



A Succinct Characterization of Period Annuli in Planar Piecewise Linear Differential Systems with a Straight Line of Nonsmoothness

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Received: 17 December 2022 / Accepted: 17 July 2023

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Abstract

We close the problem of the existence of crossing period annuli in planar piecewise linear differential systems with a straight line of nonsmoothness. In fact, a characterization for the existence of such objects is provided by means of a few basic operations on the parameters.

Keywords Piecewise linear systems · Period annuli · Poincaré half-maps · Integral characterization

Mathematics Subject Classification 34A36 · 34C25 · 37C27

Communicated by Alain Goriely.

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1 Introduction

Determining sufficient and necessary conditions for the existence of a period annulus in planar differential systems is a classical problem in qualitative theory of planar vector fields. For the particular case that the period annulus ends in a monodromic singularity, such a problem is known as *Center Problem*, which was exhaustively studied for polynomial vector fields (see, for instance, Romanovski and Shafer 2009). This problem has also been considered in the context of planar piecewise smooth differential systems (see, for instance, Coll et al. 2001, 1999; Gasull and Torregrosa 2003; Novaes and Silva 2022, 2021; Pleshkan and Sibirskii 1973).

However, due to the complexity imposed by the nonsmoothness, the center problem is not solved even for the simplest family of piecewise smooth differential systems, namely piecewise linear differential systems with two zones separated by the straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$,

$$\dot{\mathbf{x}} = \begin{cases} A_L \mathbf{x} + \mathbf{b}_L, & \text{if } x \leq 0, \\ A_R \mathbf{x} + \mathbf{b}_R, & \text{if } x \geq 0. \end{cases} \quad (1)$$

Here, $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $A_L = (a_{ij}^L)_{2 \times 2}$, $A_R = (a_{ij}^R)_{2 \times 2}$, $\mathbf{b}_L = (b_1^L, b_2^L) \in \mathbb{R}^2$, $\mathbf{b}_R = (b_1^R, b_2^R) \in \mathbb{R}^2$, and the dot denotes the derivative with respect to the independent variable t . The Filippov's convention (Filippov 1988) is assumed for trajectories of (1).

The main goal of this paper is to close the problem of the existence of crossing period annuli for system (1) by providing a characterization for the existence of such objects by means of a few basic operations on the parameters.

Since system (1) is piecewise linear, two obvious conditions implying the existence of period annuli are:

- (A) $T_L = 0$, $D_L > 0$, and $a_L < 0$; or
- (B) $T_R = 0$, $D_R > 0$, and $a_R > 0$,

where T_L , T_R and D_L , D_R denote, respectively, the traces and determinants of the matrices A_L and A_R and

$$a_L = a_{12}^L b_2^L - a_{22}^L b_1^L \quad \text{and} \quad a_R = a_{12}^R b_2^R - a_{22}^R b_1^R. \quad (2)$$

Indeed, condition (A) implies that system (1) has a linear center (and so a period annulus) in the half-plane $\{(x, y) \in \mathbb{R} : x < 0\}$, and condition (B) implies that system (1) has a linear center (and so a period annulus) in the half-plane $\{(x, y) \in \mathbb{R} : x > 0\}$.

Apart the trivial cases above, system (1) admits period annuli whose orbits cross the separation line Σ . Regarding those period annuli we may quote the following papers. In Freire et al. (2012), sufficient conditions were provided for piecewise linear systems of kind (1) formed by two foci and without sliding set to have a global center around the origin. In Buzzi et al. (2013) it was classified the centers at infinity for piecewise linear perturbations of linear centers. In Medrado and Torregrosa (2015), it was established sufficient conditions in order for a monodromic singularity at the separation line Σ

to be a center. Finally, in Freire et al. (2021), the authors have characterized when systems of kind (1), formed by two foci, have a center at infinity.

In this paper, we present a general and concise characterization of the existence of a crossing period annulus for system (1). This characterization will be given in terms of their parameters and, unlike the mentioned papers, regardless the local nature of each linear system.

Notice that the existence of a crossing periodic orbit for system (1) implies trivially the existence of the Poincaré half-maps associated with Σ . In turn, such maps exist if, and only if, the following set of conditions hold:

$$(H) : \begin{cases} a_{12}^L a_{12}^R > 0; \\ a_L \leq 0 \text{ and } 4D_L - T_L^2 > 0, \text{ or } a_L > 0; \\ a_R \geq 0 \text{ and } 4D_R - T_R^2 > 0, \text{ or } a_R < 0. \end{cases}$$

Indeed, taking into account the direction of the flow on the separation line $x = 0$, it is straightforward to see that the inequality $a_{12}^L a_{12}^R > 0$ is necessary for the existence of crossing periodic solutions. The other two conditions will be discussed below (see Propositions 1 and 2).

Now, we present the main result of this paper.

Theorem 1 *Consider the planar piecewise linear differential system (1). Let T_L, T_R and D_L, D_R be, respectively, the traces and determinants of the matrices A_L and A_R and let a_L and a_R be the values given in expression (2). Denote*

$$\xi_0 := a_R T_L - a_L T_R, \quad \xi_\infty := T_L^2 D_R - T_R^2 D_L, \quad \text{and} \quad \beta := a_{12}^L b_1^R - b_1^L a_{12}^R. \quad (3)$$

Then, the differential system (1) has a crossing period annulus if, and only if, the condition (H) holds, $\text{sign}(T_R) = -\text{sign}(T_L)$, and $\xi_0 = \xi_\infty = \beta = 0$.

At this point, we must clarify the dynamical meanings of the values ξ_0, ξ_∞ , and β and of the relationship $\text{sign}(T_R) = -\text{sign}(T_L)$.

First, under the hypothesis $a_{12}^L a_{12}^R > 0$, system (1) has a sliding region contained in Σ and delimited by the points $(0, -b_1^L/a_{12}^L)$ and $(0, -b_1^R/a_{12}^R)$ provided that β does not vanish. Accordingly, the condition $\beta = 0$ indicates that system (1) does not have any sliding region.

Second, when system (1) does not have a sliding region, as it follows from Proposition 14 of Carmona et al. (2022), the sign of the value ξ_0 (called by ξ in that work) provides the stability of the origin of system (1) when it is a monodromic singularity. Moreover, from Proposition 15 of Carmona et al. (2022), under the assumption $\text{sign}(T_R) = -\text{sign}(T_L) \neq 0$, the sign of the value $c_\infty = T_L \xi_\infty$ determines the stability of the infinity for system (1) when it is monodromic.

Finally, since system (1) is linear on each side of the separation straight line Σ , the signs of the traces T_L and T_R determine the (area) contraction/expansion of the system on each side of Σ and so the condition $\text{sign}(T_R) = -\text{sign}(T_L)$ ensures a kind of balance between the contraction of the system in one zone and the expansion of the system in the other zone.

Theorem 1 is proven in Sect. 3. Its proof is based on a recent integral characterization for Poincaré half-maps for planar linear differential systems introduced in Carmona and Fernández-Sánchez (2021), which has been successfully used to analyze periodic behavior of piecewise linear systems (see, for instance, Carmona et al. 2021b, 2022, 2023). This characterization as well as some useful properties of the Poincaré half-maps will be introduced in Sect. 2.

2 Poincaré Half-Maps and Displacement Function: Some Preliminary Results

In this section, after introducing a canonical form for system (1) in Sect. 2.1, the definition of the Poincaré half-maps for planar linear differential systems will be presented in Sect. 2.2. Some useful properties of these maps, provided in Carmona et al. (2021a), will be collected in Sect. 2.3. In Sect. 2.4, a displacement function will be given together with some of its main features.

2.1 Canonical Form

As it was said in Introduction, the existence of crossing periodic solutions of system (1) implies straightforwardly the first condition of Hypothesis (H), that is, $a_{12}^L a_{12}^R > 0$. Moreover, under this condition, Freire et al. (2012) stated that the differential system (1) is reduced, by a homeomorphism preserving the separation line $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, into the following Liénard canonical form

$$\begin{cases} \dot{x} = T_L x - y, \\ \dot{y} = D_L x - a_L, \end{cases} \quad \text{for } x \leq 0, \quad \begin{cases} \dot{x} = T_R x - y + b, \\ \dot{y} = D_R x - a_R, \end{cases} \quad \text{for } x \geq 0, \quad (4)$$

being a_L and a_R the values given in expression (2), T_L , T_R and D_L , D_R , respectively, the traces and determinants of the matrices A_L and A_R , and $b = \beta/a_{12}^R$, where β is given in expression (3).

2.2 Integral Characterization of Poincaré Half-Maps

The periodic solutions of the piecewise linear differential system (4) are studied via two Poincaré half-maps defined on Σ : the *forward Poincaré half-map* $y_L : I_L \subset [0, +\infty) \rightarrow (-\infty, 0]$ and the *backward Poincaré half-map* $y_R^b : I_R^b \subset [b, +\infty) \rightarrow (-\infty, b]$.

On the one hand, the forward Poincaré half-map takes a point $(0, y_0)$, with $y_0 \geq 0$, and maps it to a point $(0, y_L(y_0))$ by traveling through the flow of (4) in the positive time direction. Clearly, it is determined by the left linear differential system of (4) and its formal definition will be given in Proposition 1.

On the other hand, the backward Poincaré half-map takes a point $(0, y_0)$, with $y_0 \geq b$, and maps it to a point $(0, y_R^b(y_0))$ by traveling through the flow of (4) in the negative time direction. Clearly, it is determined by the right linear differential

system of (4). Notice that the simple translation $y \mapsto y - b$ applied to this right linear system allows us to write $y_R^b(y_0) = y_R(y_0 - b) + b$ and $I_R^b = I_R + b$, where $y_R : I_R \subset [0, +\infty) \rightarrow (-\infty, 0]$ is the backward Poincaré half-map of (4) for $b = 0$, that is, $y_R = y_R^0$ and $I_R = I_R^0$. The formal definition of the map y_R and its domain I_R will be given in Proposition 2.

In Propositions 1 and 2, we will need the following concept of *Cauchy principal value*:

$$PV \left\{ \int_{y_1}^{y_0} f(y)dy \right\} := \lim_{\varepsilon \searrow 0} \left(\int_{y_1}^{-\varepsilon} f(y)dy + \int_{\varepsilon}^{y_0} f(y)dy \right),$$

for $y_1 < 0 < y_0$ and f continuous in $[y_1, y_0] \setminus \{0\}$ (see, for instance, Henrici 1988). Note that if f is also continuous at 0, then the Cauchy principal value coincides with the definite integral.

The forward Poincaré half-map y_L refers to the linear system

$$\begin{cases} \dot{x} = T_L x - y, \\ \dot{y} = D_L x - a_L, \end{cases} \tag{5}$$

which corresponds with the left linear system of (4). Thus, its definition, its domain I_L , and its analyticity are given by Theorem 19, Corollary 21, and Corollary 24 of Carmona and Fernández-Sánchez (2021). In the following proposition, we summarize the mentioned results. (see Carmona et al. 2022, [Theorem 1]).

Proposition 1 *The forward Poincaré half-map y_L is well defined if, and only if, $a_L \leq 0$ and $4D_L - T_L^2 > 0$, or $a_L > 0$. In this case, $I_L := [\lambda_L, \mu_L) \neq \emptyset$ and the following statements hold:*

- (a) *The right endpoint μ_L of the interval I_L is the smallest strictly positive root of the polynomial $W_L(y) = D_L y^2 - a_L T_L y + a_L^2$, if it exists. Otherwise, $\mu_L = +\infty$.*
- (b) *The left endpoint λ_L of the interval I_L is greater than or equal to zero. If $\lambda_L > 0$, then $y_L(\lambda_L) = 0$, $a_L < 0$, $4D_L - T_L^2 > 0$, and $T_L < 0$. Moreover, if $y_L(\lambda_L) < 0$, then $\lambda_L = 0$ and $a_L < 0$, $4D_L - T_L^2 > 0$, and $T_L > 0$.*
- (c) *The polynomial W_L verifies $W_L(y) > 0$ for $y \in \text{ch}(I_L \cup y_L(I_L)) \setminus \{0\}$, where $\text{ch}(\cdot)$ denotes the convex hull of a set.*
- (d) *The forward Poincaré half-map y_L is the unique function $y_L : I_L \subset [0, +\infty) \rightarrow (-\infty, 0]$ that satisfies*

$$PV \left\{ \int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy \right\} = q_L(a_L, T_L, D_L) \text{ for } y_0 \in I_L, \tag{6}$$

where

$$q_L(a_L, T_L, D_L) = \begin{cases} 0 & \text{if } a_L > 0, \\ \frac{\pi T_L}{D_L \sqrt{4D_L - T_L^2}} & \text{if } a_L = 0, \\ \frac{2\pi T_L}{D_L \sqrt{4D_L - T_L^2}} & \text{if } a_L < 0. \end{cases} \tag{7}$$

(e) *The forward Poincaré half-map y_L is analytic in $\text{Int}(I_L)$.*

On the other hand, the backward Poincaré half-map y_R refers to the linear system

$$\begin{cases} \dot{x} = T_R x - y, \\ \dot{y} = D_R x - a_R, \end{cases}$$

which corresponds with the right linear system of (4) for $b = 0$. Thus, its definition, its domain I_R , and its analyticity are obtained from Proposition 1 by means of the change of variables $(t, x) \mapsto (-t, -x)$ and taking $(a_L, D_L, T_L) = (-a_R, D_R, -T_R)$ in system (5). (see Carmona et al. 2022, [Theorem 2]).

Proposition 2 *The backward Poincaré half-map y_R is well defined if, and only if, $a_R \geq 0$ and $4D_R - T_R^2 > 0$, or $a_R < 0$. In this case, $I_R := [\lambda_R, \mu_R] \neq \emptyset$ and the following statements hold:*

- (a) *The right endpoint μ_R of its definition interval I_R is the smallest strictly positive root of the polynomial $W_R(y) = D_R y^2 - a_R T_R y + a_R^2$, if it exists. Otherwise, $\mu_R = +\infty$.*
- (b) *The left endpoint λ_R of the interval I_R is greater than or equal to zero. If $\lambda_R > 0$, then $y_R(\lambda_R) = 0$, $a_R > 0$, $4D_R - T_R^2 > 0$, and $T_R > 0$. Moreover, if $y_R(\lambda_R) < 0$, then $\lambda_R = 0$ and $a_R > 0$, $4D_R - T_R^2 > 0$, and $T_R < 0$.*
- (c) *The polynomial W_R verifies $W_R(y) > 0$ for $y \in \text{ch}(I_R \cup y_R(I_R)) \setminus \{0\}$.*
- (d) *The backward Poincaré half-map y_R is the unique function $y_R : I_R \subset [0, +\infty) \rightarrow (-\infty, 0]$ that satisfies*

$$\text{PV} \left\{ \int_{y_R(y_0)}^{y_0} \frac{-y}{W_R(y)} dy \right\} = q_R(a_R, T_R, D_R) \text{ for } y_0 \in I_R, \tag{8}$$

where

$$q_R(a_R, T_R, D_R) = \begin{cases} 0 & \text{if } a_R < 0, \\ -\frac{\pi T_R}{D_R \sqrt{4D_R - T_R^2}} & \text{if } a_R = 0, \\ -\frac{2\pi T_R}{D_R \sqrt{4D_R - T_R^2}} & \text{if } a_R > 0. \end{cases} \tag{9}$$

(e) *he backward Poincaré half-map y_R is analytic in $\text{Int}(I_R)$.*

Remark 1 Notice that the integral that appears in (6) (resp. (8)) is divergent for $a_L = 0$ (resp. $a_R = 0$). Nevertheless, in this case, the Cauchy principal value provides

$$y_L(y_0) = -e^{\frac{\pi T_L}{\sqrt{4D_L - T_L^2}}} y_0, \quad \left(\text{resp. } y_R(y_0) = -e^{\frac{-\pi T_R}{\sqrt{4D_R - T_R^2}}} y_0 \right) \quad y_0 \geq 0. \tag{10}$$

In any other case, that is, $a_L \neq 0$ (resp. $a_R \neq 0$), the Cauchy principal value can be removed because the integral is a proper integral.

2.3 Properties of Poincaré Half-Maps

Some useful properties of the Poincaré half-maps y_L and y_R will be collected in the next results. The proofs of these properties for the map y_L are given in Carmona et al. (2021a), and they can be extended to y_R by means of the change of variables $(t, x) \mapsto (-t, -x)$ and taking $(a_L, D_L, T_L) = (-a_R, D_R, -T_R)$ in system (5). The first one (Proposition 3) provides, as a direct consequence of expressions (6) and (8), the first derivative of the Poincaré half-maps. The second result (Proposition 4) establishes the relative position between the graph of the Poincaré half-maps and the bisector of the fourth quadrant. The third result (Proposition 5) gives the first coefficients of the Taylor expansions of the Poincaré half-map y_R at the origin. The last result (Proposition 6) shows the first coefficient of the Newton-Puiseux series expansion of y_L around a point $\hat{y}_0 > 0$ such that $y_L(\hat{y}_0) = 0$.

Proposition 3 *The first derivatives of the Poincaré half-maps y_L and y_R are given by*

$$y'_L(y_0) = \frac{y_0 W_L(y_L(y_0))}{y_L(y_0) W_L(y_0)} < 0 \text{ for } y_0 \in \text{int}(I_L),$$

$$y'_R(y_0) = \frac{y_0 W_R(y_R(y_0))}{y_R(y_0) W_R(y_0)} < 0 \text{ for } y_0 \in \text{int}(I_R),$$

where the polynomials W_L and W_R are given in Propositions 1 and 2, respectively.

Proposition 4 *The following statements hold.*

(a) *The forward Poincaré half-map y_L satisfies the relationship*

$$\text{sign}(y_0 + y_L(y_0)) = -\text{sign}(T_L) \text{ for } y_0 \in I_L \setminus \{0\}.$$

In addition, when $0 \in I_L$ and $y_L(0) \neq 0$ or when $T_L = 0$, then the relationship above also holds for $y_0 = 0$.

(b) *The backward Poincaré half-map y_R satisfies the relationship*

$$\text{sign}(y_0 + y_R(y_0)) = \text{sign}(T_R) \text{ for } y_0 \in I_R \setminus \{0\}.$$

In addition, when $0 \in I_R$ and $y_R(0) \neq 0$ or when $T_R = 0$, then the relationship above also holds for $y_0 = 0$.

For the sake of simplicity, the next result is only given for the map y_R , which will be used later on in the proof of Theorem 1. A version for the map y_L can be stated in an analogous way.

Proposition 5 *Assume that $0 \in I_R$ and $y_R(0) = \hat{y}_1 < 0$, then the backward Poincaré half-map y_R is a real analytic function in I_R and its Taylor expansion around the origin writes as*

$$y_R(y_0) = \hat{y}_1 + \frac{W_R(\hat{y}_1) y_0^2}{2a_R^2 \hat{y}_1} + \mathcal{O}(y_0^3).$$

Again, for the sake of simplicity, the next result is only provided for the map y_L . An analogous result for the map y_R can be also stated.

Proposition 6 *Assume that there exists a value $\widehat{y}_0 > 0$ such that $y_L(\widehat{y}_0) = 0$. Then, $a_L < 0$, $\widehat{y}_0 = \lambda_L$, that is, \widehat{y}_0 is the left endpoint of the definition interval I_1 of y_L and the Poincaré half-map y_L admits the Newton-Puiseux series expansion around the point \widehat{y}_0 given by*

$$y_L(y_0) = a_L \sqrt{\frac{2\lambda_L}{W_L(\lambda_L)}} (y_0 - \lambda_L)^{1/2} + \mathcal{O}(y_0 - \lambda_L).$$

2.4 Displacement Function

Once the Poincaré half-maps have been characterized, a displacement function can be defined for system (4).

Suppose that $I^b := I_L \cap (I_R + b) \neq \emptyset$. The displacement function δ_b is then defined in I^b as follows:

$$\begin{aligned} \delta_b : I^b &\longrightarrow \mathbb{R} \\ y_0 &\longmapsto \delta_b(y_0) := y_R^b(y_0) - y_L(y_0) = y_R(y_0 - b) + b - y_L(y_0). \end{aligned} \tag{11}$$

From Propositions 1 and 2, one has $I^b = [\lambda_b, \mu_b)$, where $\lambda_b = \max\{\lambda_L, \lambda_R + b\}$ and $\mu_b = \min\{\mu_L, \mu_R + b\}$. In addition, δ_b is continuous on I^b and analytic on $\text{Int}(I^b)$.

Remark 2 Notice that, by the continuity of δ_b on I_b and the analyticity on $\text{Int}(I_b)$, a crossing period annulus exists if, and only if, $\delta_b(y_0) = 0$ for every $y_0 \in I^b$. Obviously, in this case, the i th order derivative satisfies $\delta_b^{(i)}(y_0) = 0$ for every $y_0 \in I^b$ and $i \in \mathbb{N}$. Of course, when $y_0 = \lambda_b$, $\delta_b^{(i)}(y_0) = 0$ refers to the lateral derivative.

Now, some of the properties of δ_b (in particular, relevant expressions for the sign of the derivatives) will be stated in the next proposition. Its proof can be seen in Carmona et al. (2022).

Proposition 7 *Let us consider the displacement function given in (11) for $b = 0$. Suppose that $y_0^* \in \text{int}(I^0)$ satisfies $\delta_0(y_0^*) = 0$. Denote $y_1^* = y_R(y_0^*) = y_L(y_0^*) < 0$ and define*

$$\begin{aligned} c_0 &:= a_R a_L (a_R T_L - a_L T_R), \\ c_1 &:= a_R T_R D_L - a_L T_L D_R, \\ c_2 &:= a_L^2 D_R - a_R^2 D_L. \end{aligned} \tag{12}$$

Then, the following statements hold:

(a) *The derivative of the displacement function δ_0 defined in (11) verifies*

$$\text{sign}(\delta_0'(y_0^*)) = \text{sign}(F(y_0^*, y_1^*)), \tag{13}$$

being

$$F(y_0, y_1) = c_0 + c_1 y_0 y_1 + c_2 (y_0 + y_1). \tag{14}$$

(b) Moreover, if $\delta'_0(y_0^*) = 0$, then the second derivative of δ_0 verifies

$$\text{sign}(\delta''_0(y_0^*)) = \text{sign}(T_L(c_2 y_0^* + c_0)) = -\text{sign}(T_R(c_2 y_1^* + c_0)).$$

Remark 3 This remark is devoted to provide some useful and interesting relationships between the coefficients $\xi_0, \xi_\infty, c_0, c_1$ and c_2 (given in expressions (3) and (12)), which will be used later on.

The set of polynomial functions $\{W_L, W_R\}$, with W_L and W_R defined in Propositions 1 and 2, is linearly dependent if, and only if, $c_0 = c_1 = c_2 = 0$.

Moreover, the following equalities hold:

$$c_0 = a_R a_L \xi_0, \quad c_0 \begin{pmatrix} D_L \\ D_R \end{pmatrix} - c_2 \begin{pmatrix} -a_L T_L \\ -a_R T_R \end{pmatrix} + c_1 \begin{pmatrix} a_2^2 \\ a_2^1 \end{pmatrix} = 0, \tag{15}$$

$$T_L c_1 + a_L \xi_\infty = D_L T_R \xi_0, \quad \text{and} \quad T_R c_1 + a_R \xi_\infty = D_R T_L \xi_0. \tag{16}$$

3 Characterization of Crossing Period Annuli

This section is dedicated to the proof of Theorem 1. It starts with a result on partial necessary conditions for the existence of a crossing period annulus. In particular, this result states that if system (1) has a crossing period annulus, then it cannot have a sliding region. This result has already been obtained in Freire et al. (2021) by in the case that system (1) is formed by two foci.

Lemma 1 *If the piecewise linear differential system (1) has a crossing period annulus, then the condition (H) holds, the value β defined in (3) vanishes, and $\text{sign}(T_R) = -\text{sign}(T_L)$.*

Proof Notice that if system (1) has a crossing period annulus, then, in particular, $a^L_{12} a^R_{12} > 0$ and, therefore, system (1) can be transformed into system (4), which will also have a crossing period annulus. Hence, $I^b = [\lambda_b, \mu_b] \neq \emptyset$, the Poincaré half-maps are well defined and so, from Propositions 1 and 2, we have that Hypothesis (H) holds.

Now, we show that the existence of a crossing period annulus implies that $b = 0$ and, consequently, $\beta = 0$. Suppose, by reduction to absurdity, that system (4) has a crossing period annulus and $b \neq 0$. Let us assume that $b > 0$; otherwise, by applying the transformation $(t, y) \mapsto (-t, -y)$, we can change the sign of b . This transformation also changes the signs of T_L and T_R , but this will not be important in getting a contradiction. In the sequel, our reasoning distinguishes whether or not b belongs to the interval I^b .

On the one hand, let us consider $b \in I^b$. Then, $0 \in I_R$ and $b \in I_L$. If $y_R(0) = 0$, it follows that $\delta_b(b) = y_R(b - b) - y_L(b) + b = -y_L(b) + b > 0$ and this contradicts

the fact that $\delta_b(y_0) = 0$ for every $y_0 \in I^b$ (see Remark 2). If $y_R(0) < 0$, then, from Proposition 5, one obtains $y'_R(0) = 0$. Thus, if $y_L(b) < 0$, from Proposition 3, one get $y'_L(b) < 0$; if, on the other hand, $y_L(b) = 0$, then, from Proposition 6 (by taking $\widehat{y}_0 = b$), one gets that

$$\lim_{y_0 \searrow b} y'_L(y_0) = -\infty.$$

In both cases, $\delta'_b(b) \neq 0$ which contradicts the fact that $\delta'_b(y_0) = 0$ for every $y_0 \in I^b$ (see Remark 2).

On the other hand, consider $b \notin I^b$. We know that $\lambda_b = \max\{\lambda_L, \lambda_R + b\}$. First, let us assume that $\lambda_b = \lambda_R + b$, which implies that $\lambda_L \leq \lambda_R + b$. Taking into account that $b \notin I^b$, we have that $\lambda_R > 0$. Thus, by statement (b) of Proposition 2, we have $y_R(\lambda_R) = 0$. Since $\lambda_R + b \in I_L$, then $y_L(\lambda_R + b) \leq 0$. Hence $\delta_b(\lambda_R + b) = y_R(\lambda_R) + b - y_L(\lambda_R + b) > 0$, which contradicts the fact that $\delta_b(\lambda_R + b) = 0$. Second, let us assume that $\lambda_b = \lambda_L$, which implies that $\lambda_L \geq \lambda_R + b \geq b$. Taking into account that $b \notin I^b$, the last inequality implies, in fact, that $\lambda_L > b > 0$. Thus, by statement (b) of Proposition 1, we have $y_L(\lambda_L) = 0$ and, then, by Proposition 6,

$$\lim_{y_0 \searrow \lambda_L} y'_L(y_0) = -\infty. \tag{17}$$

From Remark 2, $y_L(\lambda_L) = 0$ implies that $y_R(\lambda_L - b) = -b < 0$ which, in turns, from statement (b) of Proposition 2 and taking into account that $\lambda_L > b$, implies that $\lambda_L - b \in \text{Int}(I_R)$. Hence, Proposition 3 implies that

$$y'_R(\lambda_L - b) < 0. \tag{18}$$

The relationships (17) and (18) contradicts the fact that $\delta'_b(\lambda_L) = 0$.

Therefore, we have shown that the existence of a crossing period annulus implies that $b = 0$ and, consequently, $\beta = 0$.

Finally, $b = 0$ implies that $y_L(y_0) = y_R(y_0)$ for every $y_0 \in I^b$ and, from Proposition 4, it follows that $\text{sign}(T_R) = -\text{sign}(T_L)$ and the proof is finished. \square

3.1 Proof of Theorem 1

Let us start by assuming that the differential system (1) has a crossing period annulus. From Lemma 1, (H) holds, $\beta = 0$, and $\text{sign}(T_L) = -\text{sign}(T_R)$. In addition, since $T_L T_R = 0$ implies that $T_L = T_R = 0$ and, therefore, $\xi_0 = \xi_\infty = 0$, then it only remains to show that $\xi_0 = \xi_\infty = 0$ for the case $T_L T_R < 0$.

Recall that, under the first condition of (H), that is, $a_{12}^L a_{12}^R > 0$, system (1) can be transformed into system (4), with $b = 0$, which will also have a crossing period annulus.

From hypothesis and taking into account Remark 2, the displacement function δ_b for $b = 0$ verifies $\delta_0(y_0) = \delta'_0(y_0) = \delta''_0(y_0) = 0$ for every $y_0 \in \text{Int}(I^0)$ and, by

means of statement (b) of Proposition 7,

$$T_L(c_2y_0 + c_0) = T_R(c_2y_0 + c_0) = 0, \quad \forall y_0 \in \text{Int}(I^0).$$

Since $T_L T_R < 0$, we have that $c_0 = c_2 = 0$. In addition, from (13) and (14), we have that $c_1 y_0 y_L(y_0) = c_1 y_0 y_R(y_0) = 0$ for every $y_0 \in \text{Int}(I^0)$ and, consequently, $c_1 = 0$.

From Remark 3, the relationship $c_0 = c_1 = c_2 = 0$ indicates that the polynomials $W_L(y) = D_L y^2 - a_L T_L y + a_L^2$ and $W_R(y) = D_R y^2 - a_R T_R y + a_R^2$ are linearly dependent. By one hand, if $a_L = 0$, then $a_R = 0$ and so $\xi_0 = 0$. Furthermore, in this case, from (10), one can see that the existence of a crossing period annulus, that is, the condition $y_L(y_0) = y_R(y_0)$ for $y_0 \geq 0$ leads, by a direct computation, to $\xi_\infty = 0$. On the other hand, if $a_L \neq 0$, then $a_R \neq 0$. Thus, since $c_0 = 0$, from (15), one gets $\xi_0 = 0$ and so any of the relationships in (16) implies $\xi_\infty = 0$, because $c_1 = 0$.

Reciprocally, consider the planar piecewise linear differential system (1) and assume that condition (H) holds, $\text{sign}(T_L) = -\text{sign}(T_R)$, and $\beta = \xi_0 = \xi_\infty = 0$. Let us show the existence of a crossing period annulus for system (4) and, consequently, for (1).

Note that if $T_L T_R = 0$, taking into account that $\text{sign}(T_L) = -\text{sign}(T_R)$, we have that $T_L = T_R = 0$. Thus, from (6) and (8), since the integrands are odd functions, it is trivial that $y_L(y_0) = y_R(y_0) = -y_0$ for every $y_0 \in I^0$. This implies the existence of a crossing period annulus. Thus, for the rest of the proof, we can assume $T_L T_R < 0$.

From (15) and (16), $\xi_0 = \xi_\infty = 0$ implies $c_0 = c_1 = 0$. Now, we show that $c_2 = 0$. Indeed, if $a_L = a_R = 0$, then $c_2 = a_L^2 D_R - a_R^2 D_L = 0$ and, otherwise, if $a_L^2 + a_R^2 \neq 0$, then the second relationship of (15) implies $c_2 = 0$.

Since $c_0 = c_1 = c_2 = 0$, from Remark 3, the polynomials $W_L(y) = D_L y^2 - a_L T_L y + a_L^2$ and $W_R(y) = D_R y^2 - a_R T_R y + a_R^2$ are linearly dependent, that is, $W_L = k W_R$. Moreover, $k > 0$. Indeed, if $a_L^2 + a_R^2 \neq 0$, then $k > 0$ immediately, otherwise, if $a_L = a_R = 0$, from (H), we have $D_L, D_R > 0$ and, again, $k > 0$.

Hence, $\text{sign}(a_L) = -\text{sign}(a_R)$, because $T_L T_R < 0$. In addition, $\xi_\infty = 0$ implies that $D_L = (T_L/T_R)^2 D_R$. Thus, $k = (T_L/T_R)^2$ and

$$D_L = k D_R, \quad T_L = -\sqrt{k} T_R, \quad \text{and} \quad a_L = -\sqrt{k} a_R.$$

Therefore,

$$\text{PV} \left\{ \int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy \right\} = \frac{1}{k} \text{PV} \left\{ \int_{y_L(y_0)}^{y_0} \frac{-y}{W_R(y)} dy \right\}$$

and the functions q_L and q_R defined in expressions (7) and (9) satisfy

$$q_L(a_L, T_L, D_L) = q_L \left(-\sqrt{k} a_R, -\sqrt{k} T_R, k D_R \right) = \frac{1}{k} q_R(a_R, T_R, D_R).$$

Now, from Propositions 1 and 2, we see that y_L and y_R have the same integral characterization and, consequently, they coincide, that is, $y_L(y_0) = y_R(y_0)$ for $y_0 \in I_L = I_R$. This implies the existence of a crossing period annulus and the proof is finished.

Acknowledgements VC is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Plan Nacional I+D+I cofinanced with FEDER funds, in the frame of the Project PGC2018-096265-B-I00. FFS is partially supported by the Ministerio de Economía y Competitividad, Plan Nacional I+D+I cofinanced with FEDER funds, in the frame of the Project MTM2017-87915-C2-1-P. VC and FFS are partially supported by the Ministerio de Ciencia e Innovación, Plan Nacional I+D+I cofinanced with FEDER funds, in the frame of the Project PID2021-123200NB-I00, the Consejería de Educación y Ciencia de la Junta de Andalucía (TIC-0130, P12-FQM-1658) and by the Consejería de Economía, Conocimiento, Empresas y Universidad de la Junta de Andalucía (US-1380740, P20-01160). DDN is partially supported by São Paulo Research Foundation (FAPESP) grants 2022/09633-5, 2021/10606-0, 2019/10269-3, and 2018/13481-0, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 438975/2018-9 and 309110/2021-1.

Author Contributions All persons who meet authorship criteria are listed as authors, and all authors certify that they have participated sufficiently in the work to take public responsibility for the content, including participation in the conceptualization, methodology, formal analysis, investigation, writing—original draft preparation and writing—review & editing.

Funding Funding for open access publishing: Universidad de Sevilla/CBUA

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest To the best of our knowledge, no conflict of interest, financial or other, exists.

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