Order
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Separability, Boxicity, and Partial Orders

José-Miguel Díaz-Báñez[1] · Paul Horn[2] · Mario A. Lopez[3] · Nestaly Marín[4] · Adriana Ramírez-Vigueras[5] · Oriol Solé-Pi[6] · Alex Stevens[2] · Jorge Urrutia[5]

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Abstract
A collection $S = \{S_i, \dots, S_n\}$ of disjoint closed convex sets in $\mathbb{R}^d$ is *separable* if there exists a direction (a non-zero vector) $\overrightarrow{v}$ of $\mathbb{R}^d$ such that the elements of S can be removed, one at a time, by translating them an arbitrarily large distance in the direction $\overrightarrow{v}$ without hitting another element of S. We say that $S_i \prec S_j$ if S_j has to be removed before we can remove S_i. The relation $\prec$ defines a partial order $P(S, \prec)$ on S which we call the *separability order* of S and $\overrightarrow{v}$. A partial order $P(X, \prec')$ on $X = \{x_1, \dots, x_n\}$ is called a *separability order* if there is a collection of convex sets S and a vector $\overrightarrow{v}$ in some $\mathbb{R}^d$ such that $x_i \prec' x_j$ in $P(X, \prec')$ if and only if $S_i \prec S_j$ in $P(S, \prec)$. We prove that every partial order is the separability order of a collection of convex sets in $\mathbb{R}^4$, and that any poset of dimension **2** is the separability order of a set of line segments in $\mathbb{R}^3$. We then study the case when the convex sets are restricted to be boxes in d-dimensional spaces. We prove that any partial order is the separability order of a family of disjoint boxes in $\mathbb{R}^d$ for some $d \le \lfloor \frac{n}{2} \rfloor + 1$. We prove that every poset of dimension **3** has a subdivision that is the separability order of boxes in $\mathbb{R}^3$, that there are partial orders of dimension **2** that cannot be realized as box separability in $\mathbb{R}^3$ and that for any d there are posets with dimension d that are separability orders of boxes in $\mathbb{R}^3$. We also prove that for any d there are partial orders with box separability dimension d; that is, d is the smallest dimension for which they are separable orders of sets of boxes in $\mathbb{R}^d$.

Keywords Partially ordered sets · Separability · Order dimension · Boxicity

1 Introduction

Let $S = \{S_1, \dots, S_n\}$ be a set of disjoint convex sets in $\mathbb{R}^d$, and $\overrightarrow{v}$ be a direction, i.e. a non-zero vector of $\mathbb{R}^d$. We say that S_j *blocks* S_i if when we translate S_i along the direction $\overrightarrow{v}$ it hits S_j, i.e. if there is a point $p \in S_i$ and a point $q \in S_j$ such that $q = p + \lambda \overrightarrow{v}$ for some $\lambda > 0$. The blocking relation on S is acyclic if there are no $S_{\sigma_0}, \dots, S_{\sigma_{r-1}}$ such that $S_{\sigma_{i+1}}$ blocks S_{σ_i}, $i = 0, \dots, r-1$, addition taken mod r.

✉ Oriol Solé-Pi
oriolandreu@ciencias.unam.mx

Extended author information available on the last page of the article

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An acyclic blocking relation induces a partial order $P(S, \prec)$, which we will call the *separability order* of S. In this order, $S_i \prec S_j$ if S_j blocks S_i or there is a sequence $S_i = S_{\sigma_1}, \ldots, S_{\sigma_t} = S_j$ of elements of S such that $S_{\sigma_{k+1}}$ blocks S_{σ_k}, $k = 1, \ldots, t-1$.

In what follows we will deal only with separable families of disjoint convex sets and assume that $\overrightarrow{v} = (0, \ldots, 0, 1)$; for this reason, and to avoid cumbersome notation, we will only refer to $P(S, \prec)$, and omit $\overrightarrow{v}$. If $S_i \prec S_j$ we will sometimes say that S_i is *below* S_j, or that S_j is *above* S_i; that is we will assume that we translate our sets *upwards* along the direction of the last coordinate axis. In particular, in $\mathbb{R}^2$ and $\mathbb{R}^3$ we will move our sets up, see Fig. 1.

Let $X = \{x_1, \ldots, x_n\}$, and $P(X, \prec')$ be a partial order on X. We say that $P(X, \prec')$ is the separability order of a family of sets $S = \{S_1, \ldots, S_n\}$ if $x_i \prec' x_j$ if and only if $S_i \prec S_j$.

It is easy to see that not all posets are separability orders of collections of *connected* sets in $\mathbb{R}^3$. Indeed, take any non-planar graph, e.g. $K_{3,3}$ and consider the partial order $P(X, \prec)$ such that the elements of X are the edges and vertices of $K_{3,3}$ in which a vertex v is smaller than an edge e if v is a vertex of e. Sinden [1] and, independently, Ehrlich et al. [2] proved that the graph obtained by inserting a vertex in the middle of each edge of any non-planar graph, e.g. $K_{3,3}$, is not an intersection graph of *connected* sets on the plane, for otherwise this would yield a planar representation of $K_{3,3}$, see Fig. 2. It follows easily that $P(X, \prec)$ is not a separability order of any family of convex sets in $\mathbb{R}^3$, for otherwise projecting these sets on the plane would yield a planar representation of $K_{3,3}$.

Sinden and Herlich et al.'s result that not all graphs are intersection graphs of connected sets on the plane does not generalize to higher dimensions. In fact, Tietze [3] proved in 1905 that any graph is the intersection graph of a family of convex sets in $\mathbb{R}^3$. Using Tietze's result, we will prove that any partial order $P(X, \prec')$ is the separability order of a set of disjoint convex sets in $\mathbb{R}^4$. We also prove that any partial order of dimension two is the separability order of line segments in $\mathbb{R}^3$.

We then turn our attention to the case when $S = \{S_1, \ldots, S_n\}$ is a set of axis-aligned *boxes* in $\mathbb{R}^k$. Our results will use Roberts [4] boxicity theorem that any graph with n vertices is the intersection graph of a set of axis-aligned boxes in $\mathbb{R}^k$ for some $k \leq n/2$.

The *box separability dimension* of a partial order $P(X, \prec)$ is the smallest integer k such that there is a set S of disjoint axis-aligned boxes in $\mathbb{R}^k$ such that $P(X, \prec)$ is the separability order of S. We prove that for any k there are partial orders such that their box separability dimension

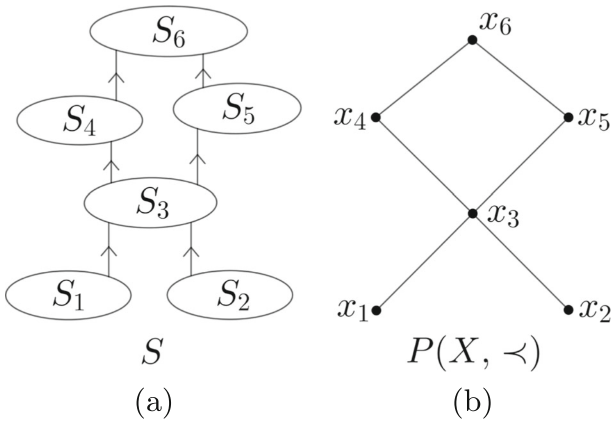

Fig. 1 **a** A family of convex sets $S = \{S_1, \ldots, S_6\}$ on the plane. **b** The separability order $P(X, \prec)$ of S

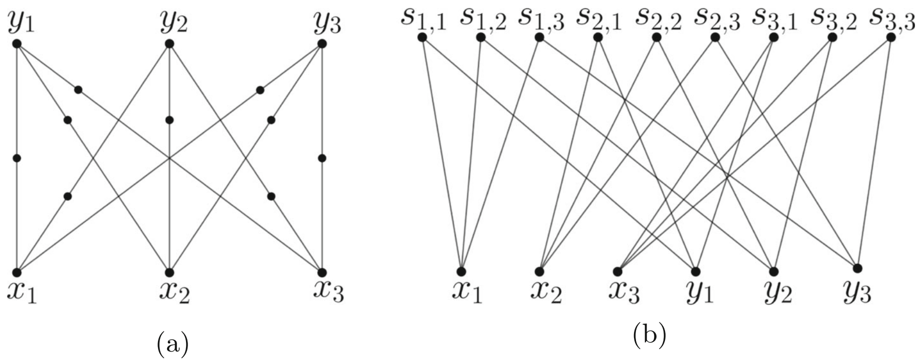

Fig. 2 **a** Graph obtained from $K_{3,3}$ by inserting a vertex $s_{i,j}$ in the middle of each edge joining x_i and y_j. **b** Covering graph of a partial order that is not a separability order of any family of convex sets in $\mathbb{R}^3$

is k, and that every partial order can be embedded, via edge subdivision, in another partial order with box separability dimension at most four. We obtain partial orders of dimension two that are not separability orders of sets of boxes in $\mathbb{R}^3$.

Using the well-known planar partial orders of Kelly [5] (see Fig. 6) we prove that for any d there is a partial order of dimension d that is the separability order of a set of boxes in $\mathbb{R}^3$.

2 Previous Work

Given a collection of disjoint convex sets S in $\mathbb{R}^d$, $d \geq 2$, it is well known that it is not always possible to assign to each of them a direction of motion such that all the sets in the collection can be separated, *one by one*, by translating each of them by an arbitrarily large distance along its assigned direction without hitting another element of S, see Dawson [6]. When such an assignment of directions is possible, we call S *separable*. Dawson also proved that given any family of m disjoint spheres in $\mathbb{R}^d$ it is always possible to remove at least $\min\{m, d+1\}$ spheres without disturbing other spheres. It is important to remark that if we allow the convex sets to move simultaneously, instead of one at a time, then any family of disjoint convex sets can be separated. An easy way to see this is to imagine that all the sets contract at the same rate, *increasing* the relative distances among them (with respect to their size).

Separability problems of families of convex sets in the plane and in $\mathbb{R}^3$ have been studied for some time in both Computational Geometry and from the point of view of ordered sets, see [7–12]. In general, separability problems involve families of disjoint convex sets in $\mathbb{R}^d$ which we want to remove, one at a time, while avoiding collisions with other elements.

It is known that any set S of convex sets in the plane is separable, and that the separability orders they generate are *truncated planar lattices*, see Rival and Urrutia [9]. If the elements of S are assigned one of m different directions of motion, $P(S, \prec)$ is called an m-directional ordered set. In [9] it is shown that every ordered set has a two-directional plane point representation using subdivisions, and that there exist posets that are not m-directional ordered sets at all.

G. X. Viennot [13] studied a combinatorial problem arising from *heap of pieces*, which may be thought of as a collection of *lego-like* blocks, piled in some way. An ordered set arises in which a block A is smaller than a block B if the removal of A from the pile involves the previous removal of B. A somewhat similar problem in which we now seek to remove sets of disjoint convex sets on the plane was studied by Díaz-Báñez et al. in [14], where the problem of removing the smallest number of convex sets enclosing a valuable object buried among them is studied.

3 Notation and Definitions

A *partially ordered set* $P(X, \prec)$, for short a *poset*, is a pair consisting of a set X, and a binary relation $\prec$ on X that is irreflexive and transitive. Let $X = \{x_1, \ldots, x_n\}$. Two elements x_i and x_j of X are *comparable* if $x_i \prec x_j$ or $x_j \prec x_i$. A *linear order* L of X is a partial order such that any pair of elements of X are comparable.

A *linear extension* L of $P(X, \prec)$ is a linear order such that for any $x_i, x_j \in X$, $x_i \prec x_j$ in $P(X, \prec)$ implies x_i is smaller than x_j in L. Given a set of linear orders $\{L_1, \ldots, L_k\}$ on a set X, the *intersection* of them, is the partial order $P(X, \prec)$ such that $x_i \prec x_j$ if and only if x_i is smaller than x_j in L_r, $r = 1, \ldots, k$. A *realizer* of $P(X, \prec)$ is a set of linear extensions $\{L_1, L_2, \ldots, L_m\}$ of X such that $\bigcap_{i=1}^{m} L_i = P(X, \prec)$. The *dimension* of $P(X, \prec)$, $dim(P(X, \prec))$, is the least number m such that $P(X, \prec)$ has a realizer of size m. Hiraguchi [15] showed that the dimension of a partial order on n elements is at most $\lfloor \frac{n}{2} \rfloor$.

A *chain* of $P(X, \prec)$ is a set $C \subseteq X$ of pairwise comparable elements in $P(X, \prec)$. An *antichain* of $P(X, \prec)$ is a set $A \subseteq X$ whose elements are pairwise incomparable in $P(X, \prec)$. The *height* of $P(X, \prec)$ is defined as the maximum length of a chain in $P(X, \prec)$. We say that a partial order is bipartite if its height is 2. The *width* of $P(X, \prec)$ is the maximum size of an antichain in $P(X, \prec)$.

The *comparability graph* of $P(X, \prec)$, denoted as G_P, is the undirected graph with vertex set X such that $(x_i, x_j) \in E(G_P)$ if and only if x_i and x_j are comparable. Given $x_i, x_k \in X$, we say that x_k covers x_i if $x_i \prec x_k$ and there is no $x_j \in X$ such that $x_i \prec x_j \prec x_k$. The *covering graph* of $P(X, \prec)$ is the subgraph G'_P of G_P in which (x_i, x_j) is an edge of G'_P if and only if x_i covers x_j or x_j covers x_i.

A *subdivision of a partial order* $P(X, \prec)$ is any other partial order that can be obtained by repeated applications of the following operation: take $x_i, x_j \in X$ such that $x_i \prec x_j$ and add an element x to X with $x_i \prec x \prec x_j$, extend this relation so that it is transitive.

A *k-dimensional box*, or k-box, is a Cartesian product of closed intervals $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_k, b_k]$, that is, the set of points $(y_1, \ldots, y_k)$ in $\mathbb{R}^k$ such that $a_i \leq y_i \leq b_i$, $i = 1, \ldots, k$.

Let G be a graph with n vertices $\{v_1, \ldots, v_n\}$. G is the intersection graph of a family of sets $\{S_1, \ldots, S_n\}$ if two vertices v_i, v_j of G are adjacent if and only if S_i and S_j intersect. We will use two classical results on the representation of graphs as intersection graphs of convex sets in Euclidean spaces.

Theorem 3.1 *(H. Tietze [3]) Any graph is the intersection graph of families of convex sets in* $\mathbb{R}^3$.

Theorem 3.2 *(Roberts [4]) Any graph with n vertices is the intersection graph of a set of k-boxes in $\mathbb{R}^k$, $k \leq \lfloor \frac{n}{2} \rfloor$.*

The smallest k for which such a set of boxes exists is called the *boxicity* of G, and is denoted as $Box(G)$.

4 Bounds on Separability Dimension

In this section we will prove the following results:

Theorem 4.1 *Any poset is the separability order of a family of convex sets in* $\mathbb{R}^4$.

Theorem 4.2 *Any poset* $P(X, \prec)$ *is the separability order of sets of boxes in* $\mathbb{R}^k$*, for some* $k \leq \lfloor n/2 \rfloor + 1$.

To prove our results we need the following lemma:

Lemma 4.1 *Suppose that the comparability graph of a poset* $P(X, \prec)$ *is the intersection graph of a family of convex sets in* $\mathbb{R}^k$*. Then* $P(X, \prec)$ *is a separability order of translates of those convex sets in* $\mathbb{R}^{k+1}$*. The same is true if the cover graph of* $P(X, \prec)$ *is an intersection graph of convex sets in* $\mathbb{R}^k$.

Proof Let $X = \{x_1, \ldots, x_n\}$, and suppose that the comparability graph G_P of $P(X, \prec)$ is the intersection graph of a family of convex sets $\{S_1, \ldots, S_n\}$ in $\mathbb{R}^k$. Let S_i be the set representing x_i, and suppose that the labels of the elements of X are such that if $x_i \prec x_j$, then $i < j$, i.e. $x_i < \ldots < x_n$ is a linear extension of $P(X, \prec)$.

Lift $S_1, \ldots, S_n$ to convex sets $S'_1, \ldots, S'_n$ in $\mathbb{R}^{k+1}$ in such a way that any point $p = (y_1, \ldots, y_k)$ that belongs to S_i is mapped to the point $p' = (y_1, \ldots, y_k, i\lambda)$, for some $\lambda > 0$, $i = 1, \ldots, n$. It is now easy to see that $P(X, \prec)$ is the separability order of $\{S'_1, \ldots, S'_n\}$.

The statement for cover graphs can be shown to be true in the same way. □

Theorems 4.1 and 4.2 follow directly applying Tietze's and Robert's results to the comparability graph of $P(X, \prec)$.

In view of Theorem 4.1, the reader may wonder if any poset is the separability order of sets of boxes of $\mathbb{R}^k$ for a fixed k. We prove next that this is not true. We present a brief argument, in the spirit of a bound of Alon and Scheinerman from [16] regarding the dimension of containment orders, showing that *any* sufficiently large class of n element posets must contain posets whose box separability dimension is large. The key observation is the following.

Observation 4.1 If $P(X, \prec)$ is an n-element partial order realized as separability of boxes in $\mathbb{R}^d$, then $\mathcal{P}(X, \prec)$ is representable as separability of boxes whose coordinates are integers in $[2n]^d$.

From this it easily follows that

Theorem 4.3 *Suppose* $\mathcal{C}$ *is a collection of n element partial orders. Then if all partial orders in* $\mathcal{C}$ *are separability orders of boxes in* $\mathbb{R}^d$*,*

$$d \geq \frac{\log |\mathcal{C}|}{2n \log(2n)}$$

Proof Suppose all posets in $\mathcal{C}$ can be realized in $\mathbb{R}^d$. Per Observation 4.1, any partial order in $\mathcal{C}$ can be represented with coordinates in $[2n]^d$. A single box is determined by two opposite corners and hence at most

$$([2n]^{2d})^n = 2n^{2dn}$$

distinct partial orders are representable in $\mathbb{R}^d$. Noting that $2n^{2dn} \geq |\mathcal{C}|$ and solving for d gives the result.

4.1 Boxicity and Box Separability Dimension

As one might expect, the box separability dimension and the boxicity of posets turn out to be closely related. Indeed, we have that:

Theorem 4.4 *Let $P(X, \prec)$ be a partial order. Then $Bsep(P(X, \prec)) = 1 + \min\{Box(G) : G'_P \subseteq G \subseteq G_P\}$, where the minimum is taken over all graphs that are simultaneously supergraphs of the covering graph and subgraphs of the comparability graph.*

Proof We first bound $Bsep(P(X, \prec))$ from above. Consider G such that $G'_P \subseteq G \subseteq G_P$. Then G can be realized as the intersection graph of boxes in $Box(G)$ dimensions. Lift the boxes using one additional dimension, as was done in the proof of Lemma 4.1. In this way $P(X, \prec)$ can be realized as box separability in $\mathbb{R}^{Box(G)+1}$.

Now we prove the matching lower bound. Suppose that $P(X, \prec)$ is the separability order of a set S of boxes in $\mathbb{R}^d$. Project the boxes of S onto the $(d-1)$-dimensional space formed by the first $d-1$ coordinates, and let G be their intersection graph. Notice that all edges in the covering graph G'_P must be present in G and, therefore, $G'_P \subseteq G$. Also note that no edges in G will be absent from the comparability graph of $\mathcal{P}$, and so $G \subseteq G_P$. Therefore $G'_P \subseteq G \subseteq G_P$ and we have $d - 1 \geq Box(G)$.

Putting everything together, $Bsep(P(X, \prec)) = 1 + \min\{Box(G) : G'_P \subseteq G \subseteq G_P\}$. □

Since $G'_P = G_P$ for all bipartite orders, we have the following corollary.

Corollary 4.1 *If $P(X, \prec)$ is bipartite, then $Bsep(P(X, \prec)) = Box(G_P) + 1$.*

4.2 Other Bounds

It was shown by Adiga et al. in [17] that $\frac{Box(G_P)}{(\chi(G_P)-1)} \leq dim(P(X, \prec)) \leq 2Box(G_P)$, where $\chi(G_P)$ denotes the chromatic number of the comparability graph G_P. We use this result, along with Corollary 4.1, to bound $Bsep(P(X, \prec))$.

Theorem 4.5 *All bipartite partial orders that can be realized as box separability in $\mathbb{R}^{d+1}$ have dimension at most $2d$. The bound is tight.*

Proof Since $P(X, \prec)$ is bipartite, we have that $\frac{dim(P(X,\prec))}{2} \leq Box(G_P) = Bsep(P(X, \prec)) - 1$. To show that the bound is tight, consider Hiraguchi's bipartite poset $H_{n,n}$ consisting of $2n$ elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ such that a_i is smaller than b_j for all $i \neq j$. It is well known that the dimension of $H_{n,n}$ is n. It was shown by Trotter [18] that the boxicity of the comparability graph of $H_{n,n}$ is $\lceil \frac{n}{2} \rceil$. It follows now that the box separability dimension of $H_{n,n}$ is $\lceil \frac{n}{2} \rceil + 1$. □

From the above paragraph, we get the following.

Corollary 4.2 *For any d there are posets such that their box separability dimension is exactly d.*

Theorem 4.6 *For any poset $P(X, \prec)$, $Bsep(P(X, \prec)) \leq dim(P(X, \prec))(\chi(G_P) - 1) + 1$.*

Proof We have $Bsep(P(X, \prec)) \leq Box(G_P) + 1 \leq dim(P(X, \prec))(\chi(G_P) - 1) + 1$, as desired.

5 Subdivision and Separability Dimension

Recall that a subdivision of a poset $P(X, \prec)$ is obtained by repeating the following operation several times: take $x_i, x_j \in X$ such that $x_i \prec x_j$ and add an element x to X with $x_i \prec x \prec x_j$. The subdivision of posets is a well studied concept, for instance it is well known that, in general subdivisions of posets may increase their dimension, as proved by Spinrad in [19]. In this section we show some somewhat surprising results. We will show that for any poset $P(X, \prec)$ it is always possible, via subdivision, to find a partial order with box separability dimension at most 4. We also prove that any poset of dimension 3 has a subdivision with box separability dimension at most 3.

Theorem 5.1 *Every poset of dimension at most* 3 *has a subdivision which can be realized as box separability in* $\mathbb{R}^3$.

Proof Let $P(X, \prec)$ be a partial order of dimension 3, and $\{L_1, L_2, L_3\}$ linear extensions of $P(X, \prec)$ such that their intersection is $P(X, \prec)$. Represent $P(X, \prec)$ as a vector dominance order in space, i.e. for each element $x_i \in X$ assign a point with integer coordinates (a_i, b_i, c_i) such that the values of a_i, b_i and c_i correspond to the position of x_i in L_i, $i = 1, 2, 3$.

For each $x_i \in X$, place a sufficiently small box B_i centered at (a_i, b_i, c_i) such that the projections onto the xy-plane of any two boxes representing incomparable elements of $P(X, \prec)$ are disjoint.

For any two elements $x_i \prec x_j$ that are adjacent in the covering graph of $P(X, \prec)$, we do the following: Consider the line segment $\ell_{i,j}$ joining (a_i, b_i, c_i) to (a_j, b_j, c_i). Notice that this segment lies in the horizontal plane that contains (a_i, b_i, c_i). Place a sufficiently large number of equidistant points $\{p_1, \ldots p_m\}$ in $\ell_{i,j}$, and then a sufficiently small box centered at each of these points such that the projection of the box B_{p_i} on the xy-plane intersects only the projection of boxes $B_{p_{i-1}}$ and $B_{p_{i+1}}$; the projections of B_{p_1} and B_{p_m} must intersect the projections of B_i and B_j respectively. We can alter the heights of the boxes slightly so that the boxes are disjoint and $B_{p_1}, \ldots, B_{p_m}$ are the steps of an ascending stair, see Fig. 3.

Clearly, if $x_i \prec x_j$ then $B_i \prec B_j$ in the separability order of the resulting collection of boxes B. All that is left to show is that no undesired comparability appears in the separability

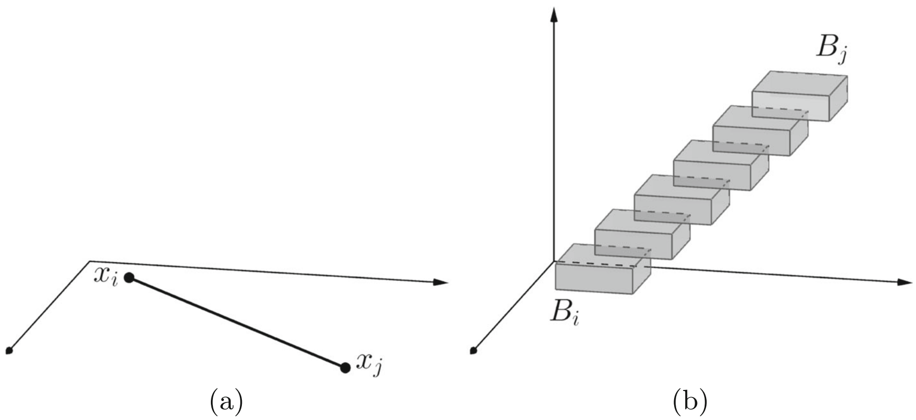

Fig. 3 **a** Two elements x_i and x_j such that $x_i \prec x_j$, and the line segment $\ell_{i,j}$ joining them. **b** A collection of boxes such that $B_i \prec B_j$ in its separability order

order. Any such comparability, if it exists, must arise from the intersection of the projections of the boxes of two stairs, as described in the last paragraph.

Consider $x_i, x_j, x_k, x_l \in X$ such that $x_i \prec x_j$ and $x_k \prec x_l$ and suppose that the projections of the segments $\ell_{i,j}$ and $\ell_{k,l}$ cross each other (if this does not happen we can easily ensure that the projections of the steps of their corresponding stairs are disjoint by taking small enough boxes). Without loss of generality, assume that $c_i < c_k$, then the stair of boxes from x_i to x_j passes below the one from x_k to x_l and, thus, $B_i \prec B_l$ in the separability order. The fact that the projections of $\ell_{i,j}$ and $\ell_{k,l}$ intersect implies that $a_i < a_l$ and $b_i < b_l$, furthermore, $c_i < c_k < c_l$, which yields $x_i \prec x_l$. This completes the proof.□

In a similar way we can now prove the next result:

Theorem 5.2 *Every poset $P(X, \prec)$ has a subdivision which can be realized as box separability in $\mathbb{R}^4$.*[1]

Proof Assume that the elements of X are labelled $x_1, \ldots, x_n$ such that if $x_i \prec x_j$ in $P(X, \prec)$, then $i < j$, i.e. $x_1 < \ldots < x_n$ is a linear extension of $P(X, \prec)$. Map the elements of X to a set of points $\{p_i = (a_i, b_i, c_i) | 1 \le i \le n\}$ in general position in $\mathbb{R}^3$, observe that no two segments joining pairs of points in this set intersect, except at a common endpoint. Consider the covering graph of $P(X, \prec)$, and map each $p_i = (a_i, b_i, c_i)$ to the point $p_i' = (a_i, b_i, c_i, i)$ in $\mathbb{R}^4$. Place a small box B_i centered at p_i' for each i. For each pair i, j with $x_i \prec x_j$ in $P(X, \prec)$ such that the edge joining them is an edge in the covering graph G'_P, consider the line segment $\ell_{i,j} \in \mathbb{R}^4$ joining p_i' to p_j'. Subdivide ℓ as in the proof of Theorem 5.1 and place a small box centered at each of the subdivision points. These boxes form a chain that implies $B_i \prec B_j$ in the separability order of the collection of boxes thus obtained. All that is left to do is show that there are no undesired comparabilities in the separability order. This follows from the observation that the projections of any two segments $\ell_{i,j}, \ell_{k,l}$ onto $\mathbb{R}^3$ do not cross. □

It is not hard to construct a representation in which each edge has been subdivided at most a constant number of times and, thus, the number of boxes is $O(n^2)$. The proof of Theorem 4.3 can easily be adapted to obtain an almost matching lower bound on the number of subdivisions required.

Theorem 5.3 *There is an n vertex partial order $\mathcal{P}(n, \prec)$ that requires at least $\Omega(\frac{n^2}{\log n})$ boxes to represent as a subdivision in $\mathbb{R}^4$.*

Proof Suppose every n vertex partial order has a subdivision which can be realized as box separability in $\mathbb{R}^4$ with at most N boxes. It is easy to see, then, that a subdivision of all n element posets can be realized with exactly N boxes. But as in the proof of Theorem 4.3, there are at most $(2N)^{8N}$ posets on N boxes representable in $\mathbb{R}^4$. Thus, $(2N)^{8N}$ must be at least the number of n element posets. As there are $2^{(1+o(1))n^2/4}$ posets on n elements, the result follows. □

Using the technique from Theorem 5.2, one can easily derive

Theorem 5.4 *Every poset $P(X, \prec)$ with a planar cover graph has a subdivision which can be realized as box separability in $\mathbb{R}^3$.*

[1] It has been called to our attention by one of the referees that using Theorem 4.4 and Lemma 12 of [20], it is possible to obtain an alternative proof of Theorem 5.2.

6 Low Dimension Representations

While posets that are separability orders of families of disjoint convex sets on the plane are well known, they are truncates planar lattices [9], not much is known on separability orders of convex sets in $\mathbb{R}^3$. We start this section studying the following question: Is it true that any poset of dimension two and three is the separability order of sets of boxes in $\mathbb{R}^3$? We construct partial orders of dimension two and three with box separability dimension larger than three. A well-known fact in the study of order dimension of posets is that if the dimension of a poset is at least two, then substituting each element by an antichain does not affect its dimension, regardless of the size of the antichains. In Fig. 4a we substituted each element of a poset by an antichain of size 3. This observation will be key in the two following results.

Theorem 6.1 *There are partial orders of dimension* 2 *that cannot be realized as box separability in* $\mathbb{R}^3$.

Proof Consider the partial order $P(X, \prec)$ with five elements shown in Fig. 4a and replace each element for an antichain with k elements for some large k (this will be made more precise in a moment), see Fig. 4b. Suppose that the resulting poset, which is of dimension 2, is realizable as box separability in $\mathbb{R}^3$.

Take any two of the antichains such that all elements in one of them cover all elements in the other and project all the corresponding boxes onto the xy-plane. Consider a graph which has a vertex for each of these projections and an edge between two vertices if at least one of corners of the two corresponding projections lies inside the other projection. Since no two elements of an antichain are comparable, the graph is bipartite and, because no corner lies in two boxes of the opposite antichain, there are no more than $4k$ edges. Thus, for any integer l, if we take k to be large enough, we can choose $2l$ vertices, l on each antichain, so that no two of them are adjacent. From now on we will ignore the boxes of these antichains that do not belong to this complete bipartite graph.

By taking a large enough k, we may repeat this process for each such pair of antichains to obtain a collection of boxes whose separability order is the result of substituting each element of $P(X, \prec)$ for an antichain of size 2 and, furthermore, no projection of one of these boxes contains a corner of the projection of another box. Observe that the two boxes in an antichain, because they both lie above or below another box (from the antichain directly

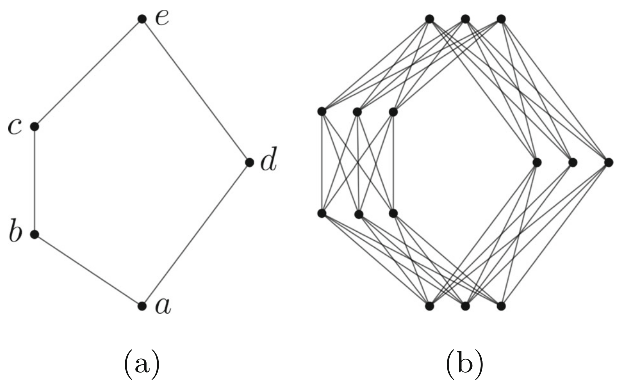

Fig. 4 **a** A partial $P(X, \prec)$ order on five elements **b** The result of substituting every element of $P(X, \prec)$ by an antichain of size 3

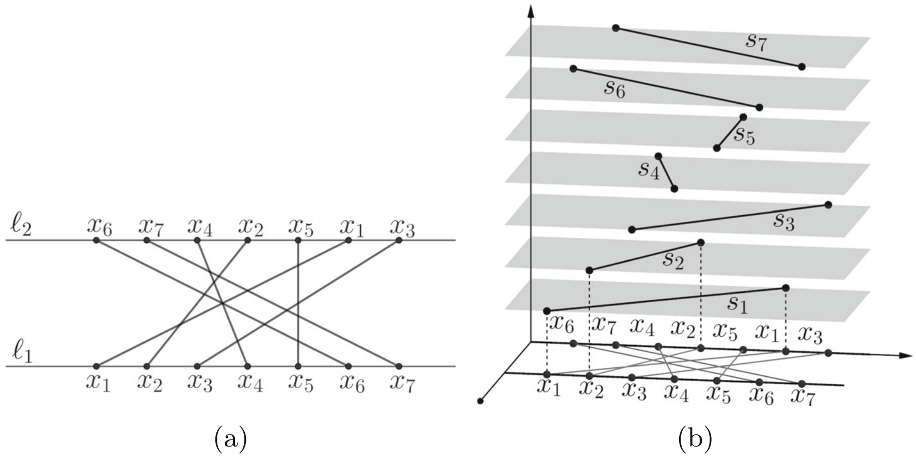

Fig. 5 **a** Representation of partial order $P(X, \prec)$ realized by the linear extensions $L_1 = \{x_1 < \ldots < x_7\}$ and $L_2 = \{x_3 < x_1 < x_5 < x_2 < x_4 < x_7 < x_6\}$, in which two segments intersect if and only if the elements represented by their corresponding endpoints are comparable. **b** Realization of $P(X, \prec)$ as separability of segments obtained by lifting the segments according to L_1

above or below), have projections either to the x-axis or the y-axis which overlap, and may be labeled as type x or y accordingly. The property that no corner lies inside another box is crucial to the previous observation. Moreover, the antichain directly above or below must be labeled oppositely, inducing a two coloring of the cover graph. This, however, leads to a contradiction as the cover graph is not bipartite. This completes the proof. □

The previous proof relies heavily on the fact that $P(X, \prec)$ is not bipartite (indeed, by Theorem 4.6, and Lemma 4.1 all bipartite posets of dimension two can be realized as box separability in $\mathbb{R}^3$). For partial orders of dimension three, we can construct a bipartite example.

Theorem 6.2 *There are bipartite partial orders of dimension* 3 *which cannot be realized as box separability or pseudosegment*[2] *separability in* $\mathbb{R}^3$.

Proof Chaplick et al. [7] constructed a finite bipartite poset of dimension 3 whose comparability graph is not the intersection graph of family of pseudosegments in $\mathbb{R}^2$. It was shown in [21] that a bipartite graph has boxicity two if and only if it is the intersection graph of a collection of vertical and horizontal segments in the plane. The result follows from Corollary 4.1. □

An alternate proof, not using the result from [21], can be obtained by of applying the technique from Theorem 6.1 to the partial order $P(X, \prec)$ mentioned above.

In some sense, Theorems 6.1 and 6.2 justify looking for other ways to represent low dimension orders in $\mathbb{R}^3$.

Theorem 6.3 *Every poset of dimension* 2 *can be realized as separability of segments in* $\mathbb{R}^3$.

Proof Let $P(X, \prec)$ be a partial order of dimension 2 and $\{L_1, L_2\}$ a realizer of $P(X, \prec)$. Consider two distinct horizontal lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$. Choose n points on ℓ_1 which represent the linear order L_1 from left to right, and n points on ℓ_2 which represent L_2, but now from

[2] A collection of pseudosegments consists of some curves, such that any two of them have at most one point in common.

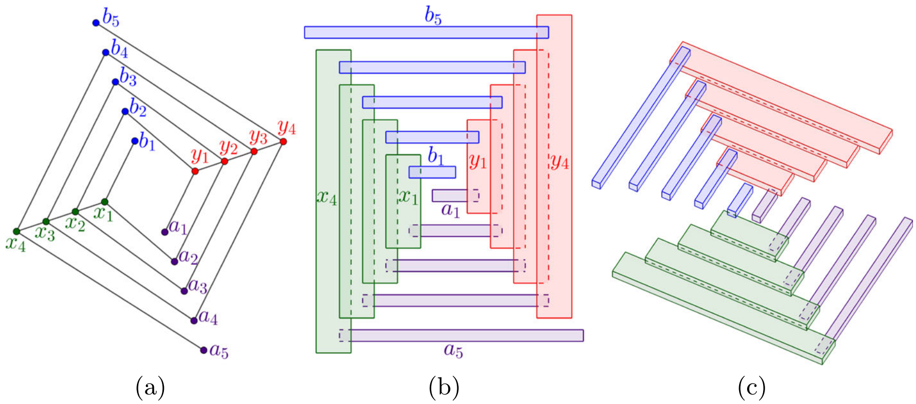

Fig. 6 **a** Kelly's construction for $d = 5$, $\mathcal{K}_5$. The set of maximal elements is $\{b_1, \ldots, b_5\}$, the set of minimal elements is $\{a_1, \ldots, a_5\}$, there is a set $\{x_1, \ldots, x_4\}$ such that $a_{i+1} \prec x_i \prec b_i$, $1 \le i \le 4$, and a set $\{y_1, \ldots, y_4\}$ such that $a_i \prec y_i \prec b_{i+1}$, $1 \le i \le 4$. **b** A set of rectangles whose intersection graph is the covering graph of $\mathcal{K}_5$. **c** Representation of $\mathcal{K}_5$ as separability of boxes in $\mathbb{R}^3$

right to left. For each $x_i \in X$, let s_i be the segment connecting the points that represent x_i in ℓ_1 and ℓ_2. Notice that two segments s_i and s_j cross each other if and only if x_i and x_j are comparable in $P(X, \prec)$. An example of this is shown in Fig. 5(a). Now Lemma 4.1 yields a realization of $P(X, \prec)$ as separability of segments. See Fig. 5(b). □

Theorem 6.4 *There is a partial order of dimension* 4 *that cannot be realized as separability of convex sets in* $\mathbb{R}^3$.

Proof As we had already noted, the partial order depicted in Fig. 2b cannot be realized as separability of connected sets in $\mathbb{R}^3$. This order has dimension 4 (see [22, 23], for example), which implies the result. □

Finally, we show the existence of some partial orders of arbitrary dimension which can be realized as box separability in $\mathbb{R}^3$.

Theorem 6.5 *For any positive integer d there is a partial order of dimension d which can be realized as box separability in* $\mathbb{R}^3$*, namely, the Kelly poset of dimension d (see* [5]*).*

Proof As in the proof of Theorem 6.3, we shall use the construction from Lemma 4.1. For any $d \ge 3$ Kelly [5] obtained a poset $\mathcal{K}_d$ with a planar covering graph such that its dimension is d. In Fig. 6(a) we show $\mathcal{K}_5$. In Fig. 6(b) we show a set of rectangles whose intersection graph is the covering graph of $\mathcal{K}_5$, and in Fig. 6(c) we show a set of boxes in $\mathbb{R}^3$ obtained from the rectangles in Fig. 6(b) whose separability order is $\mathcal{K}_5$. Our construction generalizes for $d \ge 3$. □

7 Conclusions

We have obtained several results showcasing a close relation between poset dimension and box separability dimension. In particular, Theorem 4.5 gives an upper bound on order dimension in terms of $Bsep$. Theorem 4.1 shows that every poset can be realized as separability of convex objects in $\mathbb{R}^4$, and by Theorem 6.3, orders of dimension two can be represented in $\mathbb{R}^3$ using segments. By Theorem 6.4, not all posets of dimension 4 can be realized as box separability in $\mathbb{R}^3$, it may be interesting to determine whether all posets of dimension 3 can be realized in this way.

We have also seen that, even though there are posets of dimension two that cannot be realized as box separability in $\mathbb{R}^3$, all posets of dimension at most 3 have a subdivision with $Bsep$ at most 3. A similar construction shows that, actually, all posets have a subdivision that can be realized as box separability in $\mathbb{R}^4$. During the final stages of publication, we were also able to drastically strengthen Theorem 6.1 by showing that, for every $d > 2$, there are partial orders of dimension d which have arbitrarily large box separability dimension. The proof of this result combines some of the ideas in the proof of Theorem 6.1 with the main result from [32], which implies the existence of partial orders of dimension two whose covering graphs have arbitrarily large chromatic numbers.

Regarding the complexity of finding the boxicity of a graph, it was shown by Kratochvíl in [24] that determining whether a graph has boxicity $k = 2$ is NP-hard, actually, it can be seen from the proof that this is true even for bipartite graphs. Since all bipartite graphs are comparability graphs, Corollary 4.1 implies that determining if a poset has box separability dimension 3 is NP-hard as well.

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References

1. Sinden, F.W.: Topology of thin film rc circuits. Bell Syst. Tech. J. **45**(9), 1639–1662 (1966)
2. Ehrlich, G., Even, S., Tarjan, R.E.: Intersection graphs of curves in the plane. Journal of Combinatorial Theory, Series B **21**(1), 8–20 (1976)
3. Tietze, H.: Über das problem der nachbargebiete im raum. Monatshefte für Mathematik und Physik **16**(1), 211–216 (1905)

4. Roberts, F.S.: On the boxicity and cubicity of a graph. Recent progress in combinatorics **1**, 301–310 (1969)
5. Kelly, D.: On the dimension of partially ordered sets. Discret. Math. **35**(1–3), 135–156 (1981)
6. Dawson, R.: On removing a ball without disturbing the others. Math. Mag. **57**(1), 27–30 (1984)
7. Chaplick, S., Felsner, S., Hoffmann, U., Wiechert, V.: Grid intersection graphs and order dimension. Order **35**(2), 363–391 (2018)
8. Al-Thukair, F., Pelc, A., Rival, I., Urrutia, J.: Motion planning, two-directional point representation, and ordered sets. SIAM J. Discret. Math. **4**(2), 151–163 (1991)
9. Rival, I., Urrutia, J.: Representing orders on the plane by translating convex figures. Order **4**(4), 319–339 (1988)
10. Nowakowski, R., Rival, I., Urrutia, J.: Representing orders on the plane by translating points and lines. Discret. Appl. Math. **27**(1–2), 147–156 (1990)
11. Guibas, L.J., Yao, F.F.: On translating a set of rectangles. In: Proceedings of the Twelfth Annual ACM Symposium on Theory of Computing, pp. 154–160 (1980)
12. Toussaint, G.T.: Movable separability of sets. Mach. Intelligence Pattern Recogn. **2**, 335–375 (1985)
13. Viennot, X.: Problèmes combinatoires posés par la physique statistique. Séminaire Bourbaki no **626**, 121–122 (1985)
14. Díaz-Báñez, J.M., Heredia, M.A., Peláez, C., Sellarès, J.A., Urrutia, J., Ventura, I.: Convex blocking and partial orders on the plane. Comput. Geom. **51**, 55–66 (2016)
15. Hiraguchi, T.: On the dimension of partially ordered sets. The science reports of the Kanazawa University **1**(2), 77–94 (1951)
16. Alon, N., Scheinerman, E.R.: Degrees of freedom versus dimension for containment orders. Order **5**(1), 11–16 (1988)
17. Adiga, A., Bhowmick, D., Chandran, L.S.: Boxicity and poset dimension. SIAM J. Discret. Math. **25**(4), 1687–1698 (2011)
18. Trotter, W.T., Jr.: A characterization of robert's inequality for boxicity. Discret. Math. **28**(3), 303–313 (1979)
19. Spinrad, J.P.: Edge subdivision and dimension. Order **5**(2), 143–147 (1988)
20. Francis, M.C., Mathew, R.: Boxicity of leaf powers. Graphs and Combinatorics **27**(1), 61–72 (2011)
21. Bellantoni, S.J., Hartman, I.B.-A., Przytycka, T.M., Whitesides, S.: Grid intersection graphs and boxicity. Discret. Math. **114**, 41–49 (1993)
22. Babai, L., Duffus, D.: Dimension and automorphism groups of lattices. Algebra Universalis **12**, 279–289 (1981)
23. Felsner, S., Trotter, W.T.: Posets and planar graphs. Journal of Graph Theory **49**(4), 273–284 (2005)
24. Kratochvíl, J.: A special planar satisfiability problem and a consequence of its NP-completeness. Discret. Appl. Math. **52**(3), 233–252 (1994)
25. Chandran, L.S., Francis, M.C., Sivadasan, N.: Geometric representation of graphs in low dimension using axis parallel boxes. Algorithmica **56**, 129–140 (2008)
26. Dilworth, R.P.: A decomposition theorem for partially ordered sets. In: Classic Papers in Combinatorics, pp. 139–144. Springer, Boston, MA (2009)
27. Fishburn, P.C., Trotter, W.T.: Geometric containment orders: A survey. Order **15**(2), 167–182 (1998)
28. Lee, J.G., Liu, W.-P., Nowakowski, R., Rival, I.: Dimension invariance of subdivisions. Bull. Aust. Math. Soc. **63**(1), 141–150 (2001)
29. Nowakowski, R., Rival, I., Urrutia, J.: Lattices contained in planar orders are planar. Algebra Universalis **29**(4), 580–588 (1992)
30. Schnyder, W.: Planar graphs and poset dimension. Order **5**(4), 323–343 (1989)
31. Trotter, W.T., Moore, J.I., Sumner, D.P.: The dimension of a comparability graph. Proceedings of the American Mathematical Society **60**(1), 35–38 (1976)
32. Kříž, I., Nešetřil, J.: Chromatic number of Hasse diagrams, eyebrows and dimension. Order **8**, 41–48 (1991). Springer

Authors and Affiliations

José-Miguel Díaz-Báñez[1] · Paul Horn[2] · Mario A. Lopez[3] · Nestaly Marín[4] · Adriana Ramírez-Vigueras[5] · Oriol Solé-Pi[6] · Alex Stevens[2] · Jorge Urrutia[5]

José-Miguel Díaz-Báñez
dbanez@us.es

Paul Horn
paul.horn@du.edu

Mario A. Lopez
mario.lopez@du.edu

Nestaly Marín
nestaly@ciencias.unam.mx

Adriana Ramírez-Vigueras
adriana.rv@im.unam.mx

Alex Stevens
alex.stevens@du.edu

Jorge Urrutia
urrutia@matem.unam.mx

1 Departamento de Matématica Aplicada II, Universidad de Sevilla, Seville, Spain

2 Department of Mathematics, University of Denver, Denver, CO, USA

3 Department of Computer Science, University of Denver, Denver, CO, USA

4 Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México, Mexico, Mexico

5 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico, Mexico

6 Facultad de Ciencias, Universidad Nacional Autónoma de México, Mexico, Mexico