CONTINUITY WITH RESPECT TO THE HURST PARAMETER OF SOLUTIONS TO STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY H-VALUED FRACTIONAL BROWNIAN MOTION

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Abstract. In this work, the continuity with respect to the Hurst parameter of solutions to stochastic evolution equations is studied. Compared with recent studies on such continuity property, the model here is considered in a different point of view in which the equations are of SPDEs type, the solution and the fractional Brownian motion take value on a Hilbert space. The main contribution is to investigate the existence and stability of the solution with respect to the Hurst index in the space $C([0,T];L^2(\Omega,H))$.

Keywords: evolution equations, fractional Brownian motion, Hurst parameter, continuity.

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1. Introduction

Fractional Brownian motion, a generalization of standard Brownian motion, is commonly used to model various complex phenomena in practical applications, particularly in situations where the systems are subjected to rough external forces. This motion exhibits a medium-or long-memory persistence attribute and is characterized by a positive index h (called Hurst parameter). In the case $h = \frac{1}{2}$, fractional Brownian motion becomes standard Brownian motion. When $h \neq \frac{1}{2}$, the behavior of the general motion differs greatly from the standard one and it is not categorized as either a semi-martingale or a Markov process in this case.

Recent decades have witnessed the remarkable increase in the the theory of stochastic calculus and differential equations involving fractional Brownian motion (refer [4,5,7,19–21,26] and references therein). However, as far as we know, the number of articles on the dependence on the Hurst parameter h is still limited. There remain a plethora of diverse facets on this topic that require exploration. Jolis and Viles in [13–16] have given a series of results on the stability in law with respect to the Hurst index. Some other works in this aspect can be found in [9,11,12,25]. Another impressive work in a recent year is the study of Koch and Neuenkirch in [18], where the infinite differentiability of fractional Brownian motion has been investigated.

As regards the study on the continuous dependence on the Hurst parameter for stochastic ordinary differential equations (SODEs), we can list here some related papers. In [22,23], Richard and Talay have shown the Lipchitz continuity of the smooth functionals of the SODEs

$$X_t^H = x_0 + \int_0^t b(X_s^H)ds + \int_0^t \sigma(X_s^H) \circ dB_s^H,$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ and B_t^H is the one-dimensional fractional Brownian motion taking value in $\mathbb{R}$, $H \in [\frac{1}{2}, 1)$ is the Hurst parameter. In [10], Dung and Son have studied on the Lipschitz continuity in the Hurst index of the solutions to stochastic Volterra integro-differential equations

$$x_t^H = x_0 + \int_0^t \Big(f(s, x_s^H) + \int_0^s G(s-u)g(u, x_u^H)du\Big)ds + \int_0^t \sigma(x_s^H) \circ dB_s^H,$$

where f, g, σ, G are some $\mathbb{R}$-valued functions.

Although the continuity with respect to the Hurst index for SODEs has been investigated in different works recently, as we know, such continuity result has not been studied for stochastic partial differential equations (SPDEs) driven by fractional Brown motion (taking value on an Hilbert space). Motivated by this reason, in this paper, we aim at investigating the the stability with respect to the Hurst parameter

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of the solution to the following SPDEs

$$\begin{cases} \big(\partial_t + A\big)\mathcal{X}(t) = f(t,\mathcal{X}(t))\dot{W}(t) + g(t)\dot{W}^h(t), & \text{in } D\times(0,T], \\ \mathcal{X}(t) = 0, & \text{in } \partial D\times[0,T], \\ \mathcal{X}(0) = \mathcal{X}_0, & \text{in } D, \end{cases} \tag{1}$$

which is known as evolution equations. Here, $D \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary, the Hurst parameter $h \in (\frac{1}{2}, 1)$, the operator $A : D(A) \subset H \to H$ is linear, positive-definite and self-adjoint with compact inverse on H, the two functions f, g take value on $L^2(\Omega, L_0^2)$, $\mathcal{X}_0 \in L^2(\Omega, H)$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

It is worth noting that, compared with [9, 22, 23], the mathematical model here is considered in a different point of view in which the equations are of SPDEs type, the process $\mathcal{X}(t)$ takes value on a Hilbert space, namely $H := L^2(D)$, $f(t), g(t)$ are two operators taking value on $L^2(\Omega, L_0^2)$, $W(t)$ and $W^h(t)$ are H-valued standard Brownian motion (sBm [3, 6]) and H-valued fractional Brownian motion (fBm [5, 8]) with the representations

$$W(t) = \sum_{n=1}^{\infty} w_n(t) Q^{\frac{1}{2}} e_n, \qquad W^h(t) = \sum_{n=1}^{\infty} w_n^h(t) Q^{\frac{1}{2}} e_n,$$

where $w_n(t), w_n^h(t)$ are one-dimensional sBm and fBm respectively, (e_n) is an orthonormal basis of H, Q is a positive and self-adjoint operator (called covariance) such that $Qe_n = \lambda_n e_n$ and $Tr(Q) < \infty$. Due to the difference of the model, the result in this paper shall be established in a different way. To the best our knowledge, this work is the first result on the continuity with respect to the Hurst parameter of H-valued solutions to SPDEs.

The present problem arises naturally since the Hurst index h can only be determined experimentally. Additionally, in practical phenomena, the actual value of this parameter may be unknown and we only have approximate values of this number. If the stability of the Hurst index is guaranteed, the solution of equations with Hurst parameter h could be approximated well by corresponding solutions of equations with approximate parameters $\tilde{h}$. The exact solution can be obtained by the limit of some sequence of solutions of approximate models with perturbed parameters (h_n).

The main contribution of the present paper is the stability result on the space $C([0,T]; L^2(\Omega, H))$. At first glance, it seems to be simple to establish this result. However, when constructing such continuity property, we have to undergo some difficulties raising from the complicated representation of the function K_h appearing in the Wiener integral and the fact that divergent integrals would appear easily if overestimates are used. Another worth mentioning point is that the idea here can be extended to more complicated models with more complex technique in the future, see Remark 3.1 and Remark 3.3 for detail discussions.

The organization of the paper are as follows. Section 2 is devoted to some useful properties the Beta and diagamma functions, the notations of the space L_0^2 and mild solutions. In Section 3, main results are stated, including the existence of the mild solution and its continuity with respect the Hurst parameter. In Section 4, detail proofs of the couple theorems in the main results are given.

2. Preliminaries

In this section, we collect some useful tools to estimate the Beta and diagamma functions, and recall the definition of the space L_0^2 for the sake of convenience.

Lemma 2.1 (see [1, 17])**.** *Let $\mathcal{B}(\cdot,\cdot)$ be the Beta function*

$$\mathcal{B}(z_1, z_2) = \int_0^1 \mu^{z_1-1}\mu^{z_2-1} d\mu, \qquad z_1, z_2 \in \mathbb{C} \text{ and } \Re(z_1), \Re(z_2) > 0. \tag{2}$$

Then, the following inequality holds true for all $x, y \in (0,1)$

$$\mathcal{B}(x,y) \geq \frac{x+y}{xy} - 1.$$

Lemma 2.2. *Let $\psi(\cdot)$ be the diagamma function $\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Then, the following inequality holds true for all $x > 0$*

$$\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x - \frac{1}{2x}.$$

Proof. The property in Lemma 2.2 can be verified by the fact that when $x > 0$, the two functions $x \mapsto \ln x - \frac{1}{2x} - \psi(x)$ and $x \mapsto \ln x - \frac{1}{x} + \psi(x)$ are completely monotonic, see [2]. $\square$

Definition 2.1 (see [7])**.** Define by L_0^2 be the space of all operators Φ from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\Phi\|_{L_0^2} := \Big(\sum_{n=1}^{\infty} \|\Phi Q^{\frac{1}{2}} e_n\|_H^2 \Big)^{\frac{1}{2}} < \infty.$$

Before going to state the main results of this paper, we describe here the definition of Problem (1). Inspired from [7], mild solutions to Problem (1) can be defined as in Definition 2.2. Due to the appearance of $\dot{W}(t)$ in (1), the solution here contains an additional stochastic term, which is known as the Itô integral.

Definition 2.2 (Mild solution)**.** A process $\mathcal{X}_h : [0, T] \to L^2(\Omega, H)$ is said to be a mild solution of Problem (1) if it satisfies

$$\begin{aligned} \mathcal{X}_h(t) = \exp(-At)\mathcal{X}_0 + \int_0^t \exp(-A(t-s)) f(s, \mathcal{X}_h(s)) dW(s) \\ + \int_0^t \exp(-A(t-s)) g(s) dW^h(s), \qquad \mathbb{P}-\text{a.s.} \end{aligned} \tag{3}$$

3. Existence and continuity with respect the Hurst parameter results

In this section, main results are stated, including the existence of the mild solution and its continuity with respect h. To guarantee the existence result, we need the following assumptions

(H) $\mathcal{X}_0 \in L^2(\Omega, H)$, $g \in L^1(0, T; L^2(\Omega, L_0^2))$ and there exists a positive constant $\mathcal{L}$ such that for any $\mathcal{X}, \mathcal{Y} : [0, T] \to L^2(\Omega, H)$ and $0 \le t \le T$

$$\|f(t, \mathcal{X}(t)) - f(t, \mathcal{Y}(t))\|_{L^2(\Omega, L_0^2)} \le \mathcal{L} \|\mathcal{X}(t) - \mathcal{Y}(t)\|_{L^2(\Omega, H)}.$$

Remark 3.1. In (H), we consider $\mathcal{X}_0$, f, g as functions taking value on the usual space $L^2(\Omega, H)$ and $L^2(\Omega, L_0^2)$ to guarantee the existence result on $C([0, T]; L^2(\Omega, H))$, which makes our problem become simple and easy to be handled mathematically. The spaces can be extended to more complicated cases, for instance, $L^p(\Omega, W^{k,l})$ and $L^q(\Omega, L_0^2(H_1, H_2))$ respectively (where $W^{k,l}$ is some Sobolev space, H_1, H_2 are two Hilbert scale spaces) to ensure the existence of the solution on some Hölder continuity space $C([0, T]; L^{p'}(\Omega, W^{k',l'}))$ (see to [24] for an existence result for a SPED in this space). The strategy used to extend may be to apply some calculus inequalities, some Sobolev embeddings, and stochastic tools such as the Burkholder-Davis-Gundy-type inequality, the Kahane–Khintchine inequality, etc. In the study here, we work under the simple assumption for the purpose of serving as a guide for more difficult situations in the future, which require more complicated techniques to handle.

Next, the two main theorems of this paper are stated. The detail proof of each result can be found in Section 4. It should be noted that the assumption on the operator g in the two theorems is different. If the condition (H') is satisfied then (H) is also satisfied.

Theorem 3.1. *Let $\mathcal{X}_0, f, g$ satisfy the condition (H). Then, Problem* (1) *has a unique mild solution $\mathcal{X}_h \in C([0, T]; L^2(\Omega, H))$.*

Theorem 3.2. *Let $\frac{1}{2} < k < h < 1$ and $\mathcal{X}_0, f, g$ satisfy the following condition*

(H') $\mathcal{X}_0 \in L^2(\Omega, H)$, $f \in L^1(0, T; L^2(\Omega, L_0^2))$ and $g \in L^\infty(0, T; L^2(\Omega, L_0^2))$.

Then, the mild solution $\mathcal{X}_h$ of Problem (1) *is continuous with respect to the Hurst parameter*

$$\big\|\mathcal{X}_h - \mathcal{X}_k\big\|_{C([0,T];L^2(\Omega,H))} \lesssim |h - k| \big\|g\big\|_{L^\infty(0,T;L^2(\Omega,L_0^2))}. \tag{4}$$

Remark 3.2. At first glance, it seems to be easy to prove that the solution is continuous with respect to h. However, due to the complex representation of the kernel K_h (containing several terms including the Beta function) and the fact that divergent integrals would appear if we use overestimates, it is not easy to show the continuity holds true. Furthermore, needless to say, the tool used to estimate the Wiener integral term (see Lemma 2 of [7]) can not be applied to estimate the right hand side of (5).

Remark 3.3. In this work, we establish the continuity of the solution with respect to h in the case $h \in (\frac{1}{2}, 1)$ and g is linear. More complicated situations could be considered in the future due to complex techniques and calculations, for instance, the Hölder continuity result with some order $\nu > 0$, the case when $g = g(t, \mathcal{X})$ is non-linear, and the case $h \in (0, \frac{1}{2})$.

4. Proof of main results

Proof of Theorem 3.1. Now, we briefly show that Problem (1) has a unique solution in $C([0,T];L^2(\Omega,H))$. By applying the Itô isometry and an useful tool in [7] to estimate the Wiener integral (see Lemma 2), one arrives at

$$\|\mathcal{X}_h(t)\|^2_{L^2(\Omega,H)} \lesssim \|\mathcal{X}_0\|^2_{L^2(\Omega,H)} + \mathcal{L}^2\int_0^t \mathbb{E}\|\mathcal{X}_h(s)\|^2_{L^2(\Omega,H)} + h^2(2h-1)^2T^{2(2h-1)}\|g\|^2_{L^1(0,T;L^2(\Omega,L_0^2))}.$$

The Gronwall inequality conduces us to $\mathcal{X}(t) \in L^2(\Omega,H)$ for all $t \in [0,T]$. Furthermore, the continuity with respect to t and the uniqueness of the solution can be verified easily by a quite similar technique (refer to [7], we omit the detail here). Therefore, Problem (1) has a unique mild solution $\mathcal{X}_h \in C([0,T];L^2(\Omega,H))$. $\square$

Proof of Theorem 3.2. From the mild expression as in (3), it is clear that for $\frac{1}{2} < k < h < 1$ there holds

$$\begin{aligned}\mathbb{E}\big\|\mathcal{X}_h(t)-\mathcal{X}_k(t)\big\|_H^2 &\lesssim \mathbb{E}\Big\|\int_0^t \exp(-A(t-s))\big(f(s,\mathcal{X}_h(s))-f(s,\mathcal{X}_k(s))\big)dW(s)\Big\|_H^2\\ &+\mathbb{E}\Big\|\int_0^t \exp(-A(t-s))g(s)dW^h(s)-\int_0^t \exp(-A(t-s))g(s)dW^k(s)\Big\|_H^2\\ &=: J_1+J_2. \end{aligned}\tag{5}$$

The first term in the right hand side can be estimated by applying the Itô isometry and the Lipschitz condition of f as

$$\begin{aligned}J_1 &= \int_0^t \mathbb{E}\|\exp(-A(t-s))\big(f(s,\mathcal{X}_h(s))-f(s,\mathcal{X}_k(s))\big)\|^2_{L_0^2}ds\\ &\lesssim \int_0^t \mathbb{E}\|f(s,\mathcal{X}_h(s))-f(s,\mathcal{X}_k(s))\|^2_{L_0^2}ds \lesssim \int_0^t \mathbb{E}\|\mathcal{X}_h(s)-\mathcal{X}_k(s)\|_H^2ds.\end{aligned}$$

We continue to estimate J_2. Recalling that the Wiener integral with respect the fBm has the following explicit representation

$$\int_0^t \exp(-A(t-s))g(s)dW^h(s) = \sum_{n=1}^\infty \int_0^t\int_s^t \exp(-A(t-r))g(r)Q^{\frac{1}{2}}e_n\partial_r K_h(r,s)drdw_n(s),$$

where $\partial_r K_h(r,s) = c_h\left(\frac{s}{r}\right)^{\frac{1}{2}-h}(r-s)^{h-\frac{3}{2}}$ (see [21]). By applying the Itô isometry, one obtains

$$\begin{aligned}J_2 &= \mathbb{E}\Big\|\sum_{n=1}^\infty\int_0^t\int_s^t \exp(-A(t-r))g(r)Q^{\frac{1}{2}}e_n\big(\partial_rK_h(r,s)-\partial_rK_k(r,s)\big)drdw_n(s)\Big\|_H^2\\ &= \mathbb{E}\sum_{n=1}^\infty\int_0^t\Big\|\int_s^t \exp(-A(t-r))g(r)Q^{\frac{1}{2}}e_n\big(\partial_rK_h(r,s)-\partial_rK_k(r,s)\big)dr\Big\|_H^2ds\\ &\lesssim \mathbb{E}\sum_{n=1}^\infty\int_0^t\Big[\int_s^t\big|\partial_rK_h(r,s)-\partial_rK_k(r,s)\big|\big\|\exp(-A(t-r))g(r)Q^{\frac{1}{2}}e_n\big\|_Hdr\Big]^2ds.\end{aligned}\tag{6}$$

To estimate this term, we first deal with the bias $\partial_rK_h(r,s)-\partial_rK_k(r,s)$. It is obvious to see that this term can be split into two terms as follows

$$\begin{aligned}\partial_rK_h(r,s)-\partial_rK_k(r,s) &= c_h\left(\frac{s}{r}\right)^{\frac{1}{2}-h}(r-s)^{h-\frac{3}{2}} - c_k\left(\frac{s}{r}\right)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}}\\ &= \frac{1}{\beta_h}\Big[\alpha_h\left(\frac{s}{r}\right)^{\frac{1}{2}-h}(r-s)^{h-\frac{3}{2}} - \alpha_k\left(\frac{s}{r}\right)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}}\Big]\\ &+\Big(\frac{1}{\beta_h}-\frac{1}{\beta_k}\Big)\alpha_k\left(\frac{s}{r}\right)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}},\end{aligned}\tag{7}$$

where we recall that c_h possesses the explicit representation $c_h = \sqrt{\frac{h(2h-1)}{\mathcal{B}(2-2h,h-\frac{1}{2})}} =: \frac{\alpha_h}{\beta_h}$ (see [21]), with $\alpha_h := \sqrt{h(2h-1)}$ and $\beta_h := \sqrt{\mathcal{B}(2-2h,h-\frac{1}{2})}$, $\mathcal{B}(\cdot,\cdot)$ is the Beta function. It should be noted that

$$\frac{1}{\beta_h}-\frac{1}{\beta_k} = \int_k^h \partial_x\Big(\frac{1}{\beta_x}\Big)dx = \int_k^h \frac{\Psi(x)}{2\beta_x}dx,$$

where we define $\Psi(x) := 2\psi(2-2x) - \psi\big(\frac{3}{2}-x\big) - \psi\big(x-\frac{1}{2}\big)$, for $x \in (\frac{1}{2},1)$. Since $\mathcal{B}(x,y) \geq \frac{x+y}{xy} - 1$ for all $x,y \in (0,1)$, see Lemma 2.1, one has an upper bound for $\frac{1}{\beta_x}$ as

$$\frac{1}{\beta_x} = \frac{1}{\sqrt{\mathcal{B}(2-2x, x-\frac{1}{2})}} \leq \sqrt{\frac{2(1-x)(2x-1)}{4x^2-8x+5}} < \frac{1}{2}. \tag{8}$$

which helps us to obtain

$$\Big|\frac{1}{\beta_h} - \frac{1}{\beta_k}\Big| \lesssim \Big|\int_k^h \psi(2-2x)dx\Big| + \Big|\int_k^h \psi\Big(\frac{3}{2}-x\Big)dx\Big| + \Big|\int_k^h \psi\Big(x-\frac{1}{2}\Big)dx\Big| =: I_1 + I_2 + I_3.$$

Applying the property $\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x - \frac{1}{2x}$ for all $x > 0$ (Lemma 2.2), one arrives at

$$\begin{aligned} I_1 &\leq \int_k^h \big|\psi(2-2x)\big|dx \lesssim \int_k^h \Big(-\ln(2-2x) - \frac{1}{2-2x}\Big)dx \leq \int_k^h \frac{1}{2x-2}dx \\ &\leq \ln(2h-2) - \ln(2k-2) \lesssim h-k, \end{aligned}$$

where we note that $0 < 2-2x < 1$ due to $x \in (\frac{1}{2},1)$. By exactly the same technique, one can verify that $I_2 + I_3 \lesssim h-k$. As a result, one obtains the continuity of the function $\frac{1}{\beta_h}$ as

$$\Big|\frac{1}{\beta_h} - \frac{1}{\beta_k}\Big| \lesssim h-k. \tag{9}$$

On the other hand, we know that there exists a constant $\hbar \in (\frac{1}{2},1)$ such that

$$\begin{aligned} &\Big|\alpha_h\Big(\frac{s}{r}\Big)^{\frac{1}{2}-h}(r-s)^{h-\frac{3}{2}} - \alpha_k\Big(\frac{s}{r}\Big)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}}\Big| = (h-k)\Big|\partial_h\Big(\alpha_\hbar\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}}\Big)\Big| \\ &= (h-k)\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}}\Big|\frac{2\hbar(2\hbar-1)\big(\ln(r-s)+\ln r-\ln s\big)+4\hbar-1}{2\sqrt{\hbar(2\hbar-1)}}\Big| \\ &\lesssim (h-k)\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}}\Big(T\sqrt{\hbar(2\hbar-1)} + \sqrt{\frac{\hbar}{2\hbar-1}} + \sqrt{\frac{2\hbar-1}{\hbar}}\Big) \\ &\lesssim (h-k)\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}}, \qquad \text{for } 0 \leq s < r \leq T, \end{aligned} \tag{10}$$

where the hidden constant does not depend on h,k,r,s. Now, combining (7)-(10) and noting that $\hbar \in (\frac{1}{2},1)$, one deduces that

$$\big|\partial_r K_h(r,s) - \partial_r K_k(r,s)\big| \lesssim (h-k)\Big|\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}} + \Big(\frac{s}{r}\Big)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}}\Big| \tag{11}$$

Applying the above estimate and the Hölder inequality, one arrives at

$$\begin{aligned} J_2 &\lesssim (h-k)^2\mathbb{E}\sum_{n=1}^\infty \int_0^t \Big[\int_s^t \Big(\Big(\frac{s}{r}\Big)^{\frac{1}{2}-\hbar}(r-s)^{\hbar-\frac{3}{2}} + \Big(\frac{s}{r}\Big)^{\frac{1}{2}-k}(r-s)^{k-\frac{3}{2}}\Big)\times \\ &\qquad\qquad \times \big\|\exp(-A(t-r))g(r)Q^{\frac{1}{2}}e_n\big\|_H dr\Big]^2 ds \\ &\lesssim (h-k)^2\mathbb{E}\sum_{n=1}^\infty \int_0^t s^{1-2\hbar}\Big[\int_s^t \Big((r-s)^{\hbar-\frac{3}{2}} + (r-s)^{k-\frac{3}{2}}\Big)dr\times \\ &\qquad\qquad \times \int_s^t \Big((r-s)^{\hbar-\frac{3}{2}} + (r-s)^{k-\frac{3}{2}}\Big)\big\|g(r)Q^{\frac{1}{2}}e_n\big\|_H^2 dr\Big]ds \\ &\lesssim (h-k)^2\big\|g\big\|^2_{L^\infty(0,T;L^2(\Omega,L_0^2))}\int_0^t s^{1-2\hbar}\Big((t-s)^{\hbar-\frac{1}{2}} + (t-s)^{k-\frac{1}{2}}\Big)^2 ds. \end{aligned}$$

Since $t \leq T$ and $\hbar,k \in (\frac{1}{2},1)$, one can verify that $\int_0^t s^{1-2\hbar}\Big((t-s)^{\hbar-\frac{1}{2}} + (t-s)^{k-\frac{1}{2}}\Big)^2 ds \lesssim t \leq T$. Therefore, we conclude that there exists a positive constant C independent of h,k,t such that

$$\mathbb{E}\|\mathcal{X}_h(t) - \mathcal{X}_k(t)\|_H^2 \leq C\Big[(h-k)^2\big\|g\big\|^2_{L^\infty(0,T;L^2(\Omega,L_0^2))} + \int_0^t \mathbb{E}\|\mathcal{X}_h(s) - \mathcal{X}_k(s)\|_H^2 ds\Big].$$

The Grönwall inequality allows us to obtain

$$\big\|\mathcal{X}_h(t) - \mathcal{X}_k(t)\big\|^2_{L^2(\Omega,H))} \leq Ce^{Ct}(h-k)^2\big\|g\big\|^2_{L^\infty(0,T;L^2(\Omega,L_0^2))}.$$

Since $e^{Ct} \leq e^{CT}$, the above assertion implies that (4) holds true. The proof is complete. $\square$

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