CONTINUITY AND TOPOLOGICAL STRUCTURAL STABILITY FOR NONAUTONOMOUS RANDOM ATTRACTORS

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Abstract. In this work we study the continuity and topological structural stability of attractors for nonautonomous random differential equations obtained by small bounded random perturbations of autonomous semilinear problems. First, we study the existence and permanence of unstable sets of hyperbolic solutions. Then, we use this to establish the lower semicontinuity of nonautonomous random attractors and to show that the gradient structure persists under nonautonomous random perturbations. Finally, we apply the abstract results in a stochastic differential equation and in a damped wave equation with a perturbation on the damping.

1. Introduction

In this paper we study autonomous attractors under nonautonomous random perturbations. Our goal is to provide conditions to conclude continuity and topological structural stability of nonautonomous random attractors. We consider an autonomous semilinear problem in a Banach space X

$$\dot{y} = By + f_0(y), \quad t > 0, \ y(0) = y \in X, \tag{1.1}$$

and its nonautonomous random perturbations of the type

$$\dot{y} = By + f_\eta(t, \theta_t\omega, y), \ t > \tau, \ y(\tau) = y_\tau \in X, \ \eta \in (0, 1], \tag{1.2}$$

where B generates a C^0-semigroup $\{e^{At} : t \geq 0\} \subset \mathcal{L}(X)$, and $\theta_t : \Omega \to \Omega$ is a random flow defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume that problem (1.1) generates a (nonlinear) semigroup $\{T(t) : t \geq 0\}$, and that (1.2) generates a (nonlinear) nonautonomous random dynamical system (ψ_η, Θ), for each $\eta \in [0, 1]$, and that all these dynamical systems have attractors, see [39, 40, 38] and the references therein for general theory and examples.

One of our goals is to establish continuity of this family of attractors. This is done by proving *upper* and *lower semicontinuity*. On the one hand, **upper semicontinuity** means that the perturbed attractors do not become suddenly much larger than the limiting attractor (non-explosion). On the other hand, **lower semicontinuity** means that the perturbed attractors do not become suddenly much smaller than the limiting attractor (non-implosion). For an introduction to

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the notion of continuity of attractors see [29, Chapter 3] for global and pullback attractors, and [33, Section 4.10] or [6, Chapter 8] for global attractors.

For nonautonomous (deterministic) dynamical systems the continuity of attractors is very well studied, see for instance [11, 30, 28, 37]. In the nonautonomous random setting, the upper semicontinuity was proved in several examples, see [7, 39, 38] and the references therein. However, the lower semicontinuity is more difficult to attain due to the fact that one has to prove that the *inner structure* of the limiting attractor is "preserved" under perturbation, in order to ensure that the perturbed attractor occupies a region 'as large as' the region occupied by the limiting attractor. More precisely, the typical conditions one has to assume is that the limiting attractor is the union of the unstable sets of the equilibria and then give conditions to ensure that these equilibria and their unstable sets 'persist' under perturbation, see [3, 5, 14, 34] for the lower semicontinuity of global attractors, and [28, 30, 37] for the lower semicontinuity of pullback attractors and [11] for the lower semicontinuity of uniform attractors. In [30] the authors study the permanence of hyperbolic global solutions and of their corresponding unstable and stable sets, in the nonautonomous setting, and in [28] the authors prove a general result on the lower semicontinuity of pullback attractors allowing the limiting pullback attractor to be given as the closure of a countable (possibly infinite) union of unstable sets of hyperbolic global solutions.

Thus, to prove lower semicontinuity in a nonautonomous random framework we follow this latter method and prove that the inner structure persists under perturbations. However, this is not expected to happen for general types of noises. Actually, some works show that the presence of an additive noise destroys the continuity of the attractors [8, 32], see also [15] for a complementary study of such problems. Hence, to obtain our results we will consider small bounded random perturbations as the one introduced in [16], where the authors studied the existence and permanence of hyperbolic solutions for (1.2) assuming that the perturbations are uniformly bounded in time. Now, inspired by the results in [30], we study the existence and continuity of the unstable sets associated with this hyperbolic solutions, and we use these results to conclude the lower semicontinuity for the attractors of $\{(\psi_\eta, \Theta) : \eta \in [0, 1]\}$, see Theorem 5.1. In our proofs, we show how to control the random parameter using techniques of deterministic dynamical systems.

The idea of reproducing the internal structure in the perturbed attractor is not only important to show continuity of attractors, but is also crucial to prove that the dynamics are preserved under perturbation. For instance, in [27] the authors provide conditions (permanence of the inner structure) to prove that dynamically gradient semigroups are stable under perturbation. We refer to this property as **topological structural stability**. Gradient dynamical systems were widely studied in the past years, see [1, 10, 11, 12, 30, 18] for deterministic dynamical systems, and [23, 36] for random dynamical systems. In this work we obtain a result on the topological structural stability for nonautonomous random differential equations, see Theorem 6.3. This will be also a consequence of the careful study of the internal structure of these attractors.

We also obtain stronger results on the continuity and topological structural stability of nonautonomous random attractors for the case when the random perturbations are uniformly bounded with respect to the random parameter, see Remark

5.4 and Remark 6.4 for more details. Moreover, see [9, 25] for examples of this types of noises.

We provide two applications of our abstract results. First, for a family of Stratonovich stochastic differential equations with a nonautonomous multiplicative white noise

$$dy = Bydt + f_0(y)dt + \eta\kappa(t)y \circ dW_t,\ t \geq \tau,\ y(\tau) = y_\tau \in X, \tag{1.3}$$

where $\eta \in [0, 1]$, and κ is a real function that "controls" the growth of the noise in time, see Subsection 7.1. Finally, a nonautonomous random perturbation on the damping of a damped wave equation with Dirichlet boundary condition

$$u_{tt} + \beta_\eta(t, \theta_t\omega)u_t - \Delta u = f(u),\ t \geq \tau,\ \eta \in [0, 1], \tag{1.4}$$

where $\{\theta_t : \Omega \to \Omega : t \in \mathbb{R}\}$ is a random flow in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and β_η converges to β as $\eta \to 0$ for some $\beta > 0$, see Subsection 7.2.

Next, we describe how the paper is organized. In Section 2, we recall some basic concepts of the theory of nonautonomous and random dynamical systems. Then, in Section 3, we present the results on the permanence of hyperbolic solutions and in Section 4, we obtain the existence and continuity of local unstable sets associated with these solutions. In Section 5, we prove our result on the continuity of nonautonomous random attractors. In Section 6, we provide a result on the topological structural stability. Finally, in Section 7, we present applications to differential equations.

2. Preliminaries

First, we introduce the notion of *nonautonomous random dynamical systems* in a complete separable metric space (X, d).

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that a family of maps $\{\theta_t : \Omega \to \Omega : t \in \mathbb{R}\}$ is a **random flow** if*

(1) $\theta_0 = Id_\Omega$;
(2) $\theta_{t+s} = \theta_t \circ \theta_s$, *for all* $t, s \in \mathbb{R}$;
(3) $\theta_t : \Omega \to \Omega$ *is measurable and* $\mathbb{P}\theta_t^{-1} = \mathbb{P}$ *for all* $t \in \mathbb{R}$.

Definition 2.2. *Let $\{\theta_t : \Omega \to \Omega : t \in \mathbb{R}\}$ be a random flow. Define $\Theta_t(\tau, \omega) := (t + \tau, \theta_t\omega)$ for each $(\tau, \omega) \in \mathbb{R} \times \Omega$, and $t \in \mathbb{R}$. We say that a family of maps $\{\psi(t, \tau, \omega) : X \to X; (t, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega\}$ is a **nonautonomous random dynamical system (co-cycle)** driven by Θ if*

(1) *the mapping $\mathbb{R}^+ \times \Omega \times X \ni (t, \omega, x) \mapsto \psi(t, \tau, \omega)x \in X$ is measurable for each fixed $\tau \in \mathbb{R}$;*
(2) $\psi(0, \tau, \omega) = Id_X$, *for each* $(\tau, \omega) \in \mathbb{R} \times \Omega$;
(3) $\psi(t + s, \tau, \omega) = \psi(t, \Theta_s(\tau, \omega)) \circ \psi(s, \tau, \omega)$, *for every $t, s \geq 0$ in $\mathbb{R}$, and* $(\tau, \omega) \in \mathbb{R} \times \Omega$;
(4) $\psi(t, \tau, \omega) : X \to X$ *is a continuous map for each* $(t, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega$.

We usually denote by $(\psi, \Theta)_{(X, \mathbb{R}\times\Omega)}$, or (ψ, Θ), the co-cycle ψ driven by Θ.

Remark 2.3. *We will write $\omega_\tau := (\tau, \omega) \in \mathbb{R} \times \Omega$, and $\Theta_t(\omega_\tau) := (t + \tau, \theta_t\omega) = (\theta_t\omega)_{\tau+t}$.*

Throughout this work we will assume that a nonautonomous random dynamical system (ψ, Θ) satisfies

$$\mathbb{R}^+ \times X \ni (t,x) \mapsto \psi(t,\omega_\tau)x \in X \text{ is continuous, for each } \omega_\tau \in \mathbb{R}\times\Omega. \tag{2.1}$$

This assumption is sensible in the applications, e.g., when the co-cycle is induced by a well-posed stochastic/random differential equation. Hence, we can associate our co-cycle with a family of *evolution processes*. Recall that:

Definition 2.4. *Let $\mathcal{S} = \{S(t,s);\, t \geq s\}$ be a family of continuous operators from X into itself. We say that $\mathcal{S}$ is an **evolution process** in X if $S(t,t) = Id_X$, for all $t \in \mathbb{R}$, $S(t,s)S(s,\tau) = S(t,\tau)$, for $t \geq s \geq \tau$, and the mapping $\{(t,s) \in \mathbb{R}^2;\, t \geq s\} \times X \ni (t,s,x) \mapsto S(t,s)x$ is continuous.*

Remark 2.5. *Let $(\psi,\Theta)_{(X,\mathbb{R}\times\Omega)}$ be a nonautonomous random dynamical system which satisfies* (2.1)*. Then, for each $\omega_\tau \in \mathbb{R}\times\Omega$, we define the following evolution process*

$$\Psi_{\omega_\tau} := \{\psi(t-s, \Theta_s\omega_\tau)\,;\, t \geq s\}.$$

Definition 2.6. *Let $K : \Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say that K is **measurable** if the mapping $\Omega \ni \omega \mapsto d(x, K(\omega))$ is $(\mathcal{F}, \mathcal{B}_\mathbb{R})$-measurable for every fixed $x \in X$.*

In Definition 2.6, we used that X is a complete separable metric space, see [31, Chapter III].

Definition 2.7. *Let $\hat{A} = \{\mathcal{A}(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$ be a family of nonempty subsets of X. We say that $\hat{A}$ is a **nonautonomous random attractor** for (ψ,Θ) if the following conditions are fulfilled:*

1. *$\mathcal{A}(\omega_\tau)$ is compact, for every $\omega_\tau \in \mathbb{R}\times\Omega$;*
2. *the set-valued mapping $\omega \mapsto \mathcal{A}(\tau,\omega)$ is measurable, for each $\tau \in \mathbb{R}$;*
3. *$\hat{A}$ is invariant, i.e., $\psi(t,\omega_\tau)\mathcal{A}(\omega_\tau) = \mathcal{A}(\Theta_t\omega_\tau)$ for every $t \geq 0$ and $\omega_\tau \in \mathbb{R}\times\Omega$;*
4. *$\hat{A}$ pullback attracts every bounded subset of X, i.e., for every bounded subset B of X and $\omega_\tau \in \mathbb{R}\times\Omega$,*
$$\lim_{t\to+\infty} dist(\psi(t,\Theta_{-t}\omega_\tau)B, \mathcal{A}(\omega_\tau)) = 0,$$
where $dist(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b)$ is the usual Hausdorff semi-distance;
5. *$\hat{A}$ is the minimal closed family that pullback attracts bounded subsets of X, i.e., if $\{F(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$ is a family of closed subsets of X that pullback attracts every bounded subset of X, then $\mathcal{A}(\omega_\tau) \subset F(\omega_\tau)$, for every $\omega_\tau \in \mathbb{R}\times\Omega$.*

For existence of nonautonomous random attractors and applications to differential equations, see Wang [39] and the references therein.

Since we will associate our co-cycle (ψ,Θ) with a family of evolution processes as in Remark 2.5, we recall the notion of *pullback attractors*.

Definition 2.8. *Let $\mathcal{S} = \{S(t,s) : t \geq s\}$ be an evolution process in X and $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ be a family of nonempty subsets of X. We say that $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a **pullback attractor** for $\mathcal{S}$ if*

1. *$\mathcal{A}(t)$ is compact, for every $t \in \mathbb{R}$;*
2. *$\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is invariant, i.e., $S(t,s)\mathcal{A}(s) = \mathcal{A}(t)$, $\forall\, t \geq s$;*

(3) $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *pullback attracts every bounded subset of* X*, i.e., for every bounded subset* B *of* X*,*

$$\lim_{s\to-\infty} dist(S(t,s)B, \mathcal{A}(t)) = 0;$$

(4) $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *is the minimal closed family that pullback attracts bounded subsets of* X*.*

There are several works that deal with the existence and continuity (upper and lower semicontinuity) of pullback attractors, we refer the reader to [26, 17, 29, 12], where many other references to earlier results can be found.

Remark 2.9. *Let* (ψ, Θ) *be a nonautonomous random dynamical system with a nonautonomous random attractor* $\{\mathcal{A}(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$*. Then, for each* ω_τ *fixed, the evolution process* Ψ_{ω_τ} *has a pullback attractor given by* $\{A(\Theta_t\omega_\tau) : t \in \mathbb{R}\}$*.*

Finally, we recall the definition of the *unstable set* for a global solution ξ of an evolution process, which was introduced in [30].

Definition 2.10. *Let* $\mathcal{S} = \{S(t,s) : t \geq s\}$ *be an evolution process, and* $\xi : \mathbb{R} \to X$ *be a* ***global solution*** *of* $\mathcal{S}$*, i.e.,* $S(t,s)\xi(s) = \xi(t)$*, for every* $t \geq s$*. The* ***unstable set*** *of* ξ *is defined as*

$$W^u(\xi) = \Big\{(t,z) \in \mathbb{R}\times X : \text{ there is a global solution } \zeta \text{ of } \mathcal{S} \text{ such that } \zeta(t) = z, \text{ and } \lim_{s\to-\infty} \|\zeta(s)-\xi(s)\|_X = 0\Big\}.$$

The section of $W^u(\xi)$ **at time** $t \in \mathbb{R}$ *is denoted by* $W^u(\xi)(t) = \{z \in X : (t,z) \in W^u(\xi)\}$*.*

Remark 2.11. *Let* $\mathcal{S} = \{S(t,s) : t \geq s\}$ *be an evolution process with a pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *such that* $\cup_{t\leq 0}\mathcal{A}(t)$ *is bounded. In this case*

$$\mathcal{A}(t) = \bigcup\{W^u(\xi)(t) : \xi \text{ is a backwards-bounded solution}\}, \ \forall t \in \mathbb{R}, \tag{2.2}$$

where ξ *is backwards-bounded means that the set* $\xi(-\infty, 0]$ *is bounded. Therefore, it is natural to search for the minimal collection of backwards-bounded solutions whose unstable sets form the attractor. Of course many backwards-bounded solutions have the same unstable set, and thus it is natural to seek for backward-separated solutions, see* [29, Section 3.3] *for more details. In Section 6, we will provide conditions to obtain a distinguished set of backwards-bounded global solutions that forms the nonautonomous random attractor. These conditions rely on the hyperbolicity, which we will study in the following sections. It is through this distinguished set that we will be able to address the lower semicontinuity of nonautonomous random attractors.*

3. Permanence of random hyperbolic solutions

In this section we recall some results on the existence and continuity of hyperbolic solutions for nonautonomous random differential equations obtained in [16]. As we will see further, these results are crucial to obtain the lower semicontinuity and topological structural stability of attractors.

As in [16, Section 3], problems (1.1) and (1.2) can be seen as the following family of semilinear differential equations on a separable Banach space X

$$\dot{y} = By + f_0(y), \quad y(0) = y_0, \tag{3.1}$$

$$\dot{y} = By + f_\eta(\Theta_t\omega_\tau, y), \quad y(0) = y_0, \tag{3.2}$$

where $\{\Theta_t : t \in \mathbb{R}\}$ is a driving flow given by $\Theta_t(\omega_\tau) := (t+\tau, \theta_t\omega)$ for every $\omega_\tau = (\tau, \omega) \in \mathbb{R} \times \Omega$.

We suppose that $f_\eta(\omega_\tau, \cdot) \in C^1(X)$, for every $\eta \in [0,1]$, $\omega_\tau \in \mathbb{R}\times\Omega$, and that

$$\lim_{\eta\to 0} \sup_{(t,x)\in\mathbb{R}\times B(0,r)} \Big\{ \|f_\eta(\Theta_t\omega_\tau, x) - f_0(x)\|_X + \|(f_\eta)_x(\Theta_t\omega_\tau, x) - f_0'(x)\|_{\mathcal{L}(X)} \Big\} = 0, \tag{3.3}$$

for all $r \geq 0$ and $\omega_\tau \in \mathbb{R}\times\Omega$, where $(f_\eta)_x(\omega_\tau, \cdot) : X \to \mathcal{L}(X)$ is the derivative map of $f_\eta(\omega_\tau, \cdot) : X \to X$. This ensures local well-posedness and differentiability with respect to the initial conditions of (3.1) and (3.2), for each $\omega_\tau \in \mathbb{R}\times\Omega$. We also assume that for each equilibrium $y^* \in X$ of 3.1, i.e. $f_0(y^*) = -By^*$, there exists $r_0 > 0$ such that

$$\rho(\epsilon) := \sup_{x\in B_{r_0}(y^*)} \sup_{\|h\|\leq\epsilon} \left\{ \frac{\|f_0(x+h) - f_0(x) - f_0'(x)h\|_X}{\|h\|_X} \right\} \to 0, \text{ as } \epsilon \to 0. \tag{3.4}$$

Additionally, we assume global well-posedness, so that (3.1) is associated with a semigroup $\mathcal{T}_0 = \{T_0(t) : t \geq 0\}$, and that (3.2) is associated with nonautonomous random dynamical system (ψ_η, Θ), for each $\eta \in [0,1]$.

Remark 3.1. *The global existence can be obtained by proving that the solutions do not explode in finite time, see instance* [35, Theorem 3.3.4 and Corollary 3.3.5] *or* [29, Section 6.8]*. In particular, this is achieved when we consider dissipative nonlinearities, such as the ones in our applications, see* (7.7) *for the gradient system and* (7.10) *for the damped wave equation. Moreover, these conditions are those used to obtain the existence of attractors.*

We say that a map $\xi : \mathbb{R}\times\Omega \to X$ is a **global solution** for (ψ, Θ) if

$$\psi(t, \omega_\tau)\xi(\omega_\tau) = \xi(\Theta_t\omega_\tau), \text{ for every } t \geq 0.$$

Then, for each ω_τ fixed, the mapping $\mathbb{R} \ni t \mapsto \xi(\Theta_t\omega_\tau)$ defines a global solution for the evolution process $\{\psi(t-s, \Theta_s\omega_\tau) : t \geq s\}$.

We are interested in the global solutions that are *hyperbolic*, and to define hyperbolic solutions we need to recall the concept of exponential dichotomy. First, recall the definition of Θ-*invariance*:

Definition 3.2. *A map $M : \mathbb{R}\times\Omega \to \mathbb{R}$ is said to be* ***Θ-invariant*** *if for each $\omega_\tau \in \mathbb{R}\times\Omega$ we have that $M(\Theta_t\omega_\tau) = M(\omega_\tau)$, for every $t \in \mathbb{R}$.*

Definition 3.3. *A linear nonautonomous random dynamical system (φ, Θ) such that $\varphi(t,\tau,\omega) \in \mathcal{L}(X)$, for all $(t,\tau,\omega) \in \mathbb{R}^+\times\mathbb{R}\times\Omega$, is said to admit an* ***exponential dichotomy*** *if there exists a subset $\tilde{\Omega}$ of Ω which $\theta_t\tilde{\Omega} = \tilde{\Omega}$ and $\mathbb{P}(\tilde{\Omega}) = 1$, and a family of projections, $\Pi^s := \{\Pi^s(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\tilde{\Omega}\}$ such that*

(1) *the map $\Pi^s(\tau, \cdot) : \tilde{\Omega} \to \mathcal{L}(X)$ is strongly measurable, for each $\tau \in \mathbb{R}$;*
(2) *$\Pi^s(\Theta_t\omega_\tau)\varphi(t,\omega_\tau) = \varphi(t,\omega_\tau)\Pi^s(\omega_\tau)$, for every $t \in \mathbb{R}^+$ and $\omega_\tau \in \mathbb{R}\times\tilde{\Omega}$;*
(3) *$\varphi(t,\omega_\tau) : R(\Pi^u(\omega_\tau)) \to R(\Pi^u(\Theta_t\omega_\tau))$ is an isomorphism, where $\Pi^u(\omega_\tau) := Id_X - \Pi^s(\omega_\tau)$, for all $\omega_\tau \in \mathbb{R}\times\Omega$;*

(4) *there exist* Θ*-invariant maps* $\alpha : \mathbb{R}\times\Omega \to (0,+\infty)$ *and* $M : \mathbb{R}\times\Omega \to [1,+\infty)$ *such that*

$$\|\varphi(t,\omega_\tau)\Pi^s(\omega_\tau)\|_{\mathcal{L}(X)} \leq M(\omega_\tau)e^{-\alpha(\omega_\tau)t}, \text{ for every } t \geq 0;$$
$$\|\varphi(t,\omega_\tau)\Pi^u(\omega_\tau)\|_{\mathcal{L}(X)} \leq M(\omega_\tau)e^{\alpha(\omega_\tau)t}, \text{ for every } t \leq 0,$$

for every $\omega_\tau \in \mathbb{R}\times\tilde{\Omega}$.

In this case the function M *is called a* ***bound*** *and* α *an* ***exponent*** *for the exponential dichotomy.*

For nonautonomous random dynamical systems this notion was introduced by [16]. Also in [16] the authors proved a robustness result and as an application they established the existence and continuity of random hyperbolic solutions for (3.2).

Recall that $y_0^* \in X$ is a **hyperbolic equilibrium** for (3.1) if the linear operator $A := B + f_0'(y_0^*)$ generates a C^0-semigroup $\{e^{At} : t \geq 0\}$ that admits an exponential dichotomy. We say that ξ is a **random hyperbolic solution** of (3.2) if the linearized nonautonomous random dynamical system (φ, Θ), given by

$$\varphi(t,\omega_\tau) = e^{Bt} + \int_0^t e^{B(t-s)} D_x f_\eta(\Theta_s\omega_\tau, \xi(\Theta_s\omega_\tau))\varphi(s,\omega_\tau)ds, \forall \omega_\tau \in \mathbb{R}\times\Omega,$$

admits an exponential dichotomy.

Now, we present a result on the permanence of hyperbolic solutions, for the proof see [16, Theorem 3.9].

Theorem 3.4 (Existence and continuity of hyperbolic solutions)**.** *Let* y_0^* *be a hyperbolic equilibrium for* (3.1) *and assume that* (3.3) *and* (3.4) *hold. Given* $\epsilon > 0$ *suitable small, there exists a* Θ*-invariant map* $\eta_\epsilon : \mathbb{R}\times\Omega \to (0,1]$ *such that:*

(1) *for each* $\omega_\tau \in \mathbb{R}\times\Omega$ *fixed, given* $\eta \in (0, \eta_\epsilon(\omega_\tau)]$*, there exists a global hyperbolic solution* $\mathbb{R} \ni t \mapsto \zeta_\eta(t,\omega_\tau)$ *of the evolution process* $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ *satisfying*

$$\sup_{t\in\mathbb{R}} \|\zeta_\eta^*(t,\omega_\tau) - y_0^*\|_X < \epsilon, \tag{3.5}$$

and $\zeta_\eta(t,\omega_\tau) = \zeta_\eta(0,\Theta_t\omega_\tau)$*, for all* $t \in \mathbb{R}$.

(2) *for each* Θ*-invariant function* $\bar{\eta} : \mathbb{R}\times\Omega \to [0,1]$ *with* $\bar{\eta}(\omega_\tau) \leq \eta_\epsilon(\omega_\tau)$*, there exists a random hyperbolic solution* $\xi_{\bar{\eta}}^* : \mathbb{R}\times\Omega \to X$ *of* $(\psi_{\bar{\eta}}, \Theta)$ *defined by*

$$\xi_{\bar{\eta}}^*(\omega_\tau) := \zeta_{\bar{\eta}(\omega_\tau)}^*(0,\omega_\tau),$$

and satisfying (3.5).

Theorem 3.4 is the first step to the study of existence and continuity of unstable and stable sets, which are the main tool to conclude lower semicontinuity and topological structural stability of attractors.

Remark 3.5. *Suppose that* $\{y_1^*, \cdots, y_p^*\}$ *is a set of hyperbolic equilibria for* (3.1)*. Then there exists* $\epsilon_0 > 0$ *such that* y_i^* *is isolated in* $B(y_i^*, \epsilon_0)$ *and* $B(y_i^*, \epsilon_0) \cap B(y_j^*, \epsilon_0) = \emptyset$*,* $j \neq i$*. Theorem 3.4 guarantees that for each* $i \in \{1, \cdots, p\}$ *and* $\epsilon_0' \in (0, \epsilon_0)$ *suitable small fixed, there exits a* Θ*-invariant function* $\eta_{0,i} : \mathbb{R}\times\Omega \to (0,1]$ *satisfying the conclusions of Theorem 3.4.*

Define $\eta_0(\omega_\tau) = \min_{0\leq i\leq p}\{\eta_{0,i}(\omega_\tau)\}$*, for* $\omega_\tau \in \mathbb{R}\times\Omega$*. Let* ω_τ *be fixed, then for each* $\eta \in (0, \eta_0(\omega_\tau)]$ *there exists* $\zeta_{i,\eta}^*(\cdot,\omega_\tau)$ *is a hyperbolic solution of* $\{\psi_\eta(t -$

$s, \Theta_s\omega_\tau) : t \geq s\}$ *such that*

$$\sup_{t\in\mathbb{R}} \|\zeta^*_{i,\eta}(t,\omega_\tau) - y^*_i\|_X < \epsilon'_0, \text{ for every } i \in \{1, \cdots, p\}.$$

4. Existence and continuity of unstable sets

In this section we study the existence and continuity of unstable sets for the hyperbolic solutions obtained in Theorem 3.4[Item (1)]. Under the same assumptions as in Section 3, we will apply the techniques of the deterministic case [30] to our problem. The idea here is to revisit the proofs to track the dependence on the parameter $\omega_\tau \in \mathbb{R}\times\Omega$ in the arguments.

First, inspired by [30], we extend the concept of *unstable set* for nonautonomous random dynamical systems.

Definition 4.1. *Let (ψ, Θ) be a nonautonomous random dynamical system and $\xi^* : \mathbb{R}\times\Omega \to X$ be a random hyperbolic solution of (ψ,Θ). The* ***unstable set*** *of ξ^* is the family*

$$W^u(\xi^*) = \{W^u(\xi^*;\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\},$$

where, for each ω_τ, $W^u(\xi^*;\omega_\tau)$ is the unstable set of the hyperbolic solution $t \mapsto \xi^*(\Theta_t\omega_\tau)$ of the evolution process $\Psi_{\omega_\tau} = \{\psi(t-s,\Theta_s\omega_\tau) : t \geq s\}$. **The section of** *$W^u(\xi^*;\omega_\tau)$* **at time** *$t \in \mathbb{R}$ is denoted by*

$$W^u(\xi^*;\omega_\tau)(t) = \{z \in X : (t,z) \in W^u(\xi^*;\omega_\tau)\}.$$

Let $\delta : \mathbb{R}\times\Omega \to (0,+\infty)$ be a Θ-invariant map, a **local unstable set** *is a family $W^{u,\delta}(\xi^*) = \{W^{u,\delta}(\xi^*;\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$, where*

$$W^{u,\delta}(\xi^*;\omega_\tau) = \Big\{(t,z) \in \mathbb{R}\times X : \text{ there is a global solution } \zeta \text{ of } \Psi_{\omega_\tau} \text{ such that } \zeta(t) = z, \;\; \|\zeta(s) - \xi^*(\Theta_s\omega_\tau)\|_X \leq \delta(\omega_\tau), \; \forall s \leq t, \text{ and } \lim_{s\to-\infty} \|\zeta(s) - \xi^*(\Theta_s\omega_\tau)\|_X = 0\Big\},$$

and **the section of** *$W^{u,\delta}(\xi^*;\omega_\tau)$* **at time** *$t$ is defined by*

$$W^{u,\delta}(\xi^*;\omega_\tau)(t) = \{z \in X : (t,z) \in W^{u,\delta}(\xi^*;\omega_\tau)\}.$$

For the unstable set we have the following proposition.

Proposition 4.2. *Let (ψ,Θ) be a nonautonomous random dynamical system and $\xi^* : \mathbb{R}\times\Omega \to X$ be a random hyperbolic solution of (ψ,Θ).*

For each $\omega_\tau \in \mathbb{R}\times\Omega$ and $t \in \mathbb{R}$,

$$W^u(\xi^*;\omega_\tau)(t) = W^u(\xi^*;\Theta_t\omega_\tau)(0). \tag{4.1}$$

Moreover, if (ψ,Θ) has a nonautonomous random attractor $\hat{\mathcal{A}} = \{\mathcal{A}(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$ and ξ^* is bounded, then

$$W^u(\xi^*;\omega_\tau)(0) \subset \mathcal{A}(\omega_\tau), \; \forall\, \omega_\tau \in \mathbb{R}\times\Omega. \tag{4.2}$$

Proof. First we prove (4.1). Let $z \in W^u(\xi^*;\omega_\tau)(t)$, then there exists a global solution $\zeta : \mathbb{R} \to X$ of Ψ_{ω_τ} such that $\zeta(t) = z$ and $\|\zeta(s) - \xi^*(\Theta_s\omega_\tau)\|_X \overset{s\to-\infty}{\longrightarrow} 0$. Define, $\tilde{\zeta}(s) = \zeta(t+s)$, $s \in \mathbb{R}$, thus $\tilde{\zeta}$ is a global solution for $\Psi_{\Theta_t\omega_\tau}$ such that $\tilde{\zeta}(0) = z$ and

$$\|\tilde{\zeta}(s) - \xi^*(\Theta_s\Theta_t\omega_\tau)\|_X = \|\zeta(s+t) - \xi^*(\Theta_{s+t}\omega_\tau)\|_X \overset{s\to-\infty}{\longrightarrow} 0.$$

Therefore, $z \in W^u(\xi^*, \Theta_t\omega_\tau)(0)$. By similar arguments, we see that

$$W^u(\xi^*, \Theta_t\omega_\tau)(0) \subset W^u(\xi^*, \omega_\tau)(t),$$

which concludes the proof of (4.1).

For the second claim, let $z \in W^u(\xi^*; \omega_\tau)(0)$, then there exists a global solution $\zeta : \mathbb{R} \to X$ of Ψ_{ω_τ} such that $\zeta(0) = z$ and $\|\zeta(s) - \xi^*(\Theta_s\omega_\tau)\|_X \to 0$ as $s \to -\infty$. Since $\{\xi^*(\Theta_t\omega_\tau) : t \in (-\infty, 0]\}$ is bounded, the set $B = \zeta((-\infty, 0])$ is also bounded. Then $\hat{\mathcal{A}}$ pullback attracts B, i.e.,

$$\lim_{s\to-\infty} dist_H(\psi(-s, \Theta_s\omega_\tau)B, \mathcal{A}(\omega_\tau)) = 0. \tag{4.3}$$

Note that $\zeta(s) \in B$ and $\psi(-s, \Theta_s\omega_\tau)\zeta(s) = \zeta(0)$, for every $s \leq 0$, thus from (4.3),

$$d(z, A(\omega_\tau)) = \lim_{s\to-\infty} d(\psi(-s, \Theta_s\omega_\tau)\zeta(s), A(\omega_\tau)) = 0.$$

Therefore, $z \in \mathcal{A}(\omega_\tau)$ and the proof is complete. □

Proposition 4.2 implies that the attractor contains all the unstable sets of hyperbolic solutions. Later, in Section 6, we will give conditions under which the attractor is equal to the union of these unstable sets. Next, we prove that the local unstable sets for these hyperbolic solutions are given as a graph, following the same line of arguments presented in [30]. In fact, if ξ^*_η is a random hyperbolic solution of ψ_η, we will show that the elements in $W^{u,\delta}_\eta(\xi^*_\eta; \omega_\tau)$ will be those of the form

$$(t, \xi^*_\eta(\Theta_t\omega_\tau) + \Pi^u_\eta(\Theta_t\omega_\tau)z + \Sigma^u(\omega_\tau)(t, \Pi^u_\eta(\Theta_t\omega_\tau)z)) \in \mathbb{R} \times X, \text{ and } \|z\|_X \leq \delta(\omega_\tau),$$

where $\delta : \mathbb{R} \times \Omega \to (0, +\infty)$ is a Θ-invariant map and Σ^u some Lipschitz map. Moreover, we will obtain that as $\eta \to 0$ these local unstable sets "converges" to the unstable sets of the autonomous problem (3.1).

Let y^*_0 be a hyperbolic equilibrium for (3.1), $\epsilon_0 > 0$ be suitable small. Then by Theorem 3.4, there exists a Θ-invariant map $\eta_0 : \mathbb{R} \times \Omega \to (0, 1]$ such that for each fixed ω_τ and $\eta \in (0, \eta_0(\omega_\tau)]$, there exists $t \mapsto \xi^*_\eta(\Theta_t\omega_\tau)$ a hyperbolic solution of $\{\psi_\eta(t - s, \Theta_s\omega_\tau) : t \geq s\}$ such that $\sup_{t\in\mathbb{R}} \|y^*_0 - \xi^*_\eta(\Theta_t\omega_\tau)\|_X < \epsilon_0$. Then, the change of variables $z(t) = y(t) - \xi^*_\eta(\Theta_t\omega_\tau)$ allows us to concentrate on the existence of unstable sets of global hyperbolic solutions around the zero solution for

$$\dot{z} = Az + B_\eta(\Theta_t\omega_\tau)z + h_\eta(\Theta_t\omega_\tau, z), \quad z(0) = z_0 \in X, \tag{4.4}$$

where $A = B + f'_0(y^*_0)$, $B_\eta(\omega_\tau) = (f_\eta)_x(\omega_\tau, \xi^*_\eta(\omega_\tau)) - f'_0(y^*_0)$ and

$$\begin{aligned} h_\eta(\Theta_t\omega_\tau, z) :=& f_\eta(\Theta_t\omega_\tau, \xi^*_\eta(\Theta_t\omega_\tau) + z) - f_\eta(\Theta_t\omega_\tau, \xi^*_\eta(\Theta_t\omega_\tau)) \\ &- (f_\eta)_x(\Theta_t\omega_\tau, \xi^*_\eta(\Theta_t\omega_\tau))z. \end{aligned}$$

Thus $z = 0$ is a globally defined bounded solution for (4.4) where $h_\eta(\omega_\tau, \cdot) : X \to X$ differentiable with $h_\eta(\omega_\tau, 0) = 0$, $(h_\eta)_x(\omega_\tau, 0) = 0 \in \mathcal{L}(X)$, for all $\eta \in (0, \eta_0(\omega_\tau)]$. Similarly, for $\eta = 0$ we see that $B_0 = 0 \in \mathcal{L}(X)$ and $h_0(z) = f_0(y^*_0 + z) - f_0(y^*_0) - f'(y^*_0)z$, for $z \in X$. Thus (3.3) implies that

$$\lim_{\eta\to 0} \sup_{(t,x)\in\mathbb{R}\times B(0,r)} \Big\{ \|h_\eta(\Theta_t\omega_\tau, x) - h_0(x)\|_X + \|(h_\eta)_x(\Theta_t\omega_\tau, x) - h'_0(x)\|_{\mathcal{L}(X)} \Big\} = 0, \tag{4.5}$$

for all $r > 0$ and $\omega_\tau \in \mathbb{R} \times \Omega$.

In the proof of Theorem 3.4 (see [16, Theorem 3.9] for details) the Θ-invariant map $\eta_0 : \mathbb{R}\times\Omega \to (0,1]$ is chosen such that the linear evolution process $\{\varphi_\eta(t-s,\Theta_s\omega_\tau) : t \geq s\}$, given by

$$\varphi_\eta(t-s,\Theta_s\omega_\tau) = e^{A(t-s)} + \int_s^t e^{A(t-r)}B_\eta(\Theta_r\omega_\tau)\varphi_\eta(r-s,\Theta_s\omega_\tau)\,dr,\ t\geq s, \quad (4.6)$$

admits an exponential dichotomy with bound M_η, exponent α_η and family of projections $\{\Pi^u_\eta(t) : t\in\mathbb{R}\}$, for every $\eta \in (0,\eta_0(\omega_\tau)]$. Moreover, for each Θ-invariant function $\bar\eta : \mathbb{R}\times\Omega \to [0,1]$, with $\bar\eta(\omega_\tau) \leq \eta_0(\omega_\tau)$, the co-cycle $(\varphi_{\bar\eta},\Theta)$ admits an exponential dichotomy with bound $M_{\bar\eta}$, exponent $\alpha_{\bar\eta}$ and family of projections $\{\Pi^u_{\bar\eta}(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\Omega\}$.

If z is a solution of (4.4) we write $z^u(t) = \Pi^u_\eta(t)z(t)$ and $z^s(t) = \Pi^s_\eta(t)z(t)$, $t\in\mathbb{R}$, where $\Pi^u_\eta(t) = Id_X - \Pi^s_\eta(t)$, $t \in \mathbb{R}$. Then z^u and z^s are the solutions of

$$\begin{aligned}\dot z^u &= A_\eta(\Theta_t\omega_\tau)z^u + h^u_\eta(\Theta_t\omega_\tau, z^u(t)+z^s(t)),\\ \dot z^s &= A_\eta(\Theta_t\omega_\tau)z^s + h^s_\eta(\Theta_t\omega_\tau, z^u(t)+z^s(t)),\end{aligned} \quad (4.7)$$

where $A_\eta(\omega_\tau) = A + B_\eta(\omega_\tau)$, and $h^k_\eta(\omega_\tau,\cdot) = \Pi^k_\eta(\omega_\tau)h_\eta(\omega_\tau,\cdot)$, $\quad k = u, s$.

Since, for each ω_τ fixed, $h^k_\eta(\Theta_t\omega_\tau, 0) = 0$, with $(h^k_\eta)_x(\Theta_t\omega_\tau, 0) = 0$ and h^k_η are continuous differentiable in X, uniformly with respect to t, we obtain that given $\rho > 0$ there exists $\delta_0(\omega_\tau) > 0$ such that if $\|z\|_X, \|\tilde z\|_X \leq \delta_0(\omega_\tau)$ then

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\|h^k_\eta(\Theta_t\omega_\tau, z)\|_X \leq \rho,\\ &\sup_{t\in\mathbb{R}}\|h^k_\eta(\Theta_t\omega_\tau, z) - h^k_\eta(\Theta_t\omega_\tau,\tilde z)\| \leq \rho\|z-\tilde z\|_X, \quad k = s,u.\end{aligned} \quad (4.8)$$

Note that, from (4.5), it is possible to choose $\delta_0 : \mathbb{R}\times\Omega \to (0,+\infty)$ as a Θ-invariant function. This is one of the main differences to the deterministic case and to work with the Θ-invariance is the key to our further results.

Remark 4.3. *For each ω_τ fixed, it is possible to extend $h^u_\eta(\omega_\tau,\cdot), h^s_\eta(\omega_\tau,\cdot)$ outside the ball of radius $\delta_0(\omega_\tau)$ such that this extension satisfies both conditions in* (4.8) *for all $z, \tilde z \in X$, see* [30]*. Therefore, we obtain the existence and continuity of unstable and stable set, as a graph, for h^u_η and h^s_η satisfying* (4.8)*, for all $z, \tilde z \in X$, then, using a localization procedure, we conclude the existence and continuity of the local unstable sets, as a graph, for the case when h^k_η satisfies* (4.8) *in the ball of radius $\delta(\omega_\tau)$, for each $\omega_\tau \in \mathbb{R}\times\Omega$.*

Assuming that (4.8) holds for all $z, \tilde z \in X$, we will obtain that, for all suitably small ρ, the unstable sets are graphs of Lipschitz maps in the class defined next.

Let $\{\Pi^u(s) : s\in\mathbb{R}\}$ be a family of projections and $L > 0$. Denote by $\mathcal{LB}(L)$ a complete metric space of all bounded and globally Lipschitz continuous functions $\Sigma : \mathbb{R}\times X \to X$ such that $\mathbb{R}\times X \ni (s,z) \mapsto \Sigma(s,z) := \Sigma(s,\Pi^u(s)z) \in \Pi^s(s)X$ and

$$\begin{aligned}&\sup\left\{\|\Sigma(s,\Pi^u(s)z)\|_X; (s,z)\in\mathbb{R}\times X\right\} \leq L,\\ &\|\Sigma(s,\Pi^u(s)z) - \Sigma(s,\Pi^u(s)\tilde z)\|_X \leq L\|\Pi^u(s)z - \Pi^u(s)\tilde z\|_X,\end{aligned} \quad (4.9)$$

with distance between $\Sigma, \widetilde\Sigma \in \mathcal{LB}(L)$ given by

$$|\!|\!|\Sigma - \tilde\Sigma|\!|\!| := \sup_{(t,z)\in\mathbb{R}\times X}\|\Sigma(t,z) - \tilde\Sigma(t,z)\|_X. \quad (4.10)$$

Theorem 4.4. *Let $\omega_\tau \in \mathbb{R}\times\Omega$ be fixed, and $\eta \in [0, \eta_0(\omega_\tau)]$. Suppose that $\rho > 0$ is suitable small such that there is $L = L(\rho, \alpha_\eta, M_\eta) > 0$ satisfying*

$$\begin{aligned} &\frac{\rho M_\eta}{\alpha_\eta} \le L, \quad \frac{\rho M_\eta}{\alpha_\eta}(1+L) < 1 \\ &\frac{\rho M_\eta^2 (1+L)}{\alpha_\eta - \rho M_\eta (1+L)} \le L, \\ &\rho M_\eta + \frac{\rho^2 M_\eta^2 (1+L)(1+M_\eta)}{2\alpha_\eta - \rho M_\eta(1+L)} < \frac{\alpha_\eta}{2}. \end{aligned} \tag{4.11}$$

Then, for each $\omega_\tau \in \mathbb{R}\times\Omega$ fixed and $\eta \in (0, \eta_0(\omega_\tau)]$, there exists $\Sigma_\eta^u = \Sigma_{\eta,\omega_\tau}^u \in \mathcal{LB}(L)$, such that the unstable set of the zero solution of (4.4) is given by

$$W_\eta^u(0) = \{(s,z) \in \mathbb{R}\times X : z = \Pi_\eta^u(s)z + \Sigma_\eta^u(s, \Pi_\eta^u(s)z)\}, \tag{4.12}$$

and, for any $r > 0$ and $s \in \mathbb{R}$,

$$\sup_{t\le s} \sup_{\|z\|_X \le r} \{\|\Pi_\eta^u(t)z - \Pi_0^u z\|_X + \|\Sigma_\eta^u(t, \Pi_\eta^u(t)z) - \Sigma_0^u(\Pi_0^u z)\|_X\} \overset{\eta\to 0}{\to} 0. \tag{4.13}$$

Furthermore, if $\zeta(t) = \zeta^u(t) + \zeta^s(t)$, where $\zeta^k(t) = \Pi_\eta^k(t)\zeta(t)$, for $k = u, s$, is a backward-bounded global solution of (4.4), then there is $\gamma > 0$ such that,

$$\|\zeta^s(t) - \Sigma_\eta^u(t, \zeta^u(t))\|_X \le M_\eta e^{-\gamma(t-s)} \|\zeta^s(s) - \Sigma_\eta^u(s, \zeta^u(s))\|_X, \ t \ge s. \tag{4.14}$$

Theorem 4.4 follows directly from [30, Theorem 3.1].

From Theorems 3.4 and 4.4, we can obtain the existence and continuity of local unstable sets.

Theorem 4.5 (Existence and continuity of local unstable set)**.** *Let $\eta \in [0,1]$, and $h_\eta : \mathbb{R}\times\Omega\times X \to X$ by such that for each ω_τ, the mapping $z \mapsto h_\eta(\omega_\tau, z)$ is continuously differentiable. Consider*

$$\dot{z} = A_\eta(\Theta_t\omega_\tau)z + h_\eta(\Theta_t\omega_\tau, z), \quad \omega_\tau \in \mathbb{R}\times\Omega. \tag{4.15}$$

Assume that $h_\eta(\omega_\tau, 0) = 0$, $(h_\eta)_x(\omega_\tau, 0) = 0 \in \mathcal{L}(X)$, $h_0 : X \to X$, $A_0(\Theta_t\omega_\tau) = A$, $\{h_\eta\}_{\eta\in[0,1]}$ satisfies (4.5), and that $z_0^* = 0$ is a hyperbolic solution of (4.15) for $\eta = 0$. Then given $\epsilon_0 > 0$ suitable small, the following hold:

(1) *There exist a Θ-invariant function $\eta_0 : \mathbb{R}\times\Omega \to [0,1]$ such that $z_\eta^* = 0$ is a hyperbolic solution of (4.15), for each $\eta \in (0, \eta_0(\omega_\tau)]$. In particular, the linear evolution process $\{\varphi_\eta(t-s, \Theta_s\omega_\tau) : t \ge s\}$, associated to the linear part of (4.15) (corresponding to the linearization of $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \ge s\}$ around $\xi_\eta^*(\Theta_t\omega_\tau)$), admits an exponential dichotomy with family of projections $\{\Pi_\eta^u(s) : s \in \mathbb{R}\}$.*

(2) *The families of projections $\Pi_\eta^u = \{\Pi_\eta^u(s) : s \in \mathbb{R}\}$, $\eta \in (0, \eta_0(\omega_\tau)]$ satisfy*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\Pi_\eta^u(t) - \Pi_0^u\|_{\mathcal{L}(X)} = 0. \tag{4.16}$$

(3) *There exist Θ-invariant function $\delta_0 : \mathbb{R}\times\Omega \to (0, +\infty)$ (independent of η) such that for each ω_τ and $\eta \in [0, \eta_0(\omega_\tau)]$, and a map*

$$\mathbb{R}\times B_X(0, \delta_0(\omega_\tau)) \ni (s,z) \mapsto \Sigma_\eta^u(s,z) := \Sigma_\eta^u(s, \Pi_\eta^u(s)z), \tag{4.17}$$

with the property: given $\delta \in (0, \delta_0(\omega_\tau))$*, there exists* $0 < \delta'' < \delta' < \delta$*,*

$$\begin{aligned}&\{\Pi_\eta^u(s)z + \Sigma_\eta^u(s, \Pi_\eta^u(s)z) \,:\, \|z\|_X \le \delta''\} \subset \\ &W_\eta^{u,\delta'}(0)(s) \subset \\ &\{\Pi_\eta^u(s)z + \Sigma_\eta^u(s, \Pi_\eta^u(s)z) \,:\, \|z\|_X \le \delta\}.\end{aligned} \tag{4.18}$$

(4) *For each* ω_τ *fixed, the family of graphs of the maps* $\{\Sigma_\eta\}_{\eta\in(0,\eta_0(\omega_\tau)]}$ *behaves continuously at* $\eta = 0$*:*

$$\sup_{t\le s} \sup_{\|z\|\le\delta_0(\omega_\tau)} \{\|\Pi_\eta^u(t) - \Pi_0^u\|_{\mathcal{L}(X)} + \|\Sigma_\eta^u(t, \Pi_\eta^u(t)z) - \Sigma_0^u(\Pi_0^u z)\|_X\} \overset{\eta\to 0}{\to} 0, \ \forall\, s \in \mathbb{R}. \tag{4.19}$$

Proof. Item (1) is a corollary of Theorem 3.4 and Item (2) follows from the continuous dependence of projections, in the sense of [29, Theorem 7.9] for evolution processes (see also [16, Theorem 2.23] for nonautonomous random dynamical systems).

By hypotheses, let $\rho > 0$ be such that there is L satisfying (4.11). Then there exists $\delta_0(\omega_\tau)$ such that (4.8) is satisfied for $z, \bar{z} \in B_X(0, \delta_0(\omega_\tau))$.

According to Remark 4.3 and Theorem 4.4, by a cut-off procedure, we obtain the desired function $\Sigma_\eta^u : \mathbb{R} \times B_X(0, \delta_0(\omega_\tau)) \to X$, for each $\eta \in (0, \eta_0(\omega_\tau)]$.

Thus, we only need to check (4.18). We claim that given $\delta \in (0, \delta_0(\omega_\tau))$, there exists $\delta' < \delta$ such that any global solution $\zeta : \mathbb{R} \to X$ of $\{\psi_\eta(t - s, \Theta_s\omega_\tau) : t \ge s\}$ on the unstable set such that $\|\zeta(s)\| \le \delta'$ must satisfy $\|\zeta(t)\| \le \delta$, for $t \le s$.

Indeed, from (4.7), $\zeta^u(t) = \Pi_\eta^u(t)\zeta(t)$ satisfies

$$\begin{aligned}\zeta^u(t) &= \varphi_\eta(t - s, \Theta_s\omega_\tau)\Pi_\eta^u(s)\zeta_0 \\ &\quad + \int_s^t \varphi_\eta(t - r, \Theta_r\omega_\tau)\Pi_\eta^u(r)h_\eta^u(\Theta_r\omega_\tau, \zeta^u(r) + \Sigma_\eta^u(r, \zeta^u(r)))dr, \ t \le s.\end{aligned}$$

Since $\{\varphi_\eta(t-s, \Theta_s\omega_\tau) : t \ge s\}$ admits an exponential dichotomy, due to Grönwall's inequality, we obtain

$$\|\zeta^u(t)\|_X \le M_\eta e^{(\alpha_\eta - \rho M_\eta(1+L))(t-s)}\|\zeta^u(s)\|_X, \quad t \le s.$$

Also, since $\|\Sigma_\eta^u(t, \zeta^u(t))\|_X \le L\|\zeta^u(t)\|_X$, $t \in \mathbb{R}$, we have that

$$\|\zeta(t)\|_X \le M_\eta^2(1 + L)e^{(\alpha_\eta - \rho M_\eta(1+L))(t-s)}\|\zeta(s)\|_X, \quad t \le s. \tag{4.20}$$

Then, taking $\delta' = \delta/M_\eta^2(1 + L)$, we see that

$$W_\eta^{u,\delta'}(0)(s) \subset \{\Pi_\eta^u(s)z + \Sigma_\eta^u(s, \Pi_\eta^u(s)z) \,:\, \|z\|_X \le \delta\}. \tag{4.21}$$

Finally, by the above argument, we also conclude that there exists $\delta'' \in (0, \delta')$ such that

$$\{\Pi_\eta^u(s)z + \Sigma_\eta^u(s, \Pi_\eta^u(s)z) \,:\, \|z\|_X \le \delta''\} \subset W_\eta^{u,\delta'}(0)(s). \tag{4.22}$$

The proof is complete. □

Remark 4.6. *We observe that, as in Theorem 3.4[Item (2)], using* Θ*-invariant functions* $\bar{\eta} : \mathbb{R} \times \Omega \to (0, 1]$ *it is possible to conclude existence of local unstable manifolds of the random hyperbolic solutions* $\xi_{\bar{\eta}}^*$ *for the nonautonomous random dynamical systems* $\psi_{\bar{\eta}}$*.*

We reinforce that these results on the existence and continuity of local unstable sets are the key to obtain lower semicontinuity and topological structural stability, as we will see in the following sections.

Remark 4.7. *We can obtain similar results concerning the existence and continuity of local stable sets following similar arguments to those presented here and* [30] *for the deterministic case.*

5. Continuity of nonautonomous random attractors

In this section we prove the continuity of attractors in the situation that the perturbed system is nonautonomous random whereas the limiting is an autonomous dynamical system which has an attractor given as union of unstable sets of hyperbolic equilibria.

First, we recall the definition of continuity of sets in a Banach space X. Let $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ be a family of subsets of a Banach space X. We say that $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is

(1) **Upper semicontinuous** at $\eta = 0$ if $\lim_{\eta\to 0} dist_H(A_\eta, A_0) = 0$.
(2) **Lower semicontinuous** at $\eta = 0$ if $\lim_{\eta\to 0} dist_H(A_0, A_\eta) = 0$.
(3) **Continuous** at $\eta = 0$ if it is upper and lower semicontinuous at $\eta = 0$.

Let Λ be a nonempty set. We say that $\{\mathcal{A}_\eta(\lambda) : \lambda \in \Lambda\}_{\eta\in[0,1]}$ is **upper (lower) semicontinuous at** $\eta = 0$ if $\{\mathcal{A}_\eta(\lambda)\}_{\eta\in[0,1]}$ is upper (lower) semicontinuous at $\eta = 0$, for each $\lambda \in \Lambda$, see [29, Chapter 3].

Now, we present a result on the continuity of attractors, as a consequence of a careful study of their internal structure, presented in the previews sections.

Theorem 5.1 (Continuity of nonautonomous random attractors)**.** *Let $\mathcal{T}_0 = \{T_0(t) : t \geq 0\}$ be the semigroup associated to* (3.1) *and (ψ_η, Θ) be the nonautonomous dynamical systems associated to* (3.2)*, and assume that conditions* (3.3) *and* (3.4) *are satisfied. Additionally, suppose that*

(a) *For each $\eta \in [0, 1]$, the co-cycle (ψ_η, Θ) has a nonautonomous random attractor $\{\mathcal{A}_\eta(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$,*

$$K(\omega_\tau) := \overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,1]} \mathcal{A}_\eta(\Theta_t\omega_\tau)} \text{ is compact}, \ \ \forall\, \omega_\tau \in \mathbb{R}\times\Omega, \text{ and}$$

$$\bigcup_{\eta\in[0,1]} \bigcup_{t\geq 0} \psi_\eta(t, \omega_\tau)K(\omega_\tau) \text{ is bounded}, \ \ \forall\, \omega_\tau \in \mathbb{R}\times\Omega; \tag{5.1}$$

(b) *$\mathcal{T}_0 = \{T_0(t) : t \geq 0\}$ is a semigroup with global attractor given by*

$$\mathcal{A}_0 = \bigcup_{j=1}^{p} W^u(y_j^*), \tag{5.2}$$

for which all the equilibria $\{y_j^* : 1 \leq j \leq p\}$ are hyperbolic.

Then given $\epsilon_0 > 0$ suitable small, there exists a Θ-invariant function $\eta_0 : \mathbb{R}\times\Omega \to (0, 1]$ such that, for each ω_τ fixed, the following hold:

(1) *For any $\eta \in (0, \eta_0(\omega_\tau)]$ and $j \in \{1, \cdots, p\}$, there exists a hyperbolic solution $\xi_{j,\eta}^*$ of $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ with*

$$\sup_j \sup_{t\in\mathbb{R}} \|\xi_{j,\eta}^*(\Theta_t\omega_\tau) - y_j^*\|_X < \epsilon_0, \tag{5.3}$$

where the linearized associated evolution process admits an exponential dichotomy with family of projections $\{\Pi^u_{j,\eta}(s) : s \in \mathbb{R}\}$.

(2) *There exists* $\delta_0(\omega_\tau) > 0$, *where* δ_0 *is* Θ*-invariant and independent of* η, *such that for each* ω_τ, $j \in \{1, \cdots, p\}$, *and* $\eta \in [0, \eta_0(\omega_\tau)]$, *there exists a map*

$$\mathbb{R} \times B_X(0, \delta_0(\omega_\tau)) \ni (s, z) \mapsto \Sigma^u_{j,\eta}(s, z) := \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z), \tag{5.4}$$

with the property: given $\delta \in (0, \delta_0(\omega_\tau))$, *there exists* $0 < \delta'' < \delta' < \delta$,

$$\begin{aligned} &\{\xi^*_{j,\eta}(s) + \Pi^u_{j,\eta}(s)z + \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z) \,:\, \|z\|_X \le \delta''\} \subset \\ &W^{u,\delta'}_{j,\eta}(\xi^*_{j,\eta})(s) \subset \\ &\{\xi^*_{j,\eta}(s) + \Pi^u_{j,\eta}(s)z + \Sigma^u_{j,\eta}(s, \Pi^u_{j,\eta}(s)z) \,:\, \|z\|_X \le \delta\}; \end{aligned} \tag{5.5}$$

(3) *The family of graphs of* $\{\Sigma^u_{j,\eta}\}_{\eta\in[0,\eta_0(\omega_\tau)]}$ *is continuous at* $\eta = 0$ *as in Theorem 4.5[Item (4)], for each* $j \in \{1, \cdots, p\}$.

(4) *For each* ω_τ, *the family of pullback attractors* $\{\mathcal{A}_\eta(\Theta_t\omega_\tau) : t \in \mathbb{R}\}_{\eta\in[0,\eta_0(\omega_\tau)]}$ *is continuous at* $\eta = 0$.

In particular, we have continuity of nonautonomous random attractors in the following sense: given $\epsilon > 0$, *there exists a* Θ*-invariant function* $\eta_\epsilon \le \eta_0$ *such that, for every* Θ*-invariant function* $\bar{\eta}$, *with* $\bar{\eta} \le \eta_\epsilon$, *we have*

$$\sup_{t\in\mathbb{R}} d_H(\mathcal{A}_{\bar{\eta}}(\Theta_t\omega_\tau), \mathcal{A}_0) < \epsilon, \quad \forall\, \omega_\tau \in \mathbb{R} \times \Omega, \tag{5.6}$$

where $\{\mathcal{A}_{\bar{\eta}}(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$ *is the nonautonomous random attractor of* $(\psi_{\bar{\eta}}, \Theta)$ *and* $d_H(A, B) = \max\{dist_H(A, B), dist_H(B, A)\}$, *for* $A, B \subset X$.

Proof. Note that, items (1)-(3) are consequences of Theorem 3.4 and Theorem 4.5, thus to conclude the proof we only need to prove Item (4).

Let $\omega_\tau \in \mathbb{R} \times \Omega$ be fixed and take $K(\omega_\tau)$ as the one in Condition **(a)**. Note that

$$\lim_{\eta\to 0} \sup_{t\in[0,c]} \sup_{\ell\in\mathbb{R}} \sup_{z\in K(\omega_\tau)} \|\psi_\eta(t, \Theta_\ell\omega_\tau)z - T_0(t)z\|_X = 0, \tag{5.7}$$

for any $c > 0$, and $\omega_\tau \in \mathbb{R} \times \Omega$. Indeed, let $z \in K(\omega_\tau)$ and $t \in [0, c]$. Then subtracting the variation of constants formula for (ψ_η, Θ) and $\mathcal{T}_0$, we have that

$$\psi_\eta(t, \omega_\tau)z - T_0(t)z = \int_0^t e^{B(t-s)} \big[f_\eta(\Theta_s\omega_\tau, \psi_\eta(s, \omega_\tau)z) - f_0(T_0(s)z)\big]\, ds. \tag{5.8}$$

Then, for some $M, \alpha > 0$,

$$\begin{aligned} \|\psi_\eta(t, \omega_\tau)z - T_0(t)z\|_X = &\\ &\int_0^t Me^{\alpha(t-s)} \|f_\eta(\Theta_s\omega_\tau, \psi_\eta(s, \omega_\tau)z) - f_\eta(\Theta_s\omega_\tau, T_0(s)z)\|_X\, ds \\ &+ \int_0^t Me^{\alpha(t-s)} \|f_\eta(\Theta_s\omega_\tau, T_0(s)z) - f_0(T_0(s)z)\|_X\, ds. \end{aligned} \tag{5.9}$$

Since $T_0([0, c] \times K(\omega_\tau))$ is compact, from (3.3), the second integral of the right-hand side goes to zero as $\eta \to 0$, uniformly for $t \in [0, c]$ and $z \in K$. For the first integral, we use (3.3) and (5.1) to obtain a Lipschitz constant of $f_\eta(\Theta_s\omega_\tau, \cdot)$ independent of η and s. Then (5.7) follows by applying a Gröwnwall's inequality.

The proof of upper semicontinuity follows from standard arguments using (5.7) and Assumption **(a)**, see [29, Chapter 3], for pullback attractors, and [22, 24, 38] for random attractors.

Now, we prove lower semicontinuity using a characterization via sequences, see [29, Lemma 3.2]. In fact, let $\omega_\tau \in \mathbb{R} \times \Omega$, $t \in \mathbb{R}$, and $x_0 \in \mathcal{A}_0$. We will show that there exist sequences $\eta_k \in (0, \eta_0(\omega_\tau)]$, with $\eta_k \to 0$, and $x_k \in \mathcal{A}_{\eta_k}(\Theta_t \omega_\tau)$ such that $x_k \to x_0$ as $k \to +\infty$.

Indeed, from (5.2), $x_0 \in W^u(y_j^*)$ for some $j \in \{1, \cdots, p\}$. By Item (3) of Theorem 4.5, there exist $0 < \delta'' < \delta' < \delta_0(\omega_\tau)$ such that

$$\begin{aligned} &W_0^{u,\delta''}(y_j^*) \subset \{y_j^* + \Pi_{j,0}^u z + \Sigma_0^u(\Pi_{j,0}^u z) \, : \, \|z\|_X \leq \delta'\}, \text{ and} \\ &\{\xi_{j,\eta}^*(r) + \Pi_{j,\eta}^u(r) z + \Sigma_{j,\eta}^u(r, \Pi_{j,\eta}^u(r) z) \, : \, \|z\|_X \leq \delta'\} \subset W_\eta^{u,\delta_0}(\xi_{j,\eta}^*)(r), \end{aligned} \tag{5.10}$$

for every $r \in \mathbb{R}$ and $\eta \in (0, \eta_0(\omega_\tau)]$. Thus there exists a global solution $\zeta : \mathbb{R} \to X$ of $\mathcal{T}_0$ such that $\zeta(0) = x_0$ and $\zeta(-s) \in W_0^{u,\delta''}(y_j^*)$, for some $s \geq 0$.

Since $\zeta(-s) \in \{y_j^* + \Pi_{j,0}^u z + \Sigma_{j,0}^u(\Pi_{j,0}^u z), \ \|z\|_X \leq \delta'\}$, by Theorem 4.5[Item (4)], there exist $\{\eta_k\} \subset (0, \eta_0(\omega_\tau)]$ and $z_k \in \{\xi_{j,\eta_k}^*(t-s) + \Pi_{j,\eta_k}^u(t-s) z + \Sigma_{j,\eta_k}^u(t - s, \Pi_{j,\eta_k}^u(t-s) z) : \ \|z\|_X \leq \delta'\}$ with $\eta_k \to 0$ and $z_k \to \zeta(-s)$ as $k \to +\infty$.

By (5.10) and Proposition 4.2, we see that $x_k = \psi_{\eta_k}(t - (t-s), \Theta_{t-s}\omega_\tau) z_k \in \mathcal{A}_{\eta_k}(\Theta_t \omega_\tau)$, for all $k \in \mathbb{N}$. Then, we use (5.7) and that $\lim_k z_k = \zeta(-s)$, to guarantee that $\lim_k x_k = x_0$, and the proof is complete. □

Remark 5.2. *Theorem 5.1 can be extended to the case where the limit is nonautonomous. The key steps for the proof will be again the* Θ*-invariance of the maps involved.*

Remark 5.3. *Alternatively, Assumption **(a)** can be replaced by the following conditions:*

- **(a.1)** *For each* $\eta \in [0,1]$*, the co-cycle* (ψ_η, Θ) *has a nonautonomous random attractor* $\{\mathcal{A}_\eta(\omega_\tau) : \omega_\tau \in \mathbb{R} \times \Omega\}$ *and*
$$\overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,1]} \mathcal{A}_\eta(\Theta_t \omega_\tau)} \ \textit{is bounded}, \ \ \forall \, \omega_\tau \in \mathbb{R} \times \Omega;$$
- **(a.2)** *The family* $\{\psi_\eta, \Theta\}_{\eta\in[0,1]}$ *is* ***collectively asymptotic compact*** *in X, i.e., for all* ω_τ*, the sequence*
$$\{\psi_{\eta_n}(t_n, \Theta_{-t_n}\omega_\tau) x_n\} \ \textit{has a convergent subsequence in } X$$
whenever $\eta_n \to 0$, $t_n \to +\infty$, *and* $\{x_n\}$ *is a bounded sequence in* X.

Additionally, if (5.7) *holds for every compact set, then the conclusions of Theorem 5.1 will still hold true. This will be the case when applying this result for damped wave equations, see Subsection 7.2.*

Remark 5.4. *Theorem 5.1 is not optimal in the sense that we cannot obtain the limit*

$$\sup_{\omega_\tau \in \mathbb{R}\times\Omega} d_H(\mathcal{A}_\eta(\omega_\tau), \mathcal{A}_0) \to 0, \quad \textit{as } \eta \to 0. \tag{5.11}$$

To obtain this conclusion one should assume

$$\sup_{\omega_\tau \in \mathbb{R}\times\Omega} \sup_{x \in B(0,r)} \Big\{ \|f_\eta(\omega_\tau, x) - f_0(x)\|_X + \|(f_\eta)_x(\omega_\tau, x) - f_0'(x)\|_{\mathcal{L}(X)} \Big\} \overset{\eta\to 0}{\to} 0, \tag{5.12}$$

for all $r \geq 0$*, instead of* (3.3)*. In this case it is possible to obtain the conclusions of Theorem 5.1 with* $\eta_0 > 0$ *and* $\delta_0 > 0$ *independent of* ω_τ*, and therefore to conclude* (5.11)*. Note that this case is similar to the deterministic case, see* [28, Theorem 3.1].

However, in the applications to check condition (5.12) *one has to assume that the noise is uniformly bounded as in Remark 7.6, see also* [9, 19] *for more examples of uniformly bounded noises. On the other hand, in Section 7 we provide an example, see Example 7.3, where conditions of Theorem 5.1 are checked, but we do not know if its possible to verify* (5.12)*.*

Now that the continuity of attractors is proved, the next step is to ensure that the gradient structure is preserved under nonautonomous random perturbations.

6. Topological structural stability

In this section we present a result on the topological structural stability of attractors for nonautonomous random dynamical systems. We study co-cycles (ψ_η, Θ) obtained by nonautonomous random perturbations of a gradient semigroup $\{T_0(t) : t \geq 0\}$.

First, we recall some basic concepts necessary to define *dynamically gradient evolution processes.* Assume that $\mathcal{S} = \{S(t,s) : t \geq s\}$ is an evolution process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Let $\widehat{E} = \{E(t) : t \in \mathbb{R}\}$ be an invariant family for $\mathcal{S}$. Given a family of open sets $\widehat{U} = \{U(t) : t \in \mathbb{R}\}$ such that $\widehat{E} \subset \widehat{U}$ (i.e., $E(t) \subset U(t)$, for every $t \in \mathbb{R}$) we say that $\widehat{E}$ is **the maximal invariant in** $\widehat{U}$ if given an invariant family $\widehat{F}$ in $\widehat{U}$, then $\widehat{F} \subset \widehat{E}$. If there is an $\epsilon_0 > 0$ such that $\widehat{E}$ is the maximal invariant family in $\{O_{\epsilon_0}(E(t)) : t \in \mathbb{R}\}$, we say that $\widehat{E}$ is an **isolated invariant family**. We say that $\{\widehat{E}_1, \cdots, \widehat{E}_p\}$ is a **disjoint collection of isolated invariant families** if $\widehat{E}_i$ is an isolated invariant family for every $0 \leq i \leq p$ and there is an $\epsilon_0 > 0$ such that $O_{\epsilon_0}(E_j(t)) \cap O_{\epsilon_0}(E_i(t)) = \emptyset$, for $i \neq j$ and every $t \in \mathbb{R}$. A **homoclinic structure** in $\{\widehat{E}_1, \cdots, \widehat{E}_p\}$ is a subcollection $\{\widehat{E}_{l_1}, \cdots, \widehat{E}_{l_k}\}$, with $k \leq p$, and a set of global solutions $\{\zeta_1, \cdots, \zeta_k\}$ of $\mathcal{S}$ in $\mathcal{A}$ which, setting $\widehat{E}_{l_k+1} = \widehat{E}_{l_1}$, satisfy

$$\lim_{t \to -\infty} d(\zeta_i(t), E_{l_i}(t)) = 0, \text{ and } \lim_{t \to +\infty} d(\zeta_i(t), E_{l_{i+1}}(t)) = 0, \tag{6.1}$$

for each $1 \leq i \leq k$, and there exists an $\epsilon > 0$ such that

$$\sup_{t \in \mathbb{R}} d(\, \zeta_i(t), \bigcup_{i=1}^{k} O_\epsilon(E_{l_i}(t))\,) > 0, \ \forall\, 1 \leq i \leq k, \text{ and } t \in \mathbb{R}. \tag{6.2}$$

Remark 6.1. *Condition* (6.2) *has a technical nature and it is used only in the case* $k = 1$ *to guarantee that the global solution* ζ_1 *is not entirely contained in* E_{l_1}*. In other words, we use* (6.2) *to ensure that if there is a global solution* $\zeta : \mathbb{R} \to X$ *such that* $\zeta(t) \in E_i(t)$ *for all* $t \in \mathbb{R}$ *for some* $i \in \{1, \ldots, n\}$*, the pair* (ζ, E_i) *does not make a homoclinic structure.*

Definition 6.2. *Let* $\mathcal{S} = \{S(t,s) : t \geq s\}$ *with a pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *which contains a disjoint collection of invariant families* $\{\widehat{E}_1, \cdots, \widehat{E}_p\}$*. We say that* $\mathcal{S} = \{S(t,s) : t \geq s\}$ *is a* ***dynamically gradient evolution process*** *with respect to* $\{\widehat{E}_1, \cdots, \widehat{E}_p\}$ *if*

- ***(G1)*** *given a global solution* $\zeta : \mathbb{R} \to X$ *of* $\mathcal{S}$ *such that* $\zeta(t) \in \mathcal{A}(t)$*, for each* $t \in \mathbb{R}$*, there exist* $i, j \in \{1, \cdots, p\}$ *so that*

$$\lim_{t \to -\infty} d(\zeta(t), E_i(t)) = 0, \textit{ and } \lim_{t \to +\infty} d(\zeta(t), E_j(t)) = 0; \tag{6.3}$$

- ***(G2)*** $\{\widehat{E}_1, \cdots, \widehat{E}_p\}$ *does not admit any homoclinic structure.*

This notion of dynamically gradient was studied for random dynamical systems in [23, 36]. For topological structural stability of deterministic autonomous or nonautonomous dynamical systems, see [1, 10, 27].

Now, we present our result on the topological structural stability for random dynamical systems.

Theorem 6.3. *Assume that hypotheses of Theorem 5.1 are fulfilled and additionally assume that* $\mathcal{T}_0 = \{T_0(t-s) : t \geq s\}$ *is a gradient evolution process with respect to* $\{y_1^*, \cdots, y_p^*\}$*, where* y_j^* *is hyperbolic, for every* $: 1 \leq j \leq p$.

Then, there exists a Θ*-invariant function* $\eta_1 : \mathbb{R} \times \Omega \to (0,1)$ *such that for each* ω_τ *fixed the evolution process* $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ *is dynamically gradient with respect to* $\{\xi_{1,\eta}^*, \cdots, \xi_{p,\eta}^*\}$, $\forall \eta \leq \eta_1(\omega_\tau)$*. Consequently,*

$$\mathcal{A}_\eta(\Theta_t\omega_\tau) = \bigcup_{j=1}^{p} W_\eta^u(\xi_{j,\eta}^*; \omega_\tau)(t), \forall\, \eta \in [0, \eta_1(\omega_\tau)]. \tag{6.4}$$

Proof. Let $\omega_\tau \in \mathbb{R} \times \Omega$ be fixed and $\eta \in (0, \eta_0(\omega_\tau)]$. Let us prove the following claim: there exists $\delta' \in (0, \delta_0(\omega_\tau))$ such that, if $\zeta_\eta : \mathbb{R} \to X$ is a global solution in $\{\mathcal{A}_\eta(\Theta_t\omega_\tau) : t \in \mathbb{R}\}$ so that

$$\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X < \delta', \quad \forall\, t \leq t_0 \quad (t \geq t_0), \text{ for some } t_0 \in \mathbb{R}, \tag{6.5}$$

then $\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X \overset{t\to-\infty}{\longrightarrow} 0 \quad (\|\zeta_\eta(t) - \xi_{j,\eta}^*(t)\|_X \overset{t\to+\infty}{\longrightarrow} 0)$.

We prove only the backwards case, the proof of the forward case will be similar using the analogous results for the stable sets. First, note that $\tilde{\zeta}(t) = \zeta_\eta(t) - \xi_{j,\eta}^*(t)$, for $t \in \mathbb{R}$, $j \in \{1, \cdots, p\}$, and $\eta \in (0, \eta_0(\omega_\tau)]$, thus we analyze the dynamics around the solution $z = 0$ of (4.4). From Theorem 4.5[Item (3)], there exists $0 < \delta' < \delta < \delta_0(\omega_\tau)$ such that

$$\{\Pi_{j,\eta}^u(s)z + \Sigma_{j,\eta}^u(s, \Pi_{j,\eta}^u(s)z) \,:\, \|z\|_X \leq \delta'\} \subset W_\eta^{u,\delta}(0)(s), \forall\, s \in \mathbb{R}. \tag{6.6}$$

Thus, (6.5) implies that $\tilde{\zeta}(t)$ is inside the $\delta_0(\omega_\tau)$-neighborhood for all $t \leq t_0$.

Hence, from (4.14) applied in the $\delta_0(\omega_\tau)$-neighborhood of $z = 0$, we must have that $\tilde{\zeta}(t_0) \in \{\Pi_{j,\eta}^u(t_0)z + \Sigma_{j,\eta}^u(t_0, \Pi_{j,\eta}^u(t_0)z) \,:\, \|z\|_X \leq \delta'\}$. Therefore, from (6.6), $\tilde{\zeta}(t_0) \in W_\eta^{u,\delta}(0)(t_0)$ and the proof of the claim is complete.

In this way the proof will be a consequence of [12, Theorem 8.14]. □

Remark 6.4. *Note that, if we assume* (5.12) *in Theorem 5.1 instead of* (3.3)*, we obtain* $\eta_1 > 0$*, independent of* ω_τ*, such that* $\{\psi_\eta(t-s, \Theta_s\omega_\tau) : t \geq s\}$ *is a dynamically gradient evolution process with respect to* $\{\xi_{1,\eta}^*, \cdots, \xi_{p,\eta}^*\}$, $\forall \eta \leq \eta_1$*. In this case this notion of dynamically gradient is compatible with the notion that appears in* [23, Definition 4.17].

Remark 6.5. *We believe that with the techniques employed in this paper it is also possible to obtain geometric structural stability, i.e., to show that Morse-Smale is stable under nonautonomous random perturbations and that there will be phase diagram isomorphism between the perturbed attractors and the limiting attractor, as we see in the deterministic case* [12, Chapter 12]*. This will be pursued in a future work.*

7. Applications to differential equations

In this section we present two applications. We first consider a semilinear differential equation with a small nonautonomous multiplicative white noise, and then we study the effect of a small bounded noise in the damping of a damped wave equation.

7.1. **Stochastic differential equations.**

We consider a family of Stratonovich stochastic differential equations in a separable Banach space X with a multiplicative noise given by

$$dy = Bydt + f(y)dt + \eta\kappa(t)y \circ dW_t,\ t > \tau,\ y(\tau) = y_\tau, \tag{7.1}$$

where B is a generator of a C^0-semigroup $\{e^{Bt} : t \geq 0\}$ on X, the family $\{W_t : t \in \mathbb{R}\}$ is the standard scalar Wiener process, see [2, 20], $\kappa : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and $\eta \in [0, 1]$. The assumptions on κ will be specified later, see (7.5). Equation (7.1) was considered in [16] to study hyperbolicity. Next, we will modify problem (7.1) to see it as a nonautonomous random differential equation satisfying the conditions of our results on the continuity and topological structure stability of attractors.

The canonical sample space of a Wiener process is $\Omega := C_0(\mathbb{R})$ the set of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ which are 0 at 0 equipped with the compact open topology. We denote $\mathcal{F}$ the associated Borel σ-algebra. Let $\mathbb{P}$ be the Wiener probability measure on $\mathcal{F}$ which is given by the distribution of a two-sided Wiener process with trajectories in $C_0(\mathbb{R})$. The flow θ is given by the Wiener shifts

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t),\ t \in \mathbb{R},\ \omega \in \Omega.$$

Lemma 7.1. *Consider the following scalar stochastic differential equation*

$$dz_t + zdt = dW_t. \tag{7.2}$$

There exists a θ-invariant subset $\tilde{\Omega} \in \mathcal{F}$, i.e. $\theta_t\Omega = \Omega$ for all $t \in \mathbb{R}$, such that $\mathbb{P}(\tilde{\Omega}) = 1$, $\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0$, $\omega \in \tilde{\Omega}$ and, for such ω, the random variable given by

$$z^*(\omega) = -\int_{-\infty}^{0} e^s\omega(s)ds$$

is well defined. Moreover, for $\omega \in \tilde{\Omega}$, the mapping $(t, \omega) \mapsto z^*(\theta_t\omega)$ is a stationary solution of (7.2) *with continuous trajectories, and*

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{t} = 0,\ \forall\, \omega \in \tilde{\Omega}. \tag{7.3}$$

For the proof of Lemma 7.1 see [21, Lemma 4.1].

From now on we will restrict the random flow $\{\theta_t : \Omega \to \Omega : t \in \mathbb{R}\}$ to the probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$, where $\tilde{\Omega} \in \mathcal{F}$ is obtained in Lemma 7.1 and $\tilde{\mathcal{F}} = \{\tilde{\Omega} \cap B : B \in \mathcal{F}\}$.

Let y be a solution for (7.1) and consider $v(t, \omega) := e^{-\eta\kappa(t)z^*(\theta_t\omega)}y(t, \omega)$. Hence, v satisfies the following nonautonomous random differential equation

$$\dot{v} = Bv + f_\eta(t, \theta_t\omega, v),\quad t > \tau, \tag{7.4}$$

where $f_\eta(t, \omega, v) := e^{-\eta\kappa(t)z^*(\omega)}f(e^{\eta\kappa(t)z^*(\omega)}v) + \eta[\kappa(t) - \dot{\kappa}(t)]z^*(\omega)v.$

Throughout this subsection we assume that κ is a continuously differentiable real function such that

$$\sup_{t\in\mathbb{R}}\{|\kappa(t)z^*(\theta_t\omega)|\}<\infty \text{ and } \sup_{t\in\mathbb{R}}\{|[\kappa(t)-\dot\kappa(t)]z^*(\theta_t\omega)|\}<\infty,\ \forall\,\omega\in\tilde\Omega. \qquad (7.5)$$

Thus, with analysis similar to that in [16, Section 3.3] we prove that the family $\{f_\eta:\eta\in[0,1]\}$ satisfies (3.3).

Remark 7.2. *The real function κ can be thought as a function that "controls" the growth of the Orstein-Uhlenbeck process $t\mapsto z^*(\theta_t\omega)$ for $\omega\in\tilde\Omega$. We have the following examples for κ:*

1. *Choose any continuously differentiable real function κ with compact support, where the support of κ is defined as $supp(\kappa)=\overline{\{t\in\mathbb{R}:\kappa(t)\neq 0\}}$.*
2. *Let $c>0$, $a\geq 1$ and take κ as any continuously differentiable real function such that $\kappa(t)=t^{-a}$, for $|t|>c$.*

More generally, κ can be any continuously differential real function such that $t\mapsto t\kappa(t)$ and $t\mapsto t[\kappa(t)-\dot\kappa(t)]$ are bounded.

At this point, one can choose any gradient semigroup associated to $\dot y=By+f(y)$ (see [12, Chapter 3] for examples) and consider the perturbation $\eta\kappa(t)y\circ dW_t$ and apply our results to the modified differential equation (7.4). In particular:

Example 7.3. *Let $F:\mathbb{R}^N\to\mathbb{R}$ be a smooth real-valued function and $f(x)=-\nabla F(x)$, $x\in\mathbb{R}^N$, and consider*

$$\dot x=f(x)+\eta\kappa(t)x\circ dW_t,\ t>0.$$

When $\eta=0$ this is called a gradient system. Then we obtain the nonautonomous random differential equations

$$\dot x=e^{-\eta\kappa(t)z^*(\theta_t\omega)}f(e^{\eta\kappa(t)z^*(\theta_t\omega)}x)+\eta[\kappa(t)-\dot\kappa(t)]z^*(\theta_t\omega)x,\ \eta\in[0,1]. \qquad (7.6)$$

Assume that there exists $R_0,\sigma>0$ such that

$$f(x)\cdot x<-\sigma,\ \textit{for all } |x|\geq R_0, \qquad (7.7)$$

and that the set $\{x\in\mathbb{R}^N:f(x)=0\}$ is finite and consist only of hyperbolic equilibria. Then, $\dot x=f(x)$ is globally well posed and its associated with a semigroup $\{T_0(t):t\geq 0\}$, which is gradient with respect to $\{x_1^*,\cdots,x_p^*\}$.

Then, the nonautonomous random dynamical systems associated to (7.6) *have attractors $\{A_\eta(\omega_\tau):\omega_\tau\in\mathbb{R}\times\tilde\Omega\}$, and this family of attractors satisfies the conclusions of Theorem 5.1 and Theorem 6.3.*

7.2. **An application to partial differential equation.** Now, we provide an application for a damped wave equation.

Consider the damped wave equation

$$u_{tt}+\beta u_t-\Delta u=f(u),\ \text{in } D \qquad (7.8)$$

with the boundary condition $u=0$, in ∂D, where D is a bounded smooth domain in $\mathbb{R}^3$, and $\beta\in(0,+\infty)$. For $f:\mathbb{R}\to\mathbb{R}$ we assume that

$$f\in C^2(\mathbb{R}),\ |f''(s)|\leq c(1+|s|),\ \forall\,s\in\mathbb{R}, \qquad (7.9)$$

for some $c>0$, and

$$\limsup_{|s|\to+\infty}\frac{f(s)}{s}\leq 0. \qquad (7.10)$$

Now, we consider a small random perturbation on the damping,

$$u_{tt}+\beta_\eta(\Theta_t\omega)u_t-\Delta u=f(u),\ \text{in } D. \tag{7.11}$$

where $\beta_\eta(\omega_\tau):=\beta+\eta|\kappa(\tau)z^*(\omega)|$, $\eta\in[0,1]$, $\omega_\tau\in\mathbb{R}\times\Omega$, and $\kappa:\mathbb{R}\to\mathbb{R}$ a continuous map such that

$$\sup_{t\in\mathbb{R}}\{|\kappa(t)z^*(\theta_t\omega)|\}<\infty,\ \text{for all } \omega\in\tilde{\Omega}, \tag{7.12}$$

where $\tilde{\Omega}$ is given in Lemma 7.1.

Remark 7.4. *The functions* $\kappa_1(t)=(1+|t|)^{-1}$*,* $\kappa_2(t)=(1+t^2)^{-1}$*, and* $\kappa_3(t)=e^{-|t|}$*, for* $t\in\mathbb{R}$*, satisfy* (7.12)*, due to* (7.3)*. See also Remark 7.2 for more examples.*

Thus there exists a Θ-invariant map $b_1:\mathbb{R}\times\tilde{\Omega}\to(0,+\infty)$ such that

$$\beta\le\beta_\eta(\Theta_t\omega_\tau)\le b_1(\omega_\tau),\ \forall\,\omega_\tau\in\mathbb{R}\times\tilde{\Omega}. \tag{7.13}$$

Indeed, note that $\beta_\eta(\omega_\tau)\le\beta_1(\omega_\tau)=\beta+|\kappa(t+\tau)z^*(\theta_t\omega)|$ and that $b_1(\omega_\tau):=\sup_{t\in\mathbb{R}}\beta_1(\Theta_t\omega_\tau)$ is finite for every $\omega_\tau\in\mathbb{R}\times\tilde{\Omega}$, due to Condition (7.12). Now, since

$$b_1(\Theta_r\omega_\tau)=\sup_{t\in\mathbb{R}}\beta_1(\Theta_t\Theta_r\omega_\tau)=\sup_{t\in\mathbb{R}}\beta_1(\Theta_{t+r}\omega_\tau),\ \forall\,r\in\mathbb{R}, \tag{7.14}$$

we see that b_1 is Θ-invariant and satisfies (7.13).

Now, from (7.11) we obtain the family of abstract evolutionary equations in $X=H^1_0(D)\times L^2(D)$

$$\dot{y}=By+F_\eta(\Theta_t\omega_\tau,y),\ \eta\in[0,1], \tag{7.15}$$

where

$$y=\begin{pmatrix}u\\ v\end{pmatrix}\in X,\quad B=\begin{pmatrix}0 & I\\ -A & 0\end{pmatrix},\quad F_\eta(\omega_\tau,y)=\begin{pmatrix}0\\ -\beta_\eta(\omega_\tau)v+f^e(u)\end{pmatrix},$$

$A:D(A)\subset L^2(D)\to L^2(D)$ is $-\Delta$ with Dirichlet boundary condition, and $f^e:H^1_0(D)\to L^2(D)$ is given by $f^e(y_1)(x)=f(y_1(x))$ for $x\in D$. Thus, conditions (7.9) and (7.10) implies local and global well-posedness and that f^e is continuously differentiable, see [4] or [29, Chapter 15] for details.

Consider the functional $V:H^1_0(D)\times L^2(D)\to\mathbb{R}$ given by

$$V_0(u,v)=\frac{1}{2}\int_D|\nabla u|^2+\frac{\beta}{2}\int_D v^2-\int_D G(u), \tag{7.16}$$

where $G(u)(x)=\int_0^{u(x)}f(s)\,ds$. Thus V_0 is a Lyapunov function relative to the set of equilibria for (7.8), which we assume that is finite. The hyperbolic equilibrium points of (7.8) are of the form $y_0^*=(u_0^*,0)$ where u_0^* is a solution of $-\Delta u=f(u)$ such that $0\notin\sigma(-\Delta+D_xf^e(u_0^*)Id_X)$. Thus (7.8) is associated with a gradient semigroup $\{T_0(t):t\ge0\}$, see [13] for conditions to obtain that this type of dynamics is generic on damped wave equations.

For each $y_0\in X$, $\omega_\tau\in\mathbb{R}\times\tilde{\Omega}$, and $\eta\in[0,1]$ Equation (7.15) possess a unique solution which can be written as

$$\psi_\eta(t,\omega_\tau)y_0=\varphi_\eta(t,\omega_\tau)y_0+\phi_\eta(t,\omega_\tau)y_0,\quad t\ge0. \tag{7.17}$$

where $\{\varphi_\eta(t,\omega):t\in[0,+\infty),\,\omega\in\tilde{\Omega}\}$ is the solution operator of (7.15) with $f=0$, and

$$\phi_\eta(t,\omega_\tau)y_0=\int_0^t\varphi_\eta(t-s,\Theta_s\omega_\tau)F(\psi_\eta(s,\omega_\tau)y_0)ds. \tag{7.18}$$

Towards the existence of attractors, we have the following lemma.

Lemma 7.5. *There exists a bounded subset B (independent of (t,ω)) which pullback attracts at time $\tau \in \mathbb{R}$, for each $\tau \le t$, every bounded subset of X under the action of $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$. In particular, $\{\psi_\eta(t-s,\Theta_s\omega_\tau) : t \ge s\}$ is strongly pullback bounded dissipative, in the sense of* [17, Definition 2.10].

Furthermore, there are $K > 0$ and a Θ-invariant function $\alpha : \mathbb{R}\times\tilde{\Omega} \to (0,+\infty)$, both independent of η, such that

$$\|\varphi_\eta(t,\omega_\tau)\|_{\mathcal{L}(X)} \le K e^{-\alpha(\omega_\tau)t},\ t \ge 0, \tag{7.19}$$

and $\phi_\eta(t,\omega_\tau)$ is a compact operator for every $(t,\omega_\tau) \in (0,+\infty)\times\mathbb{R}\times\tilde{\Omega}$. In particular, ψ_η is pullback asymptotically compact for each $\eta \in [0,1]$, in the sense of [39, Definition 2.14].

The proof of Lemma 7.5 follows step by step the arguments presented in [17, Section 2.1] (or see [29, Chapter 15] for more detailed proofs), thus it will be omitted. Thus there are nonautonomous random attractors $\{\mathcal{A}_\eta(\omega_\tau) : \omega_\tau \in \mathbb{R}\times\tilde{\Omega}\}$ for (ψ_η,Θ) for all $\eta \in [0,1]$ satisfying Condition **(a.1)** of Remark 5.3, see [39, 17]. Additionally, using arguments similar to those in [17], we see that the family $\{(\psi_\eta,\Theta)\}_{\eta\in[0,1]}$ is collectively pullback asymptotically compact at $\eta = 0$. Therefore, the conditions of Remark 5.3 are satisfied and it is possible to apply our results to conclude that the family of attractors behaves continuously (using Theorem 5.1) and that we have topological structural stability (using Theorem 6.3).

Remark 7.6. *Instead of considering $\beta_\eta(\omega_\tau) := \beta + \eta|\kappa(\tau)z^*(\omega)|$, we could have considered*

$$\tilde{\beta}_\eta(\omega) = \beta + \eta\frac{2}{\pi}\arctan\circ z^*(\omega),\ \omega \in \Omega,\ \eta \in [0,1], \tag{7.20}$$

and $\beta \in (1,+\infty)$. For these perturbations a condition as (5.12) *is verified for the symbol space Ω instead of $\mathbb{R}\times\Omega$. See for instance* [25] *where the authors study this type of perturbations.*

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