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Stability of insulating viscous jets under axial electric fields

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The stability spectrum for a jet of perfectly insulating, viscous liquid subjected to a longitudinal electric field is obtained for axisymmetric perturbations. Both dielectric forces at the interface and viscous dissipation produce lower growth factors for all the range of unstable wavelengths. If the imposed field is time varying, parametric resonances are possible but they are easily suppressed, even by a small viscosity, and the r.m.s. value of the dielectric pressure dominates over the pulsating part.

1 INTRODUCTION

The temporal instability of an infinitely extended column of viscous liquid in the absence of an electric field is known since the works of Rayleigh [1], Weber [2] and Chandrasekhar [3]. On the other hand the effect of a d.c. longitudinal electric field on a perfectly insulating, inviscid liquid column was studied by Nayyar and Murty [4]. In [5] a first extension of this work to viscous columns was presented. Here we give further results about their stability spectrum.

However, the validation of this analysis in the laboratory requires the use of a.c. fields. The reason is that perfectly insulating liquids do not exist in practice, for a residual conductivity is always observed and free charge is accumulated at the bulk and/or the free surface of the system. It is then customary the use of a.c. instead of d.c. fields. If the period of the imposed a.c. electric field is much shorter than the typical charge relaxation time, $\bar{\epsilon}/\sigma_c$, with $\bar{\epsilon}$ and σ_c the electrical permittivity and conductivity respectively, forces due to free charge accumulation are negligible.

When using a.c. fields, we open the door to the possibility of parametric resonances caused by the pulsating part of the dielectric forces acting on

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the free surface. This question is addressed by considering the effect of a.c. fields on slightly viscous jets.

2 D.C. ELECTRIC FIELDS: STABILITY ANALYSIS

For a nonconducting liquid jet, polarization forces act only at the free surface and perpendicularly to it. These two facts enable us to formulate and solve the electromechanical problem in two steps: (i) the purely hydrodynamic problem is considered, as done by the above-cited authors [1-3]; (ii) the dispersion relation, which is finally obtained from the Young–Laplace equation of normal stress balance at the free surface, is modified to account for an additional dielectric pressure term.

2.1 Hydrodynamic problem

We sketch a derivation of the dispersion relation of viscous jets (radius R, surface tension σ and dynamical viscosity μ). The linearized Navier-Stokes and continuity equations in non-dimensional form are

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + C\,\nabla^2 \mathbf{v}, \qquad \nabla \cdot \mathbf{v} = 0.$$

The scaling is based on the capillary forces: $\mathbf{r} \to \mathbf{r}/R$, $t \to t/t_c$ (with $t_c \equiv (\rho R^3/\sigma)^{1/2}$, the capillary time) and $p \to p/(\sigma/R)$. Here C, the Ohnesorge number (ratio of viscous to capillary forces), is defined as $C \equiv \mu R/\sigma t_c$. These equations are still valid for a nonzero electric field because no electric force acts in the bulk. Application of divergence and laplacian operators to the first equation eliminates the pressure field:

$$\nabla^2 \left(\nabla^2 - \frac{1}{C}\frac{\partial}{\partial t} \right) \mathbf{v} = 0.$$

This equation is solved through the decomposition $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, which yields

$$\nabla^2 \mathbf{v}_1 = 0, \quad \left(\nabla^2 - \frac{1}{C}\frac{\partial}{\partial t} \right) \mathbf{v}_2 = 0.$$

Hereafter, we will restrict ourselves to the axisymmetric case ($\mathbf{v} = u(r,z)\,\mathbf{e}_r + w(r,z)\,\mathbf{e}_z$). If the free surface is described by $F(r,z,t) \equiv r - f(z,t) = 0$, the appropriate boundary conditions are

- At the axis:

$$u(0,z) = 0, \qquad w(0,z) \text{ finite}.$$

- At the interface:

$$u(1,z) = \frac{\partial f}{\partial t}, \quad p = -f - \frac{\partial^2 f}{\partial z^2} + 2C\frac{\partial u}{\partial r}, \quad C\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = 0.$$

In the linear approximation, the problem is solved via the modal analysis $f(z,t) = \hat{f}\exp(\Omega t + ixz)$, leading to similar decompositions of the other involved quantities, $\{u,w,p\}(r,z,t) = \{\hat{u},\hat{w},\hat{p}\}(r)\exp(\Omega t + ixz)$. After some lengthy algebra we arrive to the well-known dispersion relation without electric contribution:

$$\frac{I_0(x)}{xI_1(x)}\Omega^2 + 4C\Omega\left[\frac{xI_0(x)}{I_1(x)} - \frac{1}{2}\right] + 4C^2x^2\left[x\frac{I_0(x)}{I_1(x)} - x_1\frac{I_0(x_1)}{I_1(x_1)}\right] = 1 - x^2, \quad (1)$$

where $x_1^2 = x^2 + \Omega/C$, and $I_n(x)$ are the modified Bessel functions of first kind and order n.

2.2 Electric problem

If the imposed field is nonzero we must evaluate the pressure due to polarization forces at the jet surface. The electroquasistatic field derives from a harmonic potential:

$$\mathbf{E} = -\nabla\phi, \qquad \nabla^2\phi = 0;$$

which are scaled with E_0, the electric field at infinity. Pertinent boundary conditions are (i) continuity of the potential across the free surface and (ii) continuity of the electric displacement vector, $\epsilon\mathbf{E}$. According to the above-stated modal decomposition, the perturbed electric potential is

$$\phi(r,z,t) = -z + \hat{\phi}(r)\exp(\Omega t + ixz),$$

and a solution for the function $\hat{\phi}(r)$ is readily found.

From the Maxwell stress tensor formalism, the electric pressure is

$$p_e = -\chi\,\Delta\left\{\epsilon\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{\partial f}{\partial z}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} - \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]\right\},$$

where Δ means jump across the free surface; ϵ is the dielectric constant relative to $\bar{\epsilon}_0$, the permittivity of the outer medium (i.e. $\epsilon = \epsilon_i$ inside the jet and $\epsilon = 1$ outside); and $\chi \equiv \bar{\epsilon}_0 E_0^2 R/\sigma$ is the electric Bond number. Once the solution for the potential is substituted, this expression leads to

$$p_e = \frac{\chi(\epsilon_i - 1)^2 x}{H_0(\epsilon_i, x)},$$

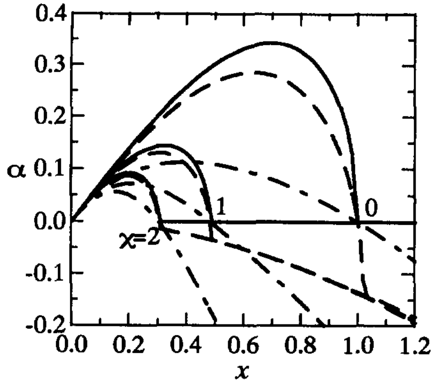

Figure 1. Growth factor versus wavenumber for a dielectric jet with $\epsilon_i = 3$ and different values of χ and C.

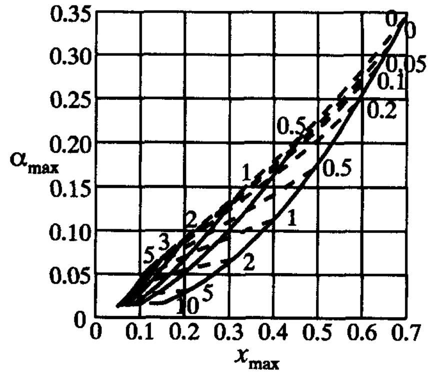

Figure 2. Maximum growth rates versus corresponding wavenumbers for constant values of χ (solid) and C (dashed lines), and $\epsilon_i = 3$. (Reprinted from [5])

where $H_0 \equiv \epsilon_i I_1(x)/I_0(x) + K_1(x)/K_0(x)$.

The dispersion relation including the electric contribution is obtained by substitution of the right-hand side of (1) by $1 - x^2 - p_e$.

2.3 Results

In general Ω is complex, $\Omega = \alpha + i\beta$. Temporal stability for a given set of parameters ϵ_i, χ, C, and a particular wavenumber x is determined by the sign of the growth factor α. The dispersion relation is analyzed using a root-finding routine in the complex plane. Figure 1 shows the wavenumber dependence of the growth factor α for some representative values of the electrical parameters ϵ_i and χ and the Ohnesorge number C. An increase in the field strength produces stabilization for all wavenumbers, which is observed as a reduction of the capillary unstable x-domain and growth rates α. Viscosity also produces stabilization, but only through a reduction of the growth rate. For each curve shown in figure 1, we find three different regimes as x increases: pure growth, pure damping, and finally, after the jump in slope, damped oscillations. Figure 2 gives a map of maximum growth factors α_{max} versus their associated wavenumbers x_{max} for a wide range of both electric and Ohnesorge numbers. Stabilization implies an increase in the size of drops after break-up, at least from estimations given by linear analysis.

3 A.C. ELECTRIC FIELDS: RESONANCES

If the imposed electric field is time varying, parametric resonances are possible. A linear analysis of this case from the dynamical formulation carried out in previous sections is very difficult for arbitrary viscosity. The reason is that an exponential time dependence of all magnitudes is assumed for the solution, which is not admissible when the electric field imposes its own time dependence $\cos(\bar{\omega}t)$. However, if viscosity is not very large we can obtain some conclusions about the importance of these resonances. The easiest way to deal with the a.c. case is via a lagrangian formulation of the dynamics of small perturbations in the amplitude of the free surface. This mathematical approach is commonly used since the works of lord Rayleigh and has been extended to low viscosity systems [6].

Perturbations of the jet are described by $r = r_S(z,t) = R_0(t) + \xi(t)\cos(kz)$. We have to construct the lagrangian ($\mathcal{L} = \mathcal{T} - \mathcal{U}_S + \mathcal{U}_E$) and dissipation ($\mathcal{R}$) functions, where $\mathcal{T}$, $\mathcal{U}_S$, and $\mathcal{U}_E$ are the kinetic, capillary, and electrostatic energies, respectively, associated with a perturbation of the jet shape. These quantities can be obtained from

$$\mathcal{T} = \frac{\rho}{2}\int_\tau v^2\,d\tau;\;\; \mathcal{U}_S = \sigma\int_S dS;\;\; \mathcal{U}_E = (\epsilon_i - 1)\frac{\bar{\epsilon}_0}{2}\int_\tau \mathbf{E}_1\cdot\mathbf{E}_0\,d\tau;\;\; \mathcal{R} = \frac{\mu}{2}\oint_S \nabla v^2\cdot d\mathbf{S}.$$

Here $\mathbf{E}_1$ is the electric field inside the jet and τ and S are jet volume and surface taken over a wavelength. Viscous effects are included in the dissipation function, the expression proposed being valid provided that the velocity field is assumed as potential. This velocity field is related to the surface perturbation through the kinematic condition $dr_S/dt = v_r$. Calculations are carried up to second order in the amplitude. Lagrange equation (e.g. see [6]) gives the governing equation for $\xi(t)$

$$\frac{I_0(x)}{xI_1(x)}\ddot{\xi} + 4C\left[x\frac{I_0(x)}{I_1(x)} - \frac{1}{2}\right]\dot{\xi} - \left[1 - x^2 - \chi\cos^2(\omega t)\frac{(\epsilon_i - 1)^2 x}{H_0(\epsilon_i, x)}\right]\xi = 0, \quad (2)$$

which has been presented in nondimensional form using the same scales as in previous sections. A new non-dimensional parameter is the imposed field frequency $\omega \equiv \bar{\omega}t_c$.

Equation (2) has periodic coefficients and can be studied on the basis of Floquet theory. Solutions may be factorized as $e^{\gamma t}g(t)$, where $g(t)$ has the same period as the imposed field, $T = 2\pi/\omega$; and γ is a complex number (the Floquet exponent) whose real part determines the stability properties. Numerical integration over a period may serve to obtain the Floquet exponents. The real part of the highest one is presented in Figure 3 as

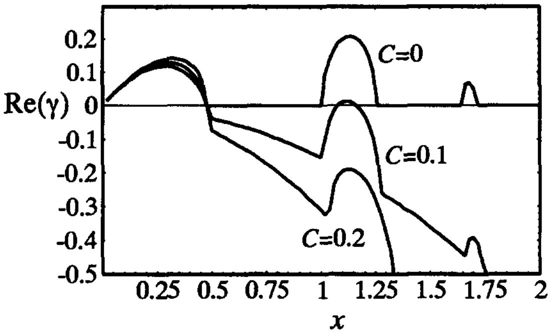

Figure 3. Highest Floquet exponent versus perturbation wavenumber, with $\epsilon_i = 3$, $\chi = 2$, $\omega = 1$, and different values of C.

a function of the wavenumber x for fixed electric parameters and three different viscosities. For $C = 0$ some resonant lobes rise on the right of the unstable capillary region. These lobes decrease for large wavenumbers and their size and extension depend mainly on the electric field amplitude. The frequency of the applied electric field affects very slightly the capillary instability domain if $\omega \geq 1$, for which the effective field amplitude is the r.m.s. value. However, the effect of viscosity is to damp the value of the Floquet exponent, which eventually lies in the negative region (stabilization), even for very moderate values of C. The observability of resonances is thus determined by competition between field strength and viscosity, being the latter very effective in suppressing this phenomenon.

In conclusion, a.c. fields behave as d.c. ones for frequencies greater than the capillary one, provided that viscosity is not too small so that parametric resonances can be avoided. The only restriction remaining in order to consider the jet as perfectly insulating and subjected to an effective d.c. field is that the period of the field must be much smaller than the charge relaxation time.

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