Periodic measures for the stochastic delay modified Swift-Hohenberg lattice systems ☆

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Abstract

In this paper, the existence and the limiting behavior of periodic measures for the periodic stochastic modified Swift-Hohenberg lattice systems with variable delays are analyzed. We first prove the existence and uniqueness of global solution when the nonlinear $\mathcal{T}$-periodic drift and diffusion terms are locally Lipchitz continuous and linearly growing. Then we show the existence of periodic measures of the system under some assumptions. Finally, by strengthening the assumptions, we prove that the set of all periodic measures is weakly compact, and we also show that every limit point of a sequence of periodic measures of the original system must be a periodic measure of the limiting system when the noise intensity tends to zero.

Keywords: periodic measures; modified Swift-Hohenberg lattice system; variable delays; limit measure
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1. Introduction

Lattice systems are commonly used for modeling in circuit theory, materials science, image processing, biology, etc. [2, 3]. However, in the actual modeling process, the time delay is inevitable and reasonable, which means that the current state depends on the past state. A time-delay system is usually described in the form of differential equations [15, 23].

The solutions and their long-term dynamics of deterministic lattice systems were studied in [12, 30] without delay and [6, 7, 13, 42, 43] with delay. The long-time behavior of stochastic lattice systems has been investigated in [5, 8, 31, 32, 35, 36, 37] without delay and [10, 16, 17, 18, 20, 33, 38, 44] with delay.

In this paper, we take into account the existence and the limiting behavior of periodic measures for the periodic stochastic delay modified Swift-Hohenberg lattice systems on the

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integer set $\mathbb{Z}$ given by:

$$\begin{cases} du_i(t) + q_1(t)\Big[\big(u_{i+2}(t) - 4u_{i+1}(t) + 6u_i(t) - 4u_{i-1}(t) + u_{i+2}(t)\big) \\ \quad + 2\big(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\big)\Big]dt + q_2(t)u_i(t)dt + q_{3,i}(t)|u_{i+1}(t) - u_i(t)|^2dt + u_i^3(t)dt \\ = f_i(t, u_i(t), u_i(t-\varrho(t)))dt + g_i(t)dt \\ \quad + \epsilon \sum_{j=1}^{\infty} \big(h_{i,j}(t) + \sigma_{i,j}(t, u_i(t), u_i(t-\varrho(t)))\big)dW_j(t), \quad t > 0, \\ u_i(s) = \varphi_i(s), \quad s \in [-\rho, 0], \ i \in \mathbb{Z}, \end{cases} \tag{1.1}$$

which is obtained by a spatial discretization of the periodic continuous modified Swift-Hohenberg equation with a variable time delay on $\mathbb{R}$:

$$\begin{cases} du(t) + q_1(t)\Delta^2u(t)dt + 2q_1(t)\Delta u(t)dt + q_2(t)u(t)dt + q_3(t)|\nabla u(t)|^2dt + u^3(t)dt \\ = f(t, u(t), u(t-\varrho(t)))dt + g(t)dt + \epsilon \sum_{j=1}^{\infty}(h_j(t) + \sigma_j(t, u(t), u(t-\varrho(t)))dW_j(t), \ t > 0, \\ u(s) = \varphi(s), \quad s \in [-\rho, 0]. \end{cases} \tag{1.2}$$

Here $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ are positive and continuous, $q_3 = (q_{3,i})_{i\in\mathbb{Z}} : \mathbb{R} \to \ell^2$ is continuous, $f_i, \sigma_{j,i} : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions for every $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, noise intensity $0 < \epsilon \leq 1$ and delay parameter $\rho > 0$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$ and $h_j : \mathbb{R} \to L^2(\mathbb{R})$ are given, and $(W_j)_{j\in\mathbb{N}}$ is a sequence of standard two-sided real-valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. Furthermore, for a given positive constant $\mathcal{T}$, all time-dependent terms of the system (1.1) are $\mathcal{T}$-periodic in time (as shown in (2.9)).

The Swift-Hohenberg equation was originally derived in the framework of fluid dynamics, e.g., simple fluid Rayleigh-Bénard convection [28], and has also been applied to chemical and ecological systems (such as neural tissues) [22]. Equation (1.2) is a usual Swift-Hohenberg equation when $q_3(t) = 0$, $f(t, u(t), u(t-\varrho(t))) = 0$ and $\epsilon = 0$. About the modified Swift-Hohenberg equation, there have been many papers discussing the existence of attractors (global, pullback, random, uniform) and bifurcations, e.g., [9, 24, 25, 27, 19, 39, 40, 41], yet there are only a few papers on the modified Swift-Hohenberg lattice system, [14] considered the deterministic case and [34] was in the stochastic case. These are both without delay. To our knowledge, there is no paper on stochastic delay modified Swift-Hohenberg lattice systems.

The existence of periodic measures for stochastic lattice systems without delay was investigated in [21, 11]. In the delay case, [17, 20] considered the periodic measure of the stochastic lattice system. The above nonlinear terms are all globally Lipschitz continuous. This paper considers variable delays, the nonlinear functions f_i and $\sigma_{j,i}$, the modified term, and the cubic terms are all locally Lipschitz continuous.

The first goal of this paper is to prove, by applying Krylov-Bogolyubov's method, the existence of periodic measures of the lattice system (1.1) for all $\epsilon \in [0, 1]$ in the space $C([-\rho, 0], \ell^2)$. We need to prove the tightness of distribution laws of solutions to (1.1), and the difficulty of proving this tightness is analogous to the fact that the Sobolev embedding is no longer compact when stochastic PDEs are over an unbounded domain. To address this difficulty, we show that the tail of the solution to (1.1) is uniformly small in $L^2(\Omega, \mathcal{F}, \ell^2)$ using the uniform tail-estimation method proposed in [29]. Furthermore, since the solution $u_t(\cdot)$ of the system (1.1) depends on the past history and is therefore non-Markov, we use the method in [26] to find the solution map for stochastic functional differential equations with finite delay possesses Markov property.

Another objective of this paper is to prove, under stronger assumptions, the limiting behavior of periodic measures of the system (1.1) when the noise intensity $\epsilon \to \epsilon_0 \in [0,1]$ (see (5.1)). We show that for $\epsilon \in [0,1]$, the set of all periodic measures of (1.1) is weakly compact, and any limit point of a tight sequence of periodic measures of the system (1.1) must be an invariant measure of the associated limiting equation (see Theorem 5.5).

The outline of this paper is as follows. In the next section, we introduce some assumptions about nonlinear and time-delay terms and prove the existence and uniqueness of solutions of (1.1). In Sect. 3, we establish uniform estimates of solutions in space $C([-\rho,0],\ell^2)$. In Sect. 4, we discuss the existence of periodic measures in $C([-\rho,0],\ell^2)$. In the last part, the limit of the periodic measure is studied when the noise intensity $\epsilon \to \epsilon_0 \in [0,1]$.

2. Well-posedness of stochastic delay modified Swift-Hohenberg lattice systems

2.1. Some assumptions

We consider a Banach space as below:

$$\ell^r = \Big\{ w = (w_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |w_i|^r < +\infty \Big\}, \quad r \geq 1, \tag{2.1}$$

with the norm $\|w\|_r^r = \sum_{i\in\mathbb{Z}} |w_i|^r$. For $r=2$, it has the norm $\|w\|^2 = \sum_{i\in\mathbb{Z}} |w_i|^2$ and inner product $(w,v) := \sum_{i\in\mathbb{Z}} w_i v_i$. In addition, we denote the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ by $C(\mathbb{R},\mathbb{R})$.

In this paper, we will show the existence of periodic measures for stochastic delay modified Swift-Hohenberg lattice systems (1.1). To that end, we need to impose the following assumptions:

A0. The delay term $\varrho(\cdot) \in C^1(\mathbb{R},[0,\rho])$ and satisfies

$$\varrho'(t) \leq \rho^* \quad \text{for some} \quad \rho^* \leq 0, \quad \forall t \geq 0. \tag{2.2}$$

A1. The same delay term as in **A0** satisfies

$$\varrho'(t) \leq \rho^* \quad \text{for some} \quad \rho^* < 1, \quad \forall t \geq 0. \tag{2.3}$$

A2. $g(\cdot) = (g_i(\cdot))_{i\in\mathbb{Z}} \in C(\mathbb{R},\ell^2)$ and $h_j(\cdot) = (h_{i,j}(\cdot))_{i\in\mathbb{Z},j\in\mathbb{N}} \in C(\mathbb{R},\ell^2)$, which means that for all $t \in \mathbb{R}$

$$\|g(t)\|^2 = \sum_{i\in\mathbb{Z}} |g_i(t)|^2 < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}} \|h_j(t)\|^2 = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{Z}} |h_{i,j}(t)|^2 < \infty. \tag{2.4}$$

A3. The nonlinear drift term $f_i \in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f_i(t,\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$; namely, for any bounded interval $I \subseteq \mathbb{R}$, there exists a constant $L_0 = L_0(I) > 0$, independent of $t \in \mathbb{R}, i \in \mathbb{Z}$ such that

$$|f_i(t,s_1,s_2) - f_i(t,s_3,s_4)| \leq L_0(|s_1 - s_3| + |s_2 - s_4|), \quad \forall t \in \mathbb{R}, s_m \in I (m = 1,2,3,4). \tag{2.5}$$

A4. For each $i \in \mathbb{Z}$, there exists $\alpha_i > 0$ such that,

$$|f_i(t, s_5, s_6)| \leq \beta_0(t)(|s_5| + |s_6|) + \alpha_i, \quad \forall t, s_5, s_6 \in \mathbb{R}, \tag{2.6}$$

where $(\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$ and $\beta_0 : \mathbb{R} \to \mathbb{R}$ is a positive continuous and non-increasing function.

A5. The nonlinear diffusion term $\sigma_{j,i} \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\sigma_{j,i}(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous; that is, for any bounded interval $I \subseteq \mathbb{R}$, there exists a constant $L_1 = L_1(I) > 0$, independent of $j \in \mathbb{N}, i \in \mathbb{Z}, t \in \mathbb{R}$ such that

$$\begin{aligned} &|\sigma_{i,j}(t, s_1, s_2) - \sigma_{i,j}(t, s_3, s_4)| \\ &\leq L_1(|s_1 - s_3| + |s_2 - s_4|), \quad \forall s_m \in I(m = 1, 2, 3, 4), t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}. \end{aligned} \tag{2.7}$$

A6. For each $t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}$, there exists $\gamma_{i,j} > 0$ such that for all $s_5, s_6 \in \mathbb{R}$

$$|\sigma_{i,j}(t, s_5, s_6)| \leq \beta_j(t)(|s_5| + |s_6|) + \gamma_{i,j}, \tag{2.8}$$

where $(\gamma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}} \in \ell^2$ and $(\beta_j(\cdot))_{j\in\mathbb{N}} : \mathbb{R} \to \ell^2$ is a positive continuous and non-increasing function.

Remarks on A0: Notice that we are assuming $\rho^* \leq 0$ in **A0**, which is more restrictive than the general case $\rho^* < 1$ in **A1**. The reason to consider this assumption is that we will be able to provide in Remark 2.3, an easy way to prove the existence and uniqueness of solutions to our problem under assumption **A0**.

Given $\mathcal{T} > 0$, we assume that all time-dependent functions in (1.1) are $\mathcal{T}$-periodic in $t \in \mathbb{R}$, which means that, for all $t \in \mathbb{R}, i \in \mathbb{Z}, j \in \mathbb{N}$,

$$\begin{cases} q_1(t + \mathcal{T}) = q_1(t), \quad q_2(t + \mathcal{T}) = q_2(t), \quad q_{3,i}(t + \mathcal{T}) = q_{3,i}(t), \quad \varrho(t + \mathcal{T}) = \varrho(t), \\ g(t + \mathcal{T}) = g(t), \quad h(t + \mathcal{T}) = h(t), \quad \beta_0(t + \mathcal{T}) = \beta_0(t), \\ f_i(t + \mathcal{T}, \cdot, \cdot) = f_i(t, \cdot, \cdot), \quad \sigma_{i,j}(t + \mathcal{T}, \cdot, \cdot) = \sigma_{i,j}(t, \cdot, \cdot), \quad \beta_j(t + \mathcal{T}) = \beta_j(t). \end{cases} \tag{2.9}$$

2.2. Existence and uniqueness of solutions

Denote by $C_\rho = C([-\rho, 0]; \ell^2)$ the Banach space of all ℓ^2-valued continuous functions on $[-\rho, 0]$ with the norm

$$\|x\|_{C_\rho} = \sup_{s\in[-\rho,0]} \|x(s)\| = \sup_{s\in[-\rho,0]} \sum_{i\in\mathbb{Z}} |x_i(s)|^2, \ \forall x \in C_\rho.$$

For any map $w : [-\rho, \infty) \to \ell^2$, we denote the delay shift (or segment of the map) by

$$w_t(s) = w(t + s), \quad \forall t \geq 0, \ s \in [-\rho, 0].$$

For convenience, define some operators from ℓ^2 to ℓ^2 as below: for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i, \tag{2.10}$$

and

$$(Du)_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}. \tag{2.11}$$

Thus, for $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$ and $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, we deduce from (2.10) that

$$A = BB^* = B^*B, \quad (Bu, v) = (u, B^*v). \tag{2.12}$$

and

$$(Au, v) = (Bu, Bv), \quad \|Bu\|^2 = \|B^* u\|^2 \le 4\|u\|^2, \quad \|Au\|^2 \le 16\|u\|^2. \tag{2.13}$$

Analogously, it follows from (2.11) that

$$(Du, v) = (Au, Av), \quad \|Du\|^2 \le 256\|u\|^2. \tag{2.14}$$

Now, consider $u = (u_i)_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}, h_j(t) = (h_{j,i}(t))_{i\in\mathbb{Z}}$, $f(t, u, v) = (f_i(t, u_i, v_i))_{i\in\mathbb{Z}}$ and $\sigma_j(t, u, v) = (\sigma_{j,i}(t, u_i, v_i))_{i\in\mathbb{Z}}$. Based on the above arguments, we study the following stochastic delay modified Swift-Hohenberg lattice system in ℓ^2 for $t > 0$:

$$\begin{cases} du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2 dt + u^3(t)dt \\ \quad = f(t, u(t), u(t-\varrho(t)))dt + g(t)dt + \epsilon \displaystyle\sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), u(t-\varrho(t)))\big)dW_j(t), \\ u(s) = \varphi(s), \quad s \in [-\rho, 0], \end{cases} \tag{2.15}$$

where $q_1, q_2 \in C(\mathbb{R}, \mathbb{R}^+), q_3 = (q_{3,i})_{i\in\mathbb{Z}} \in C(\mathbb{R}, \ell^2), \varphi = (\varphi_i)_{i\in\mathbb{Z}} \in C_\rho$.

By **A3** and **A4**, $f(t, \cdot, \cdot) : \ell^2 \times \ell^2 \to \ell^2$ is locally Lipschitz continuous and grows linearly, that is, for every $R > 0$, there exists $L_R^f > 0$ such that, for all $t \in \mathbb{R}$ and $u_1, v_1, u_2, v_2 \in \ell^2$ with $\|u_1\| \le R, \|u_2\| \le R, \|v_1\| \le R$ and $\|v_2\| \le R$,

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|^2 \le L_R^f(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \tag{2.16}$$

and for all $t \in \mathbb{R}$ and $u, v \in \ell^2$,

$$\|f(t, u, v)\|^2 \le 4\beta_0^2(t)(\|u\|^2 + \|v\|^2) + 2\|\alpha\|^2, \tag{2.17}$$

where $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$. Similarly, it follows from **A5** that $\sigma_j(t, \cdot, \cdot) : \ell^2 \times \ell^2 \to \ell^2$ is locally Lipschitz continuous for $j \in \mathbb{N}$ in the sense that, for every $R > 0$, there exists $L_R^\sigma > 0$ such that for all $t \in \mathbb{R}$ and $u_1, v_1, u_2, v_2 \in \ell^2$ with $\|u_1\| \le R, \|u_2\| \le R, \|v_1\| \le R$ and $\|v_2\| \le R$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t, u_1, v_1) - \sigma_j(t, u_2, v_2)\|^2 \le L_R^\sigma(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2). \tag{2.18}$$

For each $j \in \mathbb{N}$, we infer from **A6** that, for all $t \in \mathbb{R}$ and $u, v \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j(t, u, v)\|^2 \le 4\|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2, \tag{2.19}$$

where $\gamma = (\gamma_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}} \in \ell^2$ and $\|\beta(t)\|^2 = \sum_{j\in\mathbb{N}} |\beta_j(t)|^2$.

Definition 2.1. Suppose $\varphi \in L^2(\Omega, C_\rho)$ is $\mathcal{F}_0$-measurable. Then a continuous ℓ^2-valued stochastic process u is called a solution of lattice system (2.15) if $(u_t)_{t\ge0}$ is $\mathcal{F}_t$-adapted, $u_0 = \varphi$, for all $T > 0$,

$$u \in L^2(\Omega, C([-\rho, T], \ell^2)), \tag{2.20}$$

and, for each $t \geq 0$,

$$u(t) = \varphi(0) + \int_0^t \big(- q_1(s) Du(s) + 2q_1(s) Au(s) - q_2(s) u(s) - q_3(s)|Bu(s)|^2 - u^3(s)\big) ds$$
$$+ \int_0^t (f(s, u(s), u(s - \varrho(s))) + g(s)) ds + \epsilon \sum_{j=1}^{\infty} \int_0^t (h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s)))) dW_j(s), \tag{2.21}$$

in ℓ^2 for almost all $\omega \in \Omega$.

Now, we will show the existence and uniqueness of solutions of the system (2.15). In the particular case of $\rho^* \leq 0$ in **A0**, a short and nice proof will be given in Remark 2.3 below.

Theorem 2.2. *Suppose* **A2**-**A6** *hold and* $\varphi \in L^2(\Omega, C_\rho)$. *Then the system* (2.15) *has a unique solution* u *in the sense of Definition 2.1. In addition, if* **A1** *also holds, then for any* $T \geq 0$,

$$\mathbb{E}\big(\|u\|^2_{C([-\rho,T],\ell^2)}\big) \leq M e^{MT} \Big(\mathbb{E}(\|\varphi\|^2_{C_\rho}) + T + 1 \Big), \tag{2.22}$$

where M *is a positive constant independent of* φ *and* T.

Proof. (1) The existence and uniqueness of solution follow from a similar argument to the one used by X. Mao [23] in the case of stochastic differential equations with delay in $\mathbb{R}^n$ (see also Caraballo et al. [6] for a stochastic PDE with delay). We omit the details.

(2) Now, we prove the uniform estimates of solutions. By (2.15) and Ito's formula, we have for all $t \in [0, T]$,

$$\|u(t)\|^2 + 2\int_0^t q_1(s)\|Au(s)\|^2 ds - 4\int_0^t q_1(s)\|Bu(s)\|^2 ds + 2\int_0^t q_2(s)\|u(s)\|^2 ds$$
$$+ 2\int_0^t \|u(s)\|_4^4 ds + 2\int_0^t \big(q_3(s)|Bu(s)|^2, u(s)\big) ds$$
$$= \|u(0)\|^2 + 2\int_0^t (f(s, u(s), u(s - \varrho(s))), u(s)) ds + 2\int_0^t (g(s), u(s)) ds$$
$$+ \epsilon^2 \sum_{j=1}^{\infty} \int_0^t \|h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s)))\|^2 ds$$
$$+ 2\epsilon \sum_{j=1}^{\infty} \int_0^t \big(h_j(s) + \sigma_j(s, u(s), u(s - \varrho(s))), u(s)\big) dW_j(s). \tag{2.23}$$

By using Young's inequality, we obtain

$$- 2\int_0^t \big(q_3(s)|Bu(s)|^2, u(s)\big) ds$$
$$\leq 2\int_0^t \sum_{i \in \mathbb{Z}} q_{3,i}(s) \big| |u_{i+1}(s) - u_i(s)|^2 \big| |u_i(s)| ds$$
$$\leq 8\int_0^t \|q_3(s)\|^2 \|u(s)\|^2 ds + \frac{1}{8}\int_0^t \sum_{i \in \mathbb{Z}} \Big| u_{i+1}^2(s) - 2u_{i+1}(s)u_i(s) + u_i^2(s) \Big|^2 ds$$
$$\leq 8\int_0^t \|q_3(s)\|^2 \|u(s)\|^2 ds + 2\int_0^t \|u(s)\|_4^4 ds. \tag{2.24}$$

From (2.17) and the fact that $\beta_0 : \mathbb{R} \to \mathbb{R}$ is a positive continuous and non-increasing function as defined in **A4**, for all $t \in [0, T]$,

$$
\begin{aligned}
&2\int_0^t (f(s,u(s),u(s-\varrho(s))),u(s))ds \\
&\leq \int_0^t \frac{1}{2\beta_0(s)}\|f(s,u(s),u(s-\varrho(s)))\|^2 ds + 2\int_0^t \beta_0(s)\|u(s)\|^2 ds \\
&\leq \int_0^t \frac{4\beta_0^2(s)}{2\beta_0(s)}(\|u(s)\|^2 + \|u(s-\varrho(s))\|^2)ds + \int_0^t \frac{\|\alpha\|^2}{\beta_0(s)}ds + 2\int_0^t \beta_0(s)\|u(s)\|^2 ds \\
&\leq 4\int_0^t \beta_0(s)\|u(s)\|^2 ds + \int_0^t \frac{\|\alpha\|^2}{\beta_0(s)}ds + \frac{2}{1-\rho^*}\int_{-\rho}^t \beta_0(s)\|u(s)\|^2 ds \\
&\leq (4 + \frac{2}{1-\rho^*})\int_0^t \beta_0(s)\|u(s)\|^2 ds + \min_{t\in[0,\mathcal{T}]}\beta_0(t)\|\alpha\|^2 T + \frac{2\rho}{1-\rho^*}\|\varphi\|_{C_\rho}^2 \max_{t\in[0,\mathcal{T}]}\beta_0(t). \qquad (2.25)
\end{aligned}
$$

Using Young's inequality yields

$$
2\int_0^t (g(s),u(s))ds \leq \int_0^t \frac{1}{q_2(s)}\|g(s)\|^2 ds + \int_0^t q_2(s)\mathbb{E}(\|u(s)\|^2)ds. \qquad (2.26)
$$

By (2.19) and the fact that $(\beta_j(\cdot))_{j\in\mathbb{N}} : \mathbb{R} \to \ell^2$ is a positive continuous and non-increasing function as defined in **A6**, for all $t \in [0, T]$,

$$
\begin{aligned}
&\epsilon^2 \sum_{j=1}^\infty \int_0^t \|h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))\|^2 ds \\
&\leq 2\epsilon^2 \sum_{j=1}^\infty \int_0^t \|h_j(s)\|^2 ds + 2\epsilon^2 \sum_{j=1}^\infty \int_0^t \|\sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds \\
&\leq 2\epsilon^2 \sum_{j=1}^\infty \int_0^t \|h_j(s)\|^2 ds + 8\epsilon^2 \int_0^t \|\beta(s)\|^2(\|u(s)\|^2 + \|u(s-\varrho(s))\|^2)ds + 4\epsilon^2 \int_0^t \|\gamma\|^2 ds \\
&\leq 8\epsilon^2(1 + \frac{1}{1-\rho^*})\int_0^t \|\beta(s)\|^2\|u(s)\|^2 ds + \frac{8\epsilon^2\rho}{1-\rho^*}\|\varphi\|_{C_\rho}^2 \sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2 \\
&+ 2\epsilon^2 \int_0^t (\sum_{j=1}^\infty \|h_j(s)\|^2 + 2\|\gamma\|^2)ds. \qquad (2.27)
\end{aligned}
$$

Plugging (2.24)-(2.27) into (2.23),

$$
\begin{aligned}
\|u(t)\|^2 &\leq \|u(0)\|^2 + \int_0^t M_1(s)\|u(s)\|^2 ds + \frac{2\rho}{1-\rho^*}\left(\max_{t\in[0,\mathcal{T}]}\beta_0(t) + 4\epsilon^2 \sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\right)\|\varphi\|_{C_\rho}^2 \\
&+ \left(\min_{t\in[0,\mathcal{T}]}\beta_0(t)\|\alpha\|^2 + 4\epsilon^2\|\gamma\|^2\right)T + \int_0^t \frac{1}{q_2(s)}\|g(s)\|^2 ds + 2\epsilon^2 \int_0^t \sum_{j=1}^\infty \|h_j(s)\|^2 ds \\
&+ 2\epsilon \sum_{j=1}^\infty \int_0^t \big(h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))),u(s)\big)dW_j(s). \qquad (2.28)
\end{aligned}
$$

where $M_1(s) = 16q_1(s) + 8\|q_3(s)\|^2 + (4 + \frac{2}{1-\rho^*})\beta_0(s) + 8\epsilon^2(1 + \frac{1}{1-\rho^*})\|\beta(s)\|^2$. Then taking the expectation of (2.28), for all $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le t}\|u(r)\|^2\right) \le \mathbb{E}(\|u(0)\|^2) + \mathbb{E}\left(\int_0^t M_1(s)\|u(s)\|^2 ds\right)\\
&+ \frac{2\rho}{1-\rho^*}\left(\max_{t\in[0,\mathcal{T}]}\beta_0(t) + 4\epsilon^2 \sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\right)\mathbb{E}(\|\varphi\|^2_{C_\rho}) + \left(\min_{t\in[0,\mathcal{T}]}\beta_0(t)\|\alpha\|^2 + 4\epsilon^2\|\gamma\|^2\right)T\\
&+ \mathbb{E}\left(\int_0^t q_2^{-1}(s)\|g(s)\|^2 ds\right) + 2\epsilon^2\mathbb{E}\left(\int_0^t \sum_{j=1}^\infty \|h_j(s)\|^2 ds\right)\\
&+ 2\epsilon\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r \sum_{j=1}^\infty \big(h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))), u(s)\big) dW_j(s)\right|\right). \qquad (2.29)
\end{aligned}$$

For the last term of the right-hand side of (2.29), by the Burkhölder-Davis-Gundy inequality and (2.27), for all $t \in [0, T]$,

$$\begin{aligned}
&2\epsilon\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r \sum_{j=1}^\infty \big(h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))), u(s)\big) dW_j(s)\right|\right)\\
&\le C_0\epsilon\mathbb{E}\left(\int_0^t \sum_{j=1}^\infty \|u(s)\|^2\|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds\right)^{\frac12}\\
&\le C_0\epsilon\mathbb{E}\left(\sup_{0\le s\le t}\|u(s)\|\left(\int_0^t \sum_{j=1}^\infty \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds\right)^{\frac12}\right)\\
&\le \frac12\mathbb{E}\left(\sup_{0\le s\le t}\|u(s)\|^2\right) + \frac12 C_0^2\epsilon^2\mathbb{E}\left(\int_0^t \sum_{j=1}^\infty \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds\right)\\
&\le \frac12\mathbb{E}\left(\sup_{0\le s\le t}\|u(s)\|^2\right) + 4C_0^2\epsilon^2(1 + \frac{1}{1-\rho^*})\mathbb{E}\left(\int_0^t \|\beta(s)\|^2\|u(s)\|^2 ds\right)\\
&+ \frac{4C_0^2\epsilon^2\rho}{1-\rho^*}\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\mathbb{E}(\|\varphi\|^2_{C_\rho}) + C_0^2\epsilon^2\int_0^t (\sum_{j=1}^\infty \|h_j(s)\|^2 + 2\|\gamma\|^2) ds, \qquad (2.30)
\end{aligned}$$

where C_0 is a positive constant. Combining (2.29), (2.30) and $\|u(0)\|^2 \le \|\varphi\|^2_{C_\rho}$, we find

$$\mathbb{E}\left(\sup_{-\tau\le r\le t}\|u(r)\|^2\right) \le M_2\mathbb{E}(\|\varphi\|^2_{C_\rho}) + M_3\mathbb{E}\left(\int_0^t \sup_{0\le r\le s}\|u(s)\|^2 ds\right) + M_4 T + M_5, \qquad (2.31)$$

where

$$\begin{aligned}
&M_2 = 3 + \frac{4\rho}{1-\rho^*}\left(\max_{t\in[0,\mathcal{T}]}\beta_0(t) + (4\epsilon^2 + 2C_0^2\epsilon^2)\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2\right),\\
&M_3 = 2\max_{t\in[0,\mathcal{T}]}M_1(t) + 16C_0^2\epsilon^2(1 + \frac{1}{1-\rho^*})\sup_{t\in[0,\mathcal{T}]}\|\beta(t)\|^2,\\
&M_4 = 2\left(\min_{t\in[0,\mathcal{T}]}\beta_0(t)\|\alpha\|^2 + (4\epsilon^2 + 2C_0^2\epsilon^2)\|\gamma\|^2\right),\\
&M_5 = \max\left\{2\min_{t\in[0,\mathcal{T}]}q_2(t), 4\epsilon^2 + 2C_0^2\epsilon^2\right\}\big(\|g\|^2_{C([0,\mathcal{T}],\ell^2)} + \|h\|^2_{C([0,\mathcal{T}],\ell^2)}\big),
\end{aligned}$$

are independent of φ and T. It yields from (2.31) and the Gronwall inequality that for all $t \in [0, T]$ with $T > 0$,

$$\mathbb{E}\left(\sup_{-\tau \le r \le t} \|u(r)\|^2 \right) \le \left\{ M_2 \mathbb{E}(\|\varphi\|^2_{C_\rho}) + M_4 T + M_5 \right\} e^{M_3 T}. \tag{2.32}$$

Therefore, the conclusion (2.22) can be obtained. □

Remark 2.3. We use the method mentioned in [4, Theorem 1] to consider the existence and uniqueness of solutions to the system (2.15) in the general case that $\rho^* \le 0$ as assumed in **A0**. By **A0** we find that $\varrho(t)$ is non-increasing and non-negative, so there are only the following three possibilities:

(a) $\lim_{t \to +\infty} \varrho(t) = \gamma$ for some $\gamma > 0$.

Since $\varrho(t)$ is non-increasing, we have $\inf_{t \in [0, +\infty)} \varrho(t) = \gamma$ and $\varrho(t) \ge \gamma$ for $0 \le t \le \gamma$, so $t - \varrho(t) \le t - \gamma \le 0$ for $0 \le t \le \gamma$. Then the system (2.15) on $[0, \gamma]$ can be considered as:

$$\begin{cases} du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2 dt + u^3(t)dt \\ \quad = f(t, u(t), \varphi(t - \varrho(t)))dt + g(t)dt + \epsilon \sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), \varphi(t - \varrho(t)))\big) dW_j(t), \forall t \in [0, \gamma], \\ u(0) = \varphi(0), \end{cases} \tag{2.33}$$

which is a non-delay system. Since $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ and $q_3 = (q_{3,i})_{i \in \mathbb{Z}} : \mathbb{R} \to \ell^2$ are positive, continuous, and $\mathcal{T}$-periodic functions as defined in (2.9), we can deduce from [34, Theorem 3] that problem (2.33) has a unique solution u on $[0, \gamma]$ such that $u \in L^2(\Omega, C([0, \gamma], \ell^2))$. For all $k \ge 0$, repeating this procedure, the solution u can be extended from the interval $[k\gamma, (k+1)\gamma]$ to $[0, \infty)$, so that $u \in L^2(\Omega, C([0, T], \ell^2))$ for any $T > 0$.

(b) $\lim_{t \to +\infty} \varrho(t) = 0$, but $\varrho(t) > 0$ for any $t \ge 0$.

We choose an increasing sequence $\{t_k\}_{k \ge 0}$ such that $t_0 = 0, t_k \uparrow \infty$ and

$$\varrho(t_{k+1}) = t_{k+1} + \varrho(t_{k+1}) - t_{k+1} > t_k + \varrho(t_{k+1}) - t_{k+1} > 0,$$

which means that $t_{k+1} - \varrho(t_{k+1}) < t_k$. Similar to **(a)**, our system can be solved on $[t_k, t_{k+1}]$ for $k \ge 0$, and hence the solution u can be extended to the entire interval $[0, \infty)$.

(c) There exists $T_\rho > 0$ such that $\varrho(t) > 0$ for $t < T_\rho$, but $\varrho(t) = 0$ for $t \ge T_\rho$.

When $t < T_\rho$, we can adopt the same method as in **(b)**. When $t \ge T_\rho$, system (2.15) becomes

$$\begin{aligned} &du(t) + q_1(t)\big[Du(t) - 2Au(t)\big]dt + q_2(t)u(t)dt + q_3(t)|Bu(t)|^2 dt + u^3(t)dt \\ &\quad = f(t, u(t), u(t))dt + g(t)dt + \epsilon \sum_{j=1}^{\infty} \big(h_j(t) + \sigma_j(t, u(t), u(t))\big) dW_j(t), \forall t \in [T_\rho, \infty), \end{aligned} \tag{2.34}$$

with initial data $u(T_\rho) \in L^2(\Omega, \ell^2)$. It is easy to find that (2.34) is an equation without delay, and similar to (2.33), the existence of the solution is obvious.

3. Uniform estimates of solutions

In this part, we will show some estimates of solutions for the stochastic delay lattice system (1.1). For this purpose, we assume that if $q_1(t), \beta_0(t) \in \mathbb{R}^+$ and $q_3(t), \beta(t) \in \ell^2$ are small enough or $q_2(t)$ is large enough, there exists $p \geq 2$ such that

$$\min_{t\in[0,\mathcal{T}]} q_2(t) \geq 16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + 2^{4-\frac{2}{p}} p^{-1}(p-1)^{1-\frac{1}{p}} \max_{t\in[0,\mathcal{T}]} \beta_0(t) + 4(3p-4) \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2. \tag{3.1}$$

For all $t \in \mathbb{R}$, we set

$$\begin{cases} \Theta_1(t) = \dfrac{p}{2} q_2(t) - 8pq_1(t) - 4p\|q_3(t)\|^2 - 2^{3-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\beta_0(t) - 2p(3p-4)\|\beta(t)\|^2, \\ \Theta_2(t) = 2^{2-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\beta_0(t) + 8(\dfrac{p}{2})^{1-\frac{p}{2}}(p-1)^{\frac{p}{2}}\|\beta(t)\|^2, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \overline{\Theta}_1(t) = q_2(t) - 2q_1(t) - 8\|q_3(t)\|^2 - 4\beta_0(t) - 8\|\beta(t)\|^2, \\ \overline{\Theta}_2(t) = 2\beta_0(t) + 8\|\beta(t)\|^2. \end{cases} \tag{3.3}$$

By (3.1) and $p \geq 2$, one can obtain $\Theta_1(t) \geq 0$ directly. Since $p \geq 2$, we can verify that

$$2^{4-\frac{2}{p}}(p-1)^{1-\frac{1}{p}} \geq 4p, \quad 4p(3p-4) \geq 8p, \tag{3.4}$$

which, together with (3.1), implies $\overline{\Theta}_1(t) \geq 0$.

We also need to assume that

$$\chi = \int_0^{\mathcal{T}} \Big(\Theta_1(s) - \Theta_2(s)e^{\int_{s-\rho}^{s}\Theta_1(r)dr}\Big)ds > 0, \tag{3.5}$$

and

$$\overline{\chi} = \int_0^{\mathcal{T}} \Big(\overline{\Theta}_1(s) - \overline{\Theta}_2(s)e^{\int_{s-\rho}^{s}\overline{\Theta}_1(r)dr}\Big)ds > 0. \tag{3.6}$$

In addition, the following lemma will be very helpful in computing uniform estimates of solutions.

Lemma 3.1. *[17, Lemma 3.1] Suppose $v \in C([t_0-\tau,\infty),\mathbb{R}^+)$ is a solution of the delay inequality*

$$\begin{cases} D^+v(t) \leq -\kappa_1(t)v(t) + \kappa_2(t)v(t-\tau_0(t)) + \kappa_3, \quad t > t_0, \\ v(t_0+s) \leq \phi(s), \quad s \in [-\tau,0], \end{cases} \tag{3.7}$$

where $D^+v(t)$ is the upper right-hand Dini derivative of v at $t, t_0 \in \mathbb{R}$, $\tau > 0$ and $\kappa_3 \geq 0, \phi \in C([-\tau,0],\mathbb{R}^+), \tau_0 \in C([t_0,\infty),(0,\tau])$, $\kappa_1(t)$ and $\kappa_2(t), t \in \mathbb{R}$, are nonnegative, continuous, $\mathcal{T}$-periodic functions. Assume that the average of the function $\eta(t) = \kappa_1(t) - \kappa_2(t)e^{\int_{t-\tau}^{t}\kappa_1(r)dr}$ on $[0,\mathcal{T}]$ is positive; that is,

$$\lambda = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \eta(t)dt > 0. \tag{3.8}$$

Then, there exist positive constants $K = K(\kappa_1,\kappa_2,\tau,\mathcal{T}) > 0$ and $G = G(\kappa_1,\kappa_2,\tau,\mathcal{T}) > 0$ such that, for $t \geq t_0$,

$$v(t) \leq K\phi(0)e^{-\lambda(t-t_0)} + G(\kappa_3 + \|\kappa_2\|_{C([0,\mathcal{T}],\mathbb{R}^+)}\|\phi\|_{C([-\tau,0],\mathbb{R}^+)}). \tag{3.9}$$

We now apply Lemma 3.1 to establish the following uniform estimate.

Lemma 3.2. *Suppose* **A1-A6**, *(3.1) and (3.5) hold. If $\varphi \in L^p(\Omega, C_\rho)$ with $p \geq 2$, then for $\epsilon \in (0,1]$, the solution of system (2.15) satisfies for all $t \geq -\rho$,*

$$\mathbb{E}(\|u(t)\|^p) \leq C_1 \mathbb{E}(\|\varphi\|_{C_\rho}^p)(e^{-\lambda t} + 1) + C_1, \tag{3.10}$$

where $C_1 > 0$ may depend on p, but not on t, ϵ, ρ or φ.

Proof. For every $t \geq 0$ and $R > 0$, we define the stopping time

$$\eta_R = \inf\{s \geq t : \|u(s)\| > R\}, \tag{3.11}$$

where $\eta_R = +\infty$ if $\{s \geq t : \|u(s)\| > R\} = \emptyset$.

Given $\triangle t \geq 0$, by (2.23) and Ito's theorem in [1, P92], we have

$$\begin{aligned}
&\mathbb{E}(\|u((t+\triangle t)\wedge\eta_R)\|^p) + p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} q_1(s)\|u(s)\|^{p-2}\|Au(t)\|^2 ds\Big)\\
&- 2p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} q_1(s)\|u(s)\|^{p-2}\|Bu(s)\|^2 ds\Big) + p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} q_2(s)\|u(s)\|^p ds\Big)\\
&+ p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\|u(s)\|_4^4 ds\Big) + p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(q_3(s)|Bu(s)|^2, u(s)\big) ds\Big)\\
&= \mathbb{E}(\|u(t)\|^p) + p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(f(s,u(s),u(s-\varrho(s))), u(s)\big) ds\Big)\\
&+ p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(g(s), u(s)\big) ds\Big)\\
&+ \frac{p}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\|h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s)))\|^2 ds\Big)\\
&+ \frac{p(p-2)}{2}\epsilon^2\mathbb{E}\Big(\sum_{j=1}^{\infty}\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-4}\big|\big(h_j(s) + \sigma_j(s,u(s),u(s-\varrho(s))), u(s)\big)\big|^2 ds\Big).
\end{aligned} \tag{3.12}$$

For the last term on the left-hand side of (3.12), we obtain

$$\begin{aligned}
&- p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\big(q_3(s)|Bu(s)|^2, u(s)\big) ds\Big)\\
&\leq p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\sum_{i\in\mathbb{Z}}|q_{3,i}(s)|\big|u_{i+1}(s) - u_i(s)|^2\big|u_i(s)| ds\Big)\\
&\leq p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\Big(4\sum_{i\in\mathbb{Z}}|q_{3,i}(s)|^2|u_i(s)|^2\\
&\quad + \frac{1}{16}\sum_{i\in\mathbb{Z}}\big|u_{i+1}^2(t) - 2u_{i+1}(t)u_i(t) + u_i^2(t)\big|^2\Big) ds\Big)\\
&\leq 4p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|q_3(s)\|^2\|u(s)\|^p ds\Big) + p\mathbb{E}\Big(\int_t^{(t+\triangle t)\wedge\eta_R} \|u(s)\|^{p-2}\|u(s)\|_4^4 ds\Big).
\end{aligned} \tag{3.13}$$

For the second term on the right-hand side of (3.12), by (2.17) and Young's inequality

$$ab \leq \frac{(p-1)\varepsilon_0}{p} a^{\frac{p}{p-1}} + \frac{1}{p\varepsilon_0^{p-1}} b^p, \quad \forall \varepsilon_0 > 0,$$

we can deduce

$$\begin{aligned}
& p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \big(f(s, u(s), u(s-\varrho(s))), u(s) \big) ds \bigg) \\
& \leq p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-1} \|f(s, u(s), u(s-\varrho(s)))\| ds \bigg) \\
& \leq 2^{2-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0(s) \|u(s)\|^p ds \bigg) \\
& + 2^{4-2p-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0^{1-p}(s) \|f(s, u(s), u(s-\varrho(s)))\|^p ds \bigg) \\
& \leq 2^{2-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0(s) \|u(s)\|^p ds \bigg) + 2^{4-2p-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \times \\
& \times \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \Big[2^{2p-2} \beta_0(s) \big(\|u(s)\|^p + \|u(s-\varrho(s))\|^p \big) + 2^{p-1} \beta_0^{1-p}(s) \|\alpha\|^p \Big] ds \bigg) \\
& \leq 2^{3-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0(s) \|u(s)\|^p ds \bigg) \\
& + 2^{2-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0(s) \|u(s-\varrho(s))\|^p ds \bigg) \\
& + 2^{3-p-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0^{1-p}(s) \|\alpha\|^p ds \bigg). \qquad (3.14)
\end{aligned}$$

For the third term on the right-hand side of (3.12), we derive

$$\begin{aligned}
& p\mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \big(g(s), u(s) \big) ds \bigg) \qquad (3.15) \\
& \leq \frac{p}{4} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2(s) \|u(s)\|^p ds \bigg) + 4^{p-1} \Big(\frac{p-1}{p} \Big)^{p-1} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2^{1-p}(s) \|g(s)\|^p ds \bigg).
\end{aligned}$$

For the last two terms on the right-hand side of (3.12), by (2.19) we have

$$\begin{aligned}
& \frac{p}{2} \epsilon^2 \mathbb{E}\bigg(\sum_{j=1}^\infty \int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds \bigg) \\
& + \frac{p(p-2)}{2} \epsilon^2 \mathbb{E}\bigg(\sum_{j=1}^\infty \int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-4} \big| \big(h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))), u(s) \big) \big|^2 ds \bigg) \\
& \leq \frac{p(p-1)}{2} \epsilon^2 \mathbb{E}\bigg(\sum_{j=1}^\infty \int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds \bigg) \\
& \leq \frac{p(p-1)}{2} \epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \Big(2\|h(s)\|^2 + 2\|\sigma_j(s, u(s), u(s-\varrho(s)))\|^2 \Big) ds \bigg) \\
& \leq I_1 + I_2 + I_3, \qquad (3.16)
\end{aligned}$$

where

$$I_1 = 4p(p-1)\epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|\beta(s)\|^2 \|u(s)\|^p ds \bigg),$$

$$I_2 = 4p(p-1)\epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|\beta(s)\|^2 \|u(s)\|^{p-2} \|u(s-\varrho(s))\|^2 ds \bigg),$$

$$I_3 = p(p-1)\epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} (\|h(s)\|^2 + 2\|\gamma\|^2) ds \bigg).$$

The Young inequality yields

$$I_2 \le 2p(p-2)\epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|\beta(s)\|^2 \|u(s)\|^p ds \bigg) + 8(\frac{p}{2})^{1-\frac{p}{2}} (p-1)^{\frac{p}{2}} \epsilon^2 \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|\beta(s)\|^2 \|u(s-\varrho(s))\|^p ds \bigg).$$

Similarly,

$$I_3 \le \frac{p}{4} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2(s) \|u(s)\|^p ds \bigg) + \frac{2^{p-1}\epsilon^p (p-1)^{\frac{p}{2}} (p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2^{1-\frac{p}{2}}(s) (\|h(s)\|^2 + 2\|\gamma\|^2)^{\frac{p}{2}} ds \bigg).$$

Therefore, by $\epsilon \in (0,1]$ and (3.16),

$$\begin{aligned}
&\frac{p}{2}\epsilon^2 \mathbb{E}\bigg(\sum_{j=1}^{\infty} \int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-2} \|h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds \bigg) \\
&+ \frac{p(p-2)}{2}\epsilon^2 \mathbb{E}\bigg(\sum_{j=1}^{\infty} \int_t^{(t+\triangle t)\wedge \eta_R} \|u(s)\|^{p-4} \big| \big(h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s))), u(s)\big) \big|^2 ds \bigg) \\
&\le \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \Big[2p(3p-4)\|\beta(s)\|^2 + \frac{p}{4} q_2(s) \Big] \|u(s)\|^p \bigg) \\
&+ 8(\frac{p}{2})^{1-\frac{p}{2}} (p-1)^{\frac{p}{2}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \|\beta(s)\|^2 \|u(s-\varrho(s))\|^p ds \bigg) \\
&+ \frac{2^{p-1} (p-1)^{\frac{p}{2}} (p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2^{1-\frac{p}{2}}(s) (\|h(s)\|^2 + 2\|\gamma\|^2)^{\frac{p}{2}} ds \bigg).
\end{aligned} \tag{3.17}$$

It follows from (3.13)–(3.17) that for all $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}(\|u((t+\triangle t)\wedge \eta_R)\|^p) \\
&\le \mathbb{E}(\|u(t)\|^p) - \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \Theta_1(s) \|u(s)\|^p ds \bigg) + \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \Theta_2(s) \|u(s-\varrho(s))\|^p ds \bigg) \\
&+ 2^{3-p-\frac{2}{p}} (p-1)^{1-\frac{1}{p}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} \beta_0^{1-p}(s) \|\alpha\|^p ds \bigg) \\
&+ 4^{p-1} \Big(\frac{p-1}{p} \Big)^{p-1} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2^{1-p}(s) \|g(s)\|^p ds \bigg) \\
&+ \frac{2^{p-1} (p-1)^{\frac{p}{2}} (p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}} \mathbb{E}\bigg(\int_t^{(t+\triangle t)\wedge \eta_R} q_2^{1-\frac{p}{2}}(s) (\|h(s)\|^2 + 2\|\gamma\|^2)^{\frac{p}{2}} ds \bigg),
\end{aligned} \tag{3.18}$$

where $\Theta_1(t), \Theta_2(t)$ are defined in (3.2). It can be deduced from (3.18) that for all $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}(\|u(t+\triangle t\|^p 1_{\{t+\triangle t<\eta_R\}}) \\
&\leq \mathbb{E}(\|u(t)\|^p) - \mathbb{E}\bigg(\int_t^{t+\triangle t}\Theta_1(s)\|u(s)\|^p ds\bigg) + \mathbb{E}\bigg(\int_t^{t+\triangle t}\Theta_2(s)\|u(s-\varrho(s))\|^p ds\bigg) \\
&+ \triangle t\bigg[2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\|\alpha\|^p \min_{0\leq t\leq \mathcal{T}}\beta_0^{p-1}(t) + 4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1}\min_{0\leq t\leq \mathcal{T}} q_2^{p-1}(t)\|g\|^p_{C([0,\mathcal{T}],\ell^2)} \\
&+ \frac{2^{\frac{3p}{2}-2}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\min_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{2}-1}(t)\big(\|h\|^p_{C([0,\mathcal{T}],\ell^2)} + 2^{\frac{p}{2}}\|\gamma\|^p\big)\bigg],
\end{aligned}
\tag{3.19}
$$

Thanks to $\lim_{n\to\infty}\eta_R = +\infty$ and the continuity of solutions, we find from (3.19) that, for all $t \geq 0$,

$$
\begin{aligned}
&D^+\mathbb{E}(\|u(t)\|^p) \\
&\leq -\Theta_1(t)\mathbb{E}\big(\|u(t)\|^p\big) + \Theta_2(t)\mathbb{E}\big(\|u(t-\varrho(t))\|^p\big) + 2^{3-p-\frac{2}{p}}(p-1)^{1-\frac{1}{p}}\|\alpha\|^p\min_{0\leq t\leq \mathcal{T}}\beta_0^{p-1}(t) \\
&+ 4^{p-1}\Big(\frac{p-1}{p}\Big)^{p-1}\min_{0\leq t\leq \mathcal{T}} q_2^{p-1}(t)\|g\|^p_{C([0,\mathcal{T}],\ell^2)} \\
&+ \frac{2^{\frac{3p}{2}-2}(p-1)^{\frac{p}{2}}(p-2)^{\frac{p}{2}-1}}{p^{\frac{p}{2}-1}}\min_{0\leq t\leq \mathcal{T}} q_2^{\frac{p}{2}-1}(t)\big(\|h\|^p_{C([0,\mathcal{T}],\ell^2)} + 2^{\frac{p}{2}}\|\gamma\|^p\big).
\end{aligned}
\tag{3.20}
$$

Applying Lemma 3.1 to (3.20), then there exists $\lambda = \frac{\chi}{\mathcal{T}} > 0$ such that, for all $t \geq 0$,

$$
\mathbb{E}(\|u(t)\|^p) \leq c\|\varphi(0)\|^p e^{-\lambda t} + c(1+\|\varphi\|^p_{C_\rho}). \tag{3.21}
$$

Note that for all $t \in [-\rho, 0]$,

$$
\mathbb{E}(\|u(t)\|^p) \leq \mathbb{E}(\|\varphi\|^p_{C_\rho}), \tag{3.22}
$$

which, together with (3.21) and the fact that $\|\varphi(0)\| \leq \sup_{s\in[-\rho,0]}\|\varphi(s)\| = \|\varphi\|_{C_\rho}$, can conclude (3.10). □

As a consequence of Lemma 3.2, it can be immediately deduced that:

Lemma 3.3. *Suppose* **A1**-**A6**, (3.1) *and* (3.5) *hold. If $\varphi \in L^p(\Omega, C_\rho)$ with $p \geq 6$, then for $\epsilon \in (0,1]$, the solution of system* (2.15) *satisfies for all $t > r \geq 0$,*

$$
\mathbb{E}(\|u(t)-u(r)\|^{\frac{p}{3}}) \leq C_2\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big)(|t-r|^{\frac{p}{3}} + |t-r|^{\frac{p}{6}}), \tag{3.23}
$$

where $C_2 > 0$ is depending on p, but not on ϵ, ρ, t, r or φ.

Proof. From (2.15), we have that, for $t > r \geq 0$,

$$
\begin{aligned}
u(t)-u(r) = &-\int_r^t q_1(s)[Du(s)-2Au(s)]ds - \int_r^t q_2(s)u(s)ds \\
&-\int_r^t q_3(s)|Bu(s)|^2 ds - \int_r^t u^3(s)ds + \int_r^t \big(f(s,u(s),u(s-\varrho(s))) + g(s)\big)ds \\
&+\epsilon\sum_{j=1}^\infty\int_r^t (h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))))dW_j(s).
\end{aligned}
\tag{3.24}
$$

We infer from (3.24) that

$$\begin{aligned}
&\mathbb{E}(\|u(t)-u(r)\|^{\frac{p}{3}}) \le 7^{\frac{p}{3}-1}24^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t q_1(s)\|u(s)\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|ds\Big)^{\frac{p}{3}} \\
&+ 7^{\frac{p}{3}-1}4^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t \|q_3(s)\|\|u(s)\|^2 ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|^3 ds\Big)^{\frac{p}{3}} \\
&+ 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t \|f(s,u(s),u(s-\varrho(s)))\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}}\|g\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)} \\
&+ 7^{\frac{p}{3}-1}\epsilon^{\frac{p}{3}}\mathbb{E}\Big(\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))))dW_j(s)\Big\|^{\frac{p}{3}}\Big). \qquad (3.25)
\end{aligned}$$

For the first and second terms on the right-hand of (3.25), by using Hölder's inequality and the conclusion of Lemma 3.2, it can be concluded that, for $p \ge 6$ and $t > r \ge 0$,

$$\begin{aligned}
&7^{\frac{p}{3}-1}24^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t q_1(s)\|u(s)\|ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|ds\Big)^{\frac{p}{3}} \\
&\le 7^{\frac{p}{3}-1}24^{\frac{p}{3}}\Big(\max_{0\le t\le \mathcal{T}} q_1^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds \\
&+ 7^{\frac{p}{3}-1}\Big(\max_{0\le t\le \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds \\
&\le c_1\Big[\Big(\max_{0\le t\le \mathcal{T}} q_1^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1} + \Big(\max_{0\le t\le \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1}\Big]\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big)|t-r|^{\frac{p}{3}}. \qquad (3.26)
\end{aligned}$$

Similarly, for $p \ge 6$ and $t > r \ge 0$, the third and fourth terms on the right-hand of (3.25) satisfy

$$\begin{aligned}
&7^{\frac{p}{3}-1}4^{\frac{p}{3}}\mathbb{E}\Big(\int_r^t \|q_3(s)\|\|u(s)\|^2 ds\Big)^{\frac{p}{3}} + 7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t q_2(s)\|u(s)\|^3 ds\Big)^{\frac{p}{3}} \\
&\le 7^{\frac{p}{3}-1}4^{\frac{p}{3}}\Big(\sup_{0\le t\le \mathcal{T}} \|q_3(t)\|^{\frac{p}{p-3}}\Big)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{\frac{2p}{3}})ds \\
&+ 7^{\frac{p}{3}-1}\Big(\max_{0\le t\le \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}-1}\int_r^t \mathbb{E}(\|u(s)\|^{p})ds \qquad (3.27) \\
&\le c_2\Big[\Big(\sup_{0\le t\le \mathcal{T}} \|q_3(t)\|^{\frac{p}{p-3}}\Big)^{\frac{p}{3}-1}\big(1+\mathbb{E}(\|\varphi\|^{\frac{2p}{3}}_{C_\rho})\big) + \Big(\max_{0\le t\le \mathcal{T}} q_2^{\frac{p}{p-3}}(t)\Big)^{\frac{p}{3}-1}\big(1+\mathbb{E}(\|\varphi\|^{p}_{C_\rho})\big)\Big]|t-r|^{\frac{p}{3}}.
\end{aligned}$$

For the fifth term on the right-hand of (3.25), by (2.17) and (3.10), we have for $t > r \ge 0$,

$$\begin{aligned}
&7^{\frac{p}{3}-1}\mathbb{E}\Big(\int_r^t \|f(s,u(s),u(s-\varrho(s)))\|ds\Big)^{\frac{p}{3}} \\
&\le 7^{\frac{p}{3}-1}2^{\frac{p}{6}-1}4^{\frac{p}{3}}|t-r|^{\frac{p}{3}-1}\int_r^t \beta_0^{\frac{p}{3}}(s)\mathbb{E}(\|u(s)\|^{\frac{p}{3}})ds \\
&+ 7^{\frac{p}{3}-1}2^{\frac{p}{6}-1}4^{\frac{p}{3}}|t-r|^{\frac{p}{3}-1}\int_r^t \beta_0^{\frac{p}{3}}(s)\mathbb{E}(\|u(s-\varrho(s))\|^{\frac{p}{3}})ds + 7^{\frac{p}{3}-1}2^{\frac{p}{3}-1}|t-r|^{\frac{p}{3}}\|\alpha\|^{\frac{p}{3}} \\
&\le c_3\Big(\max_{0\le t\le \mathcal{T}} \beta_0^{\frac{p}{3}}(t)\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big) + \|\alpha\|^{\frac{p}{3}}\Big)|t-r|^{\frac{p}{3}}. \qquad (3.28)
\end{aligned}$$

By (2.19), (3.10) and the Burkholder-Davis-Gundy inequality, for all $\epsilon \in (0,1]$ and $t > r \geq 0$,

$$
\begin{aligned}
&7^{\frac{p}{3}-1}\epsilon^{\frac{p}{3}}\mathbb{E}\Big(\Big\|\sum_{j=1}^{\infty}\int_r^t (h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))))dW_j(s)\Big\|^{\frac{p}{3}}\Big)\\
&\leq C_0\mathbb{E}\Big(\int_r^t\sum_{j=1}^{\infty}\|h_j(s)+\sigma_j(s,u(s),u(s-\varrho(s))))\|^2 ds\Big)^{\frac{p}{6}}\\
&\leq C_0\mathbb{E}\Big(\int_r^t\Big[2\|h(s)\|^2+8\|\beta(s)\|^2(\|u(s)\|^2+\|u(s-\varrho(s))\|^2)+4\|\gamma\|^2\Big]ds\Big)^{\frac{p}{6}}\\
&\leq 4^{\frac{p}{6}-1}2^{\frac{p}{6}}C_0|t-r|^{\frac{p}{6}}\|h\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)}+4^{\frac{p}{6}-1}8^{\frac{p}{6}}C_0|t-r|^{\frac{p}{6}-1}\mathbb{E}\Big(\int_r^t\|\beta(s)\|^{\frac{p}{3}}\|u(s)\|^{\frac{p}{3}}ds\Big)\\
&+4^{\frac{p}{6}-1}8^{\frac{p}{6}}C_0|t-r|^{\frac{p}{6}-1}\mathbb{E}\Big(\int_r^t\|\beta(s)\|^{\frac{p}{3}}\|u(s-\varrho(s))\|^{\frac{p}{3}}ds\Big)+4^{\frac{p}{3}-1}C_0|t-r|^{\frac{p}{6}}\|\gamma\|^{\frac{p}{3}}\\
&\leq c_4\Big(\sup_{0\leq t\leq\mathcal{T}}\|\beta(t)\|^{\frac{p}{3}}\big(1+\mathbb{E}(\|\varphi\|^{\frac{p}{3}}_{C_\rho})\big)+\|h\|^{\frac{p}{3}}_{C([0,\mathcal{T}],\ell^2)}+\|\gamma\|^{\frac{p}{3}}\Big)|t-r|^{\frac{p}{6}}. \qquad (3.29)
\end{aligned}
$$

Substituting (3.26) to (3.29) into (3.25), the desired result (3.23) can be obtained. □

Next, we present uniform estimates for the tails of the solution to the system (2.15).

Lemma 3.4. *Suppose* **A1**-**A6**, (3.1) *and* (3.6) *hold. If* $\varphi \in L^2(\Omega, C_\rho)$, *then for* $\epsilon \in (0,1]$,

$$\limsup_{k\to\infty}\sup_{t\geq-\rho}\sum_{|i|\geq k}\mathbb{E}\big(|u_i(t)|^2\big)=0. \qquad (3.30)$$

Proof. Consider a smooth function $\vartheta : \mathbb{R} \to [0,1]$ such that

$$\vartheta(r)=\begin{cases}0, & \text{for } |r|\leq 1,\\ 1, & \text{for } |r|\geq 2,\end{cases} \qquad (3.31)$$

and define a constant $c_0 > 0$ such that $|\vartheta'(r)| \leq c_0$ uniformly for $r \in \mathbb{R}$.

Given $k \in \mathbb{N}$, define

$$\vartheta_k u=(\vartheta_{k,i}u_i)_{i\in\mathbb{Z}}=\Big(\vartheta(\frac{|i|}{k})u_i\Big)_{i\in\mathbb{Z}} \quad \text{for} \quad u=(u_i)_{i\in\mathbb{Z}}. \qquad (3.32)$$

By (2.15), we have

$$
\begin{aligned}
&d(\vartheta_k u(t))+q_1(t)\vartheta_k Du(t)dt-2q_1(t)\vartheta_k Au(t)dt+q_2(t)\vartheta_k u(t)dt\\
&+q_3(t)\vartheta_k|Bu(t)|^2dt+\vartheta_k u^3(t)dt \qquad (3.33)\\
&=\vartheta_k f(t,u(t),u(t-\varrho(t)))dt+\vartheta_k g(t)dt+\epsilon\sum_{j=1}^{\infty}\big(\vartheta_k h_j(t)+\vartheta_k\sigma_j(t,u(t),u(t-\varrho(t)))\big)dW_j(t).
\end{aligned}
$$

Applying Ito's formula to (3.33), and taking expectation, for all $t \geq 0$, we have

$$
\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t)\|^2) + 2\int_0^t q_1(s)\mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&- 4\int_0^t q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds + 2\int_0^t q_2(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 2\int_0^t \mathbb{E}(q_3(s)\vartheta_k|Bu(s)|^2, \vartheta_k u(s))ds + 2\int_0^t \mathbb{E}(\vartheta_k u^3(s), \vartheta_k u(s))ds \\
&= \mathbb{E}(\|\vartheta_k u(0)\|^2) + 2\int_0^t \mathbb{E}(\vartheta_k f(s, u(s), u(s-\varrho(s))), \vartheta_k u(s))ds \\
&+ 2\int_0^t \mathbb{E}(\vartheta_k g(s), \vartheta_k u(s))ds + \epsilon^2\sum_{j=1}^{\infty}\int_0^t \mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2)ds. \quad (3.34)
\end{aligned}
$$

It follows from (3.34) that, for $\triangle t \geq 0$ and $t \geq 0$,

$$
\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2) + 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&- 4\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds + 2\int_t^{t+\triangle t} q_2(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 2\int_t^{t+\triangle t} \mathbb{E}(q_3(s)|Bu(s)|^2, \vartheta_k^2 u(s))ds + 2\int_t^{t+\triangle t} \mathbb{E}(u^3(s), \vartheta_k^2 u(s))ds \quad (3.35) \\
&= \mathbb{E}(\|\vartheta_k u(t)\|^2) + 2\int_t^{t+\triangle t} \mathbb{E}(\vartheta_k f(s, u(s), u(s-\varrho(s))), \vartheta_k u(s))ds \\
&+ 2\int_t^{t+\triangle t} \mathbb{E}(\vartheta_k g(s), \vartheta_k u(s))ds + \epsilon^2\sum_{j=1}^{\infty}\int_t^{t+\triangle t} \mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2)ds.
\end{aligned}
$$

For the second term on the left-hand side of (3.35), by using the result in [37, Lemma 7], yields

$$
\begin{aligned}
&2\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&\geq 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k Au(s)\|^2)ds - \frac{136c_0}{k}\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds. \quad (3.36)
\end{aligned}
$$

For the third term on the left-hand side of (3.35), by the Young inequality, we obtain

$$
\begin{aligned}
&4\int_t^{t+\triangle t} q_1(s)\mathbb{E}\big(Au(s), \vartheta_k^2 u(s)\big)ds \\
&\leq 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k Au(s)\|^2)ds + 2\int_t^{t+\triangle t} q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds. \quad (3.37)
\end{aligned}
$$

For the penultimate term on the left-hand side of (3.35), by considering the definition of B in

(2.10), we derive

$$
\begin{aligned}
&-2\int_t^{t+\triangle t}\mathbb{E}(q_3(s)|Bu(s)|^2,\vartheta_k^2u(s))ds\\
&\leq 2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big||u_{i+1}(s)-u_i(s)|^2\big||u_i(s)|\Big)ds\\
&\leq 4\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big(u_{i+1}^2(s)+u_i^2(s)\big)|u_i(s)|\Big)ds\\
&\leq 8\int_t^{t+\triangle t}\sum_{|i|\geq k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_ku(s)\|^2)ds+2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})|u_i(s)|^4\Big)ds. \qquad (3.38)
\end{aligned}
$$

For the second and third terms on the left-hand side of (3.35), it follows from (2.17) and Young's inequality that

$$
\begin{aligned}
&2\int_t^{t+\triangle t}\mathbb{E}\big(\vartheta_kf(s,u(s),u(s-\varrho(s))),\vartheta_ku(s)\big)ds+2\int_t^{t+\triangle t}\mathbb{E}\big(\vartheta_kg(s),\vartheta_ku(s)\big)ds\\
&\leq\frac{1}{2}\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})\big(4\beta_0^2(s)(|u_i(s)|^2+|u_i(s-\varrho(s))|^2)+2|\alpha_i|^2\big)\Big)ds\\
&+2\int_t^{t+\triangle t}\beta_0(s)\mathbb{E}(\|\vartheta_ku(s)\|^2)ds+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\geq k}g_i^2(s)ds+\int_t^{t+\triangle t}q_2(s)\mathbb{E}(\|\vartheta_ku(s)\|^2)ds\\
&\leq\int_t^{t+\triangle t}\big(4\beta_0(s)+q_2(s)\big)\mathbb{E}(\|\vartheta_ku(s)\|^2)ds+2\int_t^{t+\triangle t}\beta_0(s)\mathbb{E}(\|\vartheta_ku(s-\varrho(s))\|^2)ds\\
&+\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\sum_{|i|\geq k}|\alpha_i|^2ds+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\geq k}g_i^2(s)ds. \qquad (3.39)
\end{aligned}
$$

By (2.19), for $t\geq 0$ and $\epsilon\in(0,1]$, the last term on the left-hand side of (3.35) satisfies

$$
\begin{aligned}
&\epsilon^2\sum_{j=1}^{\infty}\int_t^{t+\triangle t}\mathbb{E}(\|\vartheta_kh_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds\\
&\leq 2\epsilon^2\int_t^{t+\triangle t}\sum_{|i|\geq k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds+2\epsilon^2\int_t^{t+\triangle t}\mathbb{E}\Big(\sum_{i\in\mathbb{Z}}\vartheta^2(\frac{|i|}{k})\big(4|\beta_i(s)|^2(|u_i(s)|^2\\
&+|u_i(s-\varrho(s))|^2)+2|\gamma_i|^2\big)\Big)ds\\
&\leq 2\int_t^{t+\triangle t}\sum_{|i|\geq k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds+8\int_t^{t+\triangle t}\sum_{|i|\geq k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_ku(s)\|^2)ds\\
&+8\int_t^{t+\triangle t}\sum_{|i|\geq k}|\beta_i(s)|^2\mathbb{E}(\|\vartheta_ku(s-\varrho(s))\|^2)ds+4\int_t^{t+\triangle t}\sum_{|i|\geq k}|\gamma_i|^2ds. \qquad (3.40)
\end{aligned}
$$

Plugging (3.36)-(3.40) into (3.35),

$$
\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2)-\mathbb{E}(\|\vartheta_k u(t)\|^2)\\
&\leq -\int_t^{t+\triangle t}\Big[q_2(s)-2q_1(s)-8\sum_{|i|\geq k}|q_{3,i}(s)|^2-4\beta_0(s)-8\sum_{|i|\geq k}|\beta_i(s)|^2\Big]\mathbb{E}(\|\vartheta_k u(s)\|^2)ds\\
&\quad+\int_t^{t+\triangle t}\Big[2\beta_0(s)+8\sum_{|i|\geq k}|\beta_i(s)|^2\Big]\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds+\frac{136c_0}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\\
&\quad+\int_t^{t+\triangle t}\frac{1}{\beta_0(s)}\sum_{|i|\geq k}|\alpha_i|^2ds+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\geq k}g_i^2(s)ds+2\int_t^{t+\triangle t}\sum_{|i|\geq k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds\\
&\quad+4\int_t^{t+\triangle t}\sum_{|i|\geq k}|\gamma_i|^2ds. \qquad (3.41)
\end{aligned}
$$

Thanks to Lemma 3.2, for all $t>0$,

$$
\frac{136c_0}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\leq\frac{c_0}{k}\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\leq t\leq\mathcal{T}}q_1(t)\triangle t, \qquad (3.42)
$$

where c_0 is positive and independent of t,ϵ,k,ρ and φ. Given $\varepsilon>0$, there exists $K_1(\varepsilon)\geq 1$ such that, for all $t\geq 0,\triangle t\geq 0$ and $k\geq K_1(\varepsilon)$,

$$
\frac{136c}{k}\int_t^{t+\triangle t}q_1(s)\mathbb{E}(\|u(s)\|^2)ds\leq\varepsilon\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\leq t\leq\mathcal{T}}q_1(t)\triangle t. \qquad (3.43)
$$

Combining (3.41) and (3.43),

$$
\begin{aligned}
&\mathbb{E}(\|\vartheta_k u(t+\triangle t)\|^2)-\mathbb{E}(\|\vartheta_k u(t)\|^2)\\
&\leq -\int_t^{t+\triangle t}\overline{\Theta}_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds+\int_t^{t+\triangle t}\overline{\Theta}_2(s)\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2)ds\\
&\quad+\varepsilon\big(\mathbb{E}(\|\varphi\|^2_{C_\rho})+1\big)\max_{0\leq t\leq\mathcal{T}}q_1(t)\triangle t+\min_{0\leq t\leq\mathcal{T}}\beta_0^{-1}(t)\sum_{|i|\geq k}|\alpha_i|^2\triangle t+\int_t^{t+\triangle t}\frac{1}{q_2(s)}\sum_{|i|\geq k}g_i^2(s)ds\\
&\quad+2\int_t^{t+\triangle t}\sum_{|i|\geq k}\sum_{j=1}^{\infty}\big|h_{j,i}(s)\big|^2ds+4\sum_{|i|\geq k}|\gamma_i|^2\triangle t, \qquad (3.44)
\end{aligned}
$$

where $\overline{\Theta}_1$ and $\overline{\Theta}_2$ are defined in (3.3). Since $\alpha,\gamma\in\ell^2$, $g(\cdot),h_j(\cdot)\in C(\mathbb{R},\ell^2)$ and are $\mathcal{T}$-periodic in $t\in\mathbb{R}$, for the same $\varepsilon>0$ in (3.43), there exists $K_2(\varepsilon)\geq K_1(\varepsilon)$ such that for all $t\geq 0,\mathcal{T}\geq 0$, and $k\geq K_2(\varepsilon)$,

$$
\sum_{|i|\geq k}|\alpha_i|^2+\sum_{|i|\geq k}|\gamma_i|^2<\varepsilon, \qquad (3.45)
$$

and, for all $t\in[0,\mathcal{T}]$,

$$
\sum_{|i|\geq k}g_i^2(t)+\sum_{|i|\geq k}\sum_{j=1}^{\infty}\big|h_{j,i}(t)\big|^2<\varepsilon. \qquad (3.46)
$$

Therefore, for all $t \geq 0$ and $k \geq K_2(\varepsilon)$,

$$
\begin{aligned}
D^+\mathbb{E}(\|\vartheta_k u(t)\|^2) &\leq -\overline{\Theta}_1(t)\mathbb{E}(\|\vartheta_k u(t)\|^2) + \overline{\Theta}_2(t)\mathbb{E}(\|\vartheta_k u(t-\varrho(t))\|^2) \\
&+ \varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho}) \max_{0\leq t\leq \mathcal{T}} q_1(t) + \max_{0\leq t\leq \mathcal{T}} q_1(t) + \min_{0\leq t\leq \mathcal{T}} \beta_0^{-1}(t) + \min_{0\leq t\leq \mathcal{T}} q_2^{-1}(t) + 6\bigg). \qquad (3.47)
\end{aligned}
$$

By (3.6) and Lemma 3.1, there exists $\overline{\lambda} = \frac{\overline{\chi}}{\mathcal{T}} > 0$ such that, for all $t \geq 0$ and $k \geq K_2(\varepsilon)$,

$$
\begin{aligned}
\mathbb{E}(\|\vartheta_k u(t)\|^2) &\leq C_3\mathbb{E}(\|\vartheta_k \varphi(0)\|^2)e^{-\overline{\lambda}t} + C_4\mathbb{E}(\|\vartheta_k\varphi\|^2_{C_\rho}) \qquad (3.48)\\
&+ C_5\varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho}) \max_{0\leq t\leq \mathcal{T}} q_1(t) + \max_{0\leq t\leq \mathcal{T}} q_1(t) + \min_{0\leq t\leq \mathcal{T}} \beta_0^{-1}(t) + \min_{0\leq t\leq \mathcal{T}} q_2^{-1}(t) + 6\bigg),
\end{aligned}
$$

where $C_3, C_4, C_5 > 0$ are independent of $\epsilon, \varepsilon, \rho$.

Thanks to $\varphi \in L^2(\Omega, C_\rho)$ and the fact that $[-\rho, 0]$ is compact, we infer that the set $\{\varphi(s) \in \ell^2 : s \in [-\rho, 0]\}$ is compact, that is, for $\varepsilon > 0$, there exists $K_3(\varepsilon) \geq K_2(\varepsilon)$ such that for all $s \in [-\rho, 0]$ and $k \geq K_3(\varepsilon)$,

$$
\sum_{|i|\geq k} |\varphi_i(s)|^2 \leq \varepsilon. \qquad (3.49)
$$

Note that for all $t \in [-\rho, 0]$,

$$
\mathbb{E}(\|\vartheta_k u(t)\|^2) \leq \mathbb{E}\bigg(\sum_{|i|\geq k} |\varphi_i(s)|^2\bigg). \qquad (3.50)
$$

From (3.48) to (3.50), for all $t \geq -\rho$ and $k \geq K_3(\varepsilon)$,

$$
\begin{aligned}
&\sum_{|i|\geq 2k} \mathbb{E}(|u_i(t)|^2) \leq \mathbb{E}(\|\vartheta_k u(t)\|^2) \qquad (3.51)\\
&\leq C_3\varepsilon e^{-\overline{\lambda}t} + C_4\varepsilon + C_5\varepsilon\bigg(\mathbb{E}(\|\varphi\|^2_{C_\rho}) \max_{0\leq t\leq \mathcal{T}} q_1(t) + \max_{0\leq t\leq \mathcal{T}} q_1(t) + \min_{0\leq t\leq \mathcal{T}} \beta_0^{-1}(t) + \min_{0\leq t\leq \mathcal{T}} q_2^{-1}(t) + 6\bigg),
\end{aligned}
$$

thus the desired result is obtained. □

We improve the tail estimates given in Lemma 3.4, which is useful for the tightness of a family of probability distributions of solutions in the space C_ρ, as will be shown in Lemma 4.1.

Lemma 3.5. *Suppose* **A1**-**A6***,* (3.1)*,* (3.5) *and* (3.6) *hold. If $\varphi \in L^2(\Omega, C_\rho)$, then for $\epsilon \in (0, 1]$,*

$$
\limsup_{k\to\infty} \sup_{t\geq\rho} \mathbb{E}\bigg(\sup_{t-\rho\leq r\leq t} \sum_{|i|\geq k} |u_i(r)|^2\bigg) = 0. \qquad (3.52)
$$

Proof. Consider the same smooth function ϑ as defined in Lemma 3.4, thus also satisfying

(3.31) and (3.32). It follows from (3.33) that, for all $t \geq \rho$ and $t-\rho \leq r \leq t$,

$$
\begin{aligned}
&\|\vartheta_k u(r)\|^2 + 2\int_{t-\rho}^{r} q_1(s)\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&- 4\int_{t-\rho}^{r} q_1(s)\big(Au(s), \vartheta_k^2 u(s)\big)ds + 2\int_{t-\rho}^{r} (q_3(s)\vartheta_k|Bu(s)|^2, \vartheta_k u(s))ds \\
&+ 2\int_{t-\rho}^{r} q_2(s)\|\vartheta_k u(s)\|^2 ds + 2\int_{t-\rho}^{r} (\vartheta_k u^3(s), \vartheta_k u(s))ds \\
&= \|\vartheta_k u(t-\rho)\|^2 + 2\int_{t-\rho}^{r} (\vartheta_k f(s, u(s), u(s-\varrho(s))), \vartheta_k u(s))ds \\
&+ 2\int_{t-\rho}^{r} (\vartheta_k g(s), \vartheta_k u(s))ds + \epsilon^2 \sum_{j=1}^{\infty}\int_{t-\rho}^{r} \|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2 ds \\
&+ 2\epsilon \sum_{j=1}^{\infty}\int_{t-\rho}^{r} \big(\vartheta_k^2 u(s), h_j(s) + \sigma_j(s, u(s), u(s-\varrho(s)))\big)dW_j(s). \qquad (3.53)
\end{aligned}
$$

Similar to (3.36), for the second term on the left-hand side of (3.53), we have

$$
\begin{aligned}
&2\int_{t-\rho}^{r} q_1(s)\big(Au(s), A(\vartheta_k^2 u(s))\big)ds \\
&\geq 2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k Au(s)\|^2 ds - \frac{136c_0}{k}\int_{t-\rho}^{r} q_1(s)\|u(s)\|^2 ds. \qquad (3.54)
\end{aligned}
$$

For the third and fourth terms on the left-hand side of (3.53), the Young inequality yields

$$
4\int_{t-\rho}^{r} q_1(s)\big(Au(s), \vartheta_k^2 u(s)\big)ds \leq 2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k Au(s)\|^2 ds + 2\int_{t-\rho}^{r} q_1(s)\|\vartheta_k u(s)\|^2 ds, \quad (3.55)
$$

and

$$
\begin{aligned}
&-2\int_{t-\rho}^{r} (q_3(s)|Bu(s)|^2, \vartheta_k^2 u(s))ds \\
&\leq 4\int_{t-\rho}^{r} \Big(\sum_{i\in\mathbb{Z}} \vartheta^2(\frac{|i|}{k})|q_{3,i}(s)|\big(u_{i+1}^2(s) + u_i^2(s)\big)|u_i(s)|\Big)ds \\
&\leq 8\int_{t-\rho}^{r} \sum_{|i|\geq k} |q_{3,i}(s)|^2\|\vartheta_k u(s)\|^2 ds + 2\int_{t-\rho}^{r} \sum_{i\in\mathbb{Z}} \vartheta^2(\frac{|i|}{k})|u_i(s)|^4 ds. \qquad (3.56)
\end{aligned}
$$

Substituting (3.54)-(3.56) into (3.53), taking the supremum in $r \in [t-\rho, t]$ and its expectation, we derive for all $t \geq \rho$ and $t-\rho \leq r \leq t$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right) \leq \mathbb{E}(\|\vartheta_k u(t-\rho)\|^2) + \frac{136c_0}{k}\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds \\
&+ 2\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds + 8\int_{t-\rho}^{t}\sum_{|i|\geq k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&+ 2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k f(s,u(s),u(s-\varrho(s)))\|\|\vartheta_k u(s)\|ds\right) + 2\mathbb{E}\left(\int_{t-\rho}^{t}\|\vartheta_k g(s)\|\|\vartheta_k u(s)\|ds\right) \\
&+ \epsilon^2\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\|^2)ds \\
&+ 2\epsilon\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\left|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\big(\vartheta_k u(s),\vartheta_k h_j(s)+\vartheta_k\sigma_j(s,u(s),u(s-\varrho(s)))\big)dW_j(s)\right|\right). \qquad (3.57)
\end{aligned}$$

It can be deduced from (3.30) in Lemma 3.4 that for any $\varepsilon > 0$, there exists $K_1(\varepsilon) \geq 1$ such that, for all $s \geq -\rho$ and $k \geq K_1(\varepsilon)$,

$$\sum_{|i|\geq k}\mathbb{E}(|u_i(s)|^2) \leq \varepsilon, \qquad (3.58)$$

which means that, for all $s \geq -\rho$ and $k \geq K_1(\varepsilon)$,

$$\mathbb{E}(\|\vartheta_k u(s)\|^2) \leq \sum_{|i|\geq k}\mathbb{E}(|u_i(s)|^2) \leq \varepsilon. \qquad (3.59)$$

Considering the second term on the right-hand side of (3.57), we infer from Lemma 3.2 that there exists $M(\varphi) > 0$ such that for all $t \geq \rho$,

$$\frac{136c_0}{k}\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds \leq \frac{136c_0\rho}{k}\max_{t\in[0,\mathcal{T}]} q_1(t)M(\varphi),$$

which means that, for every $\varepsilon > 0$, there exists $K_2(\varphi,\varepsilon) \geq K_1(\varepsilon)$ such that, for all $t \geq \rho$ and $k \geq K_2(\varphi,\varepsilon)$,

$$\frac{136c_0}{k}\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|u(s)\|^2)ds \leq \varepsilon. \qquad (3.60)$$

For the third and fourth terms of the right-hand side of (3.57), by (3.59) and $q_3(\cdot) = (q_{3,i}(\cdot))_{i\in\mathbb{Z}} \in \ell^2$, there exists $K_3(\varphi,\varepsilon) > K_2(\varphi,\varepsilon)$ such that, for all $t \geq \rho$ and $k \geq K_3(\varphi,\varepsilon)$,

$$\begin{aligned}
&2\int_{t-\rho}^{t} q_1(s)\mathbb{E}(\|\vartheta_k u(s)\|^2)ds + 8\int_{t-\rho}^{t}\sum_{|i|\geq k}|q_{3,i}(s)|^2\mathbb{E}(\|\vartheta_k u(s)\|^2)ds \\
&\leq 2\rho\varepsilon\max_{t\in[0,\mathcal{T}]} q_1(t) + 8\rho\varepsilon\sup_{t\in[0,\mathcal{T}]}\sum_{|i|\geq k}|q_{3,i}(t)|^2 \leq c_5\varepsilon, \qquad (3.61)
\end{aligned}$$

where $c_5 = c_5(\rho, q_1(t), q_3(t))$. For the fifth term of the right-hand side of (3.57), by (2.17), (3.45) and (3.59), there exists $K_4(\varphi, \varepsilon) > K_3(\varphi, \varepsilon)$ such that, for all $t \geq \rho$ and $k \geq K_4(\varphi, \varepsilon)$,

$$\begin{aligned}
&2\mathbb{E}\left(\int_{t-\rho}^{t} \|\vartheta_k f(s, u(s), u(s-\varrho(s)))\| \|\vartheta_k u(s)\| ds \right) \\
&\leq \int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2) ds + 4\int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2) ds \\
&+ 4\int_{t-\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s-\varrho(s))\|^2) ds + 2\int_{t-\rho}^{t} \beta_0^{-1}(s) \sum_{|i|\geq k} |\alpha_i|^2 ds \\
&\leq 5 \max_{t\in[0,\mathcal{T}]} \beta_0(t)\varepsilon + \frac{4}{1-\rho^*} \int_{t-2\rho}^{t} \beta_0(s)\mathbb{E}(\|\vartheta_k u(s)\|^2) ds + 2\rho \min_{t\in[0,\mathcal{T}]} \beta_0^{-1}(t)\varepsilon \\
&\leq 5 \max_{t\in[0,\mathcal{T}]} \beta_0(t)\varepsilon + \frac{8\rho\varepsilon}{1-\rho^*} \max_{t\in[0,\mathcal{T}]} \beta_0(t) + 2\rho \min_{t\in[0,\mathcal{T}]} \beta_0^{-1}(t)\varepsilon \leq c_6\varepsilon,
\end{aligned} \tag{3.62}$$

where $c_6 = c_6(\rho, \beta_0(t))$. Meanwhile, by Young's inequality, (3.46) and (3.59), there exists $K_5(\varphi, \varepsilon) > K_4(\varphi, \varepsilon)$ such that, for all $t \geq \rho$ and $k \geq K_5(\varphi, \varepsilon)$,

$$\begin{aligned}
2\mathbb{E}\left(\int_{t-\rho}^{t} \|\vartheta_k g(s)\| \|\vartheta_k u(s)\| ds \right) &\leq \int_{t-\rho}^{t} \mathbb{E}(\|\vartheta_k g(s)\|^2) ds + \int_{t-\rho}^{t} \mathbb{E}(\|\vartheta_k u(s)\|^2) ds \\
&\leq \rho \sup_{t\in[0,\mathcal{T}]} \sum_{|i|\geq k} g_i^2(t) + \rho \max_{s\geq 0} \mathbb{E}(\|\vartheta_k u(s)\|^2) \leq c_7\varepsilon,
\end{aligned} \tag{3.63}$$

where $c_7 = c_7(\rho, g)$. Similar to (3.62), for the second-to-last term of the right-hand side of (3.57), by (2.19), (3.45), and (3.46), there exists $K_6(\varphi, \varepsilon) > K_5(\varphi, \varepsilon)$ such that, for all $t \geq \rho, \epsilon \in (0,1]$ and $k \geq K_6(\varphi, \varepsilon)$,

$$\begin{aligned}
&\epsilon^2 \sum_{j=1}^{\infty} \int_{t-\rho}^{t} \mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k \sigma_j(s, u(s), u(s-\varrho(s)))\|^2) ds \\
&\leq 2\rho \sup_{t\in[0,\mathcal{T}]} \sum_{|i|\geq k} \sum_{j=1}^{\infty} \left|h_{j,i}(t)\right|^2 + 8\int_{t-\rho}^{t} \sum_{|i|\geq k} |\beta_i(s)|^2 \mathbb{E}(\|\vartheta_k u(s)\|^2) ds \\
&+ \frac{8}{1-\rho^*} \int_{t-2\rho}^{t} \sum_{|i|\geq k} |\beta_i(s)|^2 \mathbb{E}(\|\vartheta_k u(s)\|^2) ds + 4\rho \sum_{|i|\geq k} |\gamma_i|^2 \\
&\leq 2\rho\varepsilon + 8\rho \sup_{t\in[0,\mathcal{T}]} \sum_{|i|\geq k} |\beta_i(t)|^2 \varepsilon + \frac{16\rho \sup_{t\in[0,\mathcal{T}]} \sum_{|i|\geq k} |\beta_i(t)|^2}{1-\rho^*} \varepsilon + 4\rho\varepsilon \leq c_8\varepsilon,
\end{aligned} \tag{3.64}$$

where $c_8 = c_8(\rho, \beta(t), h(t), \gamma)$. For the last term of the right-hand side of (3.57), by the

Burkhölder-Davis-Gundy inequality and (3.64), for all $t \geq \rho, \epsilon \in (0,1]$ and $k \geq K_6(\varphi, \varepsilon)$,

$$\begin{aligned}
&2\epsilon\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\left|\sum_{j=1}^{\infty}\int_{t-\rho}^{r}\big(\vartheta_k u(s), \vartheta_k h_j(s) + \vartheta_k\sigma_j(s, u(s), u(s-\varrho(s)))\big)dW_j(s)\right|\right)\\
&\leq C_0\mathbb{E}\left(\Big(\int_{t-\rho}^{t}\sum_{j=1}^{\infty}\big|(\vartheta_k u(s), \vartheta_k h_j(s) + \vartheta_k\sigma_j(s, u(s), u(s-\varrho(s))))\big|^2 ds\Big)^{\frac{1}{2}}\right)\\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right) + \frac{1}{2}C_0^2\sum_{j=1}^{\infty}\int_{t-\rho}^{t}\mathbb{E}(\|\vartheta_k h_j(s) + \vartheta_k\sigma_j(s, u(s), u(s-\varrho(s)))\|^2)ds\\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right) + c_9\varepsilon. \qquad (3.65)
\end{aligned}$$

From (3.60) to (3.65) we deduce that, for all $t \geq \rho$ and $k \geq K_6(\varphi, \varepsilon)$,

$$\mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\sum_{|i|\geq 2k}|u_i(r)|^2\right) \leq \mathbb{E}\left(\sup_{t-\rho\leq r\leq t}\|\vartheta_k u(r)\|^2\right) \leq c\varepsilon, \qquad (3.66)$$

where $c = \max\{c_i\}_{i=5,6,7,8,9}$. Therefore, (3.52) is completely proved. □

4. Periodic measures for stochastic delay modified Swift-Hohenberg lattice systems

In this section, we will establish the existence of periodic measures of the system (2.15) in C_ρ. Let $B_b(C_\rho)$ be the space of all bounded Borel-measurable functions on C_ρ, and $\mathscr{B}(C_\rho)$ be the Borel σ-algebra on C_ρ.

If $\phi \in B_b(C_\rho)$, then for $0 \leq r \leq t$, we define a probability transition operator with delay by

$$(p_{r,t}\phi)(\nu) = \mathbb{E}[\phi(u_t(\cdot; r, \nu))], \quad \text{for all} \quad \nu \in C_\rho. \qquad (4.1)$$

In addition, for $\Lambda \in \mathscr{B}(C_\rho)$ and $0 \leq r \leq t$, denote a transition probability function by

$$p(r, \nu; t, \Lambda) = (p_{r,t}1_\Lambda)(\nu) = \mathbb{P}\{\omega \in \Omega : u_t(\cdot; r, \nu) \in \Lambda\}, \qquad (4.2)$$

where $\nu \in C_\rho$ and 1_Λ is the indicator function of Λ. We now recall the definition of periodic measure, namely, a probability measure μ on C_ρ is called a **periodic measure** with period $\mathcal{T} > 0$ if

$$\int_{C_\rho}(p_{0,t+\mathcal{T}}\phi)(\nu)d\mu(\nu) = \int_{C_\rho}(p_{0,t}\phi)(\nu)d\mu(\nu), \quad \forall t \geq 0, \nu \in C_\rho. \qquad (4.3)$$

4.1. Tightness of a family of probability distributions

In this part, we show the tightness of a family of probability distributions of solutions to system (2.15). For this purpose, we need to recall a definition, that is, a family of probability distributions of solutions is called **tight** if for each $\varepsilon \in (0,1]$, there exists a compact set Z_ε of C_ρ such that

$$\mathbb{P}\{\omega \in \Omega : u_t(\cdot; r, \nu) \in Z_\varepsilon\} > 1 - \varepsilon, \quad \text{for all} \quad 0 \leq r \leq t \in \mathbb{R}, \nu \in C_\rho. \qquad (4.4)$$

As verified in Section 2, for any $t_0 \geq 0$, and any $\mathcal{F}_{t_0}$-measurable $\varphi \in L^2(\Omega, C_\rho)$, the system 2.15 has a unique solution $u(t, t_0, \varphi)$ for $t \in [t_0 - \rho, \infty)$. In the following lemma, we consider the initial data $\varphi = 0$ at initial time $t_0 = 0$.

Lemma 4.1. *Suppose* **A1**-**A6**, (3.1), (3.5), *and* (3.6) *hold with* $p \geq 6$. *Then, for* $\epsilon \in (0,1]$, *the distribution laws of the process* $\{u_t(\cdot;0,0)\}_{t\geq 0}$ *to the system* (2.15) *are tight on* C_ρ.

Proof. By Lemma 3.2, Lemma 3.3, Lemma 3.5 and tne Arzelà-Ascoli theorem, we can use similar arguments to thoise in [10, Lemma 4.8] to obtain the conclusion in the sense of (4.4). □

4.2. Existence of periodic measures in C_ρ

To obtain the existence of the periodic measures, we need to do some pre-preparation as follows.

We use the same method as in [34] to approximate nonlinear locally Lipschitz continuous functions $f, \sigma_j, |Bu|^2$ and u^3 by using a suitable cut-off function as follows:

$$\zeta_R(s) = \begin{cases} s, & \text{if } |s| \leq R. \\ \dfrac{Rs}{|s|}, & \text{if } |s| > R. \end{cases} \tag{4.5}$$

Then, we find that for all $s, s_1, s_2 \in \mathbb{R}$,

$$|\zeta_R(s_1) - \zeta_R(s_2)| \leq |s_1 - s_2|, \quad |\zeta_R(s)| \leq |s| \wedge R, \quad \zeta_R(0) = 0. \tag{4.6}$$

Given $j \in \mathbb{N}, R > 0$ and $i \in \mathbb{Z}$. For all $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2, v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, let

$$f^R(t,u,v) = \big(f_i(t, \zeta_R(u_i), \zeta_R(v_i))\big)_{i\in\mathbb{Z}}, \qquad \sigma_j^R(t,u,v) = \big(\sigma_{j,i}(t, \zeta_R(u_i), \zeta_R(v_i))\big)_{i\in\mathbb{Z}}.$$

It can be deduced from (2.5) that for all $u_1, u_2, v_1, v_2 \in \ell^2$,

$$\|f^R(t,u_1,v_1) - f^R(t,u_2,v_2)\|^2 \leq 2L_0^2(R)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \quad \forall t \in \mathbb{R}, \tag{4.7}$$

and by (2.6), for all $u, v \in \ell^2$,

$$\begin{aligned} \|f^R(t,u,v)\|^2 &\leq 2\|f^R(t,u,v) - f^R(t,0,0)\|^2 + 2\|f^R(t,0,0)\|^2 \\ &\leq 2L_0^2(R)(\|u - v\|^2) + 2\|\alpha\|^2, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4.8}$$

Similarly, by (2.7), for all $u_1, u_2, v_1, v_2 \in \ell^2$,

$$\sum_{j\in\mathbb{N}} \|\sigma_j^R(t,u_1,v_1) - \sigma_j^R(t,u_2,v_2)\|^2 \leq 2L_1^2(R)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \quad \forall t \in \mathbb{R}, \tag{4.9}$$

and by (2.19), for all $u, v \in \ell^2$,

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|\sigma_j^R(t,u,v)\|^2 &\leq 4\|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2 \\ &\leq 4 \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2(\|u\|^2 + \|v\|^2) + 2\|\gamma\|^2, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4.10}$$

The approximation of nonlinear terms $|Bu|^2$ and u^3 is the same as (2.25)-(2.28) in [34], that is, for all $u, v \in \ell^2$,

$$\big\||B\zeta_R(u)|^2 - |B\zeta_R(v)|^2\big\|^2 \leq 64R^2\|u - v\|^2, \quad \big\||B\zeta_R(u)|^2\big\|^2 \leq 64R^2\|u\|^2, \tag{4.11}$$

and

$$\|\zeta_R^3(u)-\zeta_R^3(v)\|^2\le 9R^4\|u-v\|^2,\quad \|\zeta_R^3(u)\|^2\le 9R^4\|u\|^2. \tag{4.12}$$

Replace now $f(t,u(t),u(t-\varrho(t))),\sigma_j(t,u(t),u(t-\varrho(t))),|Bu|^2$ and u^3 with $f^R(t,u^R(t),u^R(t-\varrho(t))),\sigma_j^R(t,u^R(t),u^R(t-\varrho(t)))$, $|B\zeta_R(u^R)|^2$ and $\zeta_R^3(u^R)$, then the existence of a unique solution u_t^R for the approximate stochastic system can be proved by [23]. We define a stopping time by

$$\varsigma_R=\inf\{t\ge 0:\|u_t^R(\cdot)\|>R\}, \tag{4.13}$$

where $\varsigma_R=+\infty$ if $\{t\ge 0:\|u_t^R(\cdot)\|>R\}=\emptyset$.

Lemma 4.2. *Suppose* **A1**-**A6** *hold. If* $\varphi\in L^2(\Omega,C_\rho)$, *then for all* $0\le s\le t$,

$$\lim_{R\to\infty}u_t^R(\cdot,s,\varphi)=u_t(\cdot,s,\varphi)\ \ \mathbb{P}\text{-}a.s., \tag{4.14}$$

where $u_t(\cdot)$ *is a solution of* (2.15).

Proof. (1) We want to verify that

$$u^{R+1}(t\wedge\varsigma_R)=u^R(t\wedge\varsigma_R)\ \text{ and }\ \varsigma_{R+1}\ge\varsigma_R\ \text{ a.s. for all }\ t\ge-\rho\ \text{ and }\ R>0. \tag{4.15}$$

Due to the delay term, it will be different from the treatment of f^R,σ_j^R in [34, Lemma 1]. We only include the proof of f^R here, and as the one for σ_j^R is similar, we omit it.

By (4.7) and the Young inequality, we have for all $t\ge 0$,

$$\begin{aligned}
&2\int_0^{t\wedge\varsigma_R}(f^{R+1}(s,u^{R+1}(s),u^{R+1}(s-\varrho(s)))-f^R(s,u^R(s),u^R(s-\varrho(s))),u^{R+1}(s)-u^R(s))ds\\
&\le(1+2L_0^2(R))\int_0^{t\wedge\varsigma_R}\|u^{R+1}(s)-u^R(s)\|^2ds\\
&+2L_0^2(R)\int_0^{t\wedge\varsigma_R}\|u^{R+1}(s-\varrho(s))-u^R(s-\varrho(s))\|^2ds\\
&\le(1+2L_0^2(R))\int_0^{t\wedge\varsigma_R}\|u^{R+1}(s)-u^R(s)\|^2ds+\frac{2L_0^2(R)}{1-\rho^*}\int_{-\rho}^{t\wedge\varsigma_R}\|u^{R+1}(s)-u^R(s)\|^2ds\\
&\le\Big(1+2L_0^2(R)+\frac{2L_0^2(R)}{1-\rho^*}\Big)\int_0^t\sup_{-\rho\le r\le s}\|u^{R+1}(r\wedge\varsigma_R)-u^R(r\wedge\varsigma_R)\|^2ds,
\end{aligned} \tag{4.16}$$

where we used the fact that $f^{R+1}(s,u^R(s),u^R(s-\varrho(s)))=f^R(s,u^R(s),u^R(s-\varrho(s)))$ due to $\|u_s^R(\cdot)\|\le R$ for all $s\in[0,\varsigma_R)$. We replace (2.41) in [34] with (4.16), then following the arguments in [34, Lemma 1] and noticing that $q_1,q_2:\mathbb{R}\to\mathbb{R}$ and $q_3=(q_{3,i})_{i\in\mathbb{Z}}:\mathbb{R}\to\ell^2$ are positive, continuous and $\mathcal{T}$-periodic functions, we can derive (4.15).

(2) Next we prove that

$$\varsigma:=\lim_{R\to+\infty}\varsigma_R=\infty,\quad \mathbb{P}\text{-almost surely.} \tag{4.17}$$

Similarly to Theorem 2.2 (2) and [34, Lemma 2], we obtain (4.17). Combining (4.15) and (4.17), using the argument of [34, Theorem 3], we can derive (4.14). □

We now prove the existence of periodic measures of stochastic delay modified Swift-Hohenberg lattice system (2.15).

Theorem 4.3. *Suppose* **A1**-**A6**, (3.1), (3.5), *and* (3.6) *hold with* $p \geq 6$. *Then, system* (2.15) *has a periodic measures on* C_ρ, *for any* $\epsilon \in (0, 1]$.

Proof. We first need to consider some properties of the transition operator $\{p_{r,t}\}_{0\leq r\leq t}$ for solutions of the system (2.15) as follows:

(1) **Feller property:** Using a similar approach to [34, Lemma 7] and combining Lemma 4.1, we realize that $\{p_{r,t}\}_{0\leq r\leq t}$ is Feller, that is, for any $0 \leq r \leq t$, if for any $\phi : C_\rho \to \mathbb{R}$ is bounded and continuous, then $p_{r,t}\phi : C_\rho \to \mathbb{R}$ is also bounded and continuous.

(2) $\mathcal{T}$**-periodic:** It follows from [17, Lemma 4.1] that $\{p_{r,t}\}_{0\leq r\leq t}$ is $\mathcal{T}$-periodic, namely, for any $0 \leq r \leq t$ and $\nu \in C_\rho$, $p(r, \nu; t, \cdot) = p(r + \mathcal{T}, \nu; t + \mathcal{T}, \cdot)$.

(3) **Markov property:** Given $r \geq 0$ and $\nu \in C_\rho$, we mainly need to prove that the solution $\{u_t(\cdot, r, \nu)\}_{r\leq t}$ of system (2.15) is a C_ρ-valued Markov process, that is, for every bounded and continuous function $\phi : C_\rho \to \mathbb{R}$ and for all $0 \leq r \leq s \leq t$,

$$\mathbb{E}(\phi(u_t(\cdot, r, \nu))|\mathcal{F}_s) = (p_{s,t}\phi)(z)|_{z=u_s(\cdot,r,\nu)}, \quad \mathbb{P}\text{-a.s.}. \tag{4.18}$$

Now we briefly provide a standard proof procedure similar to [31, Lemma 4.5]. Since f^R, σ_j^R, $|B\zeta_R(u^R)|^2$ and $\zeta_R^3(u^R)$ satisfy (4.7)-(4.12), we can obtain that for every $\mathcal{F}_0$-measurable random variable $\varphi \in L^2(\Omega, C_\rho)$,

$$\mathbb{E}\big(\phi(u_t^R(\cdot, s, \varphi))|\mathcal{F}_s\big) = \mathbb{E}\big(\phi(u_t^R(\cdot, s, z))\big)|_{z=\varphi}, \quad \mathbb{P}\text{-a.s.}, \tag{4.19}$$

which can be derived from [26, P51]. Recall that Lemma 4.2:

$$\lim_{R\to\infty} u_t^R(\cdot, s, \varphi) = u_t(\cdot, s, \varphi) \quad \mathbb{P}\text{-a.s.}. \tag{4.20}$$

From the uniqueness of the solution, for all $0 \leq r \leq s \leq t$, we have

$$u_t(\cdot, r, \varphi) = u_t(\cdot, s, u_s(\cdot, r, \varphi)) \quad \mathbb{P}\text{-a.s.}, \tag{4.21}$$

which together with (4.19)-(4.20) and the Lebesgue dominated convergence theorem, we conclude (4.18).

Then it can be inferred from (3) that if $\phi \in B_b(C_\rho)$, for any $\nu \in C_\rho$ and $0 \leq s \leq r \leq t$, $\mathbb{P}$-a.s., $(p_{s,t}\phi)(\nu) = (p_{s,r}(p_{r,t}\phi))(\nu)$. Thus the Chapman-Kolmogorov equation can be satisfied as:

$$p(s, \nu; t, \Lambda) = \int_{C_\rho} p(s, \nu; r, dy)p(r, y; t, \Lambda), \tag{4.22}$$

where $\nu \in C_\rho$ and $\Lambda \in \mathscr{B}(C_\rho)$.

Finally, based on the Krylov-Bogolyubov method and the tightness of distributions of solutions shown in Lemma 4.1, we can use the method of Theorem 4.3 in [17] to prove the existence of periodic measures in the sense of (4.3). ◻

5. Limits stability of periodic measures as noise intensity goes to zero

In this part, we consider the limiting behavior of periodic measures of (2.15) as the noise intensity $\epsilon \to 0$. To that end, we need to strengthen assumptions (3.1) as follows:

$$\min_{t\in[0,\mathcal{T}]} q_2(t) > 16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + 3\cdot 2^{3-\frac{2}{p}} p^{-1}(p-1)^{1-\frac{1}{p}} \max_{t\in[0,\mathcal{T}]} \beta_0(t)$$
$$+ 4\big[(3p-4) + 2(2-\frac{2}{p})^{\frac{p}{2}}\big] \sup_{t\in[0,\mathcal{T}]} \|\beta(t)\|^2. \qquad (5.1)$$

Then, we can still apply (5.1) to derive $\Theta_1(t) > 0$ and $\overline{\Theta}_1(t) > 0$ for $t \in \mathbb{R}$. In addition, most importantly, we can also deduce from (5.1) that

$$\Theta_1(t) > \Theta_2(t) \ \text{ and } \ \overline{\Theta}_1(t) > \overline{\Theta}_2(t), \ \forall t \in \mathbb{R}, \qquad (5.2)$$

where $\Theta_1(t), \Theta_2(t)$ and $\overline{\Theta}_1(t), \overline{\Theta}_2(t)$ are defined in (3.2) and (3.3), respectively. By (4.1) and the continuity and periodicity of Θ_1, Θ_2 and $\overline{\Theta}_1, \overline{\Theta}_2$, there exists $\varepsilon > 0$ such that $\Theta_1(t) - \Theta_2(t) \geq \varepsilon$ and $\overline{\Theta}_1(t) - \overline{\Theta}_2(t) \geq \varepsilon$. Hence there exists $h_0 > 0$ such that

$$\int_0^{\mathcal{T}} \Big(\Theta_1(s) - \Theta_2(s) e^{\int_{s-h_0}^{s} \Theta_1(r)dr}\Big) ds > 0, \qquad (5.3)$$

and

$$\int_0^{\mathcal{T}} \Big(\overline{\Theta}_1(s) - \overline{\Theta}_2(s) e^{\int_{s-h_0}^{s} \overline{\Theta}_1(r)dr}\Big) ds > 0. \qquad (5.4)$$

Noting that (5.2)-(5.4) exactly satisfy conditions in [18, Lemma 3.1], we can use this lemma so that all the estimates of Section 3 are independent of the initial data. Furthermore, to illustrate the dependence of solutions of the system (2.15) on the noise intensity ϵ, we denote it as $u_t^\epsilon(\cdot, 0, \varphi)$ in the relevant proofs that follow.

Applying the conclusion of [18, Lemma 3.1] in the proof of Lemma 3.2 immediately leads to the following Lemma.

Lemma 5.1. *Suppose* **A1**-**A6** *and* (5.1) *hold. Then, for every* $R > 0$*, there exists* $T_1 = T_1(R) > 0$ *such that the solution of the system* (2.15) *satisfies for all* $t \geq T_1$ *and* $\epsilon \in [0,1]$,

$$\mathbb{E}(\|u^\epsilon(t, 0, \varphi)\|^p) \leq C_6,$$

where $\mathbb{E}(\|\varphi\|_{C_\rho}^p) \leq R$ *and* $C_6 > 0$ *is not depending on* t, ϵ, ρ, R *and* φ.

Similar to Lemma 3.3, it can also be deduced from Lemma 5.1 that

Lemma 5.2. *Suppose* **A1**-**A6** *and* (5.1) *hold. Then for every* $R > 0$*, there exists* $T_2 = T_2(R) > 0$ *such that the solution of system* (2.15) *satisfies for all* $t \geq r \geq T_2$ *and* $\epsilon \in [0,1]$,

$$\mathbb{E}(\|u^\epsilon(t,0,\varphi) - u^\epsilon(r,0,\varphi)\|^{\frac{p}{3}}) \leq C_7(|t-r|^{\frac{p}{3}} + |t-r|^{\frac{p}{6}}),$$

where $\mathbb{E}(\|\varphi\|_{C_\rho}^p) \leq R$ *and* $C_7 > 0$ *is independent of* ϵ, ρ, R *and* φ.

By replacing Lemma 3.1 with the conclusion of [18, Lemma 3.1] and applying it to the proof of Lemma 3.5, we can derive the following tail-estimate of solutions:

Lemma 5.3. *Suppose* **A1**-**A6** *and* (5.1) *hold. Then, for every $R > 0$ and $\varepsilon > 0$, there exists $T_3 = T_3(R, \varepsilon) > 0$ and $K_1 = K_1(\varepsilon) \geq K$ such that the solution of system* (2.15) *satisfies for all $t \geq T_3$, $k \geq K_1$ and $\epsilon \in [0, 1]$,*

$$\mathbb{E}\left(\sup_{t-\rho \leq r \leq t} \sum_{|i| \geq k} |u_i^\epsilon(r, 0, \varphi)|^2 \right) \leq \varepsilon, \tag{5.5}$$

where $\varphi \in L^2(\Omega, C_\rho)$ such that $\mathbb{E}(\|\varphi\|_{C_\rho}^2) \leq R$.

We now show the convergence of solutions of the system (2.15) w.r.t. noise intensity ϵ as follows:

Lemma 5.4. *Suppose* **A1**-**A6** *and* (5.1) *hold. Then, for every compact set $\mathcal{K}$ of C_ρ, $\delta > 0$, $t \geq 0$ and $\epsilon_0 \geq 0$,*

$$\lim_{\epsilon \to \epsilon_0} \sup_{\varphi \in \mathcal{K}} \mathbb{P}(\{\omega \in \Omega : \|u_t^\epsilon(\cdot, 0, \varphi) - u_t^{\epsilon_0}(\cdot, 0, \varphi)\|_{C_\rho} \geq \delta\}) = 0. \tag{5.6}$$

Proof. For the sake of simplicity, we let $u_1(t) = u^\epsilon(t, 0, \varphi)$ and $u_2(t) = u^{\epsilon_0}(t, 0, \varphi)$ for $t \geq -\rho$. Given $T > 0$, for every compact set $\mathcal{K}$ of C_ρ, it follows from Theorem 2.2 that there exists $c = c(\mathcal{K}, T) > 0$ such that, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$,

$$\mathbb{E}\Big(\sup_{t \in [-\rho, T]} \|u_1(t)\|^2 \Big) \leq c.$$

Thus, for every $\varepsilon > 0$, there exists $n = n(\varepsilon, \mathcal{K}, T) > 0$ such that, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$,

$$\mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{t \in [-\rho, T]} \|u_1(t)\| > n \right\} \right) < \frac{1}{2}\varepsilon.$$

Given $\varphi \in \mathcal{K}$ and $\epsilon, \epsilon_0 \in (0, 1]$, we define

$$\Omega_\epsilon = \left\{ \omega \in \Omega : \sup_{t \in [-\rho, T]} \|u_1(t)\| \leq n \text{ and } \sup_{t \in [-\rho, T]} \|u_2(t)\| \leq n \right\}.$$

Then, for all $\varphi \in \mathcal{K}$ and $\epsilon \in (0, 1]$, we have $\mathbb{P}(\Omega \setminus \Omega_\epsilon) < \varepsilon$.

Given $R > 0$, we define a stopping time by

$$T_R = \inf\{t \geq 0 : \|u_1(t)\| > R \ \text{ or } \ \|u_2(t)\| > R\}. \tag{5.7}$$

Generally, $\inf \emptyset = \infty$, and we can see that $T_R \geq T$ for each $\omega \in \Omega_\epsilon$. For any $\varphi \in \mathcal{K}$ and $\delta > 0$,

$$\begin{aligned}
&\sup_{\varphi \in \mathcal{K}} \mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{t \in [-\rho, T]} \|u_1(t) - u_2(t)\| \geq \delta \right\} \right) \\
&= \sup_{\varphi \in \mathcal{K}} \mathbb{P}\left(\left\{ \omega \in \Omega_\epsilon : \sup_{t \in [-\rho, T]} \|u_1(t \wedge T_R) - u_2(t \wedge T_R)\| \geq \delta \right\} \right) \\
&+ \sup_{\varphi \in \mathcal{K}} \mathbb{P}\left(\left\{ \omega \in \Omega \setminus \Omega_\epsilon : \sup_{t \in [-\rho, T]} \|u_1(t) - u_2(t)\| \geq \delta \right\} \right) \\
&\leq \sup_{\varphi \in \mathcal{K}} \mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{t \in [-\rho, T]} \|u_1(t \wedge T_R) - u_2(t \wedge T_R)\| \geq \delta \right\} \right) + \varepsilon.
\end{aligned} \tag{5.8}$$

To complete the proof, we need to show that

$$\lim_{\epsilon\to\epsilon_0}\sup_{\varphi\in\mathcal{K}}\mathbb{P}\left(\left\{\omega\in\Omega:\sup_{t\in[-\rho,T]}\|u_1(t\wedge T_R)-u_2(t\wedge T_R)\|\ge\delta\right\}\right)=0. \tag{5.9}$$

Similar to [34, Lemma 11], we can obtain from (2.15) that, for all $t\ge 0$,

$$\begin{aligned}
&\|u_1(t\wedge T_R)-u_2(t\wedge T_R)\|^2+2\int_0^{t\wedge T_R}q_1(r)\|A\big(u_1(r)-u_2(r)\big)\|^2dr\\
&-4\int_0^{t\wedge T_R}q_1(r)\|B\big(u_1(r)-u_2(r)\big)\|^2dr+2\int_0^{t\wedge T_R}q_2(r)\|u_1(r)-u_2(r)\|^2dr\\
&+2\int_0^{t\wedge T_R}\big(u_1^3(r)-u_2^3(r),u_1(r)-u_2(r)\big)dr\\
&+2\int_0^{t\wedge T_R}\Big(q_3(r)\big(|Bu_1(r)|^2-|Bu_2(r)|^2\big),u_1(r)-u_2(r)\Big)dr\\
&\le 2\int_0^{t\wedge T_R}\big(f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dr\\
&+3\epsilon_0^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))-\sigma_j(r,u_2(r),u_2(r-\varrho(r)))\|^2dr\\
&+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))\|^2dr+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|h_j(r)\|^2dr\\
&+2|\epsilon-\epsilon_0|\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}(h_j(r),u_1(r)-u_2(r))dW_j(r)\\
&+2|\epsilon-\epsilon_0|\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}(\sigma_j(r,u_1(r),u_1(r-\varrho(r))),u_1(r)-u_2(r))dW_j(r)\\
&+2\epsilon_0\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\big(\sigma_j(r,u_1(r),u_1(r-\varrho(r)))-\sigma_j(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dW_j(r).
\end{aligned} \tag{5.10}$$

For the second-to-last term on the left-hand side of (5.10), we obtain

$$\begin{aligned}
&-2\int_0^{t\wedge T_R}\big(u_1^3(r)-u_2^3(r),u_1(r)-u_2(r)\big)dr\\
&\le 2\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}\big|(u_{1,i}(r)-u_{2,i}(r))(u_{1,i}^2(r)+u_{1,i}(r)u_{2,i}(r)+u_{2,i}^2(r))\big||u_{1,i}(r)-u_{2,i}(r)|dr\\
&\le\frac{3}{2}\int_0^{t\wedge T_R}(\|u_1(r)\|^2+\|u_2(r)\|^2)\|u_1(r)-u_2(r)\|^2dr\le 3R^2\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr.
\end{aligned} \tag{5.11}$$

For the last term on the left-hand side of (5.10), by (2.13) we have

$$
\begin{aligned}
&-2\int_0^{t\wedge T_R}\Big(q_3(r)\big(|Bu_1(r)|^2-|Bu_2(r)|^2\big),u_1(r)-u_2(r)\Big)dr\\
&\le 2\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}|q_{3,i}(r)|\Big||u_{1,i+1}(r)-u_{1,i}(r)|^2-|u_{2,i+1}(r)-u_{2,i}(r)|^2\Big||u_{1,i}(r)-u_{2,i}(r)|dr\\
&\le 8\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}|q_{3,i}(r)|^2|u_{1,i}(r)-u_{2,i}(r)|^2dr+\frac{1}{8}\int_0^{t\wedge T_R}\sum_{i\in\mathbb{Z}}\big(|u_{1,i+1}(r)-u_{1,i}(r)|\\
&+|u_{2,i+1}(r)-u_{2,i}(r)|\big)^2\Big|u_{1,i+1}(r)-u_{1,i}(r)-(u_{2,i+1}(r)-u_{2,i}(r))\Big|^2dr\\
&\le 8\int_0^{t\wedge T_R}\|q_3(r)\|^2\|u_1(r)-u_2(r)\|^2dr+\frac{1}{4}\int_0^{t\wedge T_R}(\|Bu_1(r)\|^2+\|Bu_2(r)\|^2)\|Bu_1(r)-Bu_2(r)\|^2\\
&\le 8\int_0^{t\wedge T_R}(\|q_3(r)\|^2+R^2)\|u_1(r)-u_2(r)\|^2dr. \qquad (5.12)
\end{aligned}
$$

Combining (5.7) and (2.16), for the first term on the right-hand side of (5.10), we can deduce

$$
\begin{aligned}
&2\int_0^{t\wedge T_R}\big(f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r))),u_1(r)-u_2(r)\big)dr\\
&\le\int_0^{t\wedge T_R}\|f(r,u_1(r),u_1(r-\varrho(r)))-f(r,u_2(r),u_2(r-\varrho(r)))\|^2dr+\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr\\
&\le L_R^f\int_0^{t\wedge T_R}(\|u_1(r)-u_2(r)\|^2+\|u_1(r-\varrho(r))-u_2(r-\varrho(r))\|^2)dr+\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr\\
&\le (L_R^f+1+\frac{L_R^f}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr. \qquad (5.13)
\end{aligned}
$$

Analogously, by (5.7), (2.18) and (2.19), for the second and third terms on the right-hand side of (5.10),

$$
\begin{aligned}
&3\epsilon_0^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))-\sigma_j(r,u_2(r),u_2(r-\varrho(r)))\|^2dr\\
&+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|\sigma_j(r,u_1(r),u_1(r-\varrho(r)))\|^2dr\\
&\le\frac{3\epsilon_0^2}{2}(L_R^\sigma+1+\frac{L_R^\sigma}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr+12(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\beta(r)\|^2\|u_1(r)\|^2dr\\
&+12(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\beta(r)\|^2\|u_1(r-\varrho(r))\|^2dr+6(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\|\gamma\|^2dr\\
&\le\frac{3\epsilon_0^2}{2}(L_R^\sigma+1+\frac{L_R^\sigma}{1-\rho^*})\int_0^{t\wedge T_R}\|u_1(r)-u_2(r)\|^2dr+\frac{12(\epsilon-\epsilon_0)^2}{1-\rho^*}\int_{-\rho}^{0}\|\beta(r)\|^2\|\varphi(r)\|^2dr\\
&+6(\epsilon-\epsilon_0)^2\int_0^{t\wedge T_R}\big[2(1+\frac{1}{1-\rho^*})R^2\|\beta(r)\|^2+\|\gamma\|^2\big]dr. \qquad (5.14)
\end{aligned}
$$

Bringing the combination of (5.12)-(5.14) into (5.10) and taking expectation, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le r\le t}\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)\\
&\le \int_0^t (16q_1(r)+8\|q_3(r)\|^2+c_{10})\mathbb{E}\Big(\sup_{0\le r\le s}\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)ds\\
&+3(\epsilon-\epsilon_0)^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|h_j(r)\|^2dr+I_4, \qquad (5.15)
\end{aligned}$$

where $c_{10}=11R^2+L_R^f+1+\frac{L_R^f}{1-\rho^*}+\frac{3\epsilon_0^2}{2}(L_R^\sigma+1+\frac{L_R^\sigma}{1-\rho^*})$ and

$$\begin{aligned}
I_4 &= 2|\epsilon-\epsilon_0|\mathbb{E}\Big(\sup_{0\le r\le t\wedge T_R}\Big|\sum_{j=1}^{\infty}\int_0^r (h_j(s),u_1(s)-u_2(s))dW_j(s)\Big|\Big)\\
&+2|\epsilon-\epsilon_0|\mathbb{E}\Big(\sup_{0\le r\le t\wedge T_R}\Big|\sum_{j=1}^{\infty}\int_0^r (\sigma_j(s,u_1(s),u_1(s-\varrho(s))),u_1(s)-u_2(s))dW_j(s)\Big|\Big)\\
&+2\epsilon_0\mathbb{E}\Big(\sup_{0\le r\le t\wedge T_R}\Big|\sum_{j=1}^{\infty}\int_0^r \big(\sigma_j(s,u_1(s),u_1(s-\varrho(s)))-\sigma_j(s,u_2(s),u_2(s-\varrho(s))),\\
&u_1(s)-u_2(s)\big)dW_j(s)\Big|\Big).
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, we derive from (5.14) that

$$\begin{aligned}
I_4 &\le \frac{3}{4}\mathbb{E}\Big(\sup_{0\le r\le t}\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)+c(\epsilon-\epsilon_0)^2C_0^2\sum_{j=1}^{\infty}\int_0^{t\wedge T_R}\|h_j(r)\|^2dr\\
&+c(\epsilon-\epsilon_0)^2C_0^2\int_0^{t\wedge T_R}\big[\|\beta(r)\|^2+\|\gamma\|^2\big]dr+c(\epsilon-\epsilon_0)^2C_0^2\int_{-\rho}^0\|\beta(r)\|^2\|\varphi(r)\|^2dr\\
&+cC_0^2\int_0^t\mathbb{E}\Big(\sup_{0\le r\le s}(\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)ds. \qquad (5.16)
\end{aligned}$$

Combining (5.15) and (5.16), for every compact set $\mathcal{K}$ of C_ρ, there exists $c_{11}=c_{11}(\mathcal{K})>0$ such that, for all $\varphi\in\mathcal{K}$ and $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le r\le t}\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big) \qquad (5.17)\\
&\le 4(16\max_{t\in[0,\mathcal{T}]}q_1(t)+8\sup_{t\in[0,\mathcal{T}]}\|q_3(t)\|^2+c_{10})\int_0^t\mathbb{E}\Big(\sup_{0\le r\le s}(\|u_1(r\wedge T_R)-u_2(r\wedge T_R)\|^2\Big)ds\\
&+c_{11}(\epsilon-\epsilon_0)^2T\Big(\sup_{r\in[0,\mathcal{T}]}\sum_{j=1}^{\infty}\|h_j(r)\|^2+\sup_{r\in[0,\mathcal{T}]}\|\beta(r)\|^2+\|\gamma\|^2\Big).
\end{aligned}$$

From the Gronwall inequality, for all $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le r\le t}\|u_1(r)-u_2(r)\|^2\Big)\\
&\le c_{11}(\epsilon-\epsilon_0)^2T\Big(\sup_{r\in[0,\mathcal{T}]}\sum_{j=1}^{\infty}\|h_j(r)\|^2+\sup_{r\in[0,\mathcal{T}]}\|\beta(r)\|^2+\|\gamma\|^2\Big)\\
&\times e^{4(16\max_{t\in[0,\mathcal{T}]}q_1(t)+8\sup_{t\in[0,\mathcal{T}]}\|q_3(t)\|^2+c_{10})T}, \qquad (5.18)
\end{aligned}$$

which means that, for all $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{-\rho\le r\le t} \|u_1(r) - u_2(r)\|^2 \Big) \\
&\le c_{11}(\epsilon - \epsilon_0)^2 T \Big(\sup_{r\in[0,\mathcal{T}]} \sum_{j=1}^{\infty} \|h_j(r)\|^2 + \sup_{r\in[0,\mathcal{T}]} \|\beta(r)\|^2 + \|\gamma\|^2 \Big) \\
&\times e^{4(16 \max_{t\in[0,\mathcal{T}]} q_1(t) + 8 \sup_{t\in[0,\mathcal{T}]} \|q_3(t)\|^2 + c_{10})T},
\end{aligned} \tag{5.19}
$$

which together with Chebyshev's inequality, we can obtain (5.9) as ϵ tends to ϵ_0. The proof can then be completed by (5.8) and (5.9). □

Given $\epsilon \in [0, 1]$, let S^ϵ be the collection of all $\mathcal{T}$-periodic measures μ^ϵ of system (2.15). Notice that all estimates in Section 3 are valid under (5.1),(5.3) and (5.4), thus for every $\epsilon \in [0, 1]$, S^ϵ is nonempty by the argument of Theorem 4.3.

Now, we show the tightness of $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ in the sense of (4.4), then use Theorem 5.1 in [17] to establish the limiting behavior of any sequence of S^ϵ for the system (2.15) as $\epsilon \to 0$.

Theorem 5.5. *Suppose* **A1**-**A6** *and* (5.1) *hold. Then,*

(i) $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ *is tight on* C_ρ.

(ii) If $\mu^{\epsilon_n} \in S^{\epsilon_n}$ *with* $\epsilon_n \to \epsilon_0 \in [0, 1]$, *then there exists a subsequence* ϵ_{n_k} *and a* $\mathcal{T}$*-periodic measures* $\mu^{\epsilon_0} \in S^{\epsilon_0}$ *such that* $\mu^{\epsilon_n} \rightharpoonup \mu^{\epsilon_0}$.

Proof. (i) Given a compact set $\widetilde{\mathcal{K}}$ of C_ρ. Based on Lemma 5.1-Lemma 5.3, using a similar approach to Lemma 4.1 , it is known that, for every $\varepsilon > 0$ and $\varphi \in C_\rho$, there exists $T_\varepsilon > 0$ such that, for all $t \ge T_\varepsilon$ and $\epsilon \in [0, 1]$,

$$
\mathbb{P}\{\omega \in \Omega : u_t^\epsilon(\cdot; 0, \varphi) \in \widetilde{\mathcal{K}}\} > 1 - \varepsilon.
$$

Then using the method of [17, Theorem 5.6], we can obtain that for any $\mu^\epsilon \in S^\epsilon$ with $\epsilon \in [0, 1]$,

$$
\mu^\epsilon(\widetilde{\mathcal{K}}) \ge 1 - \varepsilon,
$$

which implies that $\bigcup_{\epsilon\in[0,1]} S^\epsilon$ is tight in the sense of (4.4).

(ii) It is clear from (i) that $\{\mu^{\epsilon_n}\}_{n=1}^\infty$ is tight, which means that there exists a subsequence $\{\mu^{\epsilon_{n_m}}\}_{m=1}^\infty$ of $\{\mu^{\epsilon_n}\}_{n=1}^\infty$ and a probability measures μ such that $\mu_t^{\epsilon_{n_m}} \rightharpoonup \mu$ as $m \to \infty$. Then, by Lemma 5.4 and [17, Theorem 5.1], we can complete the proof. □

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References

[1] L. Arnold, Stochastic differential equations: Theory and applications, Wiley-Interscience, New York, 1974.

[2] L.O. Chua, Y. Yang, Cellular neural networks: theory, IEEE Trans. Circuits Syst. 35 (1988) 125–1272.

[3] S.N. Chow, Lattice dynamical systems Dynamical Systems, Lecture Notes in Math., vol. 1822, Springer-Verlag, Berlin, (2003) 1–102.

[4] T. Caraballo, J. Real, Partial differential equations with delayed random perturbations: existence uniqueness and stability of solutions, Stoch. Anal. Appl. 11 (1993) 497–511.

[5] T. Caraballo, F. Morillas, J. Valero, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, J. Differ. Equ. 253 (2012) 667–693.

[6] T. Caraballo, F. Morillas, J. Valero, On differential equations with delay in Banach spaces and attractors for retarded lattice dynamical systems, Discrete Cont. Dyn. Systems, Series A 34 (2014) 51–77.

[7] T. Caraballo, F. Morillas, J. Valero, Attractors for non-autonomous retarded lattice dynamical systems, Nonauton. Dyn. Syst. 2 (2015) 31–51.

[8] T. Caraballo, X. Han, B. Schmalfuß, J. Valero, Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise, Nonlinear Anal. 130 (2016) 255–278.

[9] Y. Choi, Dynamical bifurcation of the one dimensional modified swift-hohenberg equation, Bull. Korean. Math. Soc. 52 (2015) 1241–1252.

[10] Z. Chen, B. Wang, Weak mean attractors and invariant measures for stochastic Schrödinger delay lattice systems, J. Dyn. Differ. Equ. (2021), https://doi.org/10.1007/s10884-021-10085-3.

[11] Z. Chen, D. Yang, T. Zhong, Weak mean attractor and periodic measure for stochasticlattice systems driven by Lévy noises, Stoch. Anal. Appl. (2022), https://doi.org/10.1080/07362994.2022.2038624.

[12] A. Gu, P.E. Kloeden, Asymptotic behavior of a nonautonomous p-Laplacian lattice system, Int. J. Bifurc. Chaos Appl. Sci. Eng. 26 (2016) 1650174.

[13] X. Han, P.E. Kloeden, Non-autonomous lattice systems with switching effects and delayed recovery, J. Differ. Equ. 261 (2016) 2986–3009.

[14] Y. He, C. Li, J. Wang, Invariant measures and statistical solutions for the nonautonomous discrete modified Swift-Hohenberg equation, Bull. Malays. Math. Sci. Soc. 44 (2021) 3819–3837.

[15] Y. Kuang, Delay Differential Equations: with Applications in Population Dynamics, Academic Press, Boston, 1993.

[16] D. Li, L. Shi, Upper semicontinuity of random attractors of stochastic discrete complex Ginzburg-Landau equations with time-varying delays, J. Differ. Equ. Appl. 24 (2018) 872–897.

[17] D. Li, B. Wang, X. Wang, Periodic measures of stochastic delay lattice systems, J. Differ. Equ. 272 (2021) 74–104.

[18] D. Li, B. Wang, X. Wang, Limiting behavior of invariant measures of stochastic delay lattice systems, J. Dyn. Differ. Equ. 34 (2022) 1453–1487.

[19] Y. Li, H. Wu, T. Zhao, Random pullback attractor of a non-autonomous local modified stochastic Swift-Hohenberg equation with multiplicative noise, J. Math. Phys. 61 (2020) 092703.

[20] Y. Lin, Y. Li, D. Li, Periodic measures of impulsive stochastic neural networks lattice systems with delays, J. Math. Phys. 63 (2022) 122702.

[21] Y. Lin, D. Li, Periodic measures of impulsive stochastic Hopfield-type lattice systems, Stoch. Anal. Appl. 40 (2022) 914–930.

[22] R. Lefever, N. Barbier, P. Couteron, O. Lejeune, Deeply gapped vegetation patterns: On crown/root allometry, criticality and desertification, J. Theor. Biol. 261 (2009) 194–209.

[23] X. Mao, Stochastic Differential Equations and Applications, second edition, Woodhead Publishing Limited, Cambbridge, 2011.

[24] J.Y. Park, S.H. Park, Pullback attractor for a non-autonomous modified Swift-Hohenberg equation, Comput. Math. Appl. 67 (2014) 542–548.

[25] M. Polat, Global attractor for a modified Swift-Hohenberg equation, Comput. Math. Appl. 57 (2009) 62–66.

[26] S.-E.A. Mohammed, Stochastic Functional Differential Equations, Longman, Harlow/New York, 1986.

[27] L. Song, Y. Zhang, T. Ma, Global attractor of a modified Swift-Hohenberg equation in H^k spaces, Nonlinear Anal. 72 (2010) 183–191.

[28] J.B. Swift, E.C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A 15 (1977) 319–328.

[29] B. Wang, Attractors for reaction-diffusion equations in unbounded domains, Phys. D 128 (1999) 41–52.

[30] B. Wang, Dynamics of systems on infinite lattices, J. Differ. Equ. 221 (2006) 224–245.

[31] B. Wang, Dynamics of stochastic reaction-diffision lattice systems driven by nonlinear noise, J. Math. Anal. Appl. 477 (2019) 104–132.

[32] B. Wang, R. Wang, Asymptotic behavior of stochastic Schrödinger lattice systems driven by nonlinear noise, Stoch. Anal. Appl. 38 (2020) 213-237.

[33] F. Wang, Y. Li, Random attractors for multi-valued multi-stochastic delayed p-Laplace lattice equations, J. Difference Equat. Appl. 27 (2021) 1232–1258.

[34] F. Wang, T. Caraballo, Y. Li, R. Wang, Asymptotic stability of evolution systems of probability measures of stochastic discrete modified Swift-Hohenberg equations, (submitted).

[35] R. Wang, Y. Li, Regularity and backward compactness of attractors for non-autonomous lattice systems with random coefficients, Appl. Math. Comput. 354 (2019) 86–102.

[36] R. Wang, Y. Li, Asymptotic behavior of stochastic discrete wave equations with nonlinear noise and damping, J. Math. Phys. 61 (2020) 052701.

[37] R. Wang, Long-time dynamics of stochastic lattice plate equations with nonlinear noise and damping, J. Dyn. Differ. Equ. 33 (2021) 767–803.

[38] X. Wang, K. Lu, B. Wang, Exponential stability of non-autonomous stochastic delay lattice systems with multiplicative noise, J. Dyn. Differ. Equ. 28 (2016) 1309–1335.

[39] J. Xu, T. Caraballo, Long Time Behavior of Stochastic Nonlocal Partial Differential Equations and Wong–Zakai Approximations, SIAM J. Math. Anal. 54 (2022), no. 3, 2792–2844.

[40] J. Xu, Z. Zhang, T. Caraballo, Non-autonomous nonlocal partial differential equations with delay and memory, J. Differential Equations 270 (2021), 505-546.

[41] L. Xu, Q. Ma, Existence of the uniform attractors for a non-autonomous modified swift-hohenberg equation, Adv. Differ. Equ. 2015 (2015) 1–11.

[42] C. Zhao, S. Zhou, Attractors of retarded first order lattice systems, Nonlinearity 20 (2007) 1987–2006.

[43] C. Zhao, S. Zhou, Compact uniform attractors for dissipative lattice dynamical systems with delays, Discrete Contin. Dyn. Syst. 21 (2008) 643–663.

[44] C. Zhang, L. Zhao, The attractors for 2nd-order stochastic delay lattice systems, Discrete Contin. Dyn. Syst. 37 (2017) 575–590.