Existence of weak solutions to nonlocal PDEs with a generalized definition of Caputo derivative

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Abstract Some compactness criteria that are analogies of the Aubin-Lions lemma for the existence of weak solutions of nonlinear evolutionary PDEs play crucial roles for the existence of weak solutions to time fractional PDEs. Based on this fact, in this paper, we consider the existence of weak solutions to a kind of partial differential equations with Caputo time fractional differential operator of order $\gamma \in (0, 1)$ and fractional Laplacian operator $(-\Delta)^\alpha$, $\alpha \in (0, 1)$.

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1 Introduction

Memory effects are ubiquitous to model dynamical processes in materials with memory, e.g., the diffusion of flow in a rigid, isotropic, homogeneous heat conductor with linear memory [7], in physics and engineering, e.g., particles in heat bath [12, 34] and soft matter with viscoelasticity [4, 20] can possess memory effects. Fractional derivatives are used to describe the nonlocal and memory effects in time. Particularly, when the memory effects have power law kernels, we can use fractional calculus to describe them (see, for example [2, 25, 26, 29, 30, 31, 32, 33]). In general, the time fractional equations have been widely used to model anomalous diffusions exhibiting subdiffusive behavior ($\gamma \in (0, 1)$ [16]), since time fractional diffusion equations exhibit a behavior like const$\cdot t^\gamma$ for $t \to \infty$ (see, for example, [18]).

There are two types of fractional derivatives that are commonly used: The Riemann-Liouville derivatives and the Caputo ones, see [10]. Compared with Riemann-Liouville's derivative, Caputo's removes singularities at the origin and shares many similarities with the ordinary derivatives so that they are suitable for initial value problems [22]. Moreover, the alternative expressions for singular kernels enable us to find the fractional calculus more easily and motivate us to extend the Caputo derivatives to a generalized one [14], which has convolution group property. Also in

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[14], some functional analysis approaches are used to extend the traditional Caputo derivative to certain Sobolev spaces. In this way, it is convenient to define the Caputo derivatives in weak sense [14]. Based on this fact, the authors in [13] established a compactness criteria for time fractional PDEs with functional values in general Banach spaces, which are analogues to the Aubin-Lions lemma for the usual time derivative for PDEs.

In this paper, we focus on the following time fractional nonlocal reaction-diffusion equation,

$$\begin{cases} D_c^{\gamma} u + (-\Delta)^{\alpha} u = f(u) + h(t), & \text{in } \ \mathcal{O} \times (0, \infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \ \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set, D_c^{γ}, $\gamma \in (0, 1)$, is the weak time Caputo derivative of u with initial data u_0 (cf. Definition 2.3). The nonlocal operator $(-\Delta)^{\alpha}$ with $\alpha \in (0, 1)$, known as the Laplacian operator of order α, is given by means of the Fourier multiplier

$$(-\Delta)^{\alpha} u(x) := \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u))(x), \quad \xi \in \mathbb{R}^n,$$

where $\mathcal{F}$ is the Fourier transform defined by

$$(\mathcal{F}u)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx, \quad u \in \mathcal{S},$$

$\mathcal{S}$ is the Schwartz space of rapidly decaying C^{∞} functions on $\mathbb{R}^n$ and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The appearance of spatial fractional derivatives in diffusion equations are exploited for macroscopic description of transport and often lead to superdiffusion phenomenon [18]. Let us recall, for any fixed $\alpha \in (0, 1)$, the fractional Laplace operator $(-\Delta)^{\alpha}$ at the point x is defined by

$$(-\Delta)^{\alpha} u(x) := -\frac{1}{2} C(n, \alpha) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2\alpha}} dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $C(n, \alpha)$ is a positive constant depending on n and α given by

$$C(n, \alpha) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2\alpha}} d\xi \right)^{-1}, \qquad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n. \tag{1.3}$$

The operator $(-\Delta)^{\alpha}$ describes a particle jumping from one point $x \in \mathcal{O}$ to another $y \in \mathcal{O}$ with intensity proportional to $|x - y|^{-n-2\alpha}$ (see, for example, [6]). We emphasize that the Laplacian operator is nonlocal, the differential equations involving fractional Laplacian operator arise from many applications in physics and biology (see, e.g., [9, 11, 17]), and studying the solutions of such equations has been the subject of many publications (see, e.g., [3, 8, 19, 21, 23, 24] and the references therein).

Let $H^{\alpha}(\mathbb{R}^n) := W^{\alpha,2}(\mathbb{R}^n)$ be the fractional Sobolev space defined by,

$$H^{\alpha}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dx dy < \infty \right\},$$

which is equipped with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dxdy \right)^{\frac{1}{2}}.$$

From now on, we write the norm and the inner product of $L^2(\mathbb{R}^n)$ as $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. Furthermore, the dual spaces of $L^2(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n)$ are denoted by $L^2(\mathbb{R}^n)$ and $H^{-\alpha}(\mathbb{R}^n)$, respectively. We also write the Gagliardo semi-norm of $H^\alpha(\mathbb{R}^n)$ as $\|\cdot\|_{\dot{H}^\alpha(\mathbb{R}^n)}$, i.e.,

$$\|u\|^2_{\dot{H}^\alpha(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dxdy, \qquad u \in H^\alpha(\mathbb{R}^n).$$

Then for all $u \in H^\alpha(\mathbb{R}^n)$, we have $\|u\|^2_{H^\alpha(\mathbb{R}^n)} = \|u\|^2 + \|u\|^2_{\dot{H}^\alpha(\mathbb{R}^n)}$. Note that $H^\alpha(\mathbb{R}^n)$ is a Hilbert space with inner product given by

$$(u, v)_{H^\alpha(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2\alpha}} dxdy, \quad u, v \in H^\alpha(\mathbb{R}^n).$$

One can verify that [19],

$$\|u\|^2_{H^\alpha(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{C(n, \alpha)} \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)}, \quad \forall u \in H^\alpha(\mathbb{R}^n), \tag{1.4}$$

and hence $\left(\|u\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}$ is an equivalent norm of $H^\alpha(\mathbb{R}^n)$.

The definition of fractional Laplacian operator $(-\Delta)^\alpha$ (cf. (1.2)) implies that the solutions of (1.1) with integral operator are closely related to the solutions of the equation defined on the entire domain $\mathbb{R}^n$, when the boundary condition is replaced by $u = 0$ on $\mathbb{R}^n \backslash \mathcal{O}$. More precisely, for $\alpha \in (0, 1)$, we consider the problem:

$$\begin{cases} D_c^\gamma u + (-\Delta)^\alpha u = f(u) + h(t), & \text{in } \mathcal{O} \times (0, \infty), \\ u = 0, & \text{on } \mathbb{R}^n \backslash \mathcal{O} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathcal{O}. \end{cases} \tag{1.5}$$

Assume that the functions $h \in C((0, \infty); L^2(\mathbb{R}^n))$, $f \in C^1(\mathbb{R})$ and there exist positive constants α_1, $\alpha_2 > \frac{C(n,\alpha)}{2}$, β and κ, such that

$$-\kappa - \alpha_1 |s|^2 \leq f(s)s \leq \kappa - \alpha_2 |s|^2, \qquad \forall s \in \mathbb{R}. \tag{1.6}$$

$$|f'(s)| \leq \beta, \qquad \forall s \in \mathbb{R}. \tag{1.7}$$

From (1.7), we deduce that

$$|f(s) - f(r)| \leq \beta |s - r|, \qquad \forall s, r \in \mathbb{R}. \tag{1.8}$$

There are many works investigating fractional partial differential equations with usual time derivative, see [17, 27, 28, 31, 32]. Among all these previous studies, assumptions imposed on external terms, h and f, are much weaker than in the current ones due to the appearance of Caputo time fractional operator in (1.5). To prove the existence of weak solution to (1.5) by applying Fadeo-Galerkin method, the existence of solution for ODE system needs to be shown firstly (see Lemma 3.2 later). Therefore, conditions related to h and f of (1.5) are necessarily stronger in this manuscript since Theorem 4.4 in [14] plays the key role to show this result.

Our model (1.5) describes the biological phenomenon of chemotaxis with anomalous diffusion and memory effects. The rest of this paper is organized as follows. In Section 2, we recall the definition of generalized Caputo derivative and a compactness criterion which will be used later. The existence of weak solutions to problem (1.1) is shown in Section 3 by applying the Fadeo-Galerkin method.

2 Preliminaries

2.1 Time-fractional setting

We first recall the generalized definition of Caputo derivative based on a convolution group and present a compactness criterion which is useful for time fractional PDEs. As in [14], we use the following distributions $\{g_\gamma\}$ as the convolution kernels for $\gamma > -1$:

$$g_\gamma(t) := \begin{cases} \frac{\theta(t)}{\Gamma(\gamma)} t^{\gamma-1}, & \gamma > 0, \\ \delta(t), & \gamma = 0, \\ \frac{1}{\Gamma(1+\gamma)} D(\theta(t) t^\gamma), & \gamma \in (-1, 0). \end{cases}$$

Here θ is the standard Heaviside step function, $\Gamma(\cdot)$ is the gamma function, and D means the distributional derivative with respect to t.

Let us present the following notion of limit to introduce the generalized definition of Caputo derivatives. Let B be a Banach space.

Definition 2.1 *[13, Definition 2.1] For a locally integrable function $u \in L^1_{loc}((0,T);B)$, if there exists $u_0 \in B$ such that*

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \|u(s) - u_0\|_B ds = 0,$$

then we call u_0 the right limit of u at $t = 0$, denoted as $u(0+) = u_0$. Similarly, we define $u(T-)$ to be the constant $u_T \in B$ such that

$$\lim_{t \to T-} \frac{1}{T-t} \int_t^T \|u(s) - u_T\|_B ds = 0.$$

Definition 2.2 *[13, Definition 2.6] Let $\tilde{g}_\gamma(t) = g_\gamma(-t)$ and $0 < \gamma < 1$. Consider $u \in L^1_{loc}(-\infty, T]$. Given $u_T \in \mathbb{R}$, the γ-th order generalized right Caputo derivative (up to T) of u associated with u_T is a distribution in $\mathcal{D}'(\mathbb{R})$ with support in $(-\infty, T]$ given by*

$$\tilde{D}^\gamma_{c;T} u := \tilde{g}_{-\gamma} * (\theta(T-t)(u(t) - u_T)).$$

If u has a left limit at $t = T$ in the sense of Definition 2.1 and $u_T = u(T-)$, we call $\tilde{D}^{\gamma}_{c;T} u$ the right Caputo derivative (up to T).

Now fix $T > 0$, we introduce the following set,

$$\mathcal{D}' := \{v \mid v : C_c^{\infty}((-\infty, T); \mathbb{R}) \to B \text{ is a bounded linear operator}\}.$$

In other words, $\mathcal{D}'$ consists of functionals from $C_c^{\infty}((-\infty, T) : \mathbb{R})$ to B.

Definition 2.3 *[13, Definition 2.9] Let $u \in L^1_{loc}([0, T); B)$ and $u_0 \in B$. We define the weak Caputo derivative of u associated with initial data u_0 to be $D_c^{\gamma} u \in \mathcal{D}'$, such that for any test function $\varphi \in C_c^{\infty}((-\infty, T); \mathbb{R})$,*

$$\langle D_c^{\gamma} u, \varphi \rangle := \int_{-\infty}^{T} (u - u_0)\theta(t)(\tilde{D}^{\gamma}_{c;T}\varphi)dt = \int_0^T (u - u_0)\tilde{D}^{\gamma}_{c;T}\varphi dt. \tag{2.1}$$

If $u(0+) = u_0$ in the sense of Definition 2.1 under the norm of underlying Banach space B, then we call $D_c^{\gamma} u$ the Caputo derivative.

For linear evolutionary equations, establishing the existence of weak solutions is relatively easy. Indeed, one only needs the weak compactness, which is guaranteed by boundedness in reflexive spaces. However, for time fractional PDEs, the following strong compactness criteria are often needed [13].

Theorem 2.4 *[13, Theorem 4.1] Let $T > 0$, $\gamma \in (0, 1)$ and $p \in [1, \infty)$. Let M, B, Y be Banach spaces. $M \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Suppose $W \subset L^1_{loc}((0, T); M)$ satisfies the following conditions:*

(i) There exists a constant $C_1 > 0$, such that for any $u \in W$,

$$\sup_{t\in(0,T)} J_{\gamma}(\|u\|_M^p) = \sup_{t\in(0,T)} \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1} \|u\|_M^p(s)ds \le C_1.$$

(ii) There exist $r \in (\frac{p}{1+p\gamma}, \infty) \cap [1, \infty)$ and $C_3 > 0$ such that for any $u \in W$, there is an assignment of initial value u_0 to make the weak Caputo derivative satisfy

$$\|D_c^{\gamma} u\|_{L^r((0,T);Y)} \le C_3.$$

Then W is relatively compact in $L^p((0, T); B)$.

2.2 The fractional Laplacian operator

To deal with the fractional Laplacian nonlocal term, similar to [27], let $b : H^\alpha(\mathbb{R}^n) \times H^\alpha(\mathbb{R}^n) \to \mathbb{R}$ be a bilinear operator form given, for $v_1,\ v_2 \in H^\alpha(\mathbb{R}^n)$, by

$$\begin{aligned} b(v_1, v_2) = & \frac{1}{2} C(n, \alpha)(v_1, v_2) \\ & + \frac{1}{2} C(n, \alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_1(x) - v_1(y))(v_2(x) - v_2(y))}{|x - y|^{n+2\alpha}} dxdy, \quad \forall v_1, v_2 \in H^\alpha(\mathbb{R}^n), \end{aligned} \tag{2.2}$$

where $C(n, \alpha)$ is the constant as in (1.3). For convenience, we associate an operator $A : H^\alpha(\mathbb{R}^n) \to H^{-\alpha}(\mathbb{R}^n)$ with b such that

$$< A(v_1), v_2 >_{(H^{-\alpha}(\mathbb{R}^n), H^\alpha(\mathbb{R}^n))} = b(v_1, v_2), \qquad \forall v_1, v_2 \in H^\alpha(\mathbb{R}^n),$$

where $< \cdot, \cdot >_{(H^{-\alpha}(\mathbb{R}^n), H^\alpha(\mathbb{R}^n))}$ is the duality of $H^{-\alpha}(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n)$. It follows from [27] that the operator A has a family of eigenfunctions $\{w_j\}_{j=1}^\infty$ such that $\{w_j\}_{j=1}^\infty$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$. Moreover, if λ_j is the eigenvalue corresponding to w_j, i.e.,

$$Aw_j = \lambda_j w_j, \qquad j = 1, 2, \cdots,$$

then λ_j satisfies

$$0 < \frac{1}{2} C(n, \alpha) < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \quad \text{as} \quad j \to \infty.$$

2.3 Definition of weak solutions to problem (1.5)

Regarding the fractional integral, it is convenient to define $\| \cdot \|_{L^p_\gamma}$ for $\gamma \in (0, 1)$ and $p \geq 1$ as

$$\|u\|_{L^p_\gamma((0,T);M)} := \sup_{t \in (0,T)} \left(\left| \int_0^t (t - s)^{\gamma - 1} \|u\|_M^p(s) ds \right| \right)^{1/p} < \infty.$$

It is easy to check that $\| \cdot \|_{L^p_\gamma((0,T);M)}$ is a norm. A simple observation is the following.

Lemma 2.5 *[13, Lemma 4.3] Let $\gamma \in (0, 1)$. If $\|f\|_{L^p_\gamma((0,T);M)} < \infty$, then $f \in L^p((0, T); M)$.*

We will use the basis $\{w_j\}_{j=1}^\infty$ and the Galerkin method to prove the existence of weak solutions to (1.5), which is understood in the following sense.

Definition 2.6 *Let $\gamma \in (0, 1)$ and $\alpha \in (0, 1)$. We say*

$$u \in L^\infty((0, T); L^2(\mathbb{R}^n)) \cap L^2_\gamma((0, T); H^\alpha(\mathbb{R}^n))$$

with

$$D^\gamma_c u \in L^2((0, T); H^{-\alpha}(\mathbb{R}^n)),$$

is a weak solution to (1.5) *with initial data* $u_0 \in L^2(\mathbb{R}^n)$*, if*

$$\begin{aligned}\int_0^T \int_{\mathcal{O}} (u(x,t) - u_0)\tilde{D}^{\gamma}_{c;T}\varphi dxdt + \frac{C(n,\alpha)}{2}\int_0^T \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\alpha}}dxdydt \\ = \int_0^T \int_{\mathbb{R}^n} f(u)\varphi dxdt + \int_0^T \int_{\mathbb{R}^n} h(t)\varphi dxdt,\end{aligned} \tag{2.3}$$

for any $\varphi \in C_c^{\infty}([0,T) \times \mathbb{R}^n; \mathbb{R})$*. We say a weak solution is a regular weak solution if* $u(0+) = u_0$ *under* $H^{-\alpha}(\mathbb{R}^n)$ *in the sense of Definition 2.1. If* u *is a function defined on* $(0,\infty)$ *such that its restriction on any interval* $[0,T)$*,* $T > 0$*, is a (regular) weak solution, we say* u *is a global (regular) weak solution.*

3 Main result

3.1 A priori estimates.

Note that if we assume that Proposition 2.18 of [13] holds for u with $B = L^2(\mathbb{R}^n)$ and $\frac{1}{2}\|u\|^2$ is convex functional, we have

$$D_c^{\gamma}\frac{1}{2}\|u\|^2 \leq \langle u, D_c^{\gamma} u\rangle = \int_{\mathbb{R}^n} f(u)udx + \int_{\mathbb{R}^n} h(t)udx - \int_{\mathbb{R}^n} \|(-\Delta)^{\frac{\alpha}{2}}u\|^2 dx. \tag{3.1}$$

We will do estimates for the above equality. Since $h \in C((0,\infty); L^2(\mathbb{R}^n))$, by the Young inequality, we find for every positive ε,

$$\int_{\mathbb{R}^n} h(t)u(x)dx \leq \|h(t)\|\|u\| \leq \frac{1}{2\varepsilon}\|h(t)\|^2 + \frac{\varepsilon}{2}\|u\|^2. \tag{3.2}$$

By means of (1.6), we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}}u|^2 dx = \frac{C(n,\alpha)}{2}\|u\|^2_{\dot{H}^{\alpha}(\mathbb{R}^n)}, \tag{3.3}$$

and

$$\int_{\mathbb{R}^n} f(u)udx \leq \int_{\mathcal{O}} \left(\kappa - \alpha_2|u|^2\right) dx \leq \kappa|\mathcal{O}| - \alpha_2\|u\|^2, \tag{3.4}$$

respectively. Substituting (3.2)-(3.4) into (3.1), letting $\varepsilon = C(n,\alpha)$, we obtain

$$D_c^{\gamma}\frac{1}{2}\|u\|^2 + b(u,u) + (\alpha_2 - C(n,\alpha)/2)\|u\|^2 \leq \kappa|\mathcal{O}| + \frac{1}{2C(n,\alpha)}\|h\|^2,$$

which implies

$$D_c^{\gamma}\frac{1}{2}\|u\|^2 + \frac{C(n,\alpha)}{2}\|u\|^2_{H^{\alpha}(\mathbb{R}^n)} + (\alpha_2 - C(n,\alpha)/2)\|u\|^2 \leq \kappa|\mathcal{O}| + \frac{1}{2C(n,\alpha)}\|h\|^2.$$

Therefore, by [13, Lemma 2.3], we infer

$$
\begin{aligned}
&\|u\|^2 + \frac{C(n,\alpha)}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|u\|^2_{H^\alpha(\mathbb{R}^n)} ds + \frac{(2\alpha_2 - C(n,\alpha))}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|u\|^2 ds \\
&\quad \le \|u_0\|^2 + \frac{2}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \left(\kappa|\mathcal{O}| + \frac{1}{2C(n,\alpha)} \|h\|^2 \right) ds \\
&\quad = \|u_0\|^2 + \frac{2T^\gamma}{\Gamma(\gamma+1)} \kappa|\mathcal{O}| + \frac{1}{\Gamma(\gamma) C(n,\alpha)} \|h\|^2_{L^2_\gamma(0,T;L^2(\mathbb{R}^n))}.
\end{aligned}
$$

Consequently, $u \in L^\infty((0,T); L^2(\mathbb{R}^n)) \cap L^2_\gamma((0,T); H^\alpha(\mathbb{R}^n))$ by Lemma 2.5.

Let $\varphi \in L^2((0,T); H^\alpha(\mathbb{R}^n))$, the Young inequality and (1.8) imply that,

$$
\begin{aligned}
|\langle D_c^\gamma u, \varphi \rangle_{x,t}| &= |\langle -(-\Delta)^\alpha u + f(u) + h(t), \varphi \rangle_{x,t}| \\
&\le \left| \int_0^T < (-\Delta)^\alpha u, \varphi > dt \right| + \left| \int_0^T (f(u), \varphi) dt \right| + \left| \int_0^T (h(t), \varphi) dt \right| \\
&\le \frac{C(n,\alpha)}{2} \int_0^T \|u\|_{\dot{H}^\alpha(\mathbb{R}^n)} \|\varphi\|_{\mathbb{H}^\alpha(\mathbb{R}^n)} dt + \beta \int_0^T \|u\| \|\varphi\| dt + \int_0^T \|f(0)\| \|\varphi\| dt \\
&\quad + \frac{1}{2} \int_0^T \|h(t)\|^2 dt + \frac{1}{2} \int_0^T \|\varphi\|^2 dt \\
&\le \frac{C(n,\alpha)}{4} \int_0^T \|u\|^2_{H^\alpha(\mathbb{R}^n)} dt + \frac{C(n,\alpha)}{4} \int_0^T \|\varphi\|^2_{H^\alpha(\mathbb{R}^n)} dt + \frac{\beta}{2} \int_0^T \|u\|^2 dt \\
&\quad + \frac{\beta+2}{2} \int_0^T \|\varphi\|^2 dt + \frac{1}{2} \|f(0)\|^2 T + \frac{1}{2} \int_0^T \|h\|^2 dt.
\end{aligned}
\tag{3.5}
$$

Since $u \in L^\infty((0,T); L^2(\mathbb{R}^n)) \cap L^2_\gamma((0,T); H^\alpha(\mathbb{R}^n))$, it is clear that

$$
D_c^\gamma u \in L^2((0,T); H^{-\alpha}(\mathbb{R}^n)).
$$

3.2 Existence of weak solutions: A Galerkin method.

With the a priori energy estimates, the existence of weak solutions can be performed by the standard techniques, namely, the Faedo-Galerkin approximations. Note that w_j ($j \in \mathbb{N}$) are eigenfunctions of the operator A. Given $m \in \mathbb{N}$, let X_m be the space spanned by $\{w_j, j = 1, \cdots, m\}$ and $P_m : L^2(\mathbb{R}^n) \to X_m$ be the projection given by

$$
P_m u := \sum_{j=1}^m \alpha_j w_j, \qquad \forall u \in L^2(\mathbb{R}^n). \tag{3.6}
$$

From [28], we infer that the operator P_m can be extended to $H^{-\alpha}(\mathbb{R}^n)$ by

$$
P_m \phi = \sum_{j=1}^m (\phi(w_j)) w_j, \qquad \forall \phi \in H^{-\alpha}(\mathbb{R}^n).
$$

Let $u_0 = \sum_{j=1}^{\infty} a^j w_j$ in $L^2(\mathbb{R}^n)$, we consider the function

$$u_m(t) = \sum_{j=1}^{m} c_m^j w_j,$$

such that $c_m := (c_m^1, c_m^2, \cdots, c_m^m)$ is continuous in time and $u_m \in X_m$ satisfies the following equations,

$$\begin{cases} \langle D_c^\gamma u_m, w_j\rangle + < (-\Delta)^\alpha u_m, w_j >= (f(u_m), w_j) + (h(t), w_j), \qquad j = 1, 2, \cdots, m, \\ u_m(0) = \sum_{j=1}^m c_m^j(0) w_j = \sum_{j=1}^m a^j w_j. \end{cases} \tag{3.7}$$

Since c_m is continuous, $D_c^\gamma u_m$ is the Caputo derivative with natural initial value. Equation (3.7) can be reduced to the following FODE system,

$$\begin{cases} D_c^\gamma c_m^j(t) = -\lambda_j c_m^j + (f(\sum_{j=1}^m c_m^j w_j), w_j) + (h(t), w_j) + \frac{C(n,\alpha)}{2} c_m^j, \quad j = 1, 2, \cdots, m, \\ c_m^j(0) = a^j. \end{cases} \tag{3.8}$$

Namely, (3.8) is equivalent to the following fractional ordinary differential system,

$$\begin{pmatrix} D_c^\gamma c_m^1(t) \\ D_c^\gamma c_m^2(t) \\ \cdots \\ D_c^\gamma c_m^m(t) \end{pmatrix} = \begin{pmatrix} \left(-\lambda_1 + \frac{C(n,\alpha)}{2}\right) c_m^1(t) \\ \left(-\lambda_2 + \frac{C(n,\alpha)}{2}\right) c_m^2(t) \\ \cdots \\ \left(-\lambda_m + \frac{C(n,\alpha)}{2}\right) c_m^m(t) \end{pmatrix} + \begin{pmatrix} (f(\sum_{j=1}^m c_m^j(t) w_j), w_1) \\ (f(\sum_{j=1}^m c_m^j(t) w_j), w_2) \\ \cdots \\ (f(\sum_{j=1}^m c_m^j(t) w_j), w_m) \end{pmatrix} + \begin{pmatrix} (h(t), w_1) \\ (h(t), w_2) \\ \cdots \\ (h(t), w_m) \end{pmatrix} \tag{3.9}$$

with initial value $c_m(0)$.

Obviously, let $F(\cdot,\cdot) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be the right hand side of (3.9). Then (3.9) satisfies the following system written in abstract form,

$$\begin{cases} D_c^\gamma c_m(t) = F(t, c_m(t)), \\ c_m(0) = (a^1, a^2, \cdots, a^m). \end{cases} \tag{3.10}$$

To obtain the existence and uniqueness of solutions to problem (3.10), the following lemma is needed. In the sequel, we endow $c_m \in \mathbb{R}^m$ with the Euclidian norm $|c_m|_{\mathbb{R}^m} = \sqrt{\sum_j (c_m^j)^2}$. Note that the norm for c_m is not important because all norms are equivalent for finite dimensional vectors.

Lemma 3.1 *Let $0 < \gamma < 1$ and $c_m(0) \in \mathbb{R}^m$. Consider IVP (3.10), assume there exist $T > 0$, $A' > 0$ such that F is defined and continuous on $D = [0,T] \times U$, where $U \subset \mathbb{R}^m$ and $|c_m(t)|_{\mathbb{R}^m} \in [|c_m(0)|_{\mathbb{R}^m} - A', |c_m(0)|_{\mathbb{R}^m} + A']$, such that there exists $L > 0$,*

$$\sup_{0\le t\le T} |F(t, v_1) - F(t, v_2)|_{\mathbb{R}^m} \le L|v_1 - v_2|_{\mathbb{R}^m}, \qquad \forall |v_1|_{\mathbb{R}^m}, |v_2|_{\mathbb{R}^m} \in [|c_m(0)|_{\mathbb{R}^m} - A', |c_m(0)|_{\mathbb{R}^m} + A'].$$

Then, the IVP (3.10) *has a unique strong solution on* $[0, T_1)$*, and* T_1 *is given by*

$$T_1 = \min\left\{T, \sup\left\{t \geq 0 : \frac{M}{\Gamma(1+\gamma)} t^\gamma E_\gamma(Lt^\gamma) \leq A'\right\}\right\} > 0,$$

where $M = \sup_{t\in[0,T]} |F(t, c_m(0))|_{\mathbb{R}^m}$ *and*

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma+1)}$$

is the Mittag-Leffler function.

Proof. The idea follows the same lines as [13, Theorem 4.4] by replacing the norm in $\mathbb{R}$ by $\mathbb{R}^m$. □

Lemma 3.2 *Suppose* $f \in C^1(\mathbb{R})$ *fulfills* (1.6)-(1.7), $h \in C((0,\infty); L^2(\mathbb{R}^n))$ *and* $u_0 \in L^2(\mathbb{R}^n)$. *Then:*

(*i*) *For any* $m \geq 1$*, there exists a unique solution* u_m *of* (3.7) *that is continuous on* $[0, \infty)$*, satisfying*

$$\|u_m\|^2 \leq \|u_0\|^2 + \frac{2T^\gamma}{\Gamma(\gamma+1)}\left(\kappa|\mathcal{O}| + \frac{1}{4\alpha_2 - 2C(n,\alpha)}\|h\|^2\right), \tag{3.11}$$

and

$$\sup_{0\leq t\leq T}\int_0^t (t-s)^{\gamma-1}\|u_m\|^2_{H^\alpha(\mathbb{R}^n)} ds \leq \frac{\Gamma(\gamma)}{C(n,\alpha)}\|u_0\|^2 + \frac{2T^\gamma}{\gamma C(n,\alpha)}\left(\kappa|\mathcal{O}| + \frac{1}{4\alpha_2 - 2C(n,\alpha)}\|h\|^2\right); \tag{3.12}$$

(*ii*) *For any* $T > 0$*, there exist* $u \in L^\infty((0,T); L^2(\mathbb{R}^n)) \cap L^2_{loc}([0,\infty); H^\alpha(\mathbb{R}^n))$ *and a subsequence* m_k *such that*

$$u_{m_k} \to u \quad \text{in} \quad L^2_{loc}([0,\infty); L^2(\mathbb{R}^n)).$$

Moreover, u *has a weak Caputo derivative* $D_c^\gamma u \in L^2_{loc}([0,\infty); H^{-\alpha}(\mathbb{R}^n))$.

Proof. (*i*) First of all, we will state that there exists a unique strong solution to (3.7) based on Lemma 3.1. Since f satisfies (1.8) and $h \in C((0,\infty); L^2(\mathbb{R}^n))$, it is easy to check there exist $T > 0$ and $A' > 0$, such that

$$F(t, c_m) := \begin{pmatrix} \left(-\lambda_1 + \frac{C(n,\alpha)}{2}\right) c_m^1 \\ \left(-\lambda_2 + \frac{C(n,\alpha)}{2}\right) c_m^2 \\ \dots \\ \left(-\lambda_m + \frac{C(n,\alpha)}{2}\right) c_m^m \end{pmatrix} + \begin{pmatrix} (f(\sum_{j=1}^m c_m^j w_j), w_1) \\ (f(\sum_{j=1}^m c_m^j w_j), w_2) \\ \dots \\ (f(\sum_{j=1}^m c_m^j w_j), w_m) \end{pmatrix} + \begin{pmatrix} (h(t), w_1) \\ (h(t), w_2) \\ \dots \\ (h(t), w_m) \end{pmatrix},$$

which is defined and continuous on $[0,T]\times U$ and $|c_m|_{\mathbb{R}^m}\in[c_m(0)-A',c_m(0)+A']$. In addition, suppose there are two solutions c_m and d_m to system (3.7), by (1.8), we have

$$
\begin{aligned}
&|F(t,c_m)-F(t,d_m)|_{\mathbb{R}^m}\\
&\leq\sqrt{\sum_{j=1}^{m}((-\lambda_j+C(n,\alpha)/2))^2}|c_m-d_m|_{\mathbb{R}^m}+\sqrt{\sum_{i=1}^{m}|(f(\sum_{j=1}^{m}c_m^jw_j)-f(\sum_{j=1}^{m}d_m^jw_j),w_i)|^2}\\
&\leq\sqrt{\sum_{j=1}^{m}((-\lambda_j+C(n,\alpha)/2))^2}|c_m-d_m|_{\mathbb{R}^m}+\beta|c_m-d_m|_{\mathbb{R}^m}.
\end{aligned}
$$

Consequently, for all $t\in[0,T]$, there exists $L=\sqrt{\sum_{j=1}^{m}((-\lambda_j+C(n,\alpha)/2))^2}+\beta$, such that

$$
|F(t,c_m)-F(t,d_m)|_{\mathbb{R}^m}\leq L|c_m-d_m|_{\mathbb{R}^m},\qquad \forall |c_m|_{\mathbb{R}^m},|d_m|_{\mathbb{R}^m}\in[c_m(0)-A',c_m(0)+A'].
$$

Therefore, Lemma 3.1 implies the existence of a unique strong solution $c_m(t)$ to (3.7) on $[0,T^m]$, where either $T^m=\infty$ or $T^m<\infty$ and $\lim_{t\to T^m-}|c_m|_{\mathbb{R}^m}=\infty$. Furthermore, since f fulfills (1.7), for each fixed $t>0$, it follows from [1, Theorem 2.7] that

$$
\begin{aligned}
\left|\frac{\partial F(t,c_m)}{\partial c_m}\right|_{\mathbb{R}^m\times\mathbb{R}^m}&:=\sqrt{\sum_{j=1}^{m}(-\lambda_j+C(n,\alpha)/2+f')^2}\\
&\leq\sqrt{\sum_{j=1}^{m}(-\lambda_j+C(n,\alpha)/2+\beta)^2}.
\end{aligned}
\tag{3.13}
$$

Making use of the fact that $f\in C^1(\mathbb{R})$ and $h\in C((0,\infty);L^2(\mathbb{R}^n))$, it is easy to check $F:[0,T]\times U\to\mathbb{R}$ fulfills a Lipschitz condition with respect to the second variable (see, (3.13)). By [5, Theorem 6.28], we know $c_m^j\in C^1(0,T)\cap C[0,T)$ $(j=1,2,\cdots,m)$ and consequently,

$$
u_m\in C^1((0,T^m);H^\alpha(\mathbb{R}^n))\cap C([0,T^m);H^\alpha(\mathbb{R}^n)).
$$

By [13, Proposition 2.18], we obtain

$$
D_c^\gamma\left(\frac{1}{2}\|u_m\|^2\right)(t)\leq\langle u_m,D_c^\gamma u_m\rangle.
$$

Since $u_m=\sum_{j=1}^{m}c_m^j(t)w_j$, making use of (3.7), we find

$$
\langle D_c^\gamma u_m,u_m\rangle=-<(-\Delta)^\alpha u_m,u_m>+(f(u_m),u_m)+(h(t),u_m).
$$

Hence,

$$
\begin{aligned}
D_c^\gamma\left(\frac{1}{2}\|u_m\|^2\right)\leq\langle D_c^\gamma u_m,u_m\rangle\leq&-\frac{C(n,\alpha)}{2}\left(\|u_m\|^2_{\dot{H}^\alpha(\mathbb{R}^n)}+\|u_m\|^2\right)+\kappa|\mathcal{O}|\\
&+\frac{1}{4\alpha_2-2C(n,\alpha)}\|h\|^2,
\end{aligned}
$$

which implies that

$$\|u_m\|^2+\frac{C(n,\alpha)}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\|u_m\|^2_{H^\alpha(\mathbb{R}^n)}ds \le \|u_m(0)\|^2+\frac{2T^\gamma}{\Gamma(\gamma+1)}\left(\kappa|\mathcal{O}|+\frac{1}{4\alpha_2-2C(n,\alpha)}\|h\|^2\right).$$

Consequently, for every $0<T<\infty$, the first claim follows.

(ii) Pick up a test function $v\in L^2((0,T);H^\alpha(\mathbb{R}^n))$ with

$$\|v\|_{L^2((0,T);H^\alpha(\mathbb{R}^n))}\le 1.$$

Denote

$$v_m := P_m v,$$

where P_m is defined in (3.6). Then,

$$\|v_m\|_{L^2((0,T);H^\alpha(\mathbb{R}^n))}\le 1.$$

Since $v_m\in \mathrm{span}\{w_1,\cdots,w_m\}$, we have

$$\langle D_c^\gamma u_m, v\rangle = \langle D_c^\gamma u_m, v_m\rangle = -<(-\Delta)^\alpha u_m, v_m> + (f(u_m),v_m)+(h(t),v_m).$$

By similar computations as in (3.5), we find there exists a positive constant $C(T)$ depending on T such that,

$$\langle D_c^\gamma u_m, v_m\rangle \le C(T).$$

Therefore, $\|D_c^\gamma u_m\|_{L^2((0,T);H^{-\alpha}(\mathbb{R}^n))}\le C(T)$ for all $m\in\mathbb{N}$. Let us keep in mind that so far, we have proved that

$$u_m\in L^\infty_{loc}((0,\infty);L^2(\mathbb{R}^n)),\quad \sup_{t\in[0,T]} J_\gamma(\|u_m\|^2_{H^\alpha(\mathbb{R}^n)})\le C(T)\quad\text{and}\quad D_c^\gamma u_m\in L^2((0,T);H^{-\alpha}(\mathbb{R}^n)).$$

Theorem 2.4 implies there is a subsequence $\{u_{m_k}\}$ which converges in $L^p((0,T);L^2(\mathbb{R}^n))$ for any $p\in[1,\infty)$. In particular, we choose $p=2$. It follows from [13, Proposition 3.5] that u has a weak Caputo derivative with initial value u_0 such that

$$D_c^\gamma u\in L^2((0,T);H^{-\alpha}(\mathbb{R}^n)).$$

By a standard diagonal argument, u is defined on $(0,\infty)$ and $D_c^\gamma u\in L^2_{loc}((0,\infty);H^{-\alpha}(\mathbb{R}^n))$ such that

$$u_{m_k}\to u \quad\text{in}\quad L^2_{loc}([0,\infty);L^2(\mathbb{R}^n)).$$

By taking a further subsequence (relabeled the same), we can assume that u_{m_k} also converges a.e. to u in $[0,\infty)\times\mathcal{O}$. Then, by (3.11) it is easy to check that for any $0\le t_1<t_2<\infty$,

$$\int_{t_1}^{t_2}\|u_{m_k}\|^2ds\le\left(\|u_0\|^2+\frac{2T^\gamma}{\Gamma(\gamma+1)}\left(\kappa|\mathcal{O}|+\frac{1}{4\alpha_2-2C(n,\alpha)}\|h\|\right)\right)(t_2-t_1).$$

Thanks to Fatou's Lemma, we deduce that

$$\int_{t_1}^{t_2} \|u\|^2 ds \le \left(\|u_0\|^2 + \frac{2T^\gamma}{\Gamma(\gamma+1)} \left(\kappa|\mathcal{O}| + \frac{1}{4\alpha_2 - 2C(n,\alpha)} \|h\| \right) \right) (t_2 - t_1).$$

This then implies that $u \in L^\infty((0,T); L^2(\mathbb{R}^n))$ for any $T > 0$.

For any $T > 0$, since u_{m_k} is bounded in $L^2((0,T); H^\alpha(\mathbb{R}^n))$, it has a further subsequence that converges weakly in $L^2((0,T); H^\alpha(\mathbb{R}^n))$. By a standard diagonal argument, there is a subsequence that converges weakly in $L^2_{loc}((0,\infty); H^\alpha(\mathbb{R}^n))$. The limit must be u by paring with a smooth test function. Hence, $u \in L^2_{loc}((0,\infty); H^\alpha(\mathbb{R}^n))$. $\square$

Theorem 3.3 *Suppose $f \in C^1(\mathbb{R})$ fulfills* (1.6)-(1.7), *$h \in C((0,\infty); L^2(\mathbb{R}^n))$ and $u_0 \in L^2(\mathbb{R}^n)$. Then there exists a global weak solution to* (1.5) *with initial datum u_0 under Definition 2.6.*

Proof. It follows from Lemma 3.2 that there is a subsequence of $\{u_m\}$ that converges in $L^2((0,T); L^2(\mathbb{R}^n))$, and we denote the limit of this subsequence by u. Now, for any test function $\varphi \in C_c^\infty([0,T) \times \mathcal{O}; \mathbb{R})$, it can be extended to $C_c^\infty([0,T) \times \mathbb{R}^n; \mathbb{R})$ by letting $\varphi = 0$ in $[0,T) \times \mathbb{R}^n \backslash \mathcal{O}$. We expand

$$\varphi = \sum_{k=1}^{\infty} b_k w_k, \tag{3.14}$$

and define

$$\varphi_m := \sum_{k=1}^{m} b_k w_k. \tag{3.15}$$

Since φ is a smooth function in t that vanishes at T, so is φ_m, also $\tilde{D}_{c;T}^\gamma \varphi_m \to \tilde{D}_{c;T}^\gamma \varphi$ in $L^2((0,T); H^\alpha(\mathbb{R}^n))$ as $m \to \infty$.

First of all, we fix $m_0 \ge 1$ and for $m_k \ge m_0$, we have

$$\begin{aligned} \langle u_{m_k} - u_0, \tilde{D}_{c;T}^\gamma \varphi_{m_0} \rangle_{x,t} = \langle D_c^\gamma u_{m_k}, \varphi_{m_0} \rangle_{x,t} = \int_0^T \int_{\mathbb{R}^n} -(-\Delta)^\alpha u_{m_k} \varphi_{m_0} dx dt \\ + \int_0^T \int_{\mathcal{O}} f(u_{m_k}) \varphi_{m_0} dx dt + \int_0^T \int_{\mathcal{O}} h(t) \varphi_{m_0} dx dt. \end{aligned} \tag{3.16}$$

The first equality holds by integration by parts, while the second one holds because $\varphi_{m_0} \in \operatorname{span}\{w_1, w_2, \cdots, w_{m_k}\}$. Based on results in Lemma 3.2, we have

$$\begin{cases} u_{m_k} \to u \quad \text{strongly in} \ \ L^2((0,T); L^2(\mathbb{R}^n)); \\ u_{m_k}(t) \to u(t) \quad \text{strongly in} \ \ L^2(\mathbb{R}^n), \ \text{a.e.}\ t \in (0,T); \\ u_{m_k}(x,t) \to u(x,t) \quad \text{strongly in} \ \ \text{a.e}\ (x,t) \in \mathcal{O} \times (0,T); \\ f(u_{m_k}) \to \xi \quad \text{weakly in} \ \ L^2((0,T); L^2(\mathbb{R}^n)); \\ Au_{m_k} \to Au \quad \text{weakly in} \ \ L^2((0,T); H^{-\alpha}(\mathbb{R}^n)). \end{cases}$$

Proceeding as in the proof [31, Theorem 2.7], we can conclude that

$$f(u_{m_k}) \to f(u) \quad \text{weakly in} \quad L^2((0,T);L^2(\mathbb{R}^n)).$$

Taking $k \to \infty$ in (3.16), we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} (u-u_0)\tilde{D}^{\gamma}_{c;T}\varphi_{m_0} dxdt = &-\int_0^T \int_{\mathbb{R}^n} (-\Delta)^{\alpha} u\varphi_{m_0} dxdt \\ &+ \int_0^T \int_{\mathcal{O}} f(u)\varphi_{m_0} dxdt + \int_0^T \int_{\mathcal{O}} h(t)\varphi_{m_0} dxdt.\end{aligned}$$

Eventually, taking $m_0 \to \infty$, by the convergence $\varphi_m \to \varphi$ in $L^2((0,T);H^{\alpha}(\mathbb{R}^n))$, we find that the weak formulation holds. The proof of this theorem is complete. □

Data availability

The document has no associated data.

Conflict of interests

The authors declare that they do not have any conflict of interests for this paper.

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