Mixed norm spaces and $RM(p, q)$ spaces

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Abstract

In this paper we present the containment relationship between the spaces of analytic functions with average radial integrability $RM(p, q)$ and a family of mixed norm spaces.

1 Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathcal{H}(\mathbb{D})$ be the collection of the analytic functions in $\mathbb{D}$. A good number of meaningful spaces of analytic functions in $\mathbb{D}$, such as the classical Hardy spaces, may be defined in terms of integral means or in terms of the integrability with respect to the Lebesgue area measure, maybe with a certain weight (see, i.e., [6, 9–11]). The most classical examples in this situation are the Hardy spaces but also the mixed norm spaces $H^{q,p}$ defined explicitly by Flett in [7, 8] and, nowadays, widely studied (see [11]). Let us recall that a holomorphic function f in the unit disc $\mathbb{D}$ belongs to $H^{q,p}$ if

$$\left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{p/q} \, dr \right)^{1/p} < +\infty. \tag{1.1}$$

In other less studied cases, the belonging to a Banach space of analytic functions is determined by the average radial integrability. Maybe the most well-known space

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in this situation is the spaces of bounded radial variation BRV, a topic that goes back to Zygmund and where many different authors have work (see, i.e., [4, 12, 14]). This space of analytic function with bounded radial variation consists of those holomorphic functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_\theta \int_0^1 |g'(te^{i\theta})| \, dt < +\infty.$$

It is clear that any function $g \in BRV$ induces a bounded integral operator $T_g(f)(z) = \int_0^z f(\xi)g'(\xi)d\xi$ on H^∞. It was conjectured in [3] that such a boundedness only holds if $g \in BRV$ (see also [5]). In [13], it was shown that there are functions $g \notin BRV$ such that T_g is bounded on H^∞, but in case g is univalent T_g is bounded on H^∞ if and only if $g \in BRV$.

Other different situation where the radial integrability plays an important role is in the Féjer-Riesz theorem which says that if f belongs to the Hardy space H^p then

$$\sup_\theta \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right) \leq \frac{1}{2} \|f\|_{H^p}^p.$$

Recently, a family of spaces of holomorphic functions in the unit disc with average radial integrability, denoted by $RM(p,q)$, has been studied in [1, 2]. These spaces are formed by the analytic functions such that

$$\left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q} < +\infty \tag{1.2}$$

for $0 < p, q < +\infty$. If either p or q is infinity, we change the integral by the essential supremum, respectively. This family contains the classical Hardy spaces H^q (when $p = +\infty$) and Bergman spaces A^p (when $p = q$).

Looking at (1.1) and (1.2), a natural question that has been raised is the containment relationship with the mixed norm spaces $H^{q,p}$ and $RM(p,q)$. Notice that by Fubini's theorem, it is clear that $RM(p,p) = H^{p,p}$ for all $1 < p < \infty$, and by Minkowski's inequality,

$$RM(p,q) \subset H^{q,p}, \quad \text{if } p > q,$$
$$H^{q,p} \subset RM(p,q), \quad \text{if } q > p. \tag{1.3}$$

Therefore, a natural question is whether or not both embeddings in (1.3) are strict. The main result in this paper give a positive answer to this question:

Theorem 1.1 *Let $1 < p, q < +\infty$.*

(a) *If $p > q$, then $RM(p,q) \subsetneq H^{q,p}$.*
(b) *If $q > p$, then $H^{q,p} \subsetneq RM(p,q)$.*

Throughout the paper the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2 First definitions

We start by recalling the definition of the spaces with average radial integrability $RM(p, q)$. These spaces are formed by those holomorphic functions in $\mathbb{D}$ such that taking the p-norm in every radius and then the q-norm, the result is a finite number. More precisely, we give the following definition.

Definition 2.1 Let $0 < p, q < +\infty$. We define the spaces of analytic functions

$$RM(p, q) = \left\{ f \in \mathcal{H}(\mathbb{D}) \ \rho_{p,q}(f) < +\infty \right\}$$

where

$$\rho_{p,q}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{it})|^p \, dr \right)^{q/p} dt \right)^{1/q}, \quad \text{if } p, q < +\infty.$$

Remark 2.2 Notice that the analyticity is not required to define the norm. So that, we will use this definition for measurable functions with the aim of simplifying the notation and to facilitate the reading.

It is easy to see that $RM(p, q)$ endowed with the norm $\rho_{p,q}$ is a Banach space whenever $p, q \geq 1$.

One example of functions in $RM(p, q)$, that we will play an important role in the main result of this work, is the following family of functions.

Proposition 2.3 Let $0 < p, q \leq +\infty$. Let $\alpha \in \mathbb{D}$ and $\beta > \frac{1}{p} + \frac{1}{q}$, then there are constants $C_j = C_j(p, q, \beta)$, $j = 1, 2$, such that

$$C_1 (1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta} \leq \rho_{p,q}((1 - \overline{\alpha} z)^{-\beta}) \leq C_2 (1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta},$$

where we are using the main branch of the logarithm to define $w^{-\beta}$.

Proof Let $0 < p, q < +\infty$. We may assume without loss of generality that $1/2 \leq \alpha < 1$. Let us estimate the quantity $|1 - \alpha r e^{i\theta}|^2$ for points around 1. If $0 < \theta < 1 - \alpha$ and $1/2 < r < 1$, then $|1 - \alpha r e^{i\theta}|^2 \asymp (1 - r\alpha)^2$. If $1/2 > \theta > 1 - \alpha$, then

$$|1 - \alpha r e^{i\theta}|^2 \asymp \begin{cases} \theta^2, & 1 > r \geq \frac{1-\theta}{\alpha}, \\ (1 - r\alpha)^2, & 1/2 < r \leq \frac{1-\theta}{\alpha}. \end{cases}$$

First of all, let us see that $\rho_{p,q}((1 - \overline{\alpha}z)^{-\beta}) \lesssim (1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta}$ if $\beta > \frac{1}{p} + \frac{1}{q}$. Using the symmetry in θ and the monotonicity in θ and r, we have

$$\int_0^{2\pi} \left(\int_0^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta \leq 2^{2 + q/p} \pi \int_0^{1/2} \left(\int_{1/2}^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta$$

Therefore,

$$\int_0^{2\pi} \left(\int_0^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta$$

$$\leq 2^{2 + \frac{q}{p}} \pi \left(\int_0^{1-\alpha} \left(\int_{1/2}^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta + \int_{1-\alpha}^{1/2} \left(\int_{1/2}^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta \right)$$

$$\leq 2^{2 + \frac{q}{p}} \pi \int_{1-\alpha}^{1/2} \left(\int_{\frac{1-\theta}{\alpha}}^1 \frac{dr}{\theta^{\beta p}} + \int_{1/2}^{\frac{1-\theta}{\alpha}} \frac{dr}{(1 - r\alpha)^{\beta p}} \right)^{q/p} d\theta + \int_0^{1-\alpha} \left(\int_{1/2}^1 \frac{dr}{(1 - r\alpha)^{\beta p}} \right)^{q/p} d\theta$$

$$\leq 2^{2 + \frac{q}{p}} \pi \int_{1-\alpha}^{1/2} \left(\frac{\theta - (1 - \alpha)}{\alpha \theta^{\beta p}} + \frac{1}{\alpha(\beta p - 1)} \left(\frac{1}{\theta^{\beta p - 1}} - \frac{1}{(1 - \alpha/2)^{\beta p - 1}} \right) \right)^{q/p} d\theta$$

$$+ 2^{2 + \frac{q}{p}} \pi \int_0^{1-\alpha} \left(\frac{1}{\alpha(\beta p - 1)} \left(\frac{1}{(1 - \alpha)^{\beta p - 1}} - \frac{1}{(1 - \alpha/2)^{\beta p - 1}} \right) \right)^{q/p} d\theta$$

$$\leq 4\pi \left(\frac{2\beta p}{\alpha(\beta p - 1)} \right)^{q/p} \left(\int_{1-\alpha}^{1/2} \frac{1}{\theta^{\beta q - q/p}} d\theta + \frac{1}{(1 - \alpha)^{\beta q - q/p - 1}} \right)$$

$$\leq 4\pi \left(\frac{2\beta p}{\alpha(\beta p - 1)} \right)^{q/p} \left(\frac{\beta q - q/p}{\beta q - q/p - 1} \right) (1 - \alpha)^{1 + q/p - \beta q}.$$

Now, we will show that $\rho_{p,q}((1 - \overline{\alpha}z)^{-\beta}) \gtrsim (1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta}$ if $\beta > \frac{1}{p} + \frac{1}{q}$. Since for $0 < \theta < 1 - \alpha$ and $0 \leq r < 1$ one have that $|1 - \alpha re^{i\theta}| \leq \sqrt{2}(1 - \alpha r)$, we have

$$\int_0^{2\pi} \left(\int_0^1 \frac{dr}{|1 - \alpha re^{i\theta}|^{\beta p}} \right)^{q/p} d\theta \geq \frac{1}{(\sqrt{2})^{\beta q}} \int_0^{1-\alpha} \left(\int_0^1 \frac{dr}{(1 - \alpha r)^{\beta p}} \right)^{q/p} d\theta$$

$$\geq \frac{1}{(\sqrt{2})^{\beta q}(\beta p - 1)^{q/p}\alpha^{q/p}} \int_0^{1-\alpha} \left(\frac{1}{(1 - \alpha)^{\beta p - 1}} - 1 \right)^{q/p} d\theta$$

$$\geq \left(\frac{1 - (1/2)^{\beta p - 1}}{(\sqrt{2})^{\beta p}(\beta p - 1)\alpha} \right)^{q/p} (1 - \alpha)^{1 + q/p - \beta q}.$$

If p or q is ∞, the proof follows similarly. □

Now, we continue with the definition of a particular case of the mixed normed spaces. Bearing in mind the $RM(p, q)$ spaces, they are defined interchanging the order of integration, that is, they are formed by those holomorphic functions in $\mathbb{D}$ such that taking the q-norm in every circle and then the p-norm, the result is a finite number. More precisely,

Definition 2.4 Let $0 < p, q < +\infty$. We define the mixed norm spaces

$$H^{q,p} = \{f \in \mathcal{H}(\mathbb{D}) \;:\; \|f\|_{H^{q,p}} < +\infty\}$$

where

$$\|f\|_{H^{q,p}} = \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p}.$$

Remark 2.5 As in the definition of the $RM(p, q)$ spaces, the analyticity is not required to define the norm. Thus, we will use this definition for measurable functions with the aim of simplifying the notation and to facilitate the reading.

By Fubini's theorem, it is clear that

$$\|f\|_{H^{p,p}} = \rho_{p,p}(f).$$

Thus, it is a natural question to analyse what happens when $p \neq q$. This is the aim of this work.

Along the same lines as for the $RM(p, q)$ spaces, we have one example of functions that belong to these mixed norm spaces is the following family of analytic functions.

Proposition 2.6 Let $0 < p, q < +\infty$. Let $\alpha \in \mathbb{D}$ and $\beta > \frac{1}{p} + \frac{1}{q}$, then there are constants $C_j = C_j(p, q, \beta)$, $j = 1, 2$, such that

$$C_1(1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta} \leq \|(1 - \overline{\alpha}z)^{-\beta}\|_{H^{q,p}} \leq C_2(1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta},$$

where we are using the main branch of the logarithm to define $w^{-\beta}$.

3 Containment relationships

We proceed with the presentation of the containment relationship between the $RM(p, q)$ spaces and a particular case of the mixed norm spaces, which we have defined above.

Bearing in mind the standard argument of [1, Corollary 4.8, p. 29], one can prove an analogous duality result for the mixed norm spaces by means of the boundedness of the weighted Bergman projection:

Proposition 3.1 [11, Corollary 7.3.4, p. 153] *Let* $1 \leq p, q < +\infty$ *and* $1/p < \gamma + 1$. *Then the operator*

$$P_\gamma f(z) = (\gamma + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\gamma f(w)}{(1 - z\overline{w})^{2+\gamma}} \, dA(w)$$

is a bounded projection mapping $L^{q,p}$ onto $H^{q,p}$, where $L^{q,p}$ is the corresponding space of equivalence classes of measurable functions. In particular ($\gamma = 0$), we have the Bergman projection P maps $L^{q,p}$ onto $H^{q,p}$, when $1 < p < +\infty$.

The proof of the next result follows the same scheme as for the spaces $RM(p, q)$ [1, Corollary 4.8, p. 29].

Proposition 3.2 *Let $1 < p, q < +\infty$. Then $(H^{q,p})^* \cong H^{q',p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The antiisomorphism between $H^{q',p'}$ and $(H^{q,p})^*$ is given by the operator*

$$g \mapsto \lambda_g$$

where λ_g is defined by

$$\lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z), \quad f \in H^{q,p}.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 *(a)* If $p \geq q$ then $RM(p, q) \subset H^{q,p}$, because using Minkowski's integral inequality we have

$$\left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{p/q} dr \right)^{1/p} \leq \left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q}.$$

Let us see that $RM(p, q) \subsetneq H^{q,p}$ if $p > q$.
Consider the functions

$$u_\delta(z) = \frac{\delta}{(1 + \delta - z)^{1 + \frac{1}{p} + \frac{1}{q}}}, \quad z \in \mathbb{D},$$

where $0 < \delta < 1/2$. One can see that $\|u_\delta\|_{H^{q,p}} \asymp \rho_{p,q}(u_\delta) \asymp 1$, because, for $\alpha \in \mathbb{D}$,

$$\|(1 - \overline{\alpha}z)^{-1 - \frac{1}{p} - \frac{1}{q}}\|_{H^{q,p}} \asymp (1 - |\alpha|)^{-1}$$

(see Proposition 2.6) and

$$\rho_{p,q}\left((1 - \overline{\alpha}z)^{-1 - \frac{1}{p} - \frac{1}{q}} \right) \asymp (1 - |\alpha|)^{-1}$$

(see Proposition 2.3).
Let $\{\delta_n\}$ be a sequence of positive numbers such that $n^2\delta_n < \frac{1}{4}$ for all $n \geq 1$, $\delta_n/n^{2p} > (n + 1)\delta_{n+1}^{1/2}$ for all $n \geq 1$, and $\sum_{j=1}^{\infty} j^2\delta_j < 2$. Define the sets

$$A_n := \left\{ re^{i\theta} : |\theta - \theta_n| \leq n^2\delta_n \ r \in [1 - n\delta_n^{1/2}, 1 - \delta_n/n^{2p}] \right\},$$

where $\theta_1 = \delta_1$ and $\theta_n - \theta_{n-1} = \frac{1}{n^2} + (n-1)^2\delta_{n-1} + n^2\delta_n$, for $n \geq 2$. Observe that $A_n = \{re^{i\theta} \ r \in I_n, \ \theta \in J_n\}$ where the sets $\{I_n\}$ are pairwise disjoint and so are the sets $\{J_n\}$.

Let us check that $\rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)) \lesssim \frac{1}{n^2}$. Firstly, notice that

$$\rho_{p,q}\left(u_{\delta_n}(ze^{-i\theta_n})\chi_{\{0<|w|<1-n\delta_n^{1/2}\}}(z)\right)$$

$$= \left(\int_0^{2\pi}\left(\int_0^{1-n\delta_n^{1/2}}\frac{\delta_n^p}{|1+\delta_n-re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}$$

$$\leq \left(\int_0^{1-n\delta_n^{1/2}}\frac{\delta_n^p}{(1+\delta_n-r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right)^{1/p} \leq \frac{\delta_n}{(n\delta_n^{1/2}+\delta_n)^{1+\frac{1}{q}}}$$

$$\leq \frac{\delta_n}{(n\delta_n^{1/2})^{1+\frac{1}{q}}} \leq \frac{1}{n^2}. \tag{3.1}$$

In the next inequalities we use that, for $|\theta| \leq 1$ and $r \geq 1 - n\delta_n^{1/2}$

$$|1+\delta_n-re^{i\theta}|^2 = (1+\delta_n-r)^2 + 2r(1+\delta_n)(1-\cos(\theta)) \geq (1+\delta_n-r)^2 + \frac{r\theta^2}{2}$$

$$\geq (1+\delta_n-r)^2 + (1-n\delta_n^{1/2})\frac{\theta^2}{2} \geq (1+\delta_n-r)^2 + \frac{\theta^2}{4}.$$

Thus, it follows

$$\rho_{p,q}^q\left(u_{\delta_n}(ze^{-i\theta_n})\chi_{\{re^{i\theta}\,:\,1-n\delta_n^{1/2}<r<1,\,|\theta-\theta_n|>n^2\delta_n\}}(z)\right)$$

$$= 2\int_{n^2\delta_n}^{\pi}\left(\int_{1-n\delta_n^{1/2}}^1\frac{\delta_n^p}{|1+\delta_n-re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right)^{q/p}\frac{d\theta}{2\pi}$$

$$\leq 2\left(\frac{\pi-n^2\delta_n}{1-n^2\delta_n}\right)\int_{n^2\delta_n}^1\left(\int_{1-n\delta_n^{1/2}}^1\frac{\delta_n^p}{|1+\delta_n-re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right)^{q/p}\frac{d\theta}{2\pi},$$

since the inner integral is a decreasing function in θ. Therefore,

$$\rho_{p,q}^q\left(u_{\delta_n}(ze^{-i\theta_n})\chi_{\{re^{i\theta}\,:\,1-n\delta_n^{1/2}<r<1,\,|\theta-\theta_n|>n^2\delta_n\}}(z)\right)$$

$$\leq 2^{3q+1}\int_{n^2\delta_n}^{n^2\delta_n+n\delta_n^{1/2}}\left(\int_{1-n\delta_n^{1/2}}^{1+\delta_n-\theta}\frac{\delta_n^p}{(1+\delta_n-r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right.$$

$$+ \left.\int_{1+\delta_n-\theta}^1\frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\,dr\right)^{q/p}d\theta$$

$$+ 2^{3q+1} \int_{n^2\delta_n+n\delta_n^{1/2}}^{1} \left(\int_{1-n\delta_n^{1/2}}^{1} \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, dr \right)^{q/p} d\theta$$

$$\leq 2^{3q+1} \int_{n^2\delta_n}^{n^2\delta_n+n\delta_n^{1/2}} \left(\frac{1}{p\left(1+\frac{1}{q}\right)} \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{q}\right)}} + \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{q}\right)}} \right)^{q/p} d\theta$$

$$+ 2^{3q+1} \int_{n^2\delta_n+n\delta_n^{1/2}}^{1} \frac{\delta_n^{q+\frac{q}{2p}} n^{q/p}}{\theta^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, d\theta$$

$$\leq 2^{3q+1} 2^{q/p}\delta_n^q \int_{n^2\delta_n}^{n^2\delta_n+n\delta_n^{1/2}} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}} + 2^{3q+1}\delta_n^q \int_{n^2\delta_n+n\delta_n^{1/2}}^{1} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}} \, d\theta$$

$$\leq 2^{3q+1+\frac{q}{p}}\delta_n^q \int_{n^2\delta_n}^{1} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}} \, d\theta \leq 2^{3q+1+\frac{q}{p}} \frac{\delta_n^q}{(n^2\delta_n)^q} \leq 2^{3q+2}\frac{1}{n^{2q}}. \tag{3.2}$$

Similarly, we obtain

$$\rho_{p,q}^q \left(u_{\delta_n}(ze^{-i\theta_n}) \chi_{\{re^{i\theta} \,:\, 1-\frac{\delta_n}{n^{2p}}<r<1,\ |\theta-\theta_n|<n^2\delta_n\}}(z) \right)$$

$$= 2\int_0^{n^2\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{|1+\delta_n-re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, dr \right)^{q/p} \frac{d\theta}{2\pi}$$

$$\leq 2\int_0^{\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{(1+\delta_n-r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, dr \right)^{q/p} \frac{d\theta}{2\pi}$$

$$+ 2^{3q+1} \int_{\delta_n}^{n^2\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, dr \right)^{q/p} \frac{d\theta}{2\pi}$$

$$\leq 2\delta_n^q \left(\frac{\delta_n}{n^{2p}} \right)^{q/p} \int_0^{\delta_n} \frac{1}{\delta_n^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \frac{d\theta}{2\pi} + 2^{3q+1}\delta_n^q \left(\frac{\delta_n}{n^{2p}} \right)^{q/p} \int_{\delta_n}^{n^2\delta_n} \frac{1}{\theta^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \frac{d\theta}{2\pi}.$$

$$\leq \frac{8^{q+1}}{2\pi}\delta_n^q \left(\frac{\delta_n}{n^{2p}} \right)^{q/p} \frac{1}{\delta_n^{q\left(1+\frac{1}{p}\right)}} = \frac{8^{q+1}}{2\pi}\frac{1}{n^{2q}}. \tag{3.3}$$

Using inequalities (3.1), (3.2), and (3.3) we deduce

$$\|u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)\|_{H^{q,p}} \leq \rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)) \lesssim \frac{1}{n^2}.$$

Now, by the very definition of A_n, the sets $\{A_n\}$ are pairwise disjoint. Define the functions $g_n(z) = u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)$ and $f_n(z) = u_{\delta_n}(ze^{-i\theta_n})\chi_{A_n}(z)$ such that $u_{\delta_n}(ze^{-i\theta_n}) = f_n(z) + g_n(z)$. As we have seen, we have that $\rho_{p,q}(g_n) \lesssim \frac{1}{n^2}$ and

$\|g_n\|_{H^{q,p}} \lesssim \frac{1}{n^2}$ for $n \in \mathbb{N}$. In addition, one can see that $\rho_{p,q}(f_n) \asymp \|f_n\|_{H^{q,p}} \asymp 1$ because $\rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})) \asymp \|u_{\delta_n}(ze^{-i\theta_n})\|_{H^{q,p}} \asymp 1$.

Given a measure space $(\Omega, \mathcal{A}, \mu)$. We have that for any sequence of measurable functions $h_n : \Omega \to \mathbb{C}$ whose supports are pairwise disjoint, it holds that

$$\int_{\Omega} \left| \sum_n h_n(w) \right|^s d\mu(w) = \sum_n \int_{\Omega} |h_n(w)|^s \, d\mu(w)$$

for all $s > 0$. Then, using this fact twice (one in each variable, first with $s = q$ and then with $s = p/q$), we obtain

$$\left\| \sum \alpha_n f_n \right\|_{H^{q,p}} = \left(\int_0^1 \sum |\alpha_n|^p \left(\int_0^{2\pi} \left| f_n(re^{i\theta}) \right|^q \frac{d\theta}{2\pi} \right)^{p/q} dr \right)^{1/p}$$

$$= \left(\sum |\alpha_n|^p \|f_n\|_{H^{q,p}}^p \right)^{1/p} \asymp \left(\sum |\alpha_n|^p \right)^{1/p}.$$

By the same reason,

$$\rho_{p,q}\left(\sum \alpha_n f_n \right) = \left(\int_0^{2\pi} \sum |\alpha_n|^q \left(\int_0^1 \left| f_n(re^{i\theta}) \right|^p dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q}$$

$$= \left(\sum |\alpha_n|^q \rho_{p,q}^q(f_n) \right)^{1/q} \asymp \left(\sum |\alpha_n|^q \right)^{1/q}.$$

Hence, if we consider the function $F_m(z) := \sum_{n=1}^m u_{\delta_n}(ze^{-i\theta_n})$ we obtain that

$$\rho_{p,q}(F_m) \le \rho_{p,q}\left(\sum_{n=1}^m f_n \right) + \rho_{p,q}\left(\sum_{n=1}^m g_n \right) \lesssim m^{1/q} + \sum_{n=1}^m \frac{1}{n^2}$$

$$\le m^{1/q} + \frac{\pi^2}{6} \le \left(1 + \frac{\pi^2}{6} \right) m^{1/q}$$

and

$$\rho_{p,q}(F_m) \ge \rho_{p,q}\left(\sum_{n=1}^m f_n \right) - \rho_{p,q}\left(\sum_{n=1}^m g_n \right) \gtrsim m^{1/q}$$

for m big enough. So that $\rho_{p,q}(F_m) \asymp m^{1/q}$. In the same way for the norm in $H^{q,p}$, it follows that $\|F_m\|_{H^{q,p}} \asymp m^{1/p}$ using the same argument.

Therefore, if it were true that $RM(p, q) = H^{q,p}$, then we would have $\rho_{p,q}(F_m) \asymp \|F_m\|_{H^{q,p}}$. But if $p > q$ this is impossible, because this implies that $m^{1/p} \asymp m^{1/q}$ for all $m \in \mathbb{N}$. Thus, we conclude *(a)*.

(b) If $q \geq p$ then $H^{q,p} \subset RM(p,q)$, using Minkowski's integral inequality we have

$$\left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q} \leq \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{p/q} dr \right)^{1/p}.$$

Let us see that $H^{q,p} \neq RM(p,q)$ for $q > p$. Assume that $H^{q,p} = RM(p,q)$ then $(H^{q,p})^* = (RM(p,q))^*$. By [1, Corollary 4.8, p. 29] and Proposition 3.2, we have that $H^{q',p'} = RM(p',q')$ for $p' > q'$, where the identifications are given by the A^2-pairing

$$\langle f, g \rangle_{A^2} = \int_{\mathbb{D}} f(z) \overline{g(z)} \, dA(z).$$

But this contradicts *(a)*. So *(b)* holds and we are done. □

Declarations

References

1. Aguilar-Hernández, T., Contreras, M.D., Rodríguez-Piazza, L.: Average radial integrability spaces of analytic functions. J. Funct. Anal. **282**(1), 109262 (2022)
2. Aguilar-Hernández, T., Contreras, M.D., Rodríguez-Piazza, L.: Integration operators in average radial integrability spaces of analytic functions. Mediterr. J. Math. **18**, 117 (2021)
3. Anderson, A., Jovovic, M., Smith, W.: Some integral operators acting on H^∞. Integral Equ. Oper. Theory **80**, 275–291 (2014)
4. Bourgain, J.: On the radial variation of bounded analytic functions on the disc. Duke Math. J. **69**(3), 671–682 (1993)
5. Contreras, M.D., Peláez, J.A., Pommerenke, Ch., Rättyä, J.: Integral operators mapping into the space of bounded analytic functions. J. Funct. Anal. **271**, 2899–2943 (2016)
6. Duren, P.L.: Theory of H^p Spaces. Dover, New York (2000)
7. Flett, T.M.: The dual of an inequality of Hardy and Littlewood and some related inequalities. J. Math. Anal. Appl. **38**, 746–765 (1972)

8. Flett, T.M.: Lipschitz spaces of functions on the circle and the disk. J. Math. Anal. Appl. **39**, 125–158 (1972)
9. Garnett, J.: Bounded Analytic Functions. Graduate Texts in Mathematics, Springer-Verlag, Berlin (2007)
10. Hedenmalm, H., Korenblum, B., Zhu, K.: Theory of Bergman Spaces. Springer-Verlag, Berlin (2000)
11. Jevtić, M., Vukotić, D., Arsenović, M.: Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces. Springer-Verlag, Berlin (2016)
12. Rudin, W.: The radial variation of analytic functions. Duke Math. J. **22**, 235–242 (1955)
13. Smith, W., Stolyarov, D.M., Volberg, A.: Uniform approximation of Bloch functions and the boundedness of the integration operator on H^∞. Adv. Math. **314**, 185–202 (2017)
14. Zygmund, A.: On certain integrals. Trans. Am. Math. Soc. **55**, 170–204 (1944)