\varphi) \leqslant \inf\left\{\frac{1}{r+1} \;:\; r > 0 \;\text{ and }\; \frac{r+1}{r} > \rho_+\right\} = \frac{\rho_+ - 1}{\rho_+},$$

which is the right inequality in (3.1.22).

The left inequality in (3.1.22) is trivial if $\rho_- = 1$. If $\rho_- > 1$, we can take $r > 0$ such that $(r+1)/r < \rho_-$. By definition of $\rho_-$, there is a continuous increasing self-map $\psi$ of $[0, 1]$ with $\psi(x) > x$ for $0 < x < 1$ and $\psi(y) < \psi(x)$ if $y < x$ whenever $\psi(y) < 1$ and satisfying

$$\begin{aligned} &\psi(x) \geqslant \varphi(x) && \text{for } 0 \leqslant x \leqslant 1 \\ \text{and either } &\psi(x) = \frac{x}{(1 - x^{1/r})^r} && \text{on a neighborhood of } 0 \\ \text{or } &\psi(x) = 1 - \frac{1-x}{(1 + (1-x)^{1/r})^r} && \text{on a neighborhood of } 1. \end{aligned}$$

Again take $0 < c < 1$ and set $\gamma_n = \gamma_n(\psi, c)$. As in the proof of the right inequality, there is a positive $n_0$ such that either $\gamma_n \sim rn^{-r-1}$ as $n$ tends to $+\infty$ or $\gamma_n \sim r|n|^{-r-1}$ as $n$ tends to $-\infty$. Again, by Corollary 3.1.4 and Proposition 3.1.19, for any $R > 0$, we have

$$M(\mathcal{F}_0^\varphi, R) \geqslant M(\mathcal{F}_0^\psi, R) \geqslant \prod_{n=-\infty}^{\infty} (1 + \gamma_{2n} R) = M(G, R),$$

where

$$\text{where} \;\; G(z) = \prod_{n=-\infty}^{\infty} (1 + \gamma_{2n} z).$$

The asymptotic behavior of $\gamma_n$ shows that $s(G) \geqslant (r+1)^{-1}$. Since $s(\varphi) = \rho(\mathcal{F}_0^\varphi) \geqslant \rho(G) = s(G)$, we obtain

$$s(\varphi) \geqslant \sup\left\{ \frac{1}{r+1} \; : \; r > 0 \;\; \text{and} \;\; \frac{r+1}{r} < \rho_- \right\} = \frac{\rho_- - 1}{\rho_-},$$

which is the left inequality in (3.1.22). The proof is complete. □

The next corollaries follow immediately from Corollary 3.1.13, Lemma 3.1.20 and Theorems 3.1.7 and 3.1.21.

**Corollary 3.1.22.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and such that there exist*

$$\rho_0 = \lim_{x \to 0} \frac{\ln(\varphi(x) - x)}{\ln x} \qquad \text{and} \qquad \rho_1 = \lim_{x \to 1} \frac{\ln(\varphi(x) - x)}{\ln(1-x)}.$$

*Then*

$$s(\varphi) = \rho(\mathcal{F}_0^\varphi) = \frac{\rho - 1}{\rho},$$

*where $\rho = \max\{\rho_0, \rho_1\}$.*

**Corollary 3.1.23.** *Let $\varphi$ be a continuous self map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and satisfying*

$$\lim_{x \to 0} \frac{\ln(\varphi(x) - x)}{\ln x} = \lim_{x \to 1} \frac{\ln(\varphi(x) - x)}{\ln(1-x)} = 1.$$

*Then $s(\varphi) = \rho(\mathcal{F}_0^\varphi) = 0$. In particular, this holds whenever $\varphi$ is differentiable at $0$ and $1$ with $1 < \varphi'(0) \leqslant \infty$ and $\varphi'(1) < 1$.*

Now we deal with the summability of $|\lambda_n(\varphi)|$. We begin with a lemma that guarantees the summability of the inverses of the zeros of an entire function.

**Lemma 3.1.24.** *Let $\{\lambda_n\}_{n \geqslant 0}$ be the sequence of zeros of an entire function $F$ repeated accordingly to their multiplicity and where $F(0) = 1$. Assume also that there is a sequence $\{\gamma_n\}_{n \geqslant 0}$ in $(0,1)$ such that*

$$\sum_{k=0}^{\infty} \gamma_k (1 + |\ln \gamma_k|) < \infty \qquad \text{and} \qquad M(F,R) \leqslant \prod_{k=0}^{\infty} (1 + \gamma_k R) \;\; \textit{for each } R > 0. \quad (3.1.23)$$

*Then*

$$\sum_{n=0}^{\infty} \frac{1}{|\lambda_n|} < \infty.$$

*Proof.* Let $N(R)$ be the number of the zeros of $F$ in the disk $|z| \leqslant R$. By a corollary of Jensen's Theorem, see 1.4.4 or Lemma 4 in [29, p. 15], we have

$$N(R) \leqslant \ln M(F, eR), \qquad \text{for each } R > 0.$$

Using the above estimate in the last inequality below we find that

$$\sum_{|\lambda_n| \geqslant e^{-2}} \frac{1}{|\lambda_n|} = \sum_{k=0}^{\infty} \sum_{e^{k-2} \leqslant |\lambda_n| < e^{k-1}} \frac{1}{|\lambda_n|} \leqslant \sum_{k=0}^{\infty} e^{2-k} N(e^{k-1}) \leqslant e^2 \sum_{k=0}^{\infty} e^{-k} \ln M(F, e^k).$$

The second inequality in (3.1.23) implies that the last display is less than

$$e^2 \sum_{k=0}^{\infty} e^{-k} \sum_{j=0}^{\infty} \ln(1 + \gamma_j e^k) \; = e^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} e^{-k} \ln(1 + \gamma_j e^k).$$

Since $f(t) = e^{-t} \ln(1 + \gamma_j e^t)$ is decreasing on $[0, +\infty)$ for each $j \geqslant 0$, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-k} \ln(1 + \gamma_j e^k) &\leqslant \gamma_j + \int_0^{\infty} \frac{\ln(1 + \gamma_j e^x)}{e^x}\, dx \\ &= \gamma_j - \gamma_j \ln \gamma_j + (1 + \gamma_j) \ln(1 + \gamma_j) \\ &\leqslant 3\gamma_j (1 + |\ln \gamma_j|). \end{aligned}$$

Upon putting everything together, we have

$$\sum_{n=0}^{\infty} \frac{1}{|\lambda_n|} < 3e^2 \sum_{j=0}^{\infty} \gamma_j (1 + |\ln \gamma_j|)$$

and the required result follows. □

Unlike Theorem 2.2.11 next Theorem does not require the symbol to be increasing for the sequence of eigenvalues to be absolutely summable.

**Theorem 3.1.25.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and*

$$\max \left( \varlimsup_{x \to 0} \frac{\ln|\ln(\varphi(x) - x)|}{|\ln x|}, \varlimsup_{x \to 1} \frac{\ln|\ln(\varphi(x) - x)|}{|\ln(1 - x)|} \right) < 1. \tag{3.1.24}$$

*Then the sequence of eigenvalues of $V_\varphi$ is absolutely summable.*

*Proof.* The hypothesis (3.1.24) implies that there is $r > 1$ such that

$$\begin{aligned} &\varphi(x) - x = o(x^{1+1/r} e^{-x^{-1/r}}), && \text{as } x \to 0 \\ \text{and} \quad &\varphi(x) - x = o\left((1 - x)^{1+1/r} e^{-(1-x)^{-1/r}}\right), && \text{as } x \to 1. \end{aligned} \tag{3.1.25}$$

Consider

$$\alpha(x) = \begin{cases} 0, & \text{if } x = 0; \\ \Big(\ln(e^{x^{-1/r}} - 1)\Big)^{-r}, & \text{if } 0 < x \leqslant 1. \end{cases}$$

$$\text{and} \quad \beta(x) = \begin{cases} 1 - \Big(\ln(e^{(1-x)^{-1/r}} + 1)\Big)^{-r}, & \text{if } 0 \leqslant x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

One may check that $\alpha$ and $\beta$ are continuous, strictly increasing and

$$\begin{aligned} \alpha(x) - x &\sim \ rx^{1+1/r}e^{-x^{-1/r}}, && \text{as } \ x \to 0, \\ \text{and} \quad \beta(x) - x &\sim \ r(1-x)^{1+1/r}e^{-(1-x)^{-1/r}}, && \text{as } \ x \to 1. \end{aligned} \tag{3.1.26}$$

Furthermore, for $a > e$, we have

$$\alpha((\ln(a+1))^{-r}) = (\ln a)^{-r} \quad \text{and} \quad \beta(1 - (\ln a)^{-r}) = 1 - (\ln(a+1))^{-r}. \tag{3.1.27}$$

Now, from (3.1.25) and (3.1.26), it follows that there is a continuous, increasing self-map $\psi$ of $[0,1]$ with $\psi(0) = 0$ and $\psi(x) > x$ for $0 < x < 1$ satisfying

$$\begin{aligned} \psi(x) &\leqslant \varphi(x) && \text{for } \ 0 \leqslant x \leqslant 1, \\ \psi(x) &= \alpha(x) && \text{on a neighborhood of } 0 \\ \text{and} \quad \psi(x) &= \beta(x) && \text{on a neighborhood of } 1. \end{aligned}$$

Now, take $0 < c < 1$ and set $\gamma_n = \psi_n(c) - \psi_{n-1}(c)$ for each integer $n$ in $\mathbb{Z}$. Since $\psi$ is invertible in $[0,1]$, we have $\psi(\psi_n(c)) = \psi_{n+1}(c)$ for each $n$ in $\mathbb{Z}$. From the behavior of $\psi$ near 0 and 1 and (3.1.27), it follows that there are real numbers $a_-$ and $a_+$ such that $\psi_n(c) = 1 - (\ln(n + a_+))^{-r}$ and $\psi_{-n}(c) = (\ln(|n| + a_-))^{-r}$ for $n$ large enough. Therefore, an elementary computation shows that

$$\gamma_n \sim \frac{r}{|n|(\ln|n|)^{r+1}}, \qquad \text{as } \ |n| \to +\infty.$$

Hence,

$$\sum_{n=-\infty}^{\infty} \gamma_n(1 + |\ln \gamma_n|) < \infty.$$

Since $\psi(x) \leqslant \varphi(x)$, Corollary 3.1.4 and Proposition 3.1.19 show that

$$M(\mathcal{F}_0^\varphi, R) \leqslant M(\mathcal{F}_0^\psi, R) \leqslant \prod_{n\in\mathbb{Z}} (1 + \gamma_n R).$$

Finally, let $\{1/\lambda_n\}$ be the sequence of the non-null zeros of $\mathcal{F}_0^\varphi$. The above display along with Lemma 3.1.24 shows that $\{\lambda_n\}$, which is the sequence of eigenvalues of $V_\varphi$, is absolutely summable. $\square$

Theorem above shows that for a wide class of symbols $\varphi$ the sequence of eigenvalues is absolutely summable. This means that $V_\varphi$ behaves like a nuclear operator, although it is not. The 'trace' of $V_\varphi$ is exactly the upper bound found in Theorem 2.2.11 for increasing maps. This clarifies why the sum of the eigenvalues of $V_{x^\alpha}$ or $V_{1-(1-x)^{1/\alpha}}$ is equal to 1, see Theorems 2.1.10 and 2.1.11.

**Theorem 3.1.26.** *Let $\varphi$ be a continuous self map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and assume that the sequence $\{\lambda_n(\varphi)\}$ of eigenvalues of $V_\varphi$ is absolutely summable. Then,*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) = \mu(\{x \in [0,1] : \varphi(x) > x\}). \tag{3.1.28}$$

*Proof.* We assume that $0 \leqslant c = \mu(\{x \in [0,1] : \varphi(x) = x\}) < 1$, otherwise there is nothing to prove. Since $\{\lambda_n\} = \{\lambda_n(\varphi)\}$ is absolutely summable, we find that

$$G(z) = \prod_{n=0}^{\infty}(1 - \lambda_n z)$$

defines an entire function of exponential type zero. On the other hand, by Theorem 3.1.7, we have that $\mathcal{F}_0^\varphi$ and $G$ share the same zeros with the same order. Since, by Proposition 3.1.12, the entire function $\mathcal{F}_0^\varphi$ is of exponential type $c$, it follows that $H(z) = \mathcal{F}_0^\varphi(z)/G(z)$ is an entire function of exponential type $c$ with no zeros. Thus, Lemma 3.1.14 implies that $H(z) = ae^{bz}$, where $a$ and $b$ belong to $\mathbb{C}$ and $|b| = c$. Since $G(0) = \mathcal{F}_0^\varphi(0) = 1$, it follows that $a = 1$. By Corollary 3.1.4, we know that $M(\mathcal{F}_0^\varphi, R) = \mathcal{F}_0^\varphi(-R)$. In addition, $\mathcal{F}_0^\varphi(z) = G(z)H(z) = G(z)e^{bz}$ and $G$ has exponential type zero. It follows that $b$ is real and negative and, therefore, $b = -c$ and $\mathcal{F}_0^\varphi(z) = G(z)e^{-cz}$. Hence,

$$(\mathcal{F}_0^\varphi)'(0) = G'(0) - cG(0) = -c - \sum_{n=0}^{\infty} \lambda_n.$$

On the other hand, the Taylor series expansion of $\mathcal{F}_0^\varphi$ furnished by Proposition 3.1.2 implies that $(\mathcal{F}_0^\varphi)'(0) = -1$, which along with the above display implies (3.1.28). The proof is complete. □

From Theorems 2.2.11 and 3.1.26 we obtain

**Corollary 3.1.27.** *Let $\varphi$ be a continuous increasing self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Then*

$$\sum_{n=0}^{\infty} \lambda_n(\varphi) = \mu(\{x \in [0,1] : \varphi(x) > x\}).$$

The following example provides another family of symbols for which the spectrum of the corresponding composition Volterra operators can be found explicitly. It shows that the eigenvalues $\lambda_n(\varphi)$ of $V_\varphi$ for a continuous self-map $\varphi$ of $[0,1]$ with $\varphi(x) \geqslant x$ may be non-real. Moreover, in these examples, the sequence of eigenvalues is not absolutely summable. It is also worth noting that all the eigenvalues, except the corresponding one to the spectral radius, are non-real.

**Example 3.1.28.** *Assume that $0 < a < b \leqslant 1$ and let $\varphi_{a,b}$ be the continuous self-map of $[0,1]$ defined by*

$$\varphi_{a,b}(x) = \begin{cases} b - \frac{b-a}{a}x, & \text{if } 0 \leqslant x < a; \\ x, & \text{if } a \leqslant x \leqslant 1. \end{cases} \tag{3.1.29}$$

*Then*

$$\sigma(V_{\varphi_{a,b}}) = \{0\} \cup \left\{ \lambda_n = \frac{b-a}{\ln(b/a) + 2\pi n i} \right\}_{n\in\mathbb{Z}}.$$

*Furthermore, the eigenvalues $\lambda_n$ have algebraic multiplicity 1 and the corresponding eigenfunctions are*

$$f_n(x) = \begin{cases} \frac{b}{b-a} e^{(a-1)/\lambda_n} - \frac{a}{b-a} e^{\left(b-1-\frac{b-a}{a}x\right)/\lambda_n}, & \text{if } 0 \leqslant x < a; \\ e^{(x-1)/\lambda_n}, & \text{if } a \leqslant x \leqslant 1. \end{cases}$$

*Proof.* First, we will find an explicit expression for $\mathcal{F}^{\varphi_{a,b}}$. Set $g(x) = \mathcal{F}^{\varphi_{a,b}}(x,z)$ for each $z$ in $\mathbb{C}$. By (3.1.2) and (3.1.3), we have

$$\begin{cases} g'(x) = zg(\varphi_{a,b}(x)), & \text{for } 0 \leqslant x \leqslant 1, \\ g(1) = 1. \end{cases} \tag{3.1.30}$$

From (3.1.29) and (3.1.30), we see that

$$g'(x) = zg(x) \quad \text{for} \quad a \leqslant x \leqslant 1. \tag{3.1.31}$$

One checks that a solution of the Cauchy problem given by (3.1.31) and $g(1) = 1$ is

$$g(x) = e^{z(x-1)} \quad \text{for} \quad a \leqslant x \leqslant 1. \tag{3.1.32}$$

From (3.1.29) and (3.1.30) and the above display, we have

$$\begin{cases} g'(x) &= ze^{z\left(b-1-\frac{b-a}{a}x\right)} \quad \text{for} \quad 0 \leqslant x \leqslant a, \\ g(a) &= e^{z(a-1)}. \end{cases}$$

Solving the above Cauchy problem, we obtain

$$g(x) = \frac{b}{b-a} e^{z(a-1)} - \frac{a}{b-a} e^{z\left(b-1-\frac{b-a}{a}x\right)} \quad \text{for} \quad 0 \leqslant x \leqslant a.$$

Thus from (3.1.32) and the above display, we see that

$$\mathcal{F}^{\varphi_{a,b}}(x,z) = \begin{cases} \frac{b}{b-a}e^{z(a-1)} - \frac{a}{b-a}e^{z\left(b-1-\frac{b-a}{a}x\right)}, & \text{if } 0 \leqslant x \leqslant a; \\ e^{z(x-1)}, & \text{if } a < x \leqslant 1. \end{cases} \tag{3.1.33}$$

In particular,

$$\mathcal{F}_0^{\varphi_{a,b}}(z) = \frac{b}{b-a}e^{z(a-1)} - \frac{a}{b-a}e^{z(b-1)} = \frac{e^{-z}}{b-a}(be^{az} - ae^{bz}).$$

It is elementary to see that the zeros of $\mathcal{F}_0^{\varphi_{a,b}}$ are $1/\lambda_n$, $n$ in $\mathbb{Z}$, and they are simple. Thus the formulae for the eigenvalues and the eigenfunctions follow from Theorem 3.1.7. □

## 3.2 Analyticity of eigenfunctions of $V_\varphi$

This section is devoted to showing that under suitable hypotheses on the symbols, the eigenfunctions of composition Volterra operators are analytic. An example is provided in which while the symbol is analytic, the eigenfunctions are not. Recall that a function $f$ defined on a real interval $[u,v]$ is said to be *analytic* if it admits a holomorphic extension to an open set $U$ of $\mathbb{C}$ that contains $[u,v]$. This is equivalent to the fact that $f$ coincides with the sum of its Taylor series on a neighborhood of each point on $[u,v]$. We will use of a well-known criterion of analyticity. Let $f$ be in $\mathcal{C}^\infty[u,v]$ and

$$M_n(f) = \frac{1}{n!}\max_{[u,v]}|f^{(n)}|.$$

Then $f$ is analytic on $[u,v]$ if and only if

$$\overline{\lim_{n\to\infty}}\,(M_n(f))^{1/n} < \infty. \tag{3.2.1}$$

The formula for the $n$-th derivative of the composition provided by the next lemma is known as the Fao de Bruno formula, see [52, Chapter 3]. In the expressions below, each $k_j$, $1 \leqslant j \leqslant n$, is a non-negative integer.

**Lemma 3.2.1.** *Let $f$ and $g$ be in $\mathcal{C}^n[u,v]$. Then for each $u \leqslant x \leqslant v$, we have*

$$(g\circ f)^{(n)}(x) = n!\sum_{k_1+\cdots+nk_n=n}\frac{g^{(k_1+\cdots+k_n)}(f(x))}{k_1!\cdots k_n!(1!)^{k_1}\cdots(n!)^{k_n}}(f'(x))^{k_1}\cdots(f^{(n)}(x))^{k_n}. \tag{3.2.2}$$

The next lemma is an application of Lemma 3.2.1. It will be needed later.

**Lemma 3.2.2.** *For any $c$ in $\mathbb{C}$ and any positive integer $n$, we have*

$$\sum_{k_1+\cdots+nk_n=n}\frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}c^{k_1+\cdots+k_n} = c(c+1)^{n-1}.$$

*Proof.* Just consider $f(x) = (2-x)^{-1}$ and $g(x) = (c+1-cx)^{-1}$. Since $f^{(k)}(x) = k!(2-x)^{-k-1}$ and $g^{(k)}(x) = c^k k!(c+1-cx)^{-k-1}$, we see that (3.2.2) implies

$$(g \circ f)^{(n)}(1) = n! \sum_{k_1+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} c^{k_1+\cdots+k_n}. \tag{3.2.3}$$

On the other hand, we have $(g \circ f)(x) = (c+1)^{-1}\big(1 + c\big((c+2)-(c+1)x\big)^{-1}\big)$ and, therefore, $(g \circ f)^{(n)}(1) = n!c(c+1)^{n-1}$, which along with (3.2.3) implies the required equality. □

**Proposition 3.2.3.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$. Assume in addition that $\varphi$ is analytic on $[\alpha,1]$ for a certain $0 \leqslant \alpha < 1$ and*

$$\sup_{[\alpha,1]} |\varphi'| < 1. \tag{3.2.4}$$

*If $g$ is analytic on $[\alpha,1]$ and $\lambda$ belongs to $\mathbb{C}$, then each $f$ in $\mathcal{C}^1[\alpha,1]$ with $f'(x) = \lambda f(\varphi(x)) + g(x)$ for $\alpha \leqslant x \leqslant 1$, is analytic on $[\alpha,1]$.*

*Proof.* We suppose that $\lambda \neq 0$, otherwise the result is trivial. Clearly, $f$ belongs to $\mathcal{C}^\infty[\alpha,1]$, since $\varphi([\alpha,1])$ is contained in $[\alpha,1]$ and $g$ as well as $\varphi$ are analytic on $[\alpha,1]$. To prove that $f$ is analytic, it is enough to show that the sequence $\{M_n(f)\}$, relative to the interval $[\alpha,1]$, satisfies (3.2.1).

Let $r = \sup_{[\alpha,1]} |\varphi'|$. We may assume that $r > 0$, otherwise the result is trivial. We set

$$R = \sup_{n \geqslant 2} \left( \frac{M_n(\varphi)}{r} \right)^{1/(n-1)},$$

which is finite, since $\varphi$ is analytic on $[\alpha,1]$. We clearly have

$$M_n(\varphi) \leqslant rR^{n-1}, \qquad \text{for } n \geqslant 1. \tag{3.2.5}$$

On the other hand, since $g$ is analytic on $[\alpha,1]$ there are positive numbers $A$ and $B$ such that

$$M_n(g) \leqslant AB^{n+1}, \qquad \text{for } n \geqslant 0. \tag{3.2.6}$$

Set $C = \max\{B, R/(1-r)\}$ and take a positive integer $q$ such that

$$2r|\lambda| \leqslant C(q+1). \tag{3.2.7}$$

Let $L \geqslant 2A$ be a constant such that $M_n(f) \leqslant LC^n$ for $1 \leqslant n \leqslant q$. It suffices to verify that

$$M_n(f) \leqslant LC^n, \qquad \text{for each } n \geqslant 1. \tag{3.2.8}$$

Let $m \geqslant q$ be a positive integer. It is enough to show that if (3.2.8) holds for $1 \leqslant n \leqslant m$, then so does for $n = m+1$. Upon differentiating $m$ times on $f'(x) = \lambda f(\varphi(x)) + g(x)$ and using Lemma 3.2.1 we obtain

$$f^{(m+1)}(x) = g^{(m)}(x) + \sum_{k_1+\cdots+mk_m=m} \frac{m!\lambda f^{(k_1+\cdots+k_m)}(\varphi(x))}{k_1!\cdots k_m!(1!)^{k_1}\cdots(m!)^{k_m}}(\varphi'(x))^{k_1}\cdots(\varphi^{(m)}(x))^{k_m}.$$

Therefore,

$$M_{m+1}(f) \leqslant M_m(g) + \sum_{k_1+\cdots+mk_m=m} \frac{|\lambda|(k_1+\cdots+k_m)!}{(m+1)k_1!\cdots k_m!} M_{k_1+\cdots+k_m}(f)(M_1(\varphi))^{k_1}\cdots(M_m(\varphi))^{k_m}.$$

The induction hypotheses along with (3.2.6) shows that

$$M_{m+1}(f) \leqslant AB^{m+1} + \frac{L|\lambda|R^m}{m+1} \sum_{k_1+\cdots+mk_m=m} \frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}\left(\frac{rC}{R}\right)^{k_1+\cdots+k_m}.$$

Hence, by Lemma 3.2.2, we have

$$M_{m+1}(f) \leqslant AB^{m+1} + \frac{LrC|\lambda|}{m+1}(rC+R)^{m-1} \leqslant AB^{m+1} + \frac{LC^2}{2}(rC+R)^{m-1} \leqslant LC^{m+1},$$

where we have used (3.2.7) in the second inequality above and the fact that $B \leqslant C$, $rC + R \leqslant C$ and $2A \leqslant L$ in the third. Thus (3.2.8) is satisfied for each positive integer $n$. The result is proved. □

As usual, a function $f$ is said to be analytic on $(u, v]$, if it is on $[s, v]$ for each $u \leqslant s \leqslant v$.

**Lemma 3.2.4.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $\alpha < x < 1$ for some $0 \leqslant \alpha < 1$. Assume also that $\varphi$ is analytic on $(\alpha, 1]$ and $\varphi'(1) \neq 1$. If $g$ is analytic on $(\alpha, 1]$ and $\lambda$ belongs to $\mathbb{C}$, then each $f$ in $\mathcal{C}^1(\alpha, 1]$ with $f'(x) = \lambda f(\varphi(x)) + g(x)$, is analytic on $(\alpha, 1]$.*

*Proof.* We may assume that $\lambda \neq 0$. Now, observe that $\varphi'(1) < 1$. Thus we may choose $\alpha < v \leqslant 1$ such that $\sup_{[v,1]} |\varphi'| < 1$. By Proposition 3.2.3, we find that $f$ is analytic on $[v, 1]$. Set

$$u = \inf\{s \in (\alpha, 1) \ : \ f \text{ is analytic on } [s, 1]\}$$

and assume that $u > \alpha$. Then there is $\varepsilon > 0$ such that $[u-\varepsilon, u+\varepsilon]$ is contained $(\alpha, 1)$ and $\varphi([u-\varepsilon, u+\varepsilon])$ is contained in $(u, 1]$. Since $f'(x) = \lambda f(\varphi(x)) + g(x)$, we find that $f$ is analytic on $[u-\varepsilon, 1]$, which is a contradiction. □

**Lemma 3.2.5.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for some $\alpha \leqslant x < 1$. Assume also that $\varphi$ is analytic on $[\alpha, 1]$ and $\varphi'(1) \neq 1$. If $g$ is analytic on $[\alpha, 1]$ and $\lambda$ belongs to $\mathbb{C}$, then each $f$ in $\mathcal{C}^1[\alpha, 1]$, with $f'(x) = \lambda f(\varphi(x)) + g(x)$ for $\alpha \leqslant x \leqslant 1$, is analytic on $[\alpha, 1]$.*

*Proof.* Set $a = \min_{[\alpha,1]} \varphi$. The hypotheses on $\varphi$ imply that $a > \alpha$. According to Lemma 3.2.4, we find that $f$ is analytic on $(\alpha, 1]$. Since $\varphi([\alpha, 1]) = [a, 1]$ and $a > \alpha$, we obtain that $f$ is analytic on $[\alpha, 1]$ because $f'(x) = \lambda f(\varphi(x)) + g(x)$. The result is proved. □

Now, we can prove our main theorem in this section.

**Theorem 3.2.6.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and $f$ be a generalized eigenfunction of $V_\varphi$ corresponding to a non-zero eigenvalue. If $\varphi$ is analytic on $[\alpha, 1]$ for some $0 < \alpha < 1$ and $\varphi'(1) \neq 1$, then $f$ is analytic on $[\alpha, 1]$. The same is true for $\alpha = 0$, provided that $\varphi(0) > 0$.*

*Proof.* Since $f$ is a generalized eigenfunction of $V_\varphi$ corresponding to a non-zero eigenvalue $\lambda$, there are functions $f_0, f_1, \ldots, f_k = f$ in $L^2[0,1]$ such that and $(V_\varphi - \lambda I)f_0 = 0$ and $(V_\varphi - \lambda I)f_j = f_{j-1}$ for $1 \leqslant j \leqslant k$. Consequently, each $f_j$ is continuous and thus

$$F_j(x) = \int_0^x f_j(t)\, dt, \qquad 0 \leqslant j \leqslant k,$$

belongs to $\mathcal{C}^1[0,1]$. We also have

$$F_0'(x) = \lambda^{-1} F_0(\varphi(x)) \quad \text{and} \quad F_j'(x) = \lambda^{-1} F_j(\varphi(x)) + F_{j-1}'(x), \quad \text{for } 1 \leqslant j \leqslant k$$

and applying successively Lemma 3.2.5, we see that each $F_j$ for $1 \leqslant j \leqslant k$, is analytic on $[\alpha, 1]$ and, therefore, so is each $f_j$, $0 \leqslant j \leqslant k$. The result is proved. □

**Remark.** Observe that Theorem 3.2.6 holds for $\alpha = 0$ provided that $\varphi(0) > 0$, but it fails to be true if $\varphi(0) = 0$. Indeed, if $\varphi(0) = 0$, it follows that all the derivatives of each generalized eigenfunction $f$ of $V_\varphi$ vanish at 0. Thus either $f$ is identically zero or $f$ is not analytic on $[0,1]$. The next example shows that $\varphi'(1) \neq 1$ in Theorem 3.2.6 is also essential.

**Example 3.2.7.** *The map $\varphi(x) = (2 - x)^{-1}$ satisfies all the hypotheses of Theorem 3.2.6 with $\alpha = 0$, except that $\varphi'(1) = 1$. In addition, by Corollary 2.2.3, the spectral radius $r = r(V_\varphi) > 0$ is an eigenvalue. On the other hand, $f(x) = F'(x)$, where $F(x) = \mathcal{F}^\varphi(x, 1/r)$, satisfies, by Theorem 3.1.7, that $V_\varphi f = rf$, but $f$ is not analytic on $[0,1]$.*

*Proof.* If $f$ is analytic on [0,1], then so is $F$. Since $F'(x) = r^{-1}F(\varphi(x))$, by Lemma 3.2.1, we have

$$F^{(n+1)}(1) = \frac{n!}{r} \sum_{k_1+\cdots+nk_n=n} \frac{F^{(k_1+\cdots+k_n)}(1)}{k_1!\cdots k_n!(1!)^{k_1}\cdots(n!)^{k_n}} (\varphi'(1))^{k_1}\cdots(\varphi^{(n)}(1))^{k_n}.$$

Substituting the value $\varphi^{(n)}(1) = n!$ for $n \geqslant 0$, we obtain

$$F^{(n+1)}(1) = \frac{n!}{r} \sum_{k_1+\cdots+nk_n=n} \frac{F^{(k_1+\cdots+k_n)}(1)}{k_1!\cdots k_n!}, \tag{3.2.9}$$

which implies that $F^{(n)}(1) > 0$ for all $n \geqslant 0$. For each constant $C > 0$, we may choose a positive integer $q$ such that

$$(C+1)^{n-1} \geqslant C^n r(n+1), \qquad \text{for each } n \geqslant q. \tag{3.2.10}$$

Since $F^{(n)}(1) > 0$, there is $A > 0$ such that $F^{(n)}(1) \geqslant AC^n n!$ for $0 \leqslant n \leqslant q$. Suppose that $n \geqslant q$ and we have already proved that $F^{(m)}(1) \geqslant AC^m m!$ for each $m \leqslant n$. From (3.2.9), we have

$$F^{(n+1)}(1) \geqslant \frac{A(n+1)!}{r(n+1)} \sum_{k_1+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} C^{k_1+\cdots+k_n}.$$

Hence, by Lemma 3.2.2 and using 3.2.10 in the second inequality below, we obtain

$$F^{(n+1)}(1) \geqslant AC^{n+1}(n+1)! \frac{(C+1)^{n-1}}{r(n+1)C^n} \geqslant AC^{n+1}(n+1)!.$$

Thus for each $C > 0$, there is $A > 0$ such that $F^{(n)}(1) \geqslant AC^n n!$ for each $n \geqslant 0$. Therefore, $F$ is not analytic on $[0,1]$, as required. □

# Chapter 4

# Asymptotic behavior of orbits of quasi-nilpotent $V_\varphi$'s

In this Chapter, we deal with the asymptotic behavior of the norms of powers of $V_\varphi$ and norms of orbits $\{V_\varphi^n f\}_{n\geqslant 0}$ for quasi-nilpotent composition Volterra operators. In particular, it will be shown that for the most interesting class of symbols, the sequence $\{\|V_\varphi^n\|^{1/n^2}\}$ has a positive limit strictly less than one, that depends only on $\varphi'(0)$ and $\varphi'(1)$. These estimates will be particulary useful in Chapter 5 to prove and disprove supercyclicity and cyclicity respectively.

It is worth mentioning, see for instance [36], that for each bounded operator $T$ acting on a Banach space $\mathcal{B}$ we have that

$$\lim_{n\to\infty} \|T^n f\|^{1/n} = r(T)$$

for $f$ in a dense subset of $\mathcal{B}$ and

$$\overline{\lim_{n\to\infty}} \|T^n f\|^{1/n} = r(T)$$

for $f$ in a dense $G_\delta$ set in $\mathcal{B}$. Thus, if $T$ is not quasi-nilpotent, that is, the spectral radius $r(T) > 0$, the asymptotic behavior of the orbits is to some extend determined by the value of $r(T)$. If $r(T) = 0$, we know only that $\|T^n\|^{1/n}$ tends to 0 and, therefore, $\|T^n f\|^{1/n}$ tends to 0 for each $f$ in $\mathcal{B}$. If $\varphi$ is the identity map, that is, $V_\varphi = V$ is the Volterra operator, the asymptotic behavior of the orbits of $V_\varphi$ are known. Namely, Shkarin in [53] has proved, for each $f$ in $L^p[0,1]$, that

$$\lim_{n\to\infty} (n!\|V^n f\|_p)^{1/n}$$

does exist and equals $1 - \inf \operatorname{supp}(f)$. We shall see that if $\varphi(x) < x$ for $0 < x < 1$, the orbits of $V_\varphi$ tend to zero much faster than that.

## 4.1 The asymptotic behavior of $\|V_\varphi^n\|$

We will be mainly concerned with continuous strictly increasing symbols, since it is necessary for cyclicity of composition Volterra operators, see Section 5.3. However, most of the proofs in this section still work for non-increasing self-maps.

Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$ and $\varphi(1) = 1$. Let $\Omega_1(\varphi) = [0,1]$ and for each $n \geqslant 2$ consider

$$\Omega_n(\varphi) = \{x \in [0,1]^n : x_1 \leqslant \varphi(x_2), x_2 \leqslant \varphi(x_3), \dots, x_{n-1} \leqslant \varphi(x_n)\}. \qquad (4.1.1)$$

The next lemma relates the values of $\|V_\varphi^n\|$ to those of $\nu_n(\varphi) = \mu_n(\Omega_n(\varphi))$, where $\mu_n$ is the $n$-dimensional Lebesgue measure.

**Lemma 4.1.1.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$ and $\varphi(1) = 1$. Then $\nu_{n+1}(\varphi) \leqslant \|V_\varphi^n\|_p \leqslant \nu_{n-1}(\varphi)$ for each $n \geqslant 2$ and $1 \leqslant p \leqslant \infty$.*

*Proof.* Let $\mathbf{1}$ denote the function identically 1 on $[0,1]$. It is clear that

$$\|V_\varphi^n \mathbf{1}\|_\infty = \|V_\varphi^{n-1}\mathbf{1}\|_1 = (V_\varphi^n \mathbf{1})(1) = \nu_n(\varphi)$$

for each positive integer $n$. Hence,

$$\|V_\varphi^n\|_p \geqslant \|V_\varphi^n \mathbf{1}\|_p \geqslant \|V_\varphi^n \mathbf{1}\|_1 = \nu_{n+1}(\varphi).$$

We also have $\|V_\varphi^n f\|_\infty \leqslant \|V_\varphi^n \mathbf{1}\|_\infty \|f\|_\infty = \nu_n(\varphi)\|f\|_\infty$ for each $f \in L^\infty[0,1]$. Hence

$$\|V_\varphi^n f\|_p \leqslant \|V_\varphi^{n-1} V_\varphi f\|_\infty \leqslant \nu_{n-1}(\varphi)\|V_\varphi f\|_\infty \leqslant \nu_{n-1}(\varphi)\|f\|_p$$

for each $n \geqslant 2$. Thus, $\nu_{n+1}(\varphi) \leqslant \|V_\varphi^n\|_p \leqslant \nu_{n-1}(\varphi)$ for any $n \geqslant 2$. □

Before stating our main result, we need one more lemma. Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$. For each positive integer $n$ and each $0 < a < 1$, we set

$$\Omega_n^{a,0}(\varphi) = \{x \in \Omega_n(\varphi) : x_n \leqslant a\} \quad \text{and} \quad \Omega_n^{a,1}(\varphi) = \{x \in \Omega_n(\varphi) : x_1 \geqslant a\}. \qquad (4.1.2)$$

**Lemma 4.1.2.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$ and*

$$\delta_0^+ = \overline{\lim_{x\to 0}} \frac{\varphi(x)}{x}, \quad \delta_0^- = \underline{\lim_{x\to 0}} \frac{\varphi(x)}{x}, \quad \delta_1^+ = \overline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)}, \quad \delta_1^- = \underline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)}. \qquad (4.1.3)$$

*Then for each* $0 < a < 1$, *we have*

$$\varlimsup_{n\to\infty}(\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_0^+}, \qquad \varliminf_{n\to\infty}(\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \geqslant \sqrt{\delta_0^-}, \tag{4.1.4}$$

$$\varlimsup_{n\to\infty}(\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_1^+}, \qquad \varliminf_{n\to\infty}(\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} \geqslant \sqrt{\delta_1^-}. \tag{4.1.5}$$

*In particular, if* $\varphi$ *is differentiable at* 0 *and at* 1, *the derivative at* 1 *is allowed to be infinite, then*

$$\lim_{n\to\infty}(\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} = \sqrt{\varphi'(0)} \quad \text{and} \quad \lim_{n\to\infty}(\mu_n(\Omega_n^{a,1}(\varphi)))^{1/n^2} = \sqrt{1/\varphi'(1)}. \tag{4.1.6}$$

*Proof.* If $\delta_0^+ = 1$, the first inequality in (4.1.4) becomes trivial. Indeed, if we denote $u$ the identity function, we have

$$\mu_n(\Omega_n^{a,0}(\varphi)) \leqslant \mu_n(\Omega_n(\varphi)) \leqslant \mu_n(\Omega_n(\mathrm{u})) = \frac{1}{(n+1)!}. \tag{4.1.7}$$

Thus assume that $\delta_0^+ < 1$. We take an arbitrary $\delta_0^+ < b < 1$. Clearly, there is $0 < \delta < 1$ and a strictly increasing continuous self-map $\psi$ of $[0,1]$ such that $\psi(x) < x$ for $0 < x < 1$, $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$ and $\psi(x) \geqslant \varphi(x)$ for $0 \leqslant x \leqslant 1$. Since $\psi(x) < x$ for $0 < x < 1$, we find that there is a positive integer $k$ such that $\psi_k(a) \leqslant \delta$, where $\psi_k$ denotes the $k$-th iterate of $\psi$. It immediately follows that

$$\mu_n(\Omega_n^{a,0}(\psi)) \leqslant \mu_{n-k}(\Omega_{n-k}^{\delta,0}(\psi))$$

for $n > k$. Since $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$, we have

$$\mu_j(\Omega_j^{\delta,0}(\psi)) = \int_0^\delta dx_j \int_0^{bx_j} dx_{j-1} \cdots \int_0^{bx_3} dx_2 \int_0^{bx_2} dx_1 = \frac{\delta^j b^{j(j-1)/2}}{j!} \tag{4.1.8}$$

for each positive integer $j$. Since $\mu_n(\Omega_n^{a,0}(\varphi)) \leqslant \mu_n(\Omega_n^{a,0}(\psi))$ for each positive integer $n$, from the last two displays it follows that

$$\varlimsup_{n\to\infty}(\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \leqslant \sqrt{b}.$$

Since $\delta_0^+ < b < 1$ was arbitrary, the first inequality in (4.1.4) follows.

If $\delta_0^- = 0$, the second inequality in (4.1.4) is trivial. Thus assume that $\delta_0^- > 0$. We take an arbitrary $0 < b < \delta_0^-$. Clearly, there is $0 < \delta < a$ and a strictly increasing continuous self-map $\psi$ of $[0,1]$ such that $\psi(x) < x$ for $0 < x < 1$, $\psi(x) = bx$ for $0 \leqslant x \leqslant \delta$ and $\psi(x) \leqslant \varphi(x)$ for $0 \leqslant x \leqslant 1$. Since

$$\mu_n(\Omega_n^{a,0}(\varphi)) \geqslant \mu_n(\Omega_n^{a,0}(\psi)) \geqslant \mu_n(\Omega_n^{\delta,0}(\psi))$$

for each positive integer $n$, from (4.1.8) we obtain

$$\varliminf_{n\to\infty} (\mu_n(\Omega_n^{a,0}(\varphi)))^{1/n^2} \geqslant \sqrt{b}.$$

Since $0 < b < \delta_0^-$ was arbitrary, the second inequality in (4.1.4) also follows.

Finally, since $\varphi$ satisfies (4.1.5) if and only if $\phi(x) = 1-\varphi^{-1}(1-x)$ satisfies (4.1.4), the proof of the statement of the lemma is complete. □

In order to state the main result of this section, we define

$$\phi(u,v) = \begin{cases} \exp\left(\dfrac{\ln u \ln v}{2\ln(uv)}\right), & \text{if } u>0, v>0 \text{ and } (u,v)\neq(1,1); \\ \sqrt{|u-v|}, & \text{if } u=0 \text{ or } v=0; \\ 1, & \text{if } (u,v)=(1,1), \end{cases}$$

which is clearly continuous on $[0,1]^2$ and takes values in $[0,1]$.

**Theorem 4.1.3.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$ and $\delta_0^+$, $\delta_0^-$, $\delta_1^+$, $\delta_1^-$ be as in (4.1.3). Then, for $1 \leqslant p \leqslant \infty$, we have*

$$\rho_- \leqslant \varliminf_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} \leqslant \varlimsup_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} \leqslant \rho_+,$$

*where $\rho_- = \phi(\delta_0^-, \delta_1^-)$ and $\rho_+ = \phi(\delta_0^+, \delta_1^+)$. In particular, if $\varphi$ is differentiable at 0 and at 1, then*

$$\lim_{n\to\infty} \|V_\varphi^n\|^{1/n^2} = \phi(\varphi'(0), 1/\varphi'(1)).$$

*Proof.* According to Lemma 4.1.1, it is enough to show that

$$\rho_- \leqslant \varliminf_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \leqslant \varlimsup_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \leqslant \rho_+. \tag{4.1.9}$$

If $\rho_+ = 1$, the last inequality in (4.1.9) follows from the second one in (4.1.7). Thus assume that $\rho_+ < 1$. Hence, we must have $\delta_0^+ < 1$ and $\delta_1^+ < 1$. We take $\delta_0^+ < b_0 < 1$ and $\delta_1^+ < b_1 < 1$. By Lemma 4.1.2, there is $c > 0$ such that

$$\mu_n(\Omega_n^{1/2,0}(\varphi)) \leqslant c b_0^{n^2/2} \qquad \text{and} \qquad \mu_n(\Omega_n^{1/2,1}(\varphi)) \leqslant c b_1^{n^2/2} \tag{4.1.10}$$

for each positive integer $n$. Clearly,

$$\Omega_n(\varphi) \subset \bigcup_{k=0}^{n} A_k,$$

where $A_0 = \Omega_n^{1/2,0}(\varphi)$, $A_n = \Omega_n^{1/2,1}(\varphi)$ and $A_k = \Omega_{n-k}^{1/2,0}(\varphi) \times \Omega_k^{1/2,1}(\varphi)$ for $0 < k < n$. Hence,

$$\nu_n(\varphi) \leqslant \sum_{k=0}^{n} \mu_n(A_k) = \sum_{k=0}^{n} \mu_{n-k}(\Omega_{n-k}^{1/2,0}(\varphi))\mu_k(\Omega_k^{1/2,1}(\varphi)).$$

Using (4.1.10), we obtain

$$\begin{aligned}\nu_n(\varphi) &\leqslant c^2 \sum_{k=0}^{n} b_0^{(n-k)^2/2} b_1^{k^2/2} \\ &\leqslant c^2(n+1) \max_{0\leqslant k\leqslant n} b_0^{(n-k)^2/2} b_1^{k^2/2} \\ &\leqslant c^2(n+1) \left( \max_{[0,1]} b_0^{(1-x)^2/2} b_1^{x^2/2} \right)^{n^2}.\end{aligned}$$

The last maximum is attained for $x = \ln b_0 / \ln(b_0 b_1)$ and equals to $\phi(b_0, b_1)$. Therefore,

$$\overline{\lim_{n\to\infty}} (\nu_n(\varphi))^{1/n^2} \leqslant \phi(b_1, b_2).$$

Since $\delta_0^+ < b_0 < 1$ and $\delta_1^+ < b_1 < 1$ were arbitrary, we see that the last inequality in (4.1.9) is satisfied.

If $\rho_- = 0$, the first inequality in (4.1.9) is trivial. Thus assume that $\rho_- > 0$. Hence, we must have $\delta_0^- > 0$ and $\delta_1^- > 0$. We take $0 < b_0 < \delta_0^-$ and $0 < b_1 < \delta_1^-$. Let $a > 0$ be small enough to ensure that $a < \varphi(1-a)$. By Lemma 4.1.2, there is $c > 0$ such that

$$\mu_n(\Omega_n^{a,0}(\varphi)) \geqslant c b_0^{n^2/2} \qquad \text{and} \qquad \mu_n(\Omega_n^{1-a,1}(\varphi)) \geqslant c b_1^{n^2/2} \tag{4.1.11}$$

for each positive integer $n$. Choose a sequence $\{k_n\}_{n\geqslant 1}$ of positive integers such that $k_n < n$ for each $n$ and $k_n/n$ tends to $\ln b_1/\ln(b_1 b_2)$ as $n$ tends to $\infty$. Clearly,

$$\Omega_n(\varphi) \supset A = \Omega_{n-k_n}^{a,0}(\varphi) \times \Omega_{k_n}^{1-a,1}(\varphi).$$

Hence,

$$\nu_n(\varphi) \geqslant \mu_n(A) = \mu_{n-k_n}(\Omega_{n-k_n}^{a,0}(\varphi))\mu_{k_n}(\Omega_{k_n}^{1-a,1}(\varphi)).$$

Using (4.1.11), we obtain

$$\nu_n(\varphi) \geqslant c^2 b_0^{(n-k_n)^2/2} b_1^{k_n^2/2} = c^2 \left( b_0^{(1-(k_n/n))^2} b_1^{(k_n/n)^2} \right)^{n^2/2}.$$

Since $k_n/n$ tends to $\ln b_0/\ln(b_0 b_1)$, we see that

$$\lim_{n\to\infty} b_0^{(1-(k_n/n))^2} b_1^{(k_n/n)^2} = \phi(b_0, b_1).$$

From the last two displays, we obtain

$$\varliminf_{n\to\infty} (\nu_n(\varphi))^{1/n^2} \geqslant \phi(b_0, b_1).$$

Since $0 < b_0 < \delta_0^-$ and $0 < b_1 < \delta_1^-$ were arbitrary, we see that the first inequality in (4.1.9) is also satisfied. The proof is complete. □

**Corollary 4.1.4.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0, 1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. If $\varphi$ is differentiable at 0 and 1 and $\varphi'(0) = 0$, then, for $1 \leqslant p \leqslant \infty$, we have*

$$\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = \frac{1}{\sqrt{\varphi'(1)}}.$$

*If $\varphi$ is differentiable at 0 and $\varphi'(1) = \infty$, then*

$$\lim_{n\to\infty} \|V_\varphi^n\|_p^{1/n^2} = \sqrt{\varphi'(0)}.$$

Müller [36] proved that for any bounded operator $T$ on Banach space $\mathcal{B}$ and for each sequence $\{a_n\}_{n\geqslant 0}$ of positive numbers with

$$\sum_{n=0}^{\infty} a_n^{1/2} < \infty$$

there is a dense subset $E$ of $\mathcal{B}$ such that for each $x$ in $E$, we have $\|T^n x\| \geqslant a_n \|T^n\|$ for $n$ large enough. From the latter result and Theorem 4.1.3, we have

**Corollary 4.1.5.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0, 1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume also that $\varphi$ is differentiable at 0 and at 1. Then the set of $f$ in $L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, for which*

$$\lim_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} = \phi(\varphi'(0), 1/\varphi'(1))$$

*is dense in $L^p[0, 1]$.*

## 4.2 Orbits of $V_\varphi$. Upper estimate

The next lemma will be very useful to determine the cyclic properties of $V_\varphi$.

**Lemma 4.2.1.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0, 1]$ with $\varphi(x) < x$ for $0 < x < 1$, $\varphi(1) = 1$ and*

$$\delta_1^+ = \delta_1^+(\varphi) = \varlimsup_{x\to 1} \frac{1 - x}{1 - \varphi(x)}.$$

*Assume also that $f$ in $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, satisfies* $\inf \operatorname{supp}(f) > 0$. *Then,*

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{\delta_1^+}.$$

*In particular, if $\varphi$ is differentiable at 1, we have*

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_p^{1/n^2} \leqslant \sqrt{1/\varphi'(1)}.$$

*Proof.* Let $\varepsilon > 0$ be such that $f$ vanishes on $[0,\varepsilon]$. Since $V_\varphi f$ is continuous and also vanishes $[0,\varepsilon]$, there is $c > 0$ for which $|(V_\varphi f)(x)| \leqslant c\chi_{[\varepsilon,1]}(x)$ for each $0 \leqslant x \leqslant 1$, where $\chi_{[\varepsilon,1]}$ is the characteristic function of $[\varepsilon,1]$. Hence,

$$\|V_\varphi^n f\|_p \leqslant \|V_\varphi^n f\|_\infty \leqslant c\|V_\varphi^n \chi_{[\varepsilon,1]}\|_\infty = c(V_\varphi^n \chi_{[\varepsilon,1]})(1) = c\|V_\varphi^{n-1}\chi_{[\varepsilon,1]}\|_1. \tag{4.2.1}$$

Let $\Omega_n^{\varepsilon,1}(\varphi)$ be as in (4.1.2). Then $(V_\varphi^n \chi_{[\varepsilon,1]})(1) = \mu_n(\Omega_n^{\varepsilon,1}(\varphi))$ for each positive integer $n$. Therefore, by Lemma 4.1.2, we have

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n \chi_{[\varepsilon,1]}\|_1^{1/n^2} = \overline{\lim_{n\to\infty}} (\mu_n(\Omega_n^{\varepsilon,1}(\varphi)))^{1/n^2} \leqslant \sqrt{\delta_1^+}.$$

The required result follows immediately from the last two displays. □

## 4.3 The backward orbits of $V_\varphi$

In this section, we consider the asymptotic behavior of certain backward orbits of $V_\varphi$. In the next Chapter, we shall apply these results to determine the cyclic behavior of composition Volterra operators.

We begin by observing that if $S$ is a linear, not necessarily bounded, operator acting on a linear space $X$, then

$$S^\infty(X) = \bigcap_{n=0}^{\infty} S^n(X)$$

is a subspace of $X$ invariant under $S$. Moreover, it is clear that $S(S^\infty(X)) = S^\infty(X)$. Thus the restriction of $S$ to $S^\infty(X)$ is always onto. In addition, if $\ker S = \{0\}$, then the restriction of $S$ to $S^\infty(X)$ is bijective. In such a case, since $S$ is one-to-one from $S^\infty(X)$ onto itself, the backward orbits of any $x$ in $S^\infty$ are well defined. This is in particular our case for $\ker V_\varphi = \{0\}$.

Recall that $\mathcal{C}_0[0,1]$ is the subspace of $\mathcal{C}[0,1]$ of functions vanishing at 0, endowed with the supremum norm.

**Theorem 4.3.1.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume also that $\varphi$ is analytic at $0$ and $\varphi'(0) > 0$. Then for each $b > 1/\varphi'(0)$, the set*

$$F_b = \Big\{ f \in V_\varphi^\infty(\mathcal{C}_0[0,1]) \text{ such that } \varlimsup_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{b} \Big\}$$

*is a dense linear manifold of $\mathcal{C}_0[0,1]$ satisfying $V_\varphi(F_b) = V_\varphi^{-1}(F_b) = F_b$.*

The remainder of this section is devoted to proving the above theorem. As in Section 3.2, for $f \in \mathcal{C}^\infty[a,b]$, we set

$$M_n(f) = \frac{1}{n!} \max_{[a,b]} |f^{(n)}|.$$

We also need the space $\mathcal{F}[a,b]$ being

$$\{f \in \mathcal{C}^\infty[a,b] : f^{(n)}(a) = f^{(n)}(b) = 0 \text{ for each } n \geqslant 0 \text{ and } \varlimsup_{n\to\infty} (M_n(f))^{1/n^2} \leqslant 1\}.$$

By means of Leibnitz's formula one can check that if $f$ is in $\mathcal{F}[a,b]$ and $g$ in $\mathcal{C}^\infty[a,b]$ satisfies $\varlimsup_{n\to\infty}(M_n(g))^{1/n^2} \leqslant 1$, then $fg$ belongs to $\mathcal{F}[a,b]$. In particular, the space $\mathcal{F}[a,b]$ is an algebra with respect to the pointwise multiplication and is invariant under multiplication by analytic functions.

**Lemma 4.3.2.** *Assume that $-\infty < a < b < \infty$. Then*

$$h(x) = \begin{cases} e^{-\frac{1}{x-a} - \frac{1}{b-x}}, & \text{if } a < x < b; \\ 0, & \text{if } x = a \text{ or } x = b \end{cases} \tag{4.3.1}$$

*belongs to $\mathcal{F}[a,b]$.*

*Proof.* Clearly, $h$ is in $\mathcal{C}^\infty[a,b]$ with $h^{(n)}(a) = h^{(n)}(b) = 0$ for each non-negative integer $n$. Thus we need only prove that

$$\varlimsup_{n\to\infty} (M_n(h))^{1/n^2} \leqslant 1. \tag{4.3.2}$$

To this end, we estimate $M_n(g)$, where $g(x) = e^{-1/x}$ for $x > 0$ and $g(0) = 0$. By induction, one easily sees that $g^{(n)}(x) = p_n(1/x)g(x)$ for $x \neq 0$, where

$$p_0 = 1 \text{ and } p_{n+1}(t) = t^2 p_n(t) - t^2 p_n'(t), \quad \text{for each } n \geqslant 1. \tag{4.3.3}$$

Clearly, $p_n(t) = \sum_{j=0}^{2n} a_{n,j} t^j$, where the coefficients $a_{n,j}$ are real. Therefore,

$$M_n(g) = \frac{1}{n!} \sup_{x>0} |g^{(n)}(x)| \leqslant \frac{1}{n!} \sum_{j=0}^{2n} |a_{n,j}| \sup_{x>0} x^{-j} e^{-1/x} = \frac{1}{n!} \sum_{j=0}^{2n} |a_{n,j}| \sup_{x>0} x^j e^{-x}.$$

Since $\sup_{x>0} x^j e^{-x} = (j/e)^j$, we have

$$M_n(g) \leqslant \frac{1}{n!}\sum_{j=0}^{2n} |a_{n,j}| \left(\frac{j}{e}\right)^j \leqslant \frac{(2n)^{2n}}{n! e^{2n}}\sigma_n, \quad \text{where} \quad \sigma_n = \sum_{j=0}^{2n} |a_{n,j}|.$$

Using (4.3.3), we see that $\sigma_{n+1} \leqslant (2n+1)\sigma_n$ and, therefore, $\sigma_n \leqslant (2n)!/(2^n n!)$. Upon putting everything together and using Stirling's formula one sees that $M_n(g) \leqslant (2n/e)^{2n}$.

Since $h(x) = g(x-a)g(b-x)$, applying Leibnitz's formula, we see that

$$h^{(n)}(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} g^{(k)}(b-x) g^{(n-k)}(x-a), \qquad \text{for } a \leqslant x \leqslant b.$$

Therefore, using that $M_k(g) \leqslant (2k/e)^{2k}$ in the second inequality below, we have

$$M_n(h) \leqslant \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} M_k(g) M_{n-k}(g) \leqslant 2^n \max_{0\leqslant k\leqslant n} (2k)^{2k}(2n-2k)^{2n-2k} < (2n)^{2n},$$

from which (4.3.2) follows and the result is proved. □

The following lemma can also be derived from the Denjoy–Carleman Theorem, see [43, p. 380]. Here, we provide an elementary proof. We denote by $\mathcal{C}_{00}[a,b]$ the Banach space of complex continuous functions on $[a,b]$ that vanish at $a$ and $b$ endowed with the supremum norm.

**Lemma 4.3.3.** *Assume that $a < b$ are real. Then $\mathcal{F}[a,b]$ is dense in $\mathcal{C}_{00}[a,b]$ and*

$$\mathcal{F}^+[a,b] = \{f \in \mathcal{F}[a,b] \text{ such that } f(x) \geqslant 0 \text{ for each } x \in [a,b]\}$$

*is dense in*

$$\mathcal{C}_{00}^+[a,b] = \{f \in \mathcal{C}_{00}[a,b] \text{ such that } f(x) \geqslant 0 \text{ for each } x \in [a,b]\}.$$

*Proof.* Let $h$ be the function in (4.3.1), that is,

$$h(x) = \begin{cases} e^{-\frac{1}{x-a}-\frac{1}{b-x}}, & \text{if } a < x < b; \\ 0, & \text{if } x = a \text{ or } x = b. \end{cases}$$

By Lemma 4.3.2, we know that $h$ is in $\mathcal{F}[a,b]$. Since $h$ is in $\mathcal{C}_{00}[a,b]$ and $h(x) > 0$ for $a < x < b$, we see that

$$W = \{ph \text{ such that } p \text{ is a polynomial}\}$$

is dense in $\mathcal{C}_{00}[a,b]$. Also,

$$W^+ = \{g \in W \text{ such that } g(x) \geqslant 0 \text{ on } [a,b]\}$$

is dense in $\mathcal{C}_{00}^+[a,b]$. Now, the result follows because $\mathcal{F}[a,b]$ is stable with respect to multiplication by polynomials and therefore $W \subset \mathcal{F}[a,b]$ and $W^+ \subset \mathcal{F}^+[a,b]$. □

Now, we proceed to define an operator closely related to the 'inverse' of $V_\varphi$. For $0 < a < 1$, we set

$$\mathcal{E}_a = \{f \in \mathcal{C}^\infty[0,1] \text{ such that } \operatorname{supp} f \subset [0,a] \ \text{ and } \ f^{(n)}(0) = 0 \ \text{ for } \ n \geqslant 0\}. \quad (4.3.4)$$

For an analytic function $\psi : [0,a] \to [0,1]$ such that $\psi(a) \geqslant a$ and $\psi(0) = 0$ consider the operator $T_\psi : \mathcal{E}_a \to \mathcal{E}_a$ defined as

$$(T_\psi f)(x) = \begin{cases} f'(\psi(x)), & \text{if } x \in [0,a]; \\ 0, & \text{if } x \in (a,1]. \end{cases} \quad (4.3.5)$$

The requirements $\psi(a) \geqslant a$ and $\psi(0) = 0$ implies that $T_\psi$ acts from $\mathcal{E}_a$ into itself. As usual, for each pair $n$ and $l$ of non-negative integers, we write

$$(n)_l = 1, \ \text{ if } \ l = 0 \ \text{ and } \ (n)_l = (n+1)\cdots(n+l), \ \text{ if } \ l > 0.$$

**Lemma 4.3.4.** *Let $\psi$ be an analytic function from $[0,a]$ into $[0,1]$, where $0 < a < 1$, with $\psi(0) = 0$ and $\psi(a) \geqslant a$. Let $\{c_n\}_{n\geqslant 0}$ be such that $c_n \geqslant 1$ with $\lim_{n\to\infty} c_n^{1/n} = 1$ and let $\{f_n\}_{n\geqslant 0}$ be in $\mathcal{E}_a$ satisfying*

$$\beta_n = \sup_{k\geqslant 0} M_n(f_k) c_k^{-n-1} < \infty, \qquad \textit{for each } n \geqslant 0 \ \textit{ and } \ \varlimsup_{n\to\infty} \beta_n^{1/n^2} \leqslant 1.$$

*Then*

$$\varlimsup_{n\to\infty} \|T_\psi^n f_n\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}.$$

*Proof.* The proof is split into three steps.
*Step 1. Let $\{\widehat{\beta}_n\}_{n\geqslant 0}$ be a sequence such that $\{\widehat{\beta}_n^{1/n}\}$ is increasing. Assume also that $c > 0$ and $l$ is a non-negative integer. Then for $f$ in $\mathcal{E}_a$ satisfying $M_n(f) \leqslant c\,(n)_l \widehat{\beta}_n$ for each $n \geqslant 0$, we have*

$$M_n(T_\psi f) \leqslant c\,(n)_{l+1} \gamma^n \widehat{\beta}_{n+1} \left(1 + \frac{R}{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}\right)^n, \qquad \textit{for each } n \geqslant 0,$$

*where*

$$R = \sup_{n\geqslant 2} \left(\frac{M_n(\psi)}{\gamma}\right)^{1/(n-1)}. \quad (4.3.6)$$

*Proof of Step 1.* Since $\psi$ is analytic on $[0,a]$, by (3.2.1), we see that $R$ is finite. Clearly,

$$\|T_\psi f\|_\infty \leqslant \|f'\|_\infty = M_1(f) \leqslant c(1)_l \widehat{\beta}_1 = c(0)_{l+1} \widehat{\beta}_1.$$

Thus the result is true for $n = 0$. Since

$$M_n(\psi) \leqslant \gamma R^{n-1}, \qquad \text{for } \ n \geqslant 1, \quad (4.3.7)$$

using Lemma 3.2.1, for each $0 \leqslant x \leqslant a$ with $0 < \psi(x) < a$ and for each positive integer $n$, we have

$$(T_\psi f)^{(n)}(x) = (f' \circ \psi)^{(n)}(x) = n! \sum_{n=k_1+\cdots+nk_n} \frac{f^{(k_1+\cdots+k_n+1)}(\psi(x))}{k_1!\cdots k_n!(1!)^{k_1}\cdots(n!)^{k_n}} (\psi'(x))^{k_1}\cdots(\psi^{(n)}(x))^{k_n}.$$

From (4.3.5), we have $(T_\psi f)^{(n)}(x) = 0$ for $f$ in $\mathcal{E}_a$ and $\psi(x) \geqslant a$ and, therefore, we may write

$$\begin{aligned} M_n(T_\psi f) &\leqslant \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n+1)!}{k_1!\cdots k_n!} M_{k_1+\cdots+k_n+1}(f)(M_1(\psi))^{k_1}\cdots(M_n(\psi))^{k_n} \\ &\leqslant (n+1) \sum_{n=k_1+\ldots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} M_{k_1+\cdots+k_n+1}(f)(M_1(\psi))^{k_1}\cdots(M_n(\psi))^{k_n}. \end{aligned}$$

From (4.3.7) and the fact that $M_k(f) \leqslant c(k)_l \widehat{\beta}_k$, we have

$$M_n(T_\psi f) \leqslant c(n)_{l+1} \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \widehat{\beta}_{k_1+\cdots+k_n+1} \gamma^{k_1} (\gamma R)^{k_2} \cdots (\gamma R^{n-1})^{k_n}.$$

Since $\{\widehat{\beta}_k^{1/k}\}$ is increasing, it follows that $\widehat{\beta}_k \leqslant (\widehat{\beta}_m)^{k/m}$ for $1 \leqslant k \leqslant m$. Therefore,

$$M_n(T_\psi f) \leqslant c\,(n)_{l+1} R^n \widehat{\beta}_{n+1}^{1/(n+1)} \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \left( \frac{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}{R} \right)^{k_1+\cdots+k_n}.$$

Applying Lemma 3.2.2, we have

$$\begin{aligned} M_n(T_\psi f) &\leqslant c\,(n)_{l+1} \gamma R^{n-1} \widehat{\beta}_{n+1}^{2/(n+1)} \left( 1 + \frac{\gamma \widehat{\beta}_{n+1}^{1/(n+1)}}{R} \right)^{n-1} \\ &= c\,(n)_{l+1} \gamma^n \widehat{\beta}_{n+1} \left( 1 + \frac{R}{\gamma \widehat{\beta}_n^{1/(n+1)}} \right)^{n-1} \\ &\leqslant c\,(n)_{l+1} \gamma^n \widehat{\beta}_{n+1} \left( 1 + \frac{R}{\gamma \widehat{\beta}_n^{1/(n+1)}} \right)^{n}. \end{aligned}$$

The proof of Step 1 is complete.

*Step 2. Under the hypotheses of the lemma we have*

$$\widetilde{\beta}_n = \sup_{k \geqslant 0} M_n(T_\psi f_k) c_k^{-n-2}$$

*is finite for each $n \geqslant 0$ and*

$$\overline{\lim_{n\to\infty}} \widetilde{\beta}_n^{1/n^2} \leqslant 1.$$

*Proof of Step 2.* Let $\delta > 1$ be fixed. Since $\overline{\lim}_{n\to\infty}\beta_n^{1/n^2} \leqslant 1$, we may choose $C > 0$ such that $\beta_n \leqslant C\,\delta^{n^2}$ for each $n \geqslant 0$. For each $n \geqslant 0$, we set $\widehat{\beta}_n = c_k^n \delta^{n^2}$, where $c = Cc_k$. Then, for each $k \geqslant 0$, we have $M_n(f_k) \leqslant c\widehat{\beta}_n$. By step 1, we have

$$M_n(T_\psi f_k) \leqslant c(n+1)\gamma^n \widehat{\beta}_{n+1}\left(1 + \frac{R}{\gamma\widehat{\beta}_{n+1}^{1/(n+1)}}\right)^n, \qquad \text{for each } n \geqslant 0.$$

Upon substituting the values of $c$ and $\widehat{\beta}_{n+1}$, we obtain

$$\begin{aligned} M_n(T_\psi f_k) \leqslant & C\,(n+1)c_k^{n+2}\gamma^n\delta^{(n+1)^2}\left(1+\frac{R}{\gamma c_k\delta^{n+1}}\right)^n \\ \leqslant & C\,(n+1)c_k^{n+2}\gamma^n\delta^{(n+1)^2}\left(1+\frac{R}{\gamma\delta^{n+1}}\right)^n. \end{aligned}$$

Therefore,

$$\widetilde{\beta}_n \leqslant C\,(n+1)\gamma^n\delta^{(n+1)^2}\left(1+\frac{R}{\gamma\delta^{n+1}}\right)^n.$$

In particular, $\overline{\lim}_{n\to\infty}\widetilde{\beta}_n^{1/n^2} \leqslant \delta$. Since $\delta > 1$ was arbitrary, the proof of Step 2 is complete.

*Step 3. The conclusion of the lemma holds.*

*Proof of Step 3.* Let $\delta > 1$ be fixed. Since $\psi(0) = 0$ and $\psi(a) \geqslant a$, we see that $\gamma = \|\psi'\|_\infty \geqslant 1$. Thus using that $c_j \geqslant 1$, we may take a positive integer $l$ such that

$$\delta^{n/2}\left(1+\frac{R}{\gamma(\gamma\delta)^m\delta^{(n+1)/4}c_j}\right)^n \leqslant \delta^n, \qquad \text{for each } m \geqslant l \text{ and } n, j \geqslant 0. \tag{4.3.8}$$

Indeed, it is enough to take $l$ with $\delta^l \geqslant R/(\sqrt{\delta}-1)$. As in the proof of Step 2, there is $C > 0$ such that for $0 \leqslant k \leqslant l$, we have

$$M_n(T_\psi^k f_j) \leqslant C c_j^{n+k+1}\delta^{k/4}(n)_k(\gamma\delta)^{k(n+(k-1)/2)}\delta^{n^2/4}, \qquad \text{for each } n, j \geqslant 0. \tag{4.3.9}$$

We will prove that (4.3.9) also holds for each $k \geqslant l+1$. Suppose that (4.3.9) is true for an integer $k = m \geqslant l$. For $k = m$, we can rewrite (4.3.9) as

$$M_n(T_\psi^m f_j) \leqslant c\,(n)_m\widehat{\beta}_n,$$

where $c = C\,c_j^{m+1}\delta^{m/4}(\gamma\delta)^{m(m-1)/2}$ and $\widehat{\beta}_n = c_j^n(\gamma\delta)^{mn}\delta^{n^2/4}$. Applying Step 1, we have

$$M_n(T_\psi^{m+1} f_j) \leqslant c\,(n)_{m+1}\gamma^n\widehat{\beta}_{n+1}\left(1+\frac{R}{\gamma\widehat{\beta}_{n+1}^{1/(n+1)}}\right)^n,$$

which is equal to

$$Cc_j^{m+n+1}\delta^{(m+1)/4}(n)_{m+1}(\gamma\delta)^{m(m-1)/2}\gamma^n(\gamma\delta)^{mn+m}\delta^{n^2/4}\delta^{n/2}\left(1+\frac{R}{\gamma c_j(\gamma\delta)^m\delta^{(n+1)/4}}\right)^n.$$

Since $m \geqslant l$, we may use (4.3.8) to obtain

$$\begin{aligned} M_n(T_\psi^{m+1}f) &\leqslant Cc_j^{m+n+1}\delta^{(m+1)/4}(n)_{m+1}(\gamma\delta)^{m(m-1)/2}\gamma^n(\gamma\delta)^{mn+m}\delta^{n^2/4}\delta^n \\ &= Cc_j^{n+m+1}\delta^{(m+1)/4}(n)_{m+1}(\gamma\delta)^{mn+m(m-1)/2+n+m}\delta^{n^2/4} \\ &= C\delta^{(m+1)/4}(n)_{m+1}(\gamma\delta)^{(m+1)(n+(m/2))}\delta^{n^2/4}, \end{aligned}$$

which is (4.3.9) for $k = m+1$. Thus (4.3.9) holds for all non-negative integers $k$, $n$ and $j$. For $n = 0$ and $j = k$, we find that (4.3.9) implies that

$$\|T_\psi^k f_k\|_\infty \leqslant Cc_k^{k+1}\delta^{k/4}k!(\gamma\delta)^{k(k-1)/2}.$$

Since $c_k^{1/k}$ tends to 1, we obtain

$$\varlimsup_{k\to\infty}\|T_\psi^k f_k\|_\infty^{1/k^2} \leqslant \sqrt{\gamma\delta}.$$

Since $\delta > 1$ was arbitrary, it follows

$$\varlimsup_{k\to\infty}\|T_\psi^k f_k\|_\infty^{1/k^2} \leqslant \sqrt{\gamma}.$$

which is the required result. The proof of Step 3 and that of the statement of the Lemma is complete. □

Observe that the formula for the adjoint of $V_\varphi$ is

$$(V_\varphi^\star f)(x) = \int_{\varphi_{-1}(x)}^1 f(t)\,dt.$$

that, as an operator, has sense on $L^p[0,1]$ for $1 \leqslant p \leqslant \infty$. Indeed, the adjoint of $V_\varphi^\star$ acting on $L^1[0,1]$ is $V_\varphi$ acting on $L^\infty[0,1]$. The next lemma, which will be very useful, describes the behavior of the supports of the iterates $\{V_\varphi^n f\}$ and $\{V_\varphi^{\star n} f\}$. The proof, which is straightforward, is omitted.

**Lemma 4.3.5.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ and $\varphi(1) = 1$. Assume also that $f$ is in $L^1[0,1]$. Then we have*

(a) $\inf\operatorname{supp}(V_\varphi f) = \varphi_{-1}(\inf\operatorname{supp}(f))$.

(b) $\sup\operatorname{supp}(V_\varphi f) \in \{1, \varphi_{-1}(\sup\operatorname{supp}(f))\}$.

(c) $\inf\operatorname{supp}(V_\varphi^\star f) \in \{0, \varphi(\inf\operatorname{supp}(g))\}$.

(d) $\sup\operatorname{supp}(V_\varphi^\star g) = \varphi(\sup\operatorname{supp}(g))$.

(e) $\sup\operatorname{supp}(V_\varphi^n f)$ *tends to* 1 *and* $\inf\operatorname{supp}(V_\varphi^{\star n} g)$ *tends to* 0 *as* $n$ *tends to* $\infty$.

When dealing with supercyclicity of $V_\varphi$ in Section 5.3, we will need special dense subsets of $\mathcal{C}_0[0,1]$. For each $0 < a < 1$, we set

$$\mathcal{C}_a = \{f \in \mathcal{C}_0[0,1] \text{ such that } \sup\operatorname{supp}(f) \leqslant a\}.$$

We have,

**Lemma 4.3.6.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $0 < a < 1$. Then*

$$Z = \operatorname{span}\left(\bigcup_{n=0}^{\infty} V_\varphi^n(\mathcal{C}_a)\right)$$

*is dense in $\mathcal{C}_0[0,1]$.*

*Proof.* It is enough to prove that $Z$ is dense in $L^2[0,1]$. Indeed, once this is proved, the result follows because $V_\varphi$ acting from $L^2[0,1]$ into $\mathcal{C}_0[0,1]$ is bounded with dense range and the image of a dense set under an operator with dense range is itself dense and $Z$ is invariant under $V_\varphi$.

Thus assume that $Z$ is not dense in $L^2[0,1]$. Then there is a non-zero $g$ in $L^2[0,1]$ such that $\langle V_\varphi^n f, g\rangle = \langle f, V_\varphi^{\star n} g\rangle = 0$ for each $f$ in $\mathcal{C}_a$ and for each non-negative integer $n$. This means that $\inf\operatorname{supp}(V_\varphi^{\star n} g) \geqslant a$ for each non-negative integer $n$, which is impossible, since by Lemma 4.3.5, we have $\inf\operatorname{supp}(V_\varphi^{\star n} g)$ tends to 0 as $n$ tend to $\infty$, which proves the result. □

For $0 < a < 1$, we shall write

$$\mathcal{F}_a = \left\{f \in \mathcal{E}_a \text{ such that } \varlimsup_{n\to\infty} (M_n(f))^{1/n^2} \leqslant 1\right\}, \tag{4.3.10}$$

where $\mathcal{E}_a$ is the one defined in (4.3.4). That is, $f$ belongs to $\mathcal{F}_a$ if and only if $f$ belongs to $\mathcal{C}^\infty[0,1]$, $\operatorname{supp}(f) \subseteq [0,a]$ and the restriction of $f$ to $[0,a]$ belongs to $\mathcal{F}[0,a]$.

**Lemma 4.3.7.** *Let $\psi$ be analytic from $[0,a]$ into $[0,1]$, where $0 < a < 1$, with $\psi(0) = 0$ and $\psi(a) \geqslant a$. Let $T_\psi$ be the operator on $\mathcal{E}_a$ defined in (4.3.5) and $C_\psi$ be the operator on $\mathcal{E}_a$ defined as*

$$(C_\psi f)(x) = \begin{cases} f(\psi(x)), & \text{if } x \in [0,a]; \\ 0, & \text{if } x \in (a,1]. \end{cases}$$

*Then $\mathcal{F}_a$ is invariant under $C_\psi$ as well as under $T_\psi$.*

*Proof.* Let $\gamma > 1$ be fixed. If $f$ is in $\mathcal{F}_a$, then there is $c \geqslant 1$ such that $M_n(f) \leqslant c^n \gamma^{n^2}$ for each positive integer $n$. Since $\psi$ is analytic, by (3.2.1), we see that the value $R$

$$R = \sup_{n \geqslant 2} \left( \frac{M_n(\psi)}{\gamma} \right)^{1/(n-1)}$$

is finite. Now, from Lemma 3.2.1, it follows that

$$M_n(C_\psi f) \leqslant \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1! \cdots k_n!} M_{k_1+\cdots+k_n}(f)(M_1(\psi))^{k_1} \cdots (M_n(\psi))^{k_n}.$$

Using that $M_m(\psi) \leqslant \gamma R^{m-1}$ and $M_k(f) \leqslant c^k \gamma^{k^2}$, we obtain

$$\begin{aligned} M_n(C_\psi f) &\leqslant \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1! \cdots k_n!} c^{k_1+\cdots+k_n} \gamma^{(k_1+\ldots+k_n)^2} \gamma^{k_1} (\gamma R)^{k_2} \cdots (\gamma R^{n-1})^{k_n} \\ &\leqslant R^n \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1! \cdots k_n!} \left( \frac{c\gamma^{n+1}}{R} \right)^{k_1+\cdots+k_n}. \end{aligned}$$

Upon applying Lemma 3.2.2, we obtain

$$M_n(C_\psi f) \leqslant R^n \frac{c\gamma^{n+1}}{R} \left( 1 + \frac{c\gamma^{n+1}}{R} \right)^{n-1} \leqslant (R + c\gamma^{n+1})^n.$$

Therefore, it follows that

$$\varlimsup_{n\to\infty} (M_n(C_\psi f))^{1/n^2} \leqslant \gamma.$$

Since $\gamma > 1$ was arbitrary, we see that

$$\varlimsup_{n\to\infty} (M_n(C_\psi f))^{1/n^2} \leqslant 1$$

and, therefore, $C_\psi f$ belongs to $\mathcal{F}_a$. Finally, it is clear that $\mathcal{F}_a$ is also invariant under the derivative operator $Df = f'$. Since $T_\psi = C_\psi D$, the result follows. □

The next lemma is needed not only to prove Theorem 4.3.1, but also to show the non-cyclicity of certain composition Volterra operators in Section 5.3.

**Lemma 4.3.8.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume also that $\varphi$ is analytic on $[0, \varphi_{-1}(a)]$, where $0 < a < 1$, with $\varphi'(x) > 0$ for $0 \leqslant x \leqslant \varphi_{-1}(a)$. Then $\mathcal{F}_a$ is contained in $V_\varphi^\infty(\mathcal{C}_0[0,1])$ and*

$$\varlimsup_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma} \quad \text{for each } f \in \mathcal{F}_a,$$

*where $\gamma = \max_{[0,\varphi^{-1}(a)]} 1/\varphi'$.*

*Proof.* We set $\psi = \varphi_{-1}$ for the inverse of $\varphi$. Clearly, $\psi$ is analytic on $[0, a]$ and $\max_{[0,a]} |\psi'| = \gamma$. It is easy to check that

$$(Sf)(x) = \begin{cases} \dfrac{f'(\psi(x))}{\varphi'(\psi(x))}, & \text{if } x \in (0, a); \\ 0, & \text{otherwise} \end{cases} \tag{4.3.11}$$

acts from $\mathcal{E}_a$ into $\mathcal{E}_a$ and that $V_\varphi S f = f$ for each $f$ in $\mathcal{E}_a$. Therefore, it follows that

$$\mathcal{F}_a \subset \mathcal{E}_a \subset V_\varphi^\infty(\mathcal{C}_0[0,1])$$

and the operator $S$ defined in (4.3.11) coincides with the restriction to $\mathcal{E}_a$ of $V_\varphi^{-1}$ acting on $V_\varphi^\infty(\mathcal{C}_0[0,1])$.

Now consider the operator $T_\psi$ acting on $\mathcal{E}_a$ as defined in (4.3.5). One easily sees that $C_\psi S f = T_\psi C_\psi f$ for each $f$ in $\mathcal{E}_a$, where $C_\psi$ is defined as $(C_\psi f)(x) = f(\psi(x))$. Hence $C_\psi S^n f = T_\psi^n C_\psi f$ for each $f$ in $\mathcal{E}_a$ and each non-negative integer $n$. Thus

$$\|V_\varphi^{-n} f\|_\infty = \|S^n f\|_\infty = \|C_\psi S^n f\|_\infty = \|T_\psi^n C_\psi f\|_\infty \quad \text{for } f \in \mathcal{E}_a \text{ and } n \geqslant 0. \tag{4.3.12}$$

Now, if $f$ belongs to $\mathcal{F}_a$, then, by Lemma 4.3.7, we have $C_\psi f$ belongs to $\mathcal{F}_a$. Hence

$$\varlimsup_{n\to\infty} (M_n(C_\psi f))^{1/n^2} \leqslant 1.$$

Applying Lemma 4.3.4 with $c_n = 1$ and $f_n = C_\psi f$ for each $n \geqslant 0$, we obtain $\varlimsup_{n\to\infty} \|T_\psi^n C_\psi f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}$. Therefore, using (4.3.12), we have

$$\varlimsup_{n\to\infty} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma},$$

which is the required conclusion. □

Now, we have all necessary tools to prove Theorem 4.3.1.

*Proof of Theorem 4.3.1.* One easily checks that $F_b$ is linear and that $V_\varphi(F_b)$ and $V_\varphi^{-1}(F_b)$ are contained in $F_b$, which implies that $V_\varphi(F_b) = F_b = V_\varphi^{-1}(F_b)$. Thus we need only prove that $F_b$ is dense in $\mathcal{C}_0[0,1]$.

Set $\psi = \varphi_{-1}$ for the inverse of $\varphi$. Since $b > 1/\varphi'(0)$, we may choose $0 < a < 1$ such that $\varphi$ is analytic on $[0, \varphi_{-1}(a)]$ and $\varphi'(x) > 0$ for each $0 \leqslant x \leqslant \varphi_{-1}(a)$ and

$$\gamma = \max_{[0,\varphi_{-1}(a)]} \frac{1}{|\varphi'|} = \max_{[0,a]} |\psi'| \leqslant b.$$

By Lemma 4.3.8, we have $\mathcal{F}_a \subset V_\varphi^\infty(\mathcal{C}_0[0,1])$ and

$$\varlimsup_{n\to\infty} \|V_\varphi^n f\|_\infty^{1/n^2} \leqslant \sqrt{\gamma} \leqslant \sqrt{b}, \text{ for each } f \in \mathcal{F}_a.$$

Thus we find that $\mathcal{F}_a \subset F_b$. Since $F_b$ is invariant under $V_\varphi$, we have

$$\operatorname{span}\left(\bigcup_{n=0}^{\infty} V_\varphi^n(\mathcal{F}_a)\right) \subseteq F_b.$$

By Lemma 4.3.3, we know that $\mathcal{F}_a$ is dense in the subspace $\mathcal{C}_a$ of functions in $\mathcal{C}_0[0,1]$ that vanish on $[a,1]$. Therefore, since $V_\varphi$ is bounded, it follows from the above display that

$$\overline{\operatorname{span}}\left(\bigcup_{n=0}^{\infty} V_\varphi^n(\mathcal{C}_a)\right) \subseteq \overline{F}_b,$$

where the closures are taken in $\mathcal{C}_0[0,1]$. We may conclude from Lemma 4.3.6 that the left hand side in the above display coincides with $\mathcal{C}_0[0,1]$ and, therefore, $F_b$ is dense in $\mathcal{C}_0[0,1]$. The proof is complete. □

## 4.4 Orbits of $V_\varphi$. Lower estimate

The following theorem provides a lower estimate for orbits of $V_\varphi$ under certain regularity hypotheses on $\varphi$.

**Theorem 4.4.1.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and analytic at $1$ with $\varphi(1) = 1$. Then, for each non-zero $f$ in $L^1[0,1]$, we have*

$$\varliminf_{n\to\infty} \|V_\varphi^n f\|_1^{1/n^2} \geqslant \frac{1}{\sqrt{\varphi'(1)}}. \tag{4.4.1}$$

*Proof.* Recall from Section 2.1 that the adjoint $V_\varphi^\star$ that acts on $L^\infty[0,1]$ is $V_\varphi^\star = UV_\phi U$, where $U$ is the involutive isometry defined by $(Ug)(x) = g(1-x)$ and $\phi(x) = 1 - \varphi_{-1}(1-x)$. Since $\varphi$ is analytic at 1, then so is $\psi = \phi^{-1}$ at 0. We take $\gamma > \psi'(0) \geqslant 1$. Next, we take $0 < a < 1$ such that $\psi$ is analytic on $[0,a]$ and

$$\sup_{[0,a]} \psi' \leqslant \gamma.$$

By Lemma 4.3.5, we have $\sup\operatorname{supp}(V_\varphi^k g)$ tends to 1 as $k$ tends to $\infty$ for each non-zero $g$ in $L^1[0,1]$. Therefore, for each non-zero $g$ in $L_1[0,1]$, we have $\sup\operatorname{supp}(V_\varphi^k g) > 1-a$ for all $k$ large enough. Observe also that for any positive integer $k$ and for any non-zero $g$ in $L^1[0,1]$ the inequality in (4.4.1) is satisfied for $f = g$ if and only if it is satisfied for $f = V_\varphi^k g$. Since the range of $V_\varphi$ is contained in $\mathcal{C}_0[0,1]$, we see that it is enough to show the inequality in (4.4.1) for each $f$ in $\mathcal{C}_0[0,1]$ with $\sup\operatorname{supp}(f) > 1-a$.

Thus assume that $f$ in $\mathcal{C}_0[0,1]$ has $\sup\operatorname{supp}(f) > 1-a$. We may take $1-a < b < 1$ and $\delta > 0$ such that $1-a < b-\delta < b+\delta \leqslant 1$ and $f(b) \neq 0$. By Lemma 4.3.3,

$\mathcal{F}^+[1-b-\delta, 1-b+\delta]$ is dense in $\mathcal{C}_{00}^+[1-b-\delta, 1-b+\delta]$. In particular, there is $g_1$ in $\mathcal{F}[1-b-\delta, 1-b+\delta]$ such that $g_1(x) \geqslant 0$ for each $1-b-\delta \leqslant x \leqslant 1-b+\delta$ and

$$\int_{1-b-\delta}^{1-b+\delta} g_1(x)\,dx = 1.$$

We may think of $g_1$ as defined on the whole real line, by just making $g_1$ equal to 0 outside of $[1-b-\delta, 1-b+\delta]$. Now, consider $g_n(x) = n g_1(nx - (1-b)(n-1))$ for $n \geqslant 1$. In this way, $\{g_n\}_{n\geqslant 1}$ is a positive summability kernel at 0, see [25, pp. 9–10]. Since $\operatorname{supp}(g_n) \subseteq [1-b-\delta/n, 1-b+\delta/n] \subset [0,a]$, we may regard $\{g_n\}$ as a sequence in $\mathcal{E}_a$. Now, set $f_n(x) = g_n(\psi(x))$ and consider

$$R = \sup_{n\geqslant 2} \left(\frac{M_n(\psi)}{\gamma}\right)^{1/(n-1)},$$

where

$$M_n(\psi) = \frac{1}{n!} \sup_{[0,a]} |\psi^{(n)}|.$$

By Lemma 3.2.1, we find that

$$M_n(f_k) \leqslant k \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} M_{k_1+\cdots+k_n}(g_1)(M_1(\psi))^{k_1}\cdots(M_n(\psi))^{k_n} k^{k_1+\cdots+k_n}.$$

Since $M_n(\psi) \leqslant \gamma R^{n-1}$, setting $\alpha_n = \max_{0\leqslant j\leqslant n} M_j(g_1)$, we have

$$\begin{aligned} M_n(f_k) &\leqslant kR^n \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} M_{k_1+\cdots+k_n}(g_1) \left(\frac{k\gamma}{R}\right)^{k_1+\cdots+k_n} \\ &\leqslant kR^n\alpha_n \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \left(\frac{k\gamma}{R}\right)^{k_1+\cdots+k_n}. \end{aligned}$$

Applying Lemma 3.2.2, we obtain

$$M_n(f_k) \leqslant \alpha_n k^2 R^{n-1} \left(1 + \frac{k\gamma}{R}\right)^{n-1} = \alpha_n k^2 (R+k\gamma)^{n-1} \leqslant \alpha_n (R+k\gamma)^{n+1}.$$

Since $g_1$ belongs to $F_{\psi(a)}$, we have $\overline{\lim}_{n\to\infty} \alpha_n^{1/n^2} \leqslant 1$ and, therefore, all the hypotheses of Lemma 4.3.4 with $c_k = R + k\gamma$ are fulfilled. Thus

$$\overline{\lim_{n\to\infty}} \|T_\psi^n f_n\|_\infty^{1/n^2} \leqslant \sqrt{\gamma}, \tag{4.4.2}$$

where $T_\psi$ is as in (4.3.5). Since

$$V_\phi^{-1} f = C_\phi T_\psi C_\phi^{-1} f = C_\phi T_\psi C_\psi f, \qquad \text{for each } f \in \mathcal{E}_a,$$

we find that $g_n = V_\phi^n C_\phi T_\psi^n C_\psi g_n = V_\phi^n C_\phi T_\psi^n f_n$. Using that $U$ is involutive and $V_\varphi^\star = UV_\phi U$, we see that $Ug_n = V_\varphi^{\star n} UC_\phi T_\psi^n f_n$. Therefore,

$$\|V_\varphi^n f\|_1 \geqslant \frac{|\langle V_\varphi^n f, UC_\phi T_\psi^n f_n\rangle|}{\|UC_\phi T_\psi^n f_n\|_\infty} \geqslant \frac{|\langle f, V_\varphi^{\star n} UC_\phi T_\psi^n f_n\rangle|}{\|UC_\phi T_\psi^n f_n\|_\infty} = \frac{|\langle f, Ug_n\rangle|}{\|T_\psi^n f_n\|_\infty}.$$

Since $\{g_n\}$ is a positive summability kernel at $1-b$, then so is $\{Ug_n\}$ at $b$. Therefore, $|\langle f, Ug_n\rangle|$ converges to $|f(b)| \neq 0$. Thus,

$$\|V_\varphi^n f\|_1 \geqslant \frac{|f(b)|}{\|T_\psi^n f_n\|_\infty}(1 + o(1)), \quad \text{as} \quad n \to \infty.$$

The above display along with (4.4.2) implies that

$$\varliminf_{n\to\infty} \|V_\varphi^n f\|_p^{1/n^2} \geqslant \frac{1}{\sqrt{\gamma}}.$$

Since $\gamma > \psi'(0) = \varphi'(1)$ was arbitrary, the result follows. □

From Theorem 4.4.1 and Corollary 4.1.4, we immediately have

**Corollary 4.4.2.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(1) = 1$ and $\varphi(x) < x$ for $0 < x < 1$. Assume also that $\varphi$ is analytic at $1$ and differentiable at $0$ with $\varphi'(0) = 0$. Then for each non-zero $f$ in $L^p[0,1]$, $1 \leqslant p \leqslant \infty$, we have*

$$\lim_{n\to\infty} \|V_\varphi f\|_p^{1/n^2} = \frac{1}{\sqrt{\varphi'(1)}}.$$

# Chapter 5

# Cyclicity of composition Volterra operators

In this Chapter we begin to study cyclicity of composition Volterra operators. In the first section of the chapter, Section 5.1, we prove that the constant function 1 is cyclic for $V_\varphi$ if and only if the eigenfunctions of the adjoint $V_\varphi^\star$ span $L^2[0,1]$. It follows that there are symbols $\varphi$ with $\varphi(x) < x$ for $0 < x < 1$ for which $V_\varphi$ has the constant function 1 as a cyclic vector. This, in relation with Theorem 2.1.13, shows that the classical Volterra operator is not a limit case with respect to cyclicity.

In Section 5.2, in order to obtain, in the following section, positive results on supercyclicity of $V_\varphi$ as well as on hypercyclicity of $I + V_\varphi$, we need to extend Salas's Theorem on hypercyclicity of perturbations of the identity by backward weighted shifts, which has been crucial to solve some old open problems in hypercyclicity, see [48]. We prove a new criterion for an operator acting on a Fréchet space to be hypercyclic and another criterion that guarantees that in a given class of operators the set of hypercyclic ones is residual. The latter will be applied to several classes of operators.

In Section 5.3, we deal with supercyclicity and hypercyclicity of composition Volterra operators. Salas in [49] asked wether the classical Volterra operator is supercyclic or not, which was answered in the negative in [9]. Indeed, the Volterra operator is not even weakly supercyclic [35]. Thus the fact that there are symbols below the main diagonal that supply supercyclic composition Volterra operators is striking. Indeed, using the results of the previous two sections, we show that for every strictly increasing continuous $\varphi$ with $\varphi(x) < x$ for $0 < x \leqslant 1$ (note that $\varphi(1) < 1$), the operator $V_\varphi$ is supercyclic and the operator $I + V_\varphi$ is hypercyclic. Essentially, the only known examples of supercyclic quasi-nilpotent operators were quasi-nilpotent weighted shifts, for unilateral ones due to Hilden and Wallen [21] and for bilateral ones due to Salas, see [49]. It is also shown that there exists a continuous strictly increasing $\varphi$ with $\varphi(x) < x$ for $0 < x < 1$ such that both $V_\varphi$ and $V_\varphi^\star$ are supercyclic and both $I + V_\varphi$ and $I + V_\varphi^\star$ are hypercyclic. The class of operators for which $T$ and $T^\star$ are hypercyclic is very narrow; until now the only known examples, due to Salas [46] and [48], were bilateral weighted shifts, which are not quasinilpotent compact perturbations of the identity. Very recently, Salas [47] has also provided examples of quasinilpotent compact perturbations of the identity which are hypercyclic. In addition, it is also proved that even for certain symbols with $\varphi(1) = 1$, supercyclicity is possible. Namely, for continuous strictly increasing $\varphi$ with $\varphi(x) < x$ for $0 < x < 1$, $\varphi(1) = 1$ and analytic at 0 and at 1, it is shown that if $\varphi'(0)\varphi'(1) > 1$, then $V_\varphi$ is supercyclic and if $\varphi'(0)\varphi'(1) < 1$, then $V_\varphi$ is not even cyclic. The proofs essentially depend on the results in Chapter 4, that allow us to control the behavior of the orbits of $V_\varphi$.

## 5.1 The span of the eigenfunctions of $V_\varphi$

The next theorem establishes when the eigenvectors of $V_\varphi$ span $L^2[0,1]$ in the case that $\mathcal{F}_0^\varphi$ has order less than 1/2. It turns out that the density of the span of generalized eigenfunctions of $V_\varphi$ is equivalent to the cyclicity of the constant function 1 for $V_\psi$, where $\psi(x) = 1 - \varphi(1-x)$ for $0 \leqslant x \leqslant 1$.

Let $\mathcal{H}^0_{1/2}(\mathbb{C})$ denote the space of entire functions of order strictly less than 1/2 or of order 1/2 and type 0. In other words $F$ belongs to $\mathcal{H}^0_{1/2}(\mathbb{C})$ if and only if

$$\lim_{R\to\infty} \frac{\ln M(F,R)}{\sqrt{R}} = 0.$$

**Theorem 5.1.1.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) \geqslant x$ for $0 \leqslant x \leqslant 1$ and set $\psi(x) = 1 - \varphi(1-x)$. If the span of the generalized eigenvectors of $V_\varphi$ is dense in $L^2[0,1]$, then the constant function 1 is cyclic for $V_\psi$. The converse is also true, provided that $\mathcal{F}_0^\varphi$ belongs to $\mathcal{H}^0_{1/2}(\mathbb{C})$.*

*Proof.* Recall from Proposition 3.1.2 that the map $x \mapsto \mathcal{F}_x^\varphi$ is continuous from $[0,1]$ into the space of entire functions. Hence, for each non-null $h$ in $L^2[0,1]$, we find that

$$G^h(z) = \langle \mathcal{F}^\varphi(\varphi(\cdot), z), h\rangle = \int_0^1 \mathcal{F}^\varphi(\varphi(x), z)\overline{h(x)}\,dx, \qquad z \in \mathbb{C}$$

is an entire function. By Proposition 3.1.2, the Taylor coefficients of $G^h$ are given by

$$G_n^h = (-1)^{n-1}\langle UV_\psi^n 1, h\rangle = (-1)^{n-1}\langle V_\psi^n 1, Uh\rangle, \tag{5.1.1}$$

where $(Uf)(x) = f(1-x)$.

Proceeding by contradiction, suppose now that the constant function 1 is not cyclic for $V_\psi$. Then there is a non-zero $h$ in $L^2[0,1]$ such that $\langle V_\psi^n 1, Uh\rangle = 0$ for each $n \geqslant 0$ and, therefore, $G^h = 0$. Thus, since from (3.1.2) we know that

$$\frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z) = z\mathcal{F}^\varphi(\varphi(x), z),$$

it follows that

$$\int_0^1 \frac{\partial \mathcal{F}^\varphi}{\partial x}(x,z)\overline{h(x)}dx \equiv 0.$$

Upon differentiating in the above display with respect to $z$, we obtain

$$\int_0^1 \frac{\partial^{k+1} \mathcal{F}^\varphi}{\partial x \partial z^k}(x,z)\overline{h(x)}dx \equiv 0, \qquad \text{for each } k \geqslant 0.$$

Since, by Theorem 3.1.7, the generalized eigenfunctions of $V_\varphi$ belong to

$$\operatorname{span}\left\{\frac{\partial^{k+1} \mathcal{F}^\varphi}{\partial x \partial z^k}(x,z) : k = 0, 1, \dots\right\},$$

it follows that $h$ is orthogonal to each generalized eigenfunction of $V_\varphi$ and, therefore, the span of the generalized eigenfunctions is not dense in $L^2[0,1]$, a contradiction.

Suppose now that $\mathcal{F}_0^\varphi$ belongs to $\mathcal{H}_{1/2}^0(\mathbb{C})$ and the constant function 1 is cyclic for $V_\psi$. If the span of the generalized eigenfunctions of $V_\varphi$, is not dense in $L^2[0,1]$, then there is a non-null function $h$ in $L^2[0,1]$ such that $h$ is orthogonal to each generalized eigenfunction of $V_\varphi$. In particular, by Theorem 3.1.7, we have that each zero of $\mathcal{F}_0^\varphi$ is also a zero of $G^h$ of, at least, the same multiplicity. Hence, $H(z) = G^h(z)/\mathcal{F}_0^\varphi(z)$ is an entire function. On the other hand, Corollary 3.1.4 implies

$$M(G^h, R) \leqslant \int_0^1 M(\mathcal{F}_{\varphi(x)}^\varphi, R)|h(x)|\,dx \leqslant \int_0^1 M(\mathcal{F}_0^\varphi, R)|h(x)|\,dx = M(\mathcal{F}_0^\varphi, R)\|h\|_1.$$

Therefore, $G^h$ is in $\mathcal{H}_{1/2}^0(\mathbb{C})$, and so is $H$. Again, by Corollary 3.1.4 for each $R > 0$, we have

$$|G^h(-R)| \leqslant M(G^h, R) \leqslant M(\mathcal{F}_0^\varphi, R)\|h\|_1 = \mathcal{F}_0^\varphi(-R)\|h\|_1.$$

Hence, $|H(z)| \leqslant \|h\|_1$ for each $z$ real and negative. Since $H$ is in $\mathcal{H}_{1/2}^0(\mathbb{C})$, Theorem 1.4.6, which is a consequence of the Phragmén-Lindelöf Theorem, see [29, Theorem 22, p. 50], implies that $H$ is bounded on $\mathbb{C}$, and thus it must be constant. Hence $G^h = c\mathcal{F}_0^\varphi$, where $c$ is a constant.

Now, for $0 < x \leqslant 1$ set $\phi(x) = \inf\{t \in [0,1] : \varphi(t) \geqslant \varphi(x)\}$. Since $\varphi(x) \geqslant \phi(x) > 0$ for $0 < x \leqslant 1$, we may apply Lemma 3.1.9, for $\alpha = 0$ and $\beta = \phi(x)$ for each $0 < x \leqslant 1$, to obtain

$$\mathcal{F}^\varphi(0, -R) \geqslant (1 + \phi(x)R)\mathcal{F}^\varphi(\varphi(x), -R), \qquad \text{for each } R > 0 \quad \text{and } 0 < x \leqslant 1.$$

The last display along with $c\mathcal{F}_0^\varphi = G^h$ implies, for each $R > 0$, that

$$\begin{aligned}|c|\mathcal{F}^\varphi(0,-R) &\leqslant \int_0^1 \mathcal{F}^\varphi(\varphi(x),-R)|h(x)|\,dx\\ &\leqslant \int_0^1 \frac{\mathcal{F}^\varphi(0,-R)}{1+\phi(x)R}|h(x)|\,dx\\ &\leqslant \mathcal{F}^\varphi(0,-R)\|h\|_2\left(\int_0^1 \frac{dx}{(1+\phi(x)R)^2}\right)^{1/2}.\end{aligned}$$

Therefore,

$$\frac{|c|}{\|h\|_{L^2}^2} \leqslant \int_0^1 \frac{dx}{(1+\phi(x)R)^2}, \qquad \text{for each } R > 0.$$

Since the integral above tends to 0 as $R$ tends to $\infty$, we see that $c = 0$. Thus $G^h$ is identically zero and so are its Taylor coefficients. Consequently, from (5.1.1), we find that $\langle V_\psi^n 1, Uh\rangle = 0$ for each $n \geqslant 0$. Since $Uh$ is different from zero, the constant function 1 cannot be cyclic for $V_\psi$, which is a contradiction. The proof is complete. □

As an immediate consequence of Theorem 5.1.1 and Corollary 3.1.22, we have

**Corollary 5.1.2.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$ and assume that*

$$\varlimsup_{x\to 0} \frac{\ln(\varphi(x)-x)}{\ln x} < 2 \quad \text{and} \quad \varlimsup_{x\to 1} \frac{\ln(\varphi(x)-x)}{\ln(1-x)} < 2.$$

*Then the constant function 1 is cyclic for $V_\psi$, where $\psi(x) = 1-\varphi(1-x)$, if and only if the span of the generalized eigenfunctions of $V_\varphi$ is dense in $L^2[0,1]$.*

In particular, the above corollary applies to $\psi(x) = 1-(1-x)^{1/2}$, as mentioned in subsection 2.1.5. Using Theorem 3.1.23, we also have

**Corollary 5.1.3.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(x) > x$ for $0 < x < 1$. Assume also that $\varphi$ is differentiable at 0 and at 1 with $1 < \varphi'(0) \leqslant \infty$ and $\varphi'(1) < 1$. Then the constant function 1 is cyclic for $V_\psi$, where $\psi(x) = 1-\varphi(1-x)$, if and only if the span of the eigenfunctions of $V_\varphi$ is dense in $L^2[0,1]$.*

## 5.2 Dense generalized kernels

In the next section, we will prove that if $\varphi$ is continuous, strictly increasing and satisfies $\varphi(x) < x$ for $0 < x \leqslant 1$, then $V_\varphi$ is supercyclic and $I + V_\varphi$ is hypercyclic when $V_\varphi$ acts on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}_0[0,1]$. To do this, we will adopt a general point of view. We will show that if $T$ is a continuous operator on a separable complete metrizable topological vector space $X$ such that the linear manifold defined by

$$\mathrm{Ker}^{\dagger}(T) = \mathrm{span}\left(\bigcup_{n=1}^{\infty}(T^n(X) \cap \ker T^n)\right)$$

is dense in $X$, then the operator $I + T$ is hypercyclic. We will also show that $T$ is supercyclic. Actually, a little better results are obtained: the operator $I+T$ is strongly hereditarily hypercyclic and $T$ is strongly hereditarily supercyclic. This general point of view causes a minimal extra effort and avoids the repetition of some arguments. The methods we use allow us to conclude, under certain hypotheses, that an operator acting on a Fréchet space is hypercyclic and that the set of hypercyclic operators, in a given class of operators, is residual. The latter will be applied to several classes of operators.

Although all vector spaces in this section are supposed to be over $\mathbb{C}$, all the proofs equally work for real vector spaces. Recall that an *$\mathcal{F}$-space* is a complete metrizable topological vector space. The space of continuous operators on a topological vector space $X$ will be denoted by $\mathcal{L}(X)$.

Recall that a continuous operator $T$ acting on a topological vector space $X$ is said to be *hypercyclic* if there is $x$ in $X$ such that the orbit of $x$ under $T$, that is, $\{T^n x\}_{n\geqslant 0}$ is dense in $X$ and it is said to be *supercyclic* if there is $x$ in $X$ such that the projective orbit

$$\{\lambda T^n \text{ such that } \lambda \in \mathbb{C},\, n = 0, 1, \ldots\}$$

is dense in $X$. We say that $T$ is *hereditarily hypercyclic* if there is a subsequence $\{n_k\}$, such that for each subsequence $\{n_{k_i}\}$ of $\{n_k\}$, there is $x$ such that $\{T^{n_{k_i}}x\}$ is dense in $X$. If the sequence $\{n_k\}$ is the sequence of all positive integers, we say that $T$ is *strongly hereditarily hypercyclic*. Similarly, one can define *hereditarily* and *strongly hereditarily supercyclic*. We remark here that in [1] strongly hereditarily hypercyclic are simply called hereditarily hypercyclic. Here, we use the terminology as in [11, 13]

A bounded operator $T$ acting on a locally convex topological vector space is called *weakly hypercyclic* or *weakly supercyclic* if it is hypercyclic or supercyclic with respect to the weak topology. Observe that by Mazur's Theorem the norm closure and the weak closure of convex sets coincide, weakly supercyclic operators are cyclic. One of the advantages with respect to cyclic operators is that each positive power of a weakly supercyclic operator on a Banach space is again weakly supercyclic and thus cyclic, which is not always the case of just cyclic operators. Hypercyclic and supercyclic operators have been intensely studied during the last few decades, see surveys [14, 34] and references therein.

Let $\ell^p$, $1 \leqslant p < \infty$, denote the Banach space of complex sequences that have $p$-summable modulus. Let $\{e_n\}_{n\geqslant 0}$ be the canonical basis of $\ell^p$, $1 \leqslant p < \infty$. Given a bounded sequence $\{w_n\}_{n\geqslant 1}$ of non-zero complex numbers the backward weighted shift with weight sequence $\{w_n\}$ is defined by $Te_0 = 0$ and $Te_n = w_n e_{n-1}$ for $n \geqslant 1$ .

The next theorem, due to Salas [48], is very likely the most interesting result on hypercyclicity of a fixed operator.

**Theorem 5.2.1.** Salas' Theorem. *Let $T$ be a backward weighted shift on $\ell^2$. Then the operator $I+T$ is hypercyclic.*

The next Theorem extends Salas' Theorem in several directions.

**Theorem 5.2.2.** *Let $T$ be a continuous operator on a separable $\mathcal{F}$-space $X$ such that*

$$\ker^\dagger T = \operatorname{span}\left(\bigcup_{n=1}^{\infty}(T^n(X) \cap \ker T^n)\right)$$

*is dense in $X$. Then $I+T$ is (strongly hereditarily) hypercyclic.*

**Remark.** Recall that a continuous map $T$ on a topological vector space $X$ is called *mixing* if for each pair of non-empty open sets $U, V \subseteq X$, we have $T^n(U) \cap V \neq \emptyset$ for all $n$ large enough. In [13], it is proved, in the context of Banach spaces, that $T$ is mixing if and only if $T$ is strongly hereditarily hypercyclic. Indeed, the same proof works as well for $\mathcal{F}$-spaces. Thus the conclusion of Theorem 5.2.2 is equivalent to the operator $T$ to be mixing.

Recall that the *generalized kernel* of an operator $T$ is the space

$$\ker^\star T = \bigcup_{n=1}^{\infty} \ker T^n.$$

Spectral properties of operators with dense generalized kernel can be found in [4]. It is worth mentioning that the space $\ker^\dagger T$ is contained in $T(X)$ as well as in $\ker^\star T$. Thus, any operator with dense $\ker^\dagger T$ has dense range and dense generalized kernel. Obviously, if $T$ is a (unilateral) backward weighted shift on $\ell^p$, then $\ker^\star T = \ker^\dagger T$ is the space of sequences with finite support, which is dense in $\ell^p$, $1 \leqslant p < \infty$. Hence Theorem 5.2.2 implies Salas' Theorem. It is also worth noting that if $T(\ker T^{n+1})$ is dense in $\ker T^n$ for each positive integer $n$, then $\ker^\dagger T$ is dense in $\ker^\star T$. Thus, we have,

**Corollary 5.2.3.** *Let $T$ be a continuous operator on a separable $\mathcal{F}$-space $X$ such that $\ker^\star T$ is dense in $X$ and $T(\ker T^{n+1})$ is dense in $\ker T^n$ for each positive integer $n$. Then $I+T$ is (strongly hereditarily) hypercyclic.*

The advantage of corollary above is that it is much easier to check that $T(\ker T^{n+1})$ is dense in $\ker T^n$.

A *generalized backward shift* is a continuous operator $T$ on a topological vector space $X$ such that $\ker T$ is one-dimensional and $\ker^\star T$ is dense in $X$. A dimension argument shows immediately that if $T$ is a generalized backward shift then $\ker T^n$ is $n$-dimensional and $T(\ker T^{n+1}) = \ker T^n$ for each positive integer $n$. Indeed, since $\dim(\ker T) = 1$, for each $n$-dimensional vector space $U$ we have that $n \geqslant \dim(T(U)) \geqslant n-1$. In particular, since $\ker^\star T$ is dense in $X$, $\dim(T(\ker T^n)) = n-1$. The latter fact along with $\dim(\ker T) = 1$ implies that $\dim(\ker T^n) = n$ for each positive integer $n$. From Corollary 5.2.3, we clearly have,

**Corollary 5.2.4.** *Let $X$ be a separable $\mathcal{F}$-space and $T$ in $\mathcal{L}(X)$ be a generalized backward shift. Then $I+T$ is (strongly hereditarily) hypercyclic.*

**Remark.** The fact that $I+T$ is hypercyclic for a generalized backward shift $T$ on a separable $\mathcal{F}$-space also follows from Salas's Theorem by means of a quasisimilarity argument, as already observed by several authors, see, for instance, [13].

Before proving Theorem 5.2.2, we need some preparation.

### 5.2.1 A density criterion.

René-Louis Baire (1874-1932) developed what now a dais is known as Category Theory. This Theory consists on an attempt to classify sets by its topological size, and it is widely used to prove existence. Baire defined three categories of sets in a topological space: a set is called *nowhere dense* if its closure has empty interior, if a set is a countable union of nowhere dense sets, it is said to be a *first category* and its complement is called *residual*. All the sets that are not of the first category sets are called *second category* sets. Recall now that a topological space $X$ is called a *Baire space* if for each first category set $A \subset X$ its complement $X \setminus A$ is dense in $X$. The classical Baire theorem, provides a very wide class of Baire spaces that includes the most natural and common ones. In particular, complete metric spaces are Baire.

**Baire's Theorem 5.2.5.** *Both complete metric spaces and locally compact spaces are Baire spaces.*

We need the following proposition, in which appears the concept of *second countable* space. In topology, a *second countable* space is a topological space with a countable basis.

**Proposition 5.2.6.** *Let $X$ and $Y$ be Baire topological spaces, where $Y$ is second countable. Let $\{T_n\}_{n\geqslant 0}$ be a sequence of continuous maps from $X$ to $Y$. Let $\Sigma$ be the set of $(x,y)\in X\times Y$ for which there exists a sequence $\{x_n\}_{n\geqslant 0}$ in $X$ such that $x_n\to x$ and $T_nx_n\to y$. If $\Sigma$ is dense in $X\times Y$, then for any subsequence $\{n_k\}_{k\geqslant 0}$, there is $x$ such that $\{T_{n_k}x\}_{k\geqslant 0}$ is dense in $X$.*

*Proof.* Since $\Sigma$ is dense in $X\times Y$, it is enough to apply Theorem 1 in [14, p. 348]. □

### 5.2.2 Invertible Matrices

To prove Theorem 5.2.2, we need to show that certain matrices are invertible. For each pair of positive integers $n$ and $k$, consider the $n$-square matrix

$$M_{n,k}=\left(\frac{(k+n-l)!}{(k+n-l+j-1)!}\right)_{1\leqslant j,l\leqslant n}.$$

**Lemma 5.2.7.** *For each pair $n$ and $k$ of positive integers, we have*

$$\det M_{n,k}=\frac{(n-1)!k!(k+1)!}{(k+n-1)!(k+n)!}\det M_{n-1,k+2}. \tag{5.2.1}$$

*Proof.* It is clear that (5.2.1) holds for $n=2$. Thus suppose that $n\geqslant 3$. Subtracting to each column of $M_{n,k}$, except the first, the previous one, we see that

$$\det M_{n,k}=\det\begin{pmatrix} 1 & 0 & \dots & 0\\ \frac{(k+n-1)!}{(k+n)!} & & & \\ \vdots & & N_{n,k} & \\ \frac{(k+n-1)!}{(k+2n-2)!} & & & \end{pmatrix},$$

where

$$N_{n,k}=\left(j\frac{(k+n-l-1)!}{(k+n-l+j)!}\right)_{1\leqslant j,l\leqslant n-1}.$$

Thus $\det M_{n,k}=\det N_{n,k}$. Now, dividing each $j$-th row of $N_{n,k}$ by $j$ and multiplying each $l$-th column by $(k+n-l+1)!/(k+n-l-1)!$, we obtain $M_{n-1,k+2}$. Hence

$$\det M_{n,k}=\det M_{n-1,k+2}\prod_{\substack{j=1\\ l=1}}^{n-1}\frac{j(k+n-l-1)!}{(k+n-l+1)!}=\frac{(n-1)!k!(k+1)!}{(k+n-1)!(k+n)!}\det M_{n-1,k+2}.$$

The result is proved. □

Consider now the $n$-square matrix

$$A_n = \left( \frac{1}{(j+k-1)!} \right)_{1 \leqslant j,k \leqslant n}.$$

**Lemma 5.2.8.** *For each positive integer, $A_n$ is invertible. Furthermore, $\det A_1 = 1$, $\det A_2 = -1/12$ and*

$$\det A_n = \frac{(-1)^{(n-1)n/2}}{(2n-1)!} \left( \prod_{j=1}^{2n-4} j! \right) \left( \prod_{j=n}^{2n-3} j!^{-2} \right), \qquad \textit{for} \;\; n \geqslant 3.$$

The key lemma in the proof of Salas's Theorem is Lemma 3.1 in [48] that asserts that $A_n$ is invertible for $n = 2^k$ with $k$ a positive integer. The latter is also used in [28] to prove that the operators in Salas's Theorem do satisfy Kitai's Criterion. Actually, $A_n$ is invertible for each positive integer $n$. Indeed, $\det A_n$ can be computed explicitly.

*Proof.* Let $B_n$ be the matrix obtained from $A_n$ by reversing the order of the columns of $A_n$. Clearly, $\det A_n = (-1)^{(n-1)n/2} \det B_n$. Multiplying the $j$-th column of $B_n$ by $(n-j+1)!$ for $1 \leqslant j \leqslant n$, we obtain $M_{n,1}$. Hence,

$$\det A_n = (-1)^{(n-1)n/2} \det M_{n,1} \prod_{j=1}^{n} (j!)^{-1}$$

for each positive integer $n$. Now, the result follows by applying $n-1$ times (5.2.1) and then simplifying. □

Finally, for each pair of positive integers $m$ and $n$ with $m \geqslant 2n$, we consider the $n$-square matrix

$$B_{m,n} = \left( \binom{m}{k+j-1} \right)_{1 \leqslant j,k \leqslant n},$$

where

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

**Lemma 5.2.9.** *For each pair of positive integers $m$ and $n$ with $m \geqslant 2n$, we have that $B_{m,n}$ is invertible. Furthermore,*

$$\det B_{m,n} = \det A_n \prod_{j=-n}^{n} (m+j)^{n-|j|}.$$

*Proof.* By multiplying the $j$-th column of $B_{m,n}$ by $(m-j)!/m!$ for $1 \leqslant j \leqslant n$, we obtain $P_{m,n}$, whose entries are $p_{1,k} = 1/k!$, $1 \leqslant k \leqslant n$, and

$$p_{j,k} = \frac{(m-k)!}{(k+j-1)!(m-k-j+1)!}, \qquad \text{for } j \geqslant 2.$$

Now consider $Q_{m,n}$ obtained from $P_{m,n}$ by replacing the $j$-th row $P_{[j]}$ by

$$\sum_{l=0}^{j-1} \binom{j-1}{l} P_{[l+1]}.$$

Clearly, $\det P_{m,n} = \det Q_{m,n}$. In addition, one can check that the entries of $Q_{m,n}$ are $q_{1,k} = 1/k!$, $1 \leqslant k \leqslant n$, and

$$q_{j,k} = \frac{(m+j-1)!}{m!(k+j-1)!}, \qquad \text{for } j \geqslant 2.$$

Multiplying the $j$-th row of $Q_{m,n}$ by $m!/(m+j-1)!$ for $2 \leqslant j \leqslant n$, we arrive to $A_n$. Upon putting everything together, we obtain

$$\det B_{m,n} = \left( \prod_{j=1}^{n} \frac{m!}{(m-j)!} \right) \left( \prod_{j=2}^{n} \frac{(m+j-1)!}{m!} \right) \det A_n.$$

Simplifying, the required formula for $\det B_{m,n}$ follows. □

### 5.2.3 Proof of Theorem 5.2.2

Now, we begin to prove Theorem 5.2.2. For $x$ in $\mathbb{C}^n$, $n \geqslant 1$, we denote by $x_j$ its $j$-th coordinate.

**Lemma 5.2.10.** *Let $S$ in $\mathcal{L}(\mathbb{C}^{2n})$, $n \geqslant 1$, be defined on the canonical basis $\{e_i : 1 \leqslant i \leqslant 2n\}$ by $Se_i = e_{i-1}$, $2 \leqslant i \leqslant 2n$ and $Se_1 = 0$. Then for each $m \geqslant 2n$ and each $u$ and $v$ in $\mathbb{C}^n$, there exists a unique $x = x(m)$ in $\mathbb{C}^{2n}$ such that*

(a) $x_j(m) = u_j$, *for* $1 \leqslant j \leqslant n$;

(b) $\big((I+S)^m x(m)\big)_j = v_j$, *for* $1 \leqslant j \leqslant n$.

*Furthermore,*

$$|x_{n+j}(m)| = O(m^{-j}) \quad \text{as } m \to \infty \text{ for } 1 \leqslant j \leqslant n; \tag{5.2.2}$$

$$|((I+S)^m x(m))_{n+j}| = O(m^{-j}) \quad \text{as } m \to \infty \text{ for } 1 \leqslant j \leqslant n. \tag{5.2.3}$$

*Proof.* For $y$ in $\mathbb{C}^{2n}$ and $z$ in $\mathbb{C}^n$, set

$$\widetilde{y} = (y_{n+1}, \dots, y_{2n}) \in \mathbb{C}^n \quad \text{and} \quad \widehat{z} = (z_1, \dots, z_n, 0, \dots, 0) \in \mathbb{C}^{2n}$$

and let $w(m)$ in $\mathbb{C}^n$ be defined by

$$w_j(m) = v_{n-j+1} - ((I+S)^m \widehat{u})_{n-j+1} \qquad \text{for } j = 1, \dots, n.$$

One sees that there is a unique $x(m)$ satisfying (a) and (b) if and only if the equation

$$B_{m,n}\widetilde{x} = w(m), \tag{5.2.4}$$

where $B_{m,n}$ is the matrix defined in the previous subsection, has a unique solution. Thus the first statement of the lemma follows from Lemma 5.2.9.

It remains to show that (5.2.2) and (5.2.3) also hold. To this end, first observe that

$$w_j(m) = v_{n-j+1} - \sum_{l=0}^{j-1} \binom{m}{l} u_{n-j+1+l} \quad \text{for} \quad 1 \leqslant j \leqslant n.$$

Thus

$$w_j^m = O(m^{j-1}), \qquad \text{as } m \to \infty \text{ for } 1 \leqslant j \leqslant n. \tag{5.2.5}$$

Now consider the $n$-diagonal matrix $D_{m,n}$ with entries $m^{j-1}$, $1 \leqslant j \leqslant n-1$, in the main diagonal. A computation shows that

$$B_{m,n} = mD_{m,n}C_{m,n}D_{m,n},$$

where $C_{m,n} = \{\gamma_{j,k}\}_{1\leqslant j,k\leqslant n}$ has entries

$$\gamma_{1,1} = 1 \quad \text{and} \quad \gamma_{j,k} = \frac{1}{(j+k-1)!} \prod_{l=1}^{j+k-2} \left(1 - \frac{l}{m}\right) \quad \text{for} \quad (j,k) \neq (1,1).$$

Since $B_{m,n}$ as well as $D_{m,n}$ are invertible, so is $C_{m,n}$ and (5.2.4) implies that

$$\widetilde{x}^m = B_{m,n}^{-1} w(m) = m^{-1} D_{m,n}^{-1} C_{m,n}^{-1} D_{m,n}^{-1} w(m).$$

From (5.2.5), the sequence $\{D_{m,n}^{-1} w(m)\}_{m\geqslant 2n}$ is bounded in $\mathbb{C}^n$. On the other hand, the sequence of invertible matrices $\{C_{m,n}\}_{m\geqslant 2n}$ converges to the matrix $A_n$ defined in the previous subsection, which is invertible by Lemma 5.2.8. Hence, $C_{m,n}^{-1}$ converges to $A_n^{-1}$ as $m$ tends to $\infty$ and therefore, the sequence $\{C_{m,n}^{-1} D_{m,n}^{-1} w^m\}_{m\geqslant 2n}$ is bounded in $\mathbb{C}^n$. Hence,

$$x_{n+j}(m) = \widetilde{x}_j(m) = m^{-1}(D_{m,n}^{-1} C_{m,n}^{-1} D_{m,n}^{-1} w^m)_j$$

satisfy (5.2.2) for $1 \leqslant j \leqslant n$. Finally, since

$$((I+S)^m x(m))_{n+j} = \sum_{l=0}^{n-j} \binom{m}{l} x_{n+j+l}(m) \quad \text{for} \quad 1 \leqslant j \leqslant n,$$

the estimates in (5.2.3) follow from (5.2.2) and the result is proved. □

Lemma 5.2.10 allows us to prove the following

**Lemma 5.2.11.** *Let $T$ be a continuous operator on a topological vector space $X$. Assume that $x$ belongs to $T^m(X) \cap \ker T^m$, where $m$ is a positive integer. Then there exist sequences $\{u_k\}_{k \geqslant 0}$ and $\{v_k\}_{k \geqslant 0}$ in $X$ such that*

$$u_k \to 0, \quad (I+T)^k u_k \to x, \quad v_k \to x \text{ and } (I+T)^k v_k \to 0 \qquad \text{as } n \to \infty. \tag{5.2.6}$$

*Proof.* If $x = 0$, it is enough to take $u_k = v_k = 0$. Thus, assume that $x \neq 0$. We will show that the proof reduces to the operator $S = T$ defined in Lemma 5.2.10. Let $n$ be the smallest positive integer for which $T^n x = 0$. In particular, $n \leqslant m$, which implies that $x$ belongs to $T^n(X)$. Thus we may choose $w$ in $X$ such that $T^n w = x$. We set

$$h_j = T^{2n-j} w \quad \text{for} \quad 1 \leqslant j \leqslant 2n \qquad \text{and} \quad Y = \operatorname{span}\{h_1, \ldots, h_{2n}\}.$$

In particular, we have $Th_j = h_{j-1}$, $2 \leqslant j \leqslant 2n$, and $Th_1 = T^{2n} h_{2n} = T^n x = 0$. Thus clearly, $Y$ is invariant under $T$. Since $h_1 = T^{2n-1} h_{2n} = T^{n-1} x \neq 0$, it follows that $\dim Y \geqslant 2n$ and, therefore, $\{h_1, \ldots, h_{2n}\}$ is a basis of $Y$. Let $J$ be the operator from $\mathbb{C}^{2n}$ onto $Y$ defined by $Je_k = h_k$, $1 \leqslant k \leqslant 2n$. Clearly, $T$ acting on $Y$ is similar under $J$ to $S$ acting on $\mathbb{C}^{2n}$, where $S$ is the operator defined on Lemma 5.2.10. Now, $J^{-1}x = e_n$. Thus taking, $u = (0, \ldots, 0, 1)$ in $\mathbb{C}^n$ and $v = (0, \ldots, 0)$, we find that there is a sequence $\{g_k\}_{k \geqslant 0}$ in $\mathbb{C}^{2n}$ such that $g_k \to e_n$ and $(I+S)^k g_k \to 0$ as $k \to \infty$. Applying Lemma 5.2.10 with $u = (0, \ldots, 0, 0)$ and $v = (0, \ldots, 0, 1)$, we find that there is a sequence $\{f_k\}_{k \geqslant 0}$ in $\mathbb{C}^{2n}$ such that $f_k \to 0$ and $(I+S)^k f_k \to e_m$ as $k \to \infty$. The result follows because any topological vector spaces of the same finite dimension are homeomorphic under any algebraic isomorphism. □

**Lemma 5.2.12.** *Let $T$ be a continuous operator on a topological vector space $X$. Assume that $x$ and $y$ belong to $\ker^\dagger T$. Then there exists a sequence $\{x_k\}$ in $X$ such that $x_k \to x$ and $(I+T)x_k \to y$ as $k \to \infty$.*

*Proof.* Let $\Sigma$ be the set of $(x, y)$ in $X \times X$ for which there is a sequence $\{x_n\}$ in $X$ such that $x_n$ tends to $x$ and $(I+T)^n x_n$ tends to $y$. By Lemma 5.2.11, we have

$$\ker T^n \cap T^n(X) \times \{0\} \subset \Sigma \quad \text{and} \quad \{0\} \times \ker T^n \cap T^n(X) \subset \Sigma \;\; \text{for each } n \geqslant 1.$$

On the other hand, it is clear that $\Sigma$ is a subspace of $X \times X$. From the above display, one immediately obtains that $\ker^\dagger T \times \ker^\dagger T \subseteq \Sigma$, which is what had to be shown. □

Now we are ready to prove Theorem 5.2.2.

*Proof of Theorem 5.2.2.* Let $\Sigma$ be the set of $(x, y) \in X \times X$ for which there is $\{x_n\}$ in $X$ such that $x_n \to x$ and $(I+T)^n x_n \to y$. By Lemma 5.2.12, it follows that $\Sigma$

contains $\ker^\dagger T \times \ker^\dagger T$. Since $\ker^\dagger T$ is dense in $X$, we obtain that $\Sigma$ is dense in $X \times X$. According to Theorem 5.2.6, for each subsequence $\{n_k\}$ there is $x$ in $X$ such that $\{(I+T)^{n_k}x\}$ is dense in $X$, that is, $I+T$ is strongly hereditarily hypercyclic. The proof of Theorem 5.2.2 is complete. □

### 5.2.4 Supercyclicity

For the sake of completeness, we shall prove a proposition that extends another result by Salas [49].

**Proposition 5.2.13.** *Let $X$ be a separable $\mathcal{F}$-space and $T$ in $\mathcal{L}(X)$. Assume also that $T$ has dense range and dense generalized kernel. Then $T$ is (strongly hereditarily) supercyclic.*

The advantage of Proposition 5.2.13 over Corollary 2.8 in [49] is that we avoid the existence of the local inverse.

The next criterion for an operator to be strongly hereditarily supercyclic, is analogous to one of the forms of the Supercyclicity Criterion, see [34]. Its proof is a straightforward modification of the proof of Theorem 2.2 in [10] and it is omitted.

**Theorem 5.2.14.** *Let $T$ be a continuous operator on a $\mathcal{F}$-space $X$ and $\{\lambda_k\}_{k\geqslant 0}$ be a sequence of non-zero complex numbers. Assume also that there exist dense subsets $E$ and $F$ of $X$ and mappings $S_k : F \to X$ such that $T^k S_k y \to y$ and $\lambda_k^{-1} S_k y \to 0$ for each $y \in F$ and $\lambda_k T^k x \to 0$ for each $x \in E$ as $k \to \infty$. Then $T$ is hereditarily supercyclic.*

Theorem 5.2.14 is all what we need to prove Proposition 5.2.13.

*Proof of Proposition 5.2.13.* Let $d$ be a metric that induces the topology of $X$. Let $F$ be a dense countable subset of $X$. Since $T$ has dense range, we find that $T^k(X)$ is dense in $X$ for each $k \geqslant 0$. Hence, we may choose $S_k : F \to X$ such that $d(y, T^k S_k y) < 2^{-k}$ for each $y$ in $F$ and each $k \geqslant 0$. Clearly, $T^k S_k y \to y$ for each $y$ in $F$. Since $F$ is countable and $X$ is metrizable, there is a sequence $\{\lambda_n\}_{n\geqslant 0}$ of positive numbers such that $\lambda_n^{-1} S_n y \to 0$ as $n \to \infty$ for each $y$ in $F$. Finally, $E = \ker^\star T$ is dense in $X$ and for each $y$ in $E$ we have $T^n y = 0$ for all $n$ large enough and, therefore, $\lambda_n T^n y \to 0$ as $n \to \infty$. Thus all the hypotheses of Theorem 5.2.14 are fulfilled and we conclude that $T$ is (strongly hereditarily) supercyclic. □

### 5.2.5 Residual sets of hypercyclic operators

The next proposition allows us to show that the set of hypercyclic operators in certain subsets of $\mathcal{L}(X)$ is residual in the Baire Category sense.

**Proposition 5.2.15.** *Let $X$ be a separable metrizable topological vector space and $\mathcal{M}$ a subset of $\mathcal{L}(X)$ endowed with a topology satisfying*

(i) *$\mathcal{M} \times X$ is Baire.*

(ii) *For each $n \geqslant 0$, the map $\Phi_n : \mathcal{M} \times X \to X$ defined by $\Phi_n(T, x) = T^n x$ is continuous.*

(iii) *The set $\{(T, x, T^n x)$ such that $T \in \mathcal{M}, x \in X, n \geqslant 0\}$ is dense in $\mathcal{M} \times X \times X$.*

*Then the set of hypercyclic operators $T$ in $\mathcal{M}$ is a dense $G_\delta$-set in the Baire space $\mathcal{M}$.*

*Proof.* Let $\mathcal{H}$ be the set of hypercyclic operators $T$ in $\mathcal{M}$. Let $\{U_n\}_{n\geqslant 0}$ be a basis of open sets of the topology of $X$. We set

$$\Lambda_{n,x,m} = \{T \in \mathcal{M} \text{ such that } T^n x \in U_m\}.$$

Using that the set of hypercyclic vectors of each hypercyclic operator is dense, one can check that

$$\mathcal{H} = \bigcap_{k,m=0}^{\infty} W_{k,m}, \quad \text{where} \quad W_{k,m} = \bigcup_{x\in U_k} \bigcup_{n=0}^{\infty} \Lambda_{n,x,m}$$

From (i), we see that each $\Lambda_{n,x,m}$ is open in $\mathcal{M}$ and, therefore, so is each $W_{k,m}$. Hence, $\mathcal{H}$ is a $G_\delta$-set in $\mathcal{M}$. It remains to verify that $\mathcal{H}$ is dense in $\mathcal{M}$.

From (i) through (iii), it follows that $(T, x, \Phi_n(T, x))$ is dense in $\mathcal{M} \times X \times X$. By Theorem 1 in [14], there is a $G_\delta$ dense $\Omega \subset \mathcal{M} \times X$ for which $\{\Phi_n(T, x)\}_{n\geqslant 0}$ is dense in $X$. Since the latter density means that $T$ is hypercyclic, we have that the projection from $\mathcal{M} \times X$ onto $\mathcal{M}$ takes $\Omega$ onto $\mathcal{H}$, it follows that $\mathcal{H}$ is dense in $\mathcal{M}$. □

**Remark.** Since in the previous proposition $X$ is second countable, we find that (i) is equivalent to the fact that both $\mathcal{M}$ and $X$ are Baire, see [38].

In order to see the previous proposition in action, we present the following theorem, which provides an alternative proof, involving biorthogonal sequences, of the existence of hypercyclic bounded operators on any separable infinite dimensional Banach space. A *biorthogonal sequence* in a pair $(X, X^\star)$, where $X$ is a Banach space and $X^\star$ is its dual space, is a couple of sequences $\{f_k\}$ in $X$ and $\{g_k\}$ in $X^\star$, with $\langle f_j, g_k\rangle = \delta_{jk}$. For a detailed study of biorthogonal sequences see [45, Chap. 4].

**Theorem 5.2.16.** *Let $\mathcal{B}$ be a separable infinite dimensional Banach space. Let $\mathcal{N}$ be the operator norm closure of the finite rank nilpotent operators on $\mathcal{B}$. Then the set of hypercyclic operators $T$ in $I+\mathcal{N}$ is a dense $G_\delta$-set in $I+\mathcal{N}$.*

*Proof.* Set $\mathcal{M} = I+\mathcal{N}$. Let $\varepsilon > 0$ and $(T_0, u, v)$ in $\mathcal{M} \times \mathcal{B} \times \mathcal{B}$. Since $T_0$ is in $\mathcal{M}$, there is a finite rank nilpotent operator $S$ on $\mathcal{B}$ such that $\|I+S-T_0\| < \varepsilon/2$. Since $S$ has finite rank, there are subspaces $\mathcal{B}_0$ and $\mathcal{B}_1$ of $\mathcal{B}$, with $\dim \mathcal{B}_0 < \infty$ and $u, v \in \mathcal{B}_0$, such that $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, the operator $S$ vanishes on $\mathcal{B}_1$ and $\mathcal{B}_0$ is invariant under $S$. Let $n$ be a positive integer with $S^n u = S^n v = 0$. Since $\mathcal{B}_1$ is infinite dimensional and $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, there is a biorthogonal set $\{(x_1, f_1), \ldots, (x_{2n}, f_{2n})\}$ in $\mathcal{B} \times \mathcal{B}^\star$ such that $x_j$ belongs to $\mathcal{B}_1$ and $f_j$ vanishes on $\mathcal{B}_0$. Now, for each $t > 0$ consider the operator $S_t$ acting on $\mathcal{B}$ defined by

$$S_t x = Sx + tf_1(x)u + tf_{n+1}(x)v + \sum_{j=2}^{n} t(f_j(x)x_{j-1} + f_{n+j}(x)x_{n+j-1}).$$

Clearly, each $S_t$ is of finite rank and nilpotent, and $\|S - S_t\|$ tends to 0 as $t$ tends to 0. We take $t_0 > 0$ with $\|S - S_{t_0}\| < \varepsilon/2$. Since $S_t$ on $\mathcal{B}_0$ coincides with $S$ and $S_t^n x_n = t^n u$ and $S_t^n x_{2n} = t^n v$ for each $t > 0$, we have

$$u, v \in S_{t_0}^n \mathcal{B} \cap \ker S_{t_0}^n \subset \ker^\dagger S_{t_0}.$$

By Lemma 5.2.12, there is $x$ in $\mathcal{B}$ and a positive integer $m$ such that $\|u - x\| < \varepsilon$ and $\|v - (I+S_{t_0})^m x\| < \varepsilon$. Since $\|I + S_{t_0} - T_0\| < \varepsilon$, each neighborhood of $(T_0, u, v)$ meets the set

$$\{(T, x, T^m x) \text{ such that } T \in \mathcal{M},\ x \in \mathcal{B},\ m = 1, 2, \ldots\},$$

which means that (iii) in Proposition 5.2.15 holds. Since conditions (i) and (ii) in Proposition 5.2.15 are trivially satisfied, we find that the set of hypercyclic operators in $\mathcal{M}$ is a dense $G_\delta$-set in $\mathcal{M}$. □

As shown by Salas [46] with bilateral weighted shifts, there is a hypercyclic operator on a separable Hilbert space, whose adjoint is also hypercyclic. From Theorem 5.2.16, it follows that many operators of the form $I+T$, where $T$ is a compact quasi-nilpotent operator on a separable Hilbert space, have the required behavior. The existence of such hypercyclic ‘small’ perturbations of the identity with hypercyclic adjoint has been also shown by Salas in [47]. In particular a hypercyclic operator with hypercyclic adjoint can have one-point spectrum, as it is also the case of the example provided by Salas [47, *Remark* 3.2].

**Corollary 5.2.17.** *Let $\mathcal{Q}$ be the set of compact quasi-nilpotent operators on a separable infinite dimensional Hilbert space. Then the set of $T$ in $I+\mathcal{Q}$ such that $T$ and $T^\star$ are hypercyclic is a dense $G_\delta$-set in $I+\mathcal{Q}$.*

*Proof.* Since any compact quasi-nilpotent operator in $\mathcal{Q}$ is the limit in the operator norm of finite rank nilpotent operators [19], we see that $\mathcal{Q}$ coincides with the closure $\mathcal{N}$ of the finite rank nilpotent operators, which implies that the set $\mathcal{H}$ of hypercyclic $T$ in $I+\mathcal{N}$ is a dense $G_\delta$-set in $I+\mathcal{Q}$. Since the map that to each operator $T$ assigns its adjoint $T^\star$ is an isometric isomorphism from $I+\mathcal{Q}$ into itself, we see that the set $\mathcal{H}^\star$ of $T$ in $I+\mathcal{Q}$ for which $T^\star$ is hypercyclic, is also a dense $G_\delta$-set in $I+\mathcal{Q}$. Therefore, $\mathcal{H}\cap\mathcal{H}^\star$ is also a dense $G_\delta$-set and the result follows. □

Finally, as far as we know there are no known examples of compact bilateral weighted shifts $T$ such that $I+T$ is hypercyclic on $\ell^2(\mathbb{Z})$. The problem here is that it is quite difficult to control the orbits. The next proposition shows that there are many compact bilateral weighted shifts $T$ on $\ell^2(\mathbb{Z})$ such that $I+T$ and $I+T^\star$ are hypercyclic. Recall that a bilateral weighted shift $T_w$, where $w=\{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\mathbb{C}$, acts on the canonical basis $\{e_n\}_{n\in\mathbb{Z}}$ as $Te_n = w_n e_{n-1}$ for each $n$ in $\mathbb{Z}$. Observe that

$$\|T_w\| = \|w\|_\infty.$$

**Proposition 5.2.18.** *The set of $w$ in $c_0(\mathbb{Z})$ for which $I+T_w$ and $I+T_w^\star$ are hypercyclic on $\ell^2(\mathbb{Z})$ is a dense $G_\delta$-set in $c_0(\mathbb{Z})$.*

*Proof.* Consider $\mathcal{M}=\{I+T_w \text{ such that } w\in c_0(\mathbb{Z})\}$, endowed with the distance

$$d(I+T_w, I+T_{w'}) = \|T_w - T_{w'}\|_{\mathcal{L}^2(\ell^2(\mathbb{Z}))}.$$

Since the map $\Phi$ defined as $\Phi(w)=I+T_w$ is an isometry from $c_0(\mathbb{Z})$ onto $\mathcal{M}$, we see that $\mathcal{M}$ is complete with respect to $d$.

Let $\varepsilon>0$ and $(I+T_w,u,v)$ in $\mathcal{M}\times\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})$. Since the space $c_{00}(\mathbb{Z})$ of sequences with finite support is dense in $\ell^2(\mathbb{Z})$, we may take $x=\{x_k\}$ and $y=\{y_k\}$ in $c_{00}(\mathbb{Z})$ such that $\|u-x\|<\varepsilon/2$ and $\|v-y\|<\varepsilon/2$. Let $m$ be a positive integer such that $x_k=y_k=0$ for $|k|\geqslant m$. We can take $n\geqslant m$ and $w'$ in $c_0(\mathbb{Z})$ such that

$$\|w-w'\|_\infty<\varepsilon \quad \text{and} \quad \begin{cases} w'_k\neq 0, & \text{for } k\geqslant -n,\\ w'_k=0, & \text{for } k<-n.\end{cases}$$

Since $w'_k\neq 0$ for $k\geqslant -n$, we find that $x$ and $y$ belong to $T^k_{w'}(c_{00}(\mathbb{Z}))$ for each positive integer $k$ and $T^k_{w'}x = T^k_{w'}y = 0$ for $k>m+n+1$. Therefore, $x$ and $y$ are in $\ker^\dagger T_{w'}$.

By Lemma 5.2.12, there are $x'$ in $\ell^2(\mathbb{Z})$ and a positive integer $l$ such that

$$\|x - x'\| < \varepsilon/2 \quad \text{and} \quad \|y - (I + T_{w'})^l x'\| < \varepsilon/2.$$

Upon putting everything together, we have $x'$ in $\ell^2(\mathbb{Z})$, a positive integer $k$ and $I+T_{w'}$ in $\mathcal{M}$ for which

$$\|I + T_w - (I + T_w')\| = \|w - w'\|_\infty < \varepsilon, \quad \|u - x'\| < \varepsilon \text{ and } \|v - (I + T_{w'})^l x'\| < \varepsilon.$$

Therefore, every neighborhood of $(I + T_w, u, v)$ meets

$$\{(I + T_{w'}, x, (I + T_{w'})^k x) \text{ such that } I + T_{w'} \in \mathcal{M},\ x \in \ell^2(\mathbb{Z}),\ k = 1, 2, \ldots\},$$

which means that (iii) in Proposition 5.2.15 holds. Since (i) and (ii) are trivially satisfied, Proposition 5.2.15 implies that the set of hypercyclic operators $I + T_w$ with $w$ in $c_0(\mathbb{Z})$ is a dense $G_\delta$-set in $\mathcal{M}$.

Now consider the unitary operator $U$ on $\ell^2(\mathbb{Z})$ defined on its canonical basis by $Ue_n = e_{-n}$ for each $n$ in $\mathbb{Z}$. Clearly, the map that to each operator $T$ assigns $U^\star T^\star U$ is an isometry from $\mathcal{M}$ onto itself. Since hypercyclicity is invariant under similarity, the set of $T$ in $\mathcal{M}$ for which $T^\star$ is hypercyclic is a dense $G_\delta$-set in $\mathcal{M}$. Thus the set of $T$ in $\mathcal{M}$ such that $T$ and $T^\star$ are hypercyclic is a dense $G_\delta$-set in $\mathcal{M}$. The result follows from the fact that $\Phi(w) = I + T_w$ is an isometry from $c_0(\mathbb{Z})$ onto $\mathcal{M}$. □

For sake of completeness, we end this section by providing an analog of Proposition 5.2.15 for supercyclicity. The proof is a slight modification of the one of Proposition 5.2.15 and is omitted.

**Proposition 5.2.19.** *Let $\mathcal{B}$ be a separable Banach space and let $S = \{x \in \mathcal{B} : \|x\| = 1\}$. Assume that a subset $\mathcal{M}$ in $\mathcal{L}(\mathcal{B})$ is endowed with a topology satisfying (i) through (iii) of Proposition* 5.2.15 *and*

(iv) *The set $\{(T, x, T^n x/\|T^n x\|) : T \in \mathcal{M},\ x \in S,\ T^n x \neq 0, n = 0, 1, \ldots\}$ is dense in $\mathcal{M} \times S \times S$.*

*Then the set of supercyclic operators $T$ in $\mathcal{M}$ is a dense $G_\delta$-set in the Baire space $\mathcal{M}$.*

## 5.3 Supercyclicity of $V_\varphi$ and hypercyclicity $I + V_\varphi$

In this section we shall study the supercyclicity of $V_\varphi$ as well as the hypercyclicity of $I + V_\varphi$ acting on the spaces $L^p[0,1]$, $1 \leqslant p < \infty$, and the space $\mathcal{C}_0[0,1]$ of continuous functions that vanish at 0 endowed with the supremum norm. Observe that since $V_\varphi$ is a contraction on $L^p[0,1]$, $1 \leqslant p < \infty$, or $\mathcal{C}_0[0,1]$, it cannot be weakly hypercyclic.

The following proposition states that if $V_\varphi$ is weakly supercyclic, then $\varphi(x) \leqslant x$ a.e.

**Proposition 5.3.1.** *Let $\varphi$ is a measurable self-map of $[0,1]$ with $\varphi(x) > x$ on a set of positive Lebesgue measure, then $V_\varphi$ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, is not weakly supercyclic.*

*Proof.* Since any supercyclic compact operator acting on a Banach space must be quasi-nilpotent, see [20], and the same is true for weakly supercyclic operators (the same argument works). By Corollary 2.2, $V_\varphi$ is not quasi-nilpotent and the result follows. □

In what follows, we will be considering only continuous symbols. The following lemmas describe the closure of the range of $V_\varphi$. We will denote $\overline{\operatorname{ran}}_p V_\varphi$ the closure of the range of $V_\varphi$ acting on $L^p[0,1]$. If it acts on $\mathcal{C}[0,1]$ or $\mathcal{C}_0[0,1]$ it will be denoted by $\overline{\operatorname{ran}}\, V_\varphi$ and $\overline{\operatorname{ran}}_0\, V_\varphi$, respectively.

**Lemma 5.3.2.** *Let $\varphi$ be a continuous self-map of $[0,1]$. Assume that $V_\varphi$ acts on $\mathcal{C}[0,1]$. If $\varphi$ is not strictly monotone, then the codimension of $\overline{\operatorname{ran}}\, V_\varphi$ is infinite. If $\varphi$ is strictly monotone and $\varphi(0) \neq 0$, $\varphi(1) \neq 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \mathcal{C}[0,1]$. If $\varphi$ is strictly monotone and $\varphi(0) = 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \{f \in \mathcal{C}[0,1] : f(0) = 0\}$. Finally, if $\varphi$ is strictly monotone and $\varphi(1) = 0$, then $\overline{\operatorname{ran}}\, V_\varphi = \{f \in \mathcal{C}[0,1] : f(1) = 0\}$.*

*Proof.* If $\varphi$ is not strictly monotone, then

$$A = \{(t,s) \in [0,1]^2 : t < s \quad \text{and} \quad \varphi(t) = \varphi(s)\}$$

is infinite. Since

$$\overline{\operatorname{ran}}\, V_\varphi \subseteq \{f \in \mathcal{C}[0,1] : f(t) = f(s) \quad \text{for each} \quad (t,s) \in A\}$$

and the last space has infinite codimension, we see that $\overline{\operatorname{ran}}\, V_\varphi$ has infinite codimension in $\mathcal{C}[0,1]$.

The description of $\overline{\operatorname{ran}}\, V_\varphi$ in the case when $\varphi$ is strictly monotone follows from the decomposition $V_\varphi = C_\varphi V$, where $V$ is the Volterra operator, $(C_\varphi f)(x) = f(\varphi(x))$ and the fact that the closure of the range of the Volterra operator acting on $\mathcal{C}[0,1]$ is $\mathcal{C}_0[0,1] = \{f \in \mathcal{C}[0,1] : f(0) = 0\}$. Indeed, if $\varphi(0) \neq 0$, $\varphi(1) \neq 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \mathcal{C}[0,1]$, if $\varphi(0) = 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \mathcal{C}_0[0,1]$ and finally if $\varphi(1) = 0$, then $C_\varphi(\mathcal{C}_0[0,1]) = \{f \in \mathcal{C}[0,1] : f(1) = 0\}$. □

**Lemma 5.3.3.** *Let $\varphi$ be a continuous self-map of $[0,1]$. Assume that $V_\varphi$ acts on $L^p[0,1]$ with $1 \leqslant p < \infty$. If $\varphi$ is not strictly monotone, then $\overline{\operatorname{ran}}_p V_\varphi$ has infinite codimension. If $\varphi$ is strictly monotone, then $\overline{\operatorname{ran}}_p V_\varphi = L^p[0,1]$.*

*Proof.* As in the previous lemma, let $\overline{\operatorname{ran}}\, V_\varphi$ be the closure of $V_\varphi(\mathcal{C}[0,1])$ in $\mathcal{C}[0,1]$. In order to show that $\overline{\operatorname{ran}}_p V_\varphi \cap \mathcal{C}[0,1] = \overline{\operatorname{ran}}\, V_\varphi$, first observe that both sides in the latter equality are closed subspaces in $\mathcal{C}[0,1]$ and $\overline{\operatorname{ran}}\, V_\varphi \subset \overline{\operatorname{ran}}_p V_\varphi \cap \mathcal{C}[0,1]$. On the other hand, $\mathcal{C}[0,1]$ is dense in the space $L^p[0,1]$ and $V_\varphi$ is bounded from $L^p[0,1]$ into $\mathcal{C}[0,1]$. Indeed, given a function $f$ in $L^p[0,1]$, we have

$$\|V_\varphi f\|_\infty = \sup \left| \int_0^{\varphi(x)} f(t)\, dt \right| \leqslant \int_0^1 |f(t)|\, dt \leqslant \|f\|_p. \tag{5.3.1}$$

Hence, $\operatorname{ran} V_\varphi$ is dense in $\operatorname{ran}_p V_\varphi$ in the space $\mathcal{C}[0,1]$ endowed with the supremum norm. Therefore, there is not a bounded functional over $\mathcal{C}[0,1]$, such that vanishes on $\operatorname{ran} V_\varphi$ and does not vanish on $\operatorname{ran}_p V_\varphi$, what finishes the proof of the needed equality. Now, the result follows immediately from the previous lemma and the fact that both $\mathcal{C}_0[0,1]$ and $\{f \in \mathcal{C}[0,1] : f(1) = 0\}$ are dense in $L^p[0,1]$. □

The following lemma is an immediate consequence of Lemma 5.3.2.

**Lemma 5.3.4.** *Let $\varphi$ be a continuous self-map of $[0,1]$ satisfying $\varphi(0) = 0$. Assume that $V_\varphi$ acts on $\mathcal{C}_0[0,1]$. If $\varphi$ is not strictly increasing, then $\overline{\operatorname{ran}}_0 V_\varphi$ has infinite codimension. If $\varphi$ is strictly increasing, then $\overline{\operatorname{ran}}_0 V_\varphi = \mathcal{C}_0[0,1]$.*

Now, we can use the previous lemmas to show that the cyclicity of $V_\varphi$ is a severe restriction on the inducing symbol.

**Proposition 5.3.5.** *Let $\varphi$ be a continuous self-map of $[0,1]$. Assume that $V_\varphi$ acting on $L^p[0,1]$, $1 \leqslant p < \infty$ or on $\mathcal{C}[0,1]$ is cyclic. Then $\varphi$ is strictly monotone. In addition, if $\varphi(0) = 0$ and $V_\varphi$ is cyclic when acting on $\mathcal{C}_0[0,1]$, then $\varphi$ is strictly increasing.*

*Proof.* It is well known and easy to see that if an operator is cyclic, then the codimension of the closure of its range is at most 1. Thus it remains to apply Lemmas 5.3.2, 5.3.3 and 5.3.4. □

Since weakly supercyclic operators are cyclic, as another immediate consequence of Propositions 5.3.1 and 5.3.5, we have

**Corollary 5.3.6.** *Let $\varphi$ be a continuous self-map of $[0,1]$. If $V_\varphi$ acting on $L^p[0,1]$, $1 \leqslant p < \infty$ or on $\mathcal{C}_0[0,1]$, is weakly supercyclic, then $\varphi$ is strictly increasing and $\varphi(x) \leqslant x$ for $0 \leqslant x \leqslant 1$.*

The cyclic properties that we will be considering are cyclic, weakly supercyclic, weakly hypercyclic, supercyclic and hypercyclic. Actually, the real core of the question, wether a Volterra type operator satisfies any of these properties or not, is in the friendly Hilbert space setting $L^2[0,1]$. A basic tool in the study of the cyclic properties of an operator is the Comparison Principle, for instances see [8] or [49].

**Comparison Principle 5.3.7.** *Suppose that $X$ is a linear metric space and $Y$ is a dense subspace that is itself a linear metric space with a stronger topology. Suppose that $T$ is a continuous linear transformation on $X$ that also maps the smaller space $Y$ into itself, and is continuous in the topology of this space. If $T$ is cyclic on $Y$, then it is also cyclic on $X$ and has an $X$-cyclic vector that belongs to $Y$. Furthermore, the same is true for supercyclic and hypercyclic operators.*

**Proposition 5.3.8.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(0) > 0$ and $\varphi(1) > 0$. Then $V_\varphi$ acting on $L^2[0,1]$ has a given cyclic property if and only if $V_\varphi$ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}[0,1]$ has the same cyclic property.*

*Proof.* Let $1 < p < \infty$. First, observe that $\mathcal{C}[0,1]$ is densely and continuously embedded into $L^p[0,1]$ and $L^p[0,1]$ is densely and continuously embedded into $L^1[0,1]$. The same holds true if all the spaces carry their weak topologies. Thus by the Comparison Principle, see 5.3.7, it suffices to show that if $V_\varphi$ acting on $L^1[0,1]$ has a given cyclic property, then $V_\varphi$ acting on $\mathcal{C}[0,1]$ has it.

Suppose that $V_\varphi$ acting on $L^1[0,1]$ has a given cyclic property. By Proposition 5.3.5, $\varphi$ is strictly monotone. By Lemma 5.3.2, $V_\varphi$ acting on $\mathcal{C}[0,1]$ has dense range. Indeed, from 5.3.1 we have that $V_\varphi$ is a bounded linear operator from $L^1[0,1]$ into $\mathcal{C}[0,1]$ with dense range. It follows that whenever $f$ in $L^1[0,1]$ provides a given cyclic property for $V_\varphi$ acting on $L^1[0,1]$, then $V_\varphi f$ provides the same property for $V_\varphi$ acting on $\mathcal{C}[0,1]$. □

**Proposition 5.3.9.** *Let $\varphi$ be a continuous self-map of $[0,1]$ with $\varphi(0) = 0$. Then $V_\varphi$ acting on $L^2[0,1]$ has a given cyclic property if and only if $V_\varphi$ acting on $L^p[0,1]$, $1 \leqslant p < \infty$, or on $\mathcal{C}_0[0,1]$ has the same cyclic property.*

*Proof.* Exactly as in the proof of the above proposition, it suffices to show that if $V_\varphi$ acting on $L^1[0,1]$ has a given cyclic property, then $V_\varphi$ acting on $\mathcal{C}_0[0,1]$ has it.

Suppose that $V_\varphi$ acting on $L^1[0,1]$ has a given cyclic property. By Proposition 5.3.5, $\varphi$ is strictly increasing and by Lemma 5.3.4, $V_\varphi$ acting on $\mathcal{C}_0[0,1]$ has dense range. Hence, as in the proof of the previous Proposition, $V_\varphi$ is a bounded linear operator from $L^1[0,1]$ into $\mathcal{C}_0[0,1]$ with dense range, and again it follows that whenever $f$ in $L^1[0,1]$ provides a given cyclic property for $V_\varphi$ acting on $L^1[0,1]$, then $V_\varphi f$ provides the same property for $V_\varphi$ acting on $\mathcal{C}_0[0,1]$. □

In view of propositions 5.3.8 and 5.3.9, there is no interest in studding the cyclic properties of composition Volterra operators acting on different spaces. Therefore, in the reminder, the statements concerning cyclicity will not mention the underlying spaces where the operators act. These will be understood to be the $L^p[0,1]$ spaces, for $1 \leqslant p < \infty$ and either $\mathcal{C}[0,1]$ or $\mathcal{C}_0[0,1]$, depending on the value of $\varphi$ at 0.

### 5.3.1 Supercyclicity of $V_\varphi$ and hypercyclicity of $I + V_\varphi$. Case $\varphi(1) < 1$

Although $V_\varphi$ acting on $L^2[0,1]$ cannot be weakly hypercyclic, it may happen that $I + V_\varphi$ is hypercyclic. We have,

**Theorem 5.3.10.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ such that $\varphi(x) < x$ for $0 < x \leqslant 1$. Then $V_\varphi$ is supercyclic and $I + V_\varphi$ is hypercyclic.*

*Proof.* We just need to verify that conditions of Proposition 5.2.13 and Corollary 5.2.3 are satisfied.

Clearly, the sequence $\{\varphi_n(1)\}$ is strictly decreasing and tends to zero as $n$ tends to $\infty$. The right-left inclusion in the equality

$$\ker V_\varphi^n = \{f \text{ such that } \inf\operatorname{supp}(f) \geqslant \varphi_n(1)\}$$

is clear if we observe that (a) in Lemma 4.3.5 still holds under the assumptions of the statement. The left-right inclusion is proved by induction. Let $n = 1$ and f be such that $0 \equiv V_\varphi f = C_\varphi V f$. Since the classical Volterra operator $V$ is injective, we have that $Vf_{|[0,\varphi(1)]} \equiv 0$ implies that $f_{|[0,\varphi(1)]} \equiv 0$. The second part of the induction process is just a repetition, with the only observation that a function $f$ is in $\ker V_\varphi^{n+1}$ if and only if $V_\varphi f$ belongs to $\ker V_\varphi^n$.

Since the canonical injection from $\mathcal{C}_0[0,1]$ to $L^p[0,1]$, $1 \leqslant p < \infty$, is continuous and has dense range, we restrict the rest of the proof to the space $\mathcal{C}_0[0,1]$ without lose of generality. As above, write $V_\varphi = C_\varphi V$. We have that $V$ maps densely $\ker V_\varphi^{n+1}$ into itself and $C_\varphi$ is a bijective isometry from $\ker V_\varphi^{n+1}$ to $\ker V_\varphi^n$ for each positive integer $n$.

The last condition to be checked is that $\ker^\star V_\varphi$ is dense in $\mathcal{C}_0[0,1]$, but this is straightforward since $\{\varphi_n(1)\}$ tends to 0. □

From Corollary 5.3.6, it follows that if $V_\varphi$ is weakly supercyclic, then $\varphi$ cannot fail to be strictly increasing or to have the graph below the identity function. However, $\varphi(1) < 1$ is a different issue.

### 5.3.2 Supercyclicity of $V_\varphi$. Case $\varphi(1) = 1$

Although the Volterra operator is not weakly supercyclic, see [35], there are supercyclic composition Volterra operators whose symbols are below the diagonal and take the value 1 at 1. In this subsection we will prove

**Theorem 5.3.11.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $\varphi$ is analytic at $0$ and $\varphi'(0) > \delta_1^+$, where*

$$\delta_1^+ = \delta_1^+(\varphi) = \overline{\lim_{x\to 1}} \frac{1-x}{1-\varphi(x)}.$$

*Then $V_\varphi$ is supercyclic.*

As an immediate corollary of Theorem 5.3.11, we have

**Corollary 5.3.12.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $\varphi$ is analytic at $0$ and differentiable at $1$ with $\varphi'(0)\varphi'(1) > 1$. Then $V_\varphi$ is supercyclic.*

*Proof of Theorem 5.3.11.* By Proposition 5.3.9, it is enough to show that $V_\varphi$ is supercyclic on $\mathcal{C}_0[0,1]$.

We take $b > 0$ with $1/\varphi'(0) < b < 1/\delta_1^+$ and consider the dense subspace of $\mathcal{C}_0[0,1]$ defined by

$$E = \{f \in \mathcal{C}_0[0,1] : \inf \operatorname{supp}(f) > 0\}.$$

According to Lemma 4.2.1, we have

$$\overline{\lim_{n\to\infty}} \|V_\varphi^n f\|_\infty^{1/n^2} \leqslant \sqrt{\delta_1^+} \quad \text{for each } f \in E. \tag{5.3.2}$$

On the other hand, by Theorem 4.3.1,

$$F = \left\{ f \in V_\varphi^\infty(\mathcal{C}_0[0,1]) \text{ such that } \overline{\lim_{n\to\infty}} \|V_\varphi^{-n} f\|_\infty^{1/n^2} \leqslant \sqrt{b} \right\}$$

is a dense linear subspace of $\mathcal{C}_0[0,1]$ satisfying $V_\varphi(F) = F = V_\varphi^{-1}(F)$. Let $S$ be the restriction of $V_\varphi^{-1}$ to $F$. Clearly, $V_\varphi S f = f$ for each $f$ in $F$ and

$$\overline{\lim_{n\to\infty}} \|S^n f\|_\infty^{1/n^2} \leqslant \sqrt{b} \quad \text{for each } f \in F. \tag{5.3.3}$$

Finally, take $b < c < 1/\delta_1^+$ and let $\lambda_n = c^{n^2/2}$ for $n \geqslant 0$. The above display along with (5.3.2) imply that $\lambda_n V_\varphi^n f$ tends to 0 as $n$ tends to $\infty$ for each $f$ in $E$ and $\lambda_n^{-1} S^n f$ tends to 0 for each $f$ in $F$. Upon applying Theorem 5.2.14 with $T = V_\varphi$ and $S_k = S^k$, we conclude that $V_\varphi$ acting on $\mathcal{C}_0[0,1]$ is supercyclic. □

### 5.3.3 Non-cyclicity

The next theorem complements Theorem 5.3.11.

**Theorem 5.3.13.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $\varphi$ is analytic at 1 with $\varphi'(1)\delta_0^+ < 1$, where*

$$\delta_0^+ = \delta_0^+(\varphi) = \overline{\lim_{x \to 0}} \frac{\varphi(x)}{x}.$$

*Then $V_\varphi$ is not cyclic.*

As an immediate corollary, we have

**Corollary 5.3.14.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $\varphi$ is analytic at 1 and differentiable at 0 with $\varphi'(0)\varphi'(1) < 1$. Then $V_\varphi$ is not cyclic.*

From Corollaries 5.3.12 and 5.3.14, we immediately obtain

**Corollary 5.3.15.** *Let $\varphi$ be a continuous strictly increasing self-map of $[0,1]$ with $\varphi(x) < x$ for $0 < x < 1$ and $\varphi(1) = 1$. Assume that $\varphi$ is analytic at 0 and at 1.*

(i) *If $\varphi'(0)\varphi'(1) > 1$, then $V_\varphi$ is supercyclic.*

(ii) *If $\varphi'(0)\varphi'(1) < 1$, then $V_\varphi$ is not cyclic.*

*Proof of Theorem 5.3.13.* By Proposition 5.3.9, it is enough to prove that $V_\varphi$ is not cyclic on $L^2[0,1]$.

Clearly, $\phi(x) = 1 - \varphi_{-1}(1-x)$ is continuous, strictly increasing, analytic at 0, $\phi(x) < x$ for $0 < x < 1$, $\phi(1) = 1$, $\phi'(0) = 1/\varphi'(1)$ and

$$\delta_1^+(\phi) = \overline{\lim_{x \to 1}} \frac{1-x}{1-\phi(x)} = \delta_0^+(\varphi).$$

In addition, the fact that $\varphi'(1)\delta_0^+(\varphi) < 1$ implies that $\phi'(0) > \delta_1^+(\phi)$. Thus we may choose $1 \leqslant 1/\phi'(0) < b < 1/\delta_1^+(\phi)$. Since $\phi$ is analytic at zero, there is $0 < a < 1$ such that $\phi$ is analytic on $[0, \phi^{-1}(a)]$ and

$$\max_{[0,\phi^{-1}(a)]} \frac{1}{\phi'} \leqslant b.$$

For each $n$ in $\mathbb{Z}$, we set $a_n = \phi_{-n}(a)$. We choose $a_{-1} < c < a_0$ and set $c_n = \phi_{-n}(c)$ for each $n$ in $\mathbb{Z}$. Clearly, $\{a_n\}$ and $\{c_n\}$ converge to 1 as $n$ tends to $+\infty$ and to 0 as $n$ tends to $-\infty$. Moreover, $c_n < a_n < c_{n+1}$ for each $n$ in $\mathbb{Z}$. By Lemma 4.3.3, there are non-zero functions $f_0$ in $\mathcal{F}[c_0, a_0]$ and $f_1$ in $\mathcal{F}[a_{-1}, c_0]$ that we extend to the whole interval $[0, 1]$ by defining them as zero outside their intervals of definition. By Lemma 4.3.8, we find that $f_0$ as well as $f_1$ are in $V_\phi^\infty(\mathcal{C}_0[0, 1])$, which we defined in Section 4.3, and

$$\varlimsup_{n\to\infty} \|V_\phi^{-n} f_j\|_2^{1/n^2} \leqslant \sqrt{b} \quad \text{for } j = 0, 1. \tag{5.3.4}$$

On the other hand, Lemma 4.2.1 implies that

$$\varlimsup_{n\to\infty} \|V_\phi^{n} f_j\|_2^{1/n^2} \leqslant \sqrt{\delta_1^+(\phi)} \quad \text{for } j = 0, 1. \tag{5.3.5}$$

Now, take real numbers $b < \alpha < \beta < 1/\delta_1^+(\phi)$ and set

$$z_n = \begin{cases} \alpha^{n(1-n)/2}, & \text{if } n < 0; \\ \beta^{n(n+1)/2}, & \text{if } n \geqslant 0. \end{cases}$$

From (5.3.4) and (5.3.5), and the choice of $\alpha$ and $\beta$ in the definition of the sequence $z_n$, it follows that

$$\varlimsup_{n\to\infty} \left(z_n \|V_\phi^{-n} f_j\|_2\right)^{1/n^2} \leqslant \sqrt{b/\alpha} < 1 \quad \text{for } j = 0, 1$$

and

$$\varlimsup_{n\to\infty} \left(z_n \|V_\phi^{n} f_j\|_2\right)^{1/n^2} \leqslant \sqrt{\beta\delta_1^+(\phi)} < 1 \quad \text{for } j = 0, 1.$$

$$J(x \oplus y) = \sum_{n=-\infty}^{\infty} z_n (x_n V_\phi^n f_0 + y_n V_\phi^n f_1)$$

defines a bounded operator from $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ into $L^2[0, 1]$.

We need to show that $J^\star$ has dense range. To this end, it is enough to check that $J$ is one-to-one. Let $x$ and $y$ be in $\ell^2(\mathbb{Z})$ and suppose that

$$J(x \oplus y) = 0.$$

By Lemma 4.3.5, it follows that $\inf \operatorname{supp}(V_\phi^n f_0) = c_n$ and $\inf \operatorname{supp}(V_\phi^n f_1) = a_{n-1}$ for each $n$ in $\mathbb{Z}$ and $\sup \operatorname{supp}(V_\phi^n f_0) = a_n$ and $\sup \operatorname{supp}(V_\phi^n f_1) = c_n$ for $n \leqslant 0$. Thus, for each $n \leqslant 0$, we find that $V_\phi^n f_0$ is different from zero and supported on $[c_n, a_n]$ and for each $m \neq n$, we have that $V_\phi^m f_j$ vanishes on $[c_n, a_n]$. Similarly for each $n \leqslant 0$, we find that $V_\phi^n f_1$ is different from zero and supported in $[a_{n-1}, c_n]$ and for each $m \neq n$, we have that $V_\phi^m f_j$ vanishes on $[a_{n-1}, c_n]$. It follows that $x_n = y_n = 0$ for $n \leqslant 0$. If $x \oplus y$

is different from zero, let $n$ be the minimal positive integer for which $|x_n|+|y_n|>0$. Since all $V_\phi^m f_j$ vanish on $[a_{n-1},c_n]$, except for $m=n$ and $j=1$, it follows that $y_n=0$. Similarly, $x_n=0$, a contradiction. Therefore, $J$ is one-to-one and $J^\star$ has dense range.

Let $\{e_n\}_{n\in\mathbb{Z}}$ denote the canonical basis of $\ell^2(\mathbb{Z})$ and consider the (forward) weighted shift $Se_n=w_{n+1}e_{n+1}$, with weight sequence

$$w_n=\frac{z_{n-1}}{z_n}=\begin{cases}\alpha^{n-1}, & \text{for } n\leqslant 0;\\ \beta^{-n}, & \text{for } n\geqslant 1.\end{cases}$$

We have

$$V_\phi J=J(S\oplus S),$$

what clearly implies both

$$V_\phi^n J=J(S^n\oplus S^n)\qquad \text{for } n\geqslant 1,$$

and

$$J^\star V_\phi^\star=(S^\star\oplus S^\star)J^\star.$$

From subsection 2.1.2, we know that $V_\phi^\star$ is unitarily similar under $(Uf)(x)=f(1-x)$ to $V_\varphi$. Thus assuming that $V_\varphi$ is cyclic, then so is $V_\phi^\star$. Let $f$ in $L^2[0,1]$ be cyclic for $V_\phi^\star$. Then according to the last display we have

$$\operatorname{span}\{(S^{\star n}\oplus S^{\star n})(J^\star f):n\geqslant 0\}=J^\star(\operatorname{span}\{V_\phi^{\star n}f:n\geqslant 0\}).$$

Since $J^\star$ has dense range, it follows that $J^\star f$ is cyclic for $S^\star\oplus S^\star$. Now, the operator $R$ on $\ell^2(\mathbb{Z})$ defined by $Re_n=(\alpha/\beta)^{|n(n+1)|/2}e_{-n}$, $n$ in $\mathbb{Z}$, is bounded because $\alpha<\beta$. The operator $R$ is clearly injective and self-adjoint, and one may check that $SR=RS^\star$. Hence,

$$(I\oplus R)(S^\star\oplus S^\star)=(S^\star\oplus S)(I\oplus R).$$

Therefore,

$$\operatorname{span}\{(S^\star\oplus S)^n(I\oplus R)(J^\star f):n\geqslant 0\}=(I\oplus R)(\operatorname{span}\{(S^\star\oplus S^\star)^n(J^\star f):n\geqslant 0\}).$$

Taking into account that $J^\star f$ is cyclic for $S^\star\oplus S^\star$ and that since $R$ is self-adjoint and injective, $I\oplus R$ has dense range, we see that $S^\star\oplus S$ is cyclic. Let $x\oplus y$ in $\ell^2(\mathbb{Z})\oplus\ell^2(\mathbb{Z})$ be cyclic for $S^\star\oplus S$ and consider the dual pairing

$$\langle u,v\rangle=\sum_{n\in\mathbb{Z}}u_nv_n,\qquad u,v\in\ell^2(\mathbb{Z}).$$

Since $x\oplus y$ must be different from zero, the functional

$$\Phi(u\oplus v)=\langle u,y\rangle-\langle v,x\rangle$$

on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ is non-zero. However, for each non-negative integer $n$, we have

$$\Phi((S^\star \oplus S)^n(x \oplus y)) = \langle S^{\star n}x, y\rangle - \langle S^n y, x\rangle = 0,$$

which contradicts that $x \oplus y$ is cyclic for $S^\star \oplus S$. The proof is complete. □

### 5.3.4 Residual sets of hypercyclic $I + V_\varphi$ and supercyclic $V_\varphi$

Consider the set

$$\Omega = \{\varphi \in \mathcal{C}_0[0,1] \text{ such that } 0 \leqslant \varphi(x) \leqslant x \ \text{ for } \ 0 \leqslant x \leqslant 1 \ \text{ and } \varphi \text{ is increasing}\}$$

endowed with the metric

$$d(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \max_{[3^{-n}, 1-3^{-n}]} |\varphi - \psi|.$$

It is clear that $d(\varphi_n, \varphi)$ tends to 0 if and only if $\varphi_n$ converges to $\varphi$ uniformly on $[\varepsilon, 1-\varepsilon]$ for each $0 < \varepsilon < 1/2$. It is also straightforward to see that $(\Omega, d)$ is complete. Now, consider

$$\Omega_0 = \{\varphi \in \Omega : \varphi(x) < x \text{ for } 0 < x < 1, \ \varphi(1) = 1 \text{ and } \varphi \text{ is strictly increasing}\}.$$

Clearly, $\Omega_0$ is a dense in $(\Omega, d)$. We shall see that $\Omega_0$ is also a $G_\delta$ set in $(\Omega, d)$. Indeed, we have $\Omega \setminus \Omega_0 = A \cup B \cup C$, where

$$\begin{aligned} A &= \{\varphi \in \Omega \text{ such that } \varphi(1) < 1\}, \\ B &= \{\varphi \in \Omega \text{ such that } \varphi \ \text{ is not strictly increasing}\}, \\ C &= \{\varphi \in \Omega \text{ such that there is } 0 < a < 1 \text{ for which } \varphi(a) = a\}. \end{aligned}$$

On the other hand,

$$A = \bigcup_{n=0}^{\infty} A_n, \quad B = \bigcup_{\substack{0<a<b<1 \\ a,b\in\mathbb{Q}}} B_{a,b} \quad \text{and} \quad C = \bigcup_{n=1}^{\infty} C_n,$$

where

$$\begin{aligned} A_n &= \{\varphi \in \Omega \text{ such that } \|\varphi\|_\infty \leqslant 1 - 2^{-n}\}, \\ B_{a,b} &= \{\varphi \in \Omega \text{ such that } \varphi \ \text{ is constant on } [a,b]\} \quad \text{ and} \\ C_n &= \{\varphi \in \Omega \text{ such that there is } 3^{-n} < a < 1 - 3^{-n} \text{ for which } \varphi(a) = a\}. \end{aligned}$$

One can check that $A_n$, $B_{a,b}$ and $C_n$ are all closed and with empty interior in $(\Omega, d)$ and, therefore, they all are $F_\sigma$ sets. Thus, $\Omega_0$ is a dense $G_\delta$-set in $\Omega$.

**Theorem 5.3.16.** *The set of $\varphi$ in $\Omega_0$ for which $V_\varphi$ and $V_\varphi^\star$ are supercyclic and $I+V_\varphi$ and $I+V_\varphi^\star$ are both hypercyclic is a dense $G_\delta$-set in $\Omega_0$.*

*Proof.* First, let $\mathcal{M}_1$ be the set of $\varphi$ in $\Omega$ such that $V_\varphi$ is supercyclic and $I+V_\varphi$ is hypercyclic. Since $\|V_\varphi - V_\psi\|_2 \leqslant \|\varphi-\psi\|_1$ and convergence with respect to $d$ on $\Omega$ implies $L^1$-convergence, we see that $\varphi \mapsto V_\varphi$ from $\Omega$ into $\mathcal{L}(L^2[0,1])$ is continuous. Since the sets of hypercyclic and supercyclic operators are $G_\delta$-sets with respect to the operator norm topology, we have that $\mathcal{M}_1$ is a $G_\delta$-set in $\Omega$ as the pre-image of a $G_\delta$-set with respect to a continuous map. On the other hand,

$$\{\varphi \in \Omega \text{ such that } \ \varphi(x) < x \text{ for } 0<x<1,\ \varphi(1)<1 \ \text{ and } \varphi \text{ is strictly increasing}\}$$

is clearly dense in $\Omega$ and is contained in $\mathcal{M}_1$ by Theorem 5.3.10. Thus, $\mathcal{M}_1$ is a dense $G_\delta$-set in $\Omega$. Since $\Omega_0$ is a dense $G_\delta$-set in $\Omega$, Baire's Theorem implies that $\mathcal{M}_2 = \mathcal{M}_1 \cap \Omega_0$ is a dense $G_\delta$-set in $\Omega_0$.

Now, the map $\Phi$ from $\Omega_0$ onto itself defined as $\Phi(\varphi)(x) = 1-\varphi_{-1}(1-x)$ is one-to-one. One can easily see that $\Phi$ is also a continuous involution. Since, as already mentioned many times, $V_\varphi^\star$ is similar to $V_\psi$ where $\psi = \Phi(\varphi)$, we see that $\Phi(\mathcal{M}_2)$ is exactly the set of $\varphi$ in $\Omega_0$ for which $V_\varphi^\star$ is supercyclic and $I+V_\varphi^\star$ is hypercyclic. Since $\Phi$ is an homeomorphism from $\Omega_0$ onto itself, $\Phi(\mathcal{M}_2)$ is a dense $G_\delta$-set in $\Omega_0$. Hence, $\mathcal{M} = \mathcal{M}_2 \cap \Phi(\mathcal{M}_2)$ is a dense $G_\delta$-set in $\Omega_0$ and the result follows. □

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