Ulam-Hyers stability of Caputo-type fractional fuzzy stochastic differential equations with delays

Danfeng Luo[a], Xue Wang[a], Tomás Caraballo[b], Quanxin Zhu[c,d,*]

[a]*Department of Mathematics, Guizhou University, Guiyang 550025, China*
[b]*Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, c/ Tarfia s/n, 41012 Seville, Spain*
[c]*MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China*
[d]*Key Laboratory of Control and Optimization of Complex Systems, College of Hunan Province, Hunan Normal University, Changsha 410081, China*

Abstract

In this paper, we explore a new kind of Caputo-type fractional fuzzy stochastic differential equations (FFSDEs) with delays. We establish the existence result of FFSDEs with delays by the method of upper and lower solutions, and then the uniqueness for this considered system is proved with the aid of the method of contradiction. Subsequently, we study the U-H stability with the help of Hölder's inequality and Grönwall-Bellman inequality. Finally, we demonstrate the validity of the proposed conclusions through three examples with numerical simulations.

Keywords: Fuzzy stochastic differential equations, Fractional calculus, Ulam-Hyers stability, Existence and uniqueness

1. Introduction

In 1695, L'Hospital sent a letter asking Leibniz what $\frac{d^n y}{dx^n}$ meant when $n = \frac{1}{2}$. We know that when n is a positive integer, $\frac{d^n y}{dx^n}$ represents the n-th derivative of function y. The researchers of that era basically considered the case where n was a positive integer, so Leibniz was puzzled by this sudden question. He

*Corresponding author
Email address: zqx22@126.com (Quanxin Zhu)

said a lot of vague things in his reply to L'Hospital, and he actually did not answer L'Hospital's question correctly at that time. This was the eleventh year of calculus, and Leibniz was understandably confused, and then the concept of fractional calculus (FC) was born. However, due to the limitations of cognition at that time and the lack of corresponding practical applications, the development of FC was relatively slow. Later Laplace, Abel, Liouville, Letnikov, Pincherle, Davis and many other mathematicians made many important contributions to the development of FC. As a result, the definitions of FC have become more and more perfect, and its properties have become clearer, and its applications have become more and more extensive. Fractional differential equations are widely used in many fields, such as genetic law, disease control, viscoelastic mechanics, and the number of biological populations, because it is more suitable to describe some complex systems with memory and hereditary properties, and readers can refer to [1, 2, 3, 4, 5, 6, 7, 8].

With the progress and development, the deterministic differential equations can no longer solve some problems in related fields. Many real phenomena in control theory, physics, biology, and economics are characterized by uncertainty. Most of these uncertainties have two cases, and one is stochastic uncertainty caused by random disturbance, and the other is fuzzy uncertainty caused by observation, experiment and maintenance. The method of probability theory is used in the analysis of random uncertainty, and the fuzzy set theory is used in the analysis of fuzzy uncertainty. In 1951, Itô combined probability theory and ordinary differential equations and published his book: On Stochastic Differential Equations [9]. He strictly described stochastic differential equations in mathematical language, which provided an important theoretical basis for dealing with stochastic problems in differential equations. Later in order to consider the advantages of fractional differential equations, researchers extended stochastic differential equations to fractional stochastic differential equations. In 1965, Zadeh [10] proposed fuzzy set theory, which provided a theoretical basis for dealing with fuzzy uncertainty in systems, and scholars combined fuzzy set theory and ordinary differential equations theory to consider fuzzy differential equations. In 2010, Agarwal et al. studied fractional fuzzy differential equations (FFDEs) in [11]. Nowadays, FFSDEs have received extensive attention because they can be used in the study of many physical and engineering problems affected by both randomness and ambiguity.

Since the existence and uniqueness of solutions to differential equations are the premise subject of studying other properties, they have basic significance. Therefore, the existence and uniqueness of solutions to the fuzzy stochastic differential equations (FSDEs) is also a focus to researchers. In [12], the author obtained the existence and uniqueness of solutions to FSDEs under bounded conditions and Lipschitz conditions that are weaker than the linear growth conditions. Kim [13] proved the existence of the one-dimensional FSDEs by Picard iteration under the Lipschitz conditions, and proved its uniqueness by contradictory method. Malinowski [14] used the same method to prove the existence and uniqueness of FSDEs, but Malinowski explored n-dimensional FSDEs. The papers [15, 16] also studied the existence and uniqueness to FSDEs. So far, many papers have studied the existence and uniqueness of the solutions to FSDEs, but there are relatively few papers on FFSDEs. We can only find that Priyadharsini et al. [17] proved the existence and uniqueness of the solution to the fuzzy fractional stochastic Pantograph differential system by using the compression mapping principle.

In 1940, Ulam [18] proposed a question about the stability of group homomorphism. The following year, Hyers [19] gave the first affirmative answer to this question. Since then, the mathematical community has produced U-H stability. With the further exploration of U-H stability by scholars such as Aoki, Rassias, Sahoo and Jung [20, 21, 22, 23, 24], the U-H stability theory has been gradually developed. Vu et al. made important contributions to the U-H stability of FFDEs, and they proved the U-H stability of different types of FFDEs or fractional fuzzy integral equations by successive approximation [25, 26, 27, 28]. Vu and Hoa studied μ-Ulam-Hyers-Rassias stability and Ulam-Hyers-Rassias-Mittag-Leffler stability of FFDEs using the fixed point theorem in [29], and similar results can be found in reference [30, 31, 32, 33, 34, 35, 36, 37], but there are few papers to study the U-H stability of FSDEs and FFSDEs. Therefore, there are still many questions in this field waiting to be discovered and answered by researchers.

Due to the relative lack of results on FFSDEs, and inspired by the above articles, this paper will study the existence, uniqueness and U-H stability of solutions to FFSDEs with delays:

$$\begin{cases} {}^{C}\mathscr{D}^{\beta}_{0+}y(\tau) = \mathcal{U}(\tau,\ y(\tau),\ y(\tau-\rho)) + \left\langle \mathcal{V}(\tau,\ y(\tau),\ y(\tau-\rho))\dfrac{dW(\tau)}{d\tau} \right\rangle,\ \tau \in [0,\ T], \\ y(\tau) = \mathscr{P}(\tau),\ \tau \in [-\rho,\ 0], \end{cases} \tag{1.1}$$

where ${}^{C}\mathscr{D}^{\beta}_{0+}$ denotes the Caputo fractional derivative, $\frac{1}{2} < \beta < 1$, the state vector $y \in \mathscr{F}(\mathbb{R})$ is the fuzzy stochastic process, $\mathcal{U} : [0,T] \times \mathscr{F}(\mathbb{R}) \times \mathscr{F}(\mathbb{R}) \to \mathscr{F}(\mathbb{R})$, $\mathcal{V} : [0,T] \times \mathscr{F}(\mathbb{R}) \times \mathscr{F}(\mathbb{R}) \to \mathbb{R}$ are continuous functions with respect to y that are nondecreasing, $W(\cdot)$ is one-dimensional Brownian motion on a complete probability space (Ω, A, P), $\mathscr{P} : [-\rho, 0] \longrightarrow \mathscr{F}(\mathbb{R})$ is a continuous fuzzy function with $\mathscr{P}(0) = \mathscr{P}_0$, and $\mathbb{E}d^2_\infty[\mathscr{P}_0,\ \langle 0\rangle] < \infty$.

The innovations and difficulties of this paper have at least the following aspects:

(i) Since there are few papers on FFSDEs at present for this issue, the methods for our reference are also not enough. Therefore, many problems need to be solved by referring to the methods of integer FSDEs and FFDEs. For example, when proving the existence of solutions to system (1.1), we refer to the method of proving FFDEs in article [38], and use the upper and lower solutions method to get the existence results of FFSDEs.

(ii) Although the authors also studied the existence and uniqueness of solutions to FFSDEs in [17], the authors considered the granular difference, which is different to the situation that we consider the generalized Hukuhara difference. Since the properties of generalized Hukuhara difference is not as good as that of granular difference [39], we will encounter more difficulties in the proof process.

(iii) We study the U-H stability of FFSDEs, which firstly promotes the further development of U-H stability in fuzzy space, and secondly fills the gap of U-H stability theory in the field of FFSDEs.

The remainder of this article is organized as follows: we review several definitions and lemmas related to fuzzy sets, fractional calculus, and stochastic differential equations and give an assumption that will be used throughout the paper in Section 2. The existence and uniqueness of solution to FFSDEs are investigated in Section 3. In Section 4, the U-H stability of solution to Eq. (1.1) is presented. In Section 5, we give three examples to illustrate the validity of the theoretical results.

2. Preliminaries

Let $\mathscr{K}(\mathbb{R})$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}$, then the Hausdorff metric of $\forall \mathscr{B}, \mathscr{S} \in \mathscr{K}(\mathbb{R})$ is defined as (see [14])

$$D_0(\mathscr{B}, \mathscr{S}) := \max\left\{ \sup_{b\in\mathscr{B}} \inf_{s\in\mathscr{S}} \|b-s\|, \sup_{s\in\mathscr{S}} \inf_{b\in\mathscr{B}} \|b-s\| \right\},$$

where $\|\cdot\|$ is the norm in $\mathbb{R}$, and $(\mathscr{K}(\mathbb{R}), D_0)$ is a complete metric space.

Definition 2.1. *[40] Let $\mathscr{F}(\mathbb{R})$ denote the fuzzy sets $\mathscr{X}: \mathbb{R} \to [0,1]$ such that $[\mathscr{X}]^r \in \mathscr{K}(\mathbb{R})$ for every $r \in [0,1]$, and $\mathscr{X}$ satisfies*

(1) For $0 \le a \le 1$ and $\forall \tau_1, \tau_2 \in \mathbb{R}$, $\mathscr{X}(a\tau_1 + (1-a)\tau_2) \ge \min\{\mathscr{X}(\tau_1), \mathscr{X}(\tau_2)\}$, which means that $\mathscr{X}$ is fuzzy convex in $\mathbb{R}$;

(2) There is $\tau_0 \in \mathbb{R}$ such that $\mathscr{X}(\tau_0) = 1$, which means that $\mathscr{X}$ is normal;

(3) $[\mathscr{X}]^0 = \overline{\{\tau_0 \in \mathbb{R} \mid \mathscr{X}(\tau_0) > 0\}}$ is compact;

(4) $\mathscr{X}$ is upper semi-continuous on $\mathbb{R}$.

Here $[\mathscr{X}]^r$ is the r-level set of $\mathscr{X}$ and defines as

$$[\mathscr{X}]^r = \begin{cases} \{\tau_0 \in \mathbb{R} \mid \mathscr{X}(\tau_0) \ge r\}, & \forall 0 < r \le 1, \\ \overline{\{\tau_0 \in \mathbb{R} \mid \mathscr{X}(\tau_0) > 0\}}, & r = 0. \end{cases}$$

Let $\langle\cdot\rangle : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ denote the embedding of $\mathbb{R}$ into $\mathcal{F}(\mathbb{R})$, i.e. for $\mathcal{X} \in \mathbb{R}$ we have

$$\langle\mathcal{X}\rangle(\tau) = \begin{cases} 1, & \text{if } \tau = \mathcal{X}, \\ 0, & \text{if } \tau \in \mathbb{R}\backslash\{\mathcal{X}\}. \end{cases}$$

Definition 2.2. *[40] The summation and scalar multiplication in $\mathcal{F}(\mathbb{R})$ are defined as*

$$[\mathcal{X}_1 + \mathcal{X}_2]^r = [\mathcal{X}_1]^r + [\mathcal{X}_2]^r = \{\tau_1 + \tau_2 \mid \tau_1 \in [\mathcal{X}_1]^r,\ \tau_2 \in [\mathcal{X}_2]^r\};$$
$$[k \cdot \mathcal{X}_1]^r = k[\mathcal{X}_1]^r = \{k\tau \mid \tau \in [\mathcal{X}]^r\},\ \forall r \in [0,\ 1],\ k \in \mathbb{R},$$

where $[\mathcal{X}]^r = [\underline{\mathcal{X}}(r),\ \overline{\mathcal{X}}(r)]\,(\forall r \in [0,\ 1])$ is a bounded, closed interval and the diameter of $[\mathcal{X}]^r$ is defined by $d([\mathcal{X}]^r) = \overline{\mathcal{X}}(r) - \underline{\mathcal{X}}(r)$.

Definition 2.3. *[41] Let $\mathcal{X}_1,\ \mathcal{X}_2 \in \mathcal{F}(\mathbb{R})$. If there exists a $\mathcal{X}_3 \in \mathcal{F}(\mathbb{R})$ such that $\mathcal{X}_1 = \mathcal{X}_2 + \mathcal{X}_3$, then $\mathcal{X}_3$ is the Hukuhara difference of $\mathcal{X}_1$, $\mathcal{X}_2$, and we define $\mathcal{X}_1 \ominus \mathcal{X}_2 = \mathcal{X}_3$.*

Definition 2.4. *[41] The generalized Hukuhara difference between $\mathcal{X}_1 \in \mathcal{F}(\mathbb{R})$ and $\mathcal{X}_2 \in \mathcal{F}(\mathbb{R})$ is given as:*

$$\mathcal{X}_1 \ominus_{gH} \mathcal{X}_2 = \mathcal{X}_3 \Leftrightarrow \begin{cases} (i) \quad \mathcal{X}_1 = \mathcal{X}_2 + \mathcal{X}_3,\ d([\mathcal{X}_1]^r) \geq d([\mathcal{X}_2]^r), \\ (ii) \quad \mathcal{X}_2 = \mathcal{X}_1 + (-1)\mathcal{X}_3,\ d([\mathcal{X}_1]^r) \leq d([\mathcal{X}_2]^r). \end{cases}$$

Definition 2.5. *[41] If for every $r \in [0,\ 1]$, the real function $d([\mathcal{G}(\cdot)]^r)$ is nondecreasing (nonincreasing) on $[a,\ b]$, then the fuzzy function $\mathcal{G} : [a,\ b] \to \mathcal{F}(\mathbb{R})$ is called d-increasing (d-decreasing) on $[a,\ b]$. If $\mathcal{G}$ is d-increasing or d-decreasing on $[a,\ b]$, then we say that $\mathcal{G}$ is d-monotone on $[a,\ b]$.*

Definition 2.6. *[14] For $\mathcal{X}_1,\ \mathcal{X}_2 \in \mathcal{F}(\mathbb{R})$, we denote*

$$d_\infty[\mathcal{X}_1, \mathcal{X}_2] = \sup_{0 \leq r \leq 1} D_0([\mathcal{X}_1]^r,\ [\mathcal{X}_2]^r),$$

where d_∞ is a metric in $\mathcal{F}(\mathbb{R})$. It is known that $\mathcal{F}(\mathbb{R})$ is a complete metric space with respect to d_∞. For $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3,\ \mathcal{H}_4 \in \mathcal{F}(\mathbb{R}),\ l \in \mathbb{R}$, the following properties are hold:

(1) $d_\infty(\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_2 + \mathcal{H}_3) = d_\infty(\mathcal{H}_1, \mathcal{H}_2)$;

(2) $d_\infty(\mathcal{H}_1+\mathcal{H}_2,\mathcal{H}_3+\mathcal{H}_4)\leq d_\infty(\mathcal{H}_1,\mathcal{H}_3)+d_\infty(\mathcal{H}_2,\mathcal{H}_4)$;

(3) $d_\infty(l\mathcal{H}_1,l\mathcal{H}_2)=|l|d_\infty(\mathcal{H}_1,\mathcal{H}_2)$;

(4) $d_\infty(\mathcal{H}_1\ominus\mathcal{H}_2,\langle 0\rangle)=d_\infty(\mathcal{H}_1,\mathcal{H}_2)$;

(5) $d_\infty(\mathcal{H}_1\ominus\mathcal{H}_2,\mathcal{H}_1\ominus\mathcal{H}_3)=d_\infty(\mathcal{H}_2,\mathcal{H}_3)$;

(6) $d_\infty(\mathcal{H}_1\ominus\mathcal{H}_2,\mathcal{H}_3\ominus\mathcal{H}_4)\leq d_\infty(\mathcal{H}_1,\mathcal{H}_3)+d_\infty(\mathcal{H}_2,\mathcal{H}_4)$.

Definition 2.7. *[41] The Riemann-Liouville fractional integral of order $\beta\in(0,\ 1]$ of the fuzzy function $\mathcal{G}\in L([0,\ b],\ \mathcal{F}(\mathbb{R}))$ is defined by*

$$\Im_{0^+}^{\beta}\mathcal{G}(s)=\frac{1}{\Gamma(\beta)}\int_0^s(s-t)^{\beta-1}\mathcal{G}(t)dt,\ s>0.$$

Definition 2.8. *[42] Let $\mathcal{O},\ \mathcal{Q}\in\mathcal{F}(\mathbb{R})$. We say that $\mathcal{O}\preceq\mathcal{Q}(\mathcal{O}\succeq\mathcal{Q})$ if and only if $\underline{\mathcal{O}}(r)\geq\underline{\mathcal{Q}}(r)$ and $\overline{\mathcal{O}}(r)\leq\overline{\mathcal{Q}}(r)$ $\left(\underline{\mathcal{O}}(r)\leq\underline{\mathcal{Q}}(r)\right.$ and $\left.\overline{\mathcal{O}}(r)\geq\overline{\mathcal{Q}}(r)\right)$, that is,*

$$[\mathcal{O}]^r\subseteq[\mathcal{Q}]^r([\mathcal{O}]^r\supseteq[\mathcal{Q}]^r),\quad\forall r\in[0,1].$$

Definition 2.9. *[41] The Caputo type fractional derivative of order $\beta\in(0,\ 1]$ of a d-monotone fuzzy function $\mathcal{G}\in AC([0,\ T],\ \mathcal{F}(\mathbb{R}))$ is defined by*

$${}^{C}\mathcal{D}_{0^+}^{\beta}\mathcal{G}(s)=\frac{1}{\Gamma(1-\beta)}\int_0^s(s-t)^{-\beta}\frac{d}{dt}\mathcal{G}(t)dt,$$

where $AC([0,\ T],\ \mathcal{F}(\mathbb{R}))$ represents the set of all absolute continuous fuzzy functions from $[0,\ T]$ to $\mathcal{F}(\mathbb{R})$.

Lemma 2.1. *[38] Let $\mathcal{W},\mathcal{R}\in AC([0,\ T],\ \mathcal{F}(\mathbb{R}))$. If there exists a $s_1\in(0,\ T]$ such that $\mathcal{W}(s_1)=\mathcal{R}(s_1)$ and $\mathcal{W}(s)\preceq\mathcal{R}(s)$ on $[0,\ T]$, then we have ${}^{C}\mathcal{D}_{0^+}^{\beta}\mathcal{W}(s_1)\succeq{}^{C}\mathcal{D}_{0^+}^{\beta}\mathcal{R}(s_1)$.*

Definition 2.10. *[41] A d-monotone fuzzy function $y\in AC([0,\ T],\ \mathcal{F}(\mathbb{R}))$ is a solution of the Eq. (1.1) if and only if $y\in AC([0,\ T],\ \mathcal{F}(\mathbb{R}))$ satisfies the following fractional interval integral equation*

$$\begin{aligned}y(\tau)\ominus_{gH}\mathcal{P}_0=&\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\eta)^{\beta-1}\mathcal{U}(\eta,y(\eta),y(\eta-\rho))d\eta\\&+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\eta)^{\beta-1}\mathcal{V}(\eta,y(\eta),y(\eta-\rho))dW(\eta)\right\rangle,\end{aligned}\tag{2.1}$$

and the fuzzy function $\tau \mapsto \Im^{\beta}_{0+}\mathscr{W}(\tau)$ *is d-increasing on* $[0,T]$, *where* $\mathscr{W}(\tau) = \mathscr{U}(\tau,\ y(\tau),\ y(\tau-\rho)) + \left\langle \mathscr{V}(\tau,\ y(\tau),\ y(\tau-\rho))\frac{dW(\tau)}{d\tau}\right\rangle$.

Remark 2.1. *If* $y \in AC\left([0,T],\mathscr{F}\left(\mathbb{R}\right)\right)$ *such that* $d\left([y(\tau)]^r\right) \geq d\left([\mathscr{P}_0]^r\right)$, *then* (2.1) *can be rewritten as*

$$\begin{aligned} y(\tau) =& \mathscr{P}_0 + \left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{U}(\eta,y(\eta),y(\eta-\rho))d\eta \right.\\ &\left. + \left\langle \frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{V}(\eta,y(\eta),y(\eta-\rho))dW(\eta)\right\rangle\right). \end{aligned} \tag{2.2}$$

If $y \in AC\left([0,T],\mathscr{F}\left(\mathbb{R}\right)\right)$ *such that* $d\left([y(\tau)]^r\right) \leq d\left([\mathscr{P}_0]^r\right)$, *then* (2.1) *can be rewritten as*

$$\begin{aligned} y(\tau) =& \mathscr{P}_0 \ominus (-1)\left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{U}(\eta,y(\eta),y(\eta-\rho))d\eta \right.\\ &\left. + \left\langle \frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{V}(\eta,y(\eta),y(\eta-\rho))dW(\eta)\right\rangle\right). \end{aligned} \tag{2.3}$$

Here we can represent (2.2) *and* (2.3) *as*

$$\begin{aligned} y(\tau) =& \mathscr{P}_0 \Theta \left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{U}(\eta,y(\eta),y(\eta-\rho))d\eta \right.\\ &\left. + \left\langle \frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\eta)^{\beta-1}\mathscr{V}(\eta,y(\eta),y(\eta-\rho))dW(\eta)\right\rangle\right), \end{aligned}$$

where $\Theta := \{+, \ominus(-1)\}$.

Definition 2.11. *[38] A d-monotone fuzzy function* $y^{\lambda} \in AC([-\rho,T],\mathscr{F}\left(\mathbb{R}\right))$ *is said to be a lower solution for* (1.1) *if*

$$\begin{cases} {}^{C}\mathscr{D}^{\beta}_{0+}y^{\lambda}(\tau) \preccurlyeq \mathscr{U}(\tau,\ y^{\lambda}(\tau),\ y^{\lambda}(\tau-\rho)) + \left\langle \mathscr{V}(\tau,\ y^{\lambda}(\tau),\ y^{\lambda}(\tau-\rho))\frac{dW(\tau)}{d\tau}\right\rangle,\ \tau \in [0,\ T], \\ y^{\lambda}(\tau) \preccurlyeq \mathscr{P}(\tau),\ \tau \in [-\rho,\ 0]. \end{cases} \tag{2.4}$$

A d-monotone upper solution $y^{\omega} \in AC([-\rho,T],\mathscr{F}\left(\mathbb{R}\right))$ *for* (1.1) *is defined analogously by reversing the inequalities in* (2.4).

Lemma 2.2. *[43] For* $T > 0,\ 2\beta > 1,\ y(\cdot)$ *is a stochastic process satisfying* $\int_0^T d^2_{\infty}[y(\eta),\ \langle 0\rangle]dt < \infty$. *Then we can obtain*

$$\sup_{0\leq\tau\leq T}\mathbb{E}d^2_{\infty}\left[\left\langle\int_0^{\tau}(\tau-\eta)^{\beta-1}y(\eta)dW(\eta)\right\rangle,\ \langle 0\rangle\right] \leq \frac{T^{2\beta-1}}{2\beta-1}\mathbb{E}\int_0^T d^2_{\infty}[y(\tau),\ \langle 0\rangle]d\tau.$$

Assumption 1. For $y,\ \tilde{y}, x, \widetilde{x} \in \mathcal{F}(\mathbb{R}),\ \eta \in [-\rho,\ T]$, and suppose that there exist some constants N_1, N_2 and N_3 such that

(a_1) $d_\infty^2 \left[\mathcal{P}(\eta), \langle 0\rangle\right] \leq N_1$;

(a_2) $d_\infty^2[\mathcal{U}(\eta,0,0),\langle 0\rangle] \vee d_\infty^2[\mathcal{V}(\eta,0,0),\langle 0\rangle] \leq N_2$;

(a_3) $d_\infty^2[\mathcal{U}(\eta,y(\eta),\tilde{y}(\eta)),\mathcal{U}(\eta,x(\eta),\widetilde{x}(\eta))] \vee d_\infty^2[\mathcal{V}(\eta,y(\eta),\tilde{y}(\eta)),\mathcal{V}(\eta,x(\eta),\tilde{x}(\eta))] \leq N_3\left(d_\infty^2[y(\eta),x(\eta)] + d_\infty^2[\tilde{y}(\eta),\widetilde{x}(\eta)]\right)$,

where $\vee$ is defined as: $\mathcal{K}_1 \vee \mathcal{K}_2 = \max\{\mathcal{K}_1,\ \mathcal{K}_2\}$.

3. Existence and uniqueness

For convenience, we use the notation $y(\tau) \overset{P.1}{=} \mathfrak{z}(\tau)$ instead of $P(y(\tau) = \mathfrak{z}(\tau)) = 1$, where $\tau \in [-\rho, T]$, y and $\mathfrak{z}$ are the stochastic processes. For any $x_i, y_i \in \mathcal{F}(\mathbb{R}), i = 1, 2, \ldots, p$, the following inequality is true

$$d_\infty^2\left(x_1 + x_2 + \cdots + x_p,\ y_1 + y_2 + \cdots + y_p\right) \leq p\sum_{i=1}^{p} d_\infty^2\left(x_i, y_i\right). \tag{3.1}$$

Theorem 3.1. *If **Assumption 1** and $T^{2\beta}N_3 < \dfrac{\Gamma^2(\beta)(2\beta-1)}{32}$ are true, then there exists a unique d-monotone solution $y \in [y^\lambda,\ y^\omega]$ for Eq. (1.1) in $AC([-\rho,\ T],\ \mathcal{F}(\mathbb{R}))$.*

Proof. **Step 1.** We consider the sequence of continuous fuzzy functions $\{y_n, n = 0, 1, 2, \ldots\}$ given by :

$$\begin{cases} y_0(\tau) = \mathcal{P}_0, \quad \tau \in [0, T], \\ y_0(\tau) = \mathcal{P}(\tau),\ \tau \in [-\rho, 0], \end{cases}$$

and for $n \in \mathbb{N}^+$

$$\begin{cases} y_n(\tau) = \mathcal{P}_0 \Theta \left(\dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^\tau (\tau-\eta)^{\beta-1} \mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) d\eta \right. \\ \qquad\qquad \left. + \left\langle \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^\tau (\tau-\eta)^{\beta-1} \mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) dW(\eta) \right\rangle \right), \tau \in [0, T], \\ y_n(\tau) = \mathcal{P}(\tau), \tau \in [-\rho, 0], \end{cases} \tag{3.2}$$

where the fuzzy function

$$\tau \mapsto \Im_{0+}^{\beta} \mathscr{W}_{n-1}(\tau) := \Im_{0+}^{\beta} \left[\mathcal{U}(\tau,\ y_{n-1}(\tau),\ y_{n-1}(\tau-\rho)) + \left\langle \mathcal{V}(\tau,\ y_{n-1}(\tau),\ y_{n-1}(\tau-\rho)) \frac{dW(\tau)}{d\tau} \right\rangle \right]$$

is d-increasing on $[0, T]$.

By Eq. (3.2), Lemma 2.2, Assumption 1, Hölder's inequality and inequality (3.1), we have

$$\begin{aligned}
&\mathbb{E} d_{\infty}^2 \left[y_n(\tau),\ \langle 0 \rangle \right] \\
\leq & 3\mathbb{E} d_{\infty}^2 \left[\mathscr{P}_0,\ \langle 0 \rangle \right] + 3\mathbb{E} d_{\infty}^2 \left[\frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\eta)^{\beta-1} \mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) d\eta,\ \langle 0 \rangle \right] \\
&+ 3\mathbb{E} d_{\infty}^2 \left[\left\langle \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\eta)^{\beta-1} \mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) dW(\eta) \right\rangle,\ \langle 0 \rangle \right] \\
\leq & 3N_1 + \frac{3}{\Gamma^2(\beta)} \int_0^{\tau} (\tau-\eta)^{2\beta-2} d\eta \mathbb{E} \int_0^{\tau} d_{\infty}^2 \left[\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right), \langle 0 \rangle \right] d\eta \\
&+ \frac{3}{\Gamma^2(\beta)} \sup_{0 \leq \tau \leq T} \mathbb{E} d_{\infty}^2 \left[\left\langle \int_0^{\tau} (\tau-\eta)^{\beta-1} \mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) dW(\eta) \right\rangle,\ \langle 0 \rangle \right] \\
\leq & 3N_1 + \frac{6T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_{\infty}^2 \left[\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-p)\right), \mathcal{U}(0,0,0) \right] d\eta \\
&+ \frac{6T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T \left(d_{\infty}^2 [\mathcal{U}(\eta,0,0),\ \langle 0 \rangle] + d_{\infty}^2 [\mathcal{V}(\eta,0,0),\ \langle 0 \rangle] \right) d\eta \\
&+ \frac{6T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_{\infty}^2 \left[\mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right),\ \mathcal{V}(\eta,0,0) \right] d\eta \\
\leq & 3N_1 + \frac{12T^{2\beta}N_2}{\Gamma^2(\beta)(2\beta-1)} + \frac{12T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T \left(d_{\infty}^2 \left[y_{n-1}(\eta),\ \langle 0 \rangle \right] + d_{\infty}^2 \left[y_{n-1}(\eta-\rho),\ \langle 0 \rangle \right] \right) d\eta \\
\leq & K_1 + K_2 \mathbb{E} \int_0^T \left(2d_{\infty}^2 \left[y_{n-1}(\eta),\ \langle 0 \rangle \right] + \sup_{0 \leq \eta \leq \rho} d_{\infty}^2 [\mathscr{P}(\eta-\rho),\ \langle 0 \rangle] \right) d\eta \\
\leq & K_1 + K_2 N_1 T + 2K_2 \int_0^T \mathbb{E} d_{\infty}^2 \left[y_{n-1}(\eta),\ \langle 0 \rangle \right] \\
= & L_1 + L_2 \int_0^T \mathbb{E} d_{\infty}^2 \left[y_{n-1}(\eta),\ \langle 0 \rangle \right],
\end{aligned}$$

where $K_1 = 3N_1 + \frac{12T^{2\beta}N_2}{\Gamma^2(\beta)(2\beta-1)}$, $K_2 = \frac{12T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}$, $L_1 = K_1 + K_2 N_1 T$ and $L_2 = 2K_2$.

For any integer $j \geq 1$, we have

$$\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\tau),\ \langle 0\rangle\right]\leq L_{1}+L_{2}\int_{0}^{T}\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n-1}(\eta),\ \langle 0\rangle\right]d\eta.$$

We should notice that

$$\begin{aligned}
&\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n-1}(\eta),\ \langle 0\rangle\right]\\
=&\max\left\{\mathbb{E}d_{\infty}^{2}\left[y_{0}(\eta),\ \langle 0\rangle\right],\ \mathbb{E}d_{\infty}^{2}\left[y_{1}(\eta),\ \langle 0\rangle\right],\cdots,\mathbb{E}d_{\infty}^{2}\left[y_{j-1}(\eta),\ \langle 0\rangle\right]\right\}\\
\leq&\max\left\{N_{1},\mathbb{E}d_{\infty}^{2}\left[y_{1}(\eta),\ \langle 0\rangle\right],\ldots,\mathbb{E}d_{\infty}^{2}\left[y_{j-1}(\eta),\ \langle 0\rangle\right],\mathbb{E}d_{\infty}^{2}\left[y_{j}(\eta),\ \langle 0\rangle\right]\right\}\\
\leq&N_{1}+\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\eta),\ \langle 0\rangle\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\tau),\ \langle 0\rangle\right]\\
\leq&L_{1}+L_{2}\int_{0}^{T}\left(N_{1}+\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\eta),\ \langle 0\rangle\right]\right)d\eta\\
\leq&L_{1}+L_{2}N_{1}T+L_{2}\int_{0}^{T}\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\eta),\ \langle 0\rangle\right]d\eta.
\end{aligned}$$

In terms of Grönwall-Bellman inequality,

$$\begin{aligned}
\max_{1\leq n\leq j}\mathbb{E}d_{\infty}^{2}\left[y_{n}(\tau),\ \langle 0\rangle\right]&\leq\left(L_{1}+L_{2}N_{1}T\right)\exp\left(\int_{0}^{T}L_{2}d\eta\right)\\
&=\left(L_{1}+L_{2}N_{1}T\right)e^{L_{2}T}.
\end{aligned}$$

Since the integer $j\geq 1$ is arbitrary, we can obtain

$$\mathbb{E}\sup_{0\leq\tau\leq T}d_{\infty}^{2}\left[y_{n}(\tau),\langle 0\rangle\right]\leq\left(L_{1}+L_{2}N_{1}T\right)e^{L_{2}T}.$$

Next, for $n=1$, based on inequality (3.1), Assumption 1, Hölder's inequality and Lemma 2.2, we infer that

$$\begin{aligned}
&\mathbb{E}d_{\infty}^{2}\left[y_{1}(\tau),\ y_{0}(\tau)\right]\\
=&\mathbb{E}d_{\infty}^{2}\Bigg[\mathscr{P}_{0}\Theta\left(\frac{1}{\Gamma(\beta)}\int_{0}^{\tau}(\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta,y_{1}(\eta),y_{1}(\eta-\rho)\right)d\eta\right.\\
&\left.+\left\langle\frac{1}{\Gamma(\beta)}\int_{0}^{\tau}(\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta,y_{1}(\eta),y_{1}(\eta-\rho)\right)dW(\eta)\right\rangle\right),\ \mathscr{P}_{0}\Bigg]\\
=&\mathbb{E}d_{\infty}^{2}\left[\frac{1}{\Gamma(\beta)}\int_{0}^{\tau}(\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta,y_{1}(\theta),y_{1}(\eta-\rho)\right)d\eta\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left\langle \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\eta)^{\beta-1} \mathcal{V}\left(\eta, y_1(\eta), y_1(\eta-\rho)\right) dW(\eta) \right\rangle, \ \langle 0 \rangle \Big] \\
\le & \frac{2}{\Gamma^2(\beta)} \mathbb{E} d_\infty^2 \left[\int_0^\tau (\tau-\eta)^{\beta-1} \mathcal{U}\left(\eta, y_1(\eta), y_1(\eta-\rho)\right) d\eta, \ \langle 0 \rangle \right] \\
&+ \frac{2}{\Gamma^2(\beta)} \mathbb{E} \sup_{0\le\tau\le T} d_\infty^2 \left[\left\langle \int_0^\tau (\tau-\eta)^{\beta-1} \mathcal{V}\left(\eta, y_1(\eta), y_1(\eta-p)\right) dW(\eta) \right\rangle, \ \langle 0 \rangle \right] \\
\le & \frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_\infty^2 \left[\mathcal{U}\left(\eta, y_1(\eta), y_1(\eta-\rho)\right), \ \langle 0 \rangle \right] d\eta \\
&+ \frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_\infty^2 \left[\mathcal{V}\left(\eta, y_1(\rho), y_1(\eta-\rho)\right), \ \langle 0 \rangle \right] d\eta \\
\le & \frac{4T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_\infty^2 \left[\mathcal{U}\left(\eta, y_1(\eta), y_1(\eta-\rho)\right), \mathcal{U}(\eta,0,0) \right] d\eta \\
&+ \frac{4T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T \left(d_\infty^2[\mathcal{U}(\eta,0,0), \ \langle 0 \rangle] + d_\infty^2[\mathcal{V}(\eta,0,0), \ \langle 0 \rangle] \right) d\eta \\
&+ \frac{4T^2\beta-1}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T d_\infty^2 \left[\mathcal{V}\left(\eta, y_1(\eta), y_1(\eta-\rho)\right), \mathcal{V}(\eta,0,0) \right] d\eta \\
\le & \frac{8T^{2\beta} N_2}{\Gamma^2(\beta)(2\beta-1)} + \frac{8T^{2\beta-1} N_3}{\Gamma^2(\beta)(2\beta-1)} \mathbb{E} \int_0^T \left(d_\infty^2 \left[y_1(\eta), \ \langle 0 \rangle \right] + d_\infty^2 \left[y_1(\eta-\rho), \ \langle 0 \rangle \right] \right) d\eta \\
\le & \frac{8T^{2\beta}\left(N_2 + N_1 N_3\right)}{\Gamma^2(\beta)(2\beta-1)} + \frac{16T^{2\beta-1} N_3}{\Gamma^2(\beta)(2\beta-1)} \int_0^T \mathbb{E} d_\infty^2 \left[y_1(\eta), \ \langle 0 \rangle \right] d\eta \\
\le & \frac{8T^{2\beta}\left(N_2 + N_1 N_3\right)}{\Gamma^2(\beta)(2\beta-1)} + \frac{16T^{2\beta} N_3}{\Gamma^2(\beta)(2\beta-1)} \left(L_1 + L_2 N_1 T\right) e^{L_2 T} \\
=& : M,
\end{aligned}
$$

then we have

$$
\mathbb{E} \sup_{0\le\tau\le T} d_\infty^2 \left[y_1(\tau), \ y_0(\tau) \right] \le M.
$$

Assume for some n,

$$
\mathbb{E} \sup_{0\le\tau\le T} d_\infty^2 \left[y_n(\tau), y_{n-1}(\tau) \right] \le \left(\frac{8T^{2\beta-1} N_3}{\Gamma^2(\beta)(2\beta-1)} \right)^{n-1} M T^{n-1}.
$$

For $n+1$, by using inequality (3.1), Assumption 1 (a_3), Hölder's inequality and Lemma 2.2 to yield

$$
\begin{aligned}
& \mathbb{E} d_\infty^2 \left[y_{n+1}(\tau), y_n(\tau) \right] \\
\le & 2\mathbb{E} d_\infty^2 \left(\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\eta)^{\beta-1} \left[\mathcal{U}\left(\eta, y_n(\eta), y_n(\eta-\rho)\right), \mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) \right] d\eta \right) \\
&+ 2\mathbb{E} d_\infty^2 \left\langle \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\eta)^{\beta-1} \left[\mathcal{V}\left(\eta, y_n(\eta), y_n(\eta-\rho)\right), \mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right) \right] dW(\eta) \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{2}{\Gamma^2(\beta)}\int_0^\tau (\tau-\eta)^{2\beta-2}d\eta\mathbb{E}\int_0^\tau d_\infty^2\left[\mathcal{U}\left(\eta,y_n(\eta),y_n(\eta-\rho)\right),\mathcal{U}\left(\eta,y_{n-1}(\eta),y_{n-1}(\eta-\rho)\right)\right]d\eta \\
& + \frac{2}{\Gamma^2(\beta)}\mathbb{E}\sup_{0\leq\tau\leq T}d_\infty^2\left\langle\int_0^\tau (\tau-\eta)^{\beta-1}\left[\mathcal{V}\left(\eta,y_n(\eta),y_n(\eta-\rho)\right),\mathcal{V}\left(\eta,y_{n-1}(\eta),y_{n-1}(\eta-\rho)\right)\right]dW(\eta)\right\rangle \\
\leq & \frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2\left[\mathcal{U}\left(\eta,y_n(\eta),y_n(\eta-\rho)\right),\mathcal{U}\left(\eta,y_{n-1}(\eta),y_{n-1}(\eta-\rho)\right)\right]d\eta \\
& + \frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2\left[\mathcal{V}\left(\eta,y_n(\eta),y_n(\eta-\rho)\right),\mathcal{V}(\eta),y_{n-1}(\eta),y_{n-1}(\eta-\rho))\right]d\eta \\
\leq & \frac{4T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T\left(d_\infty^2\left[y_n(\eta),y_{n-1}(\eta)\right]+d_\infty^2\left[y_n(\eta-\rho),y_{n-1}(\eta-\rho)\right]\right)d\eta \\
\leq & \frac{4T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}\left(2d_\infty^2\left[y_n(\eta),y_{n-1}(\eta)\right]+\sup_{0\leq\eta\leq\rho}d_\infty^2\left[y_n(\eta-\rho),y_{n-1}(\eta-\rho)\right]\right)d\eta \\
= & \frac{4T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}\left(2d_\infty^2\left[y_n(\eta),y_{n-1}(\eta)\right]+\sup_{0\leq\eta\leq\rho}d_\infty^2[\mathcal{P}(\eta-\rho),\mathcal{P}(\eta-\rho)]\right)d\eta \\
= & \frac{8T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}d_\infty^2\left[y_n(\eta),y_{n-1}(\eta)\right]d\eta.
\end{aligned}
$$

So we get

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq\tau\leq T}d_\infty^2\left[y_{n+1}(\tau),y_n(\tau)\right] &\leq \frac{8T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}\sup_{0\leq\eta\leq T}\left[y_n(\eta),y_{n-1}(\eta)\right]d\eta \\
&\leq \left(\frac{8T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)^n MT^n \\
&= \left(\frac{8T^{2\beta}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)^n M.
\end{aligned}
$$

Indeed, by using the Chebyshev's inequality and the above inequality to yield

$$
\begin{aligned}
P\left(\sup_{0\leq\tau\leq T}d_\infty^2\left[y_n(\tau),y_{n-1}(\tau)\right]>\frac{1}{4^n}\right) &\leq 4^n\mathbb{E}\sup_{0\leq\tau\leq T}d_\infty^2\left[y_n(\tau),y_{n-1}(\tau)\right] \\
&\leq \left(\frac{32T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)^{n-1}4MT^{n-1} \\
&= 4M\left(\frac{32T^{2\beta}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)^{n-1}.
\end{aligned}
$$

Since the series $\sum_{n=1}^{\infty}\left(\frac{32T^{2\beta}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)^{n-1}<\infty$, by the Borel-Cantelli lemma, we can get

$$
P\left(\sup_{0\leq\tau\leq T}d_\infty^2\left[y_n(\tau),y_{n-1}(\tau)\right]>\frac{1}{2^n}\text{ infinitely often }\right)=0.
$$

Hence similarly as in [14], we obtain that there exists a continuous fuzzy stochastic process y such that $\lim_{n\to\infty} d_\infty\left[y_n(\tau), y(\tau)\right] \overset{P.1}{=} 0, \tau \in [0,T]$. And then we can verify that $\lim_{n\to\infty} \mathbb{E}d_\infty^2\left[y_n(\tau), y(\tau)\right] \overset{P.1}{=} 0, \tau \in [0,T]$.

Then, for $\tau \in [0,T]$ we prove that y satisfies system (1.1). By the same technique as $\mathbb{E}d_\infty^2\left[y_{n+1}(\tau), y_n(\tau)\right]$, it is immediate to obtain

$$\begin{aligned}
&\mathbb{E}d_\infty^2\Bigg[\mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)d\eta\right.\\
&\quad\left.+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)dW(\eta)\right\rangle\right),\\
&\quad\mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\right.\\
&\quad\left.+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\right)\Bigg]\\
=&\mathbb{E}d_\infty^2\Bigg[\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)d\eta\\
&\quad+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)dW(\eta)\right\rangle,\\
&\quad\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\\
&\quad+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\Bigg]\\
\leq&2\mathbb{E}d_\infty^2\Bigg[\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)d\eta,\\
&\quad\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\Bigg]\\
&\quad+2\mathbb{E}d_\infty^2\Bigg[\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)dW(\eta)\right\rangle,\\
&\quad\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\Bigg]\\
\leq&\frac{8T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T \mathbb{E}d_\infty^2\left[y_{n-1}(\eta), y(\eta)\right]d\eta \to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

And since

$$\begin{aligned}
&\mathbb{E}d_\infty^2\Bigg[y(\eta),\ \mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\right.\\
&\quad\left.+\left\langle\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\right)\Bigg]\\
\leq&2\mathbb{E}d_\infty^2\left[y(\eta),\ y_n(\eta)\right]+2\mathbb{E}d_\infty^2\Bigg[\mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\right.
\end{aligned}$$

$$+\left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\Bigg),$$
$$\mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)d\eta\right.$$
$$\left.+\left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y_{n-1}(\eta), y_{n-1}(\eta-\rho)\right)dW(\eta)\right\rangle\right)\Bigg] \to 0, \quad \text{as } n\to\infty,$$

we get

$$\mathbb{E}d_\infty^2\Bigg[y(\eta),\ \mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\right.$$
$$\left.+\left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\right)\Bigg] = 0,$$

which means that

$$\mathbb{E}d_\infty\Bigg[y(\eta),\ \mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right)d\eta\right.$$
$$\left.+\left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right)dW(\eta)\right\rangle\right)\Bigg] \overset{P.1}{=} 0,\ \eta\in[0,\ T],$$

where the fuzzy function $\tau\mapsto \Im_{0+}^{\beta}\mathcal{W}(\tau)$ is d-increasing.

Step 2. We prove that $y\in[y^\lambda,\ y^\omega]$. For $\forall\varepsilon>0$, we consider

$$y_\varepsilon^\omega(\tau) = y^\omega(\tau) + \varepsilon(\rho+\tau),$$

and

$$y_\varepsilon^\lambda(\tau) + \varepsilon(\rho+\tau) = y^\lambda(\tau).$$

Then, we have

$$y_\varepsilon^\omega(\tau) \succ y^\omega(\tau),\ \tau\in[-\rho, T],$$

and

$$y_\varepsilon^\lambda(\tau) \prec y^\lambda(\tau), \tau\in[-\rho, T].$$

Thus, we can get

$$\begin{cases} y_\varepsilon^\lambda(\tau)\preceq y^\lambda(\tau)\preceq y^\omega(\tau)\preceq y_\varepsilon^\omega(\tau), t\in[-\rho,0],\\ y_\varepsilon^\lambda(\tau)\preceq y^\lambda(\tau)\preceq y^\omega(\tau)\preceq y_\varepsilon^\omega(\tau), t\in[0,T],\end{cases}$$

and

$$y_\varepsilon^\lambda(0)\preceq y^\lambda(0)\preceq y^\omega(0)\preceq y_\varepsilon^\omega(0),$$

where $y^{\lambda}(\tau),\ y^{\omega}(\tau)$ are the lower and upper solutions of the Eq. (1.1).

Hence, we derive that

$$y_{\varepsilon}^{\lambda}(\tau) \preceq y^{\lambda}(\tau) \preceq y(\tau) \preceq y^{\omega}(\tau) \preceq y_{\varepsilon}^{\omega}(\tau), \tau \in [-\rho, 0],$$

and

$$y_{\varepsilon}^{\lambda}(0) \preceq y(0) \preceq y_{\varepsilon}^{\omega}(0),$$

where $y(\tau)$ is a solution of the Eq. (1.1).

Next, we need to prove that

$$y_{\varepsilon}^{\lambda}(\tau) \prec y(\tau) \prec y_{\varepsilon}^{\omega}(\tau),\ \tau \in [0, T].$$

If the above assertion does not hold, then there exists $\tau_1 \in (0, T)$ such that

$$y\left(\tau_1\right) = y_{\varepsilon}^{\lambda}\left(\tau_1\right), \quad y_{\varepsilon}^{\lambda}(\tau) \prec y(\tau) \prec y_{\varepsilon}^{\omega}(\tau),\ \tau \in [0,\ T] \setminus \left\{\tau_1\right\}. \tag{3.3}$$

Therefore, based on Lemma 2.2, we infer that ${}^{C}\mathscr{D}_{0+}^{\beta} y\left(\tau_1\right) \succeq {}^{C}\mathscr{D}_{0+}^{\beta} y_{\varepsilon}^{\omega}\left(\tau_1\right)$, which yields

$$\begin{aligned}
{}^{C}\mathscr{D}_{0+}^{\beta} y_{\varepsilon}^{\omega}\left(\tau_1\right) &= \mathcal{U}(\tau_1,\ y_{\varepsilon}^{\omega}\left(\tau_1\right),\ y_{\varepsilon}^{\omega}(\tau_1 - \rho)) + \mathcal{V}(\tau_1,\ y_{\varepsilon}^{\omega}\left(\tau_1\right),\ y_{\varepsilon}^{\omega}(\tau_1 - \rho)) \frac{dW(\tau_1)}{d\tau_1} \\
&\preceq {}^{C}\mathscr{D}_{0+}^{\beta} y\left(\tau_1\right) \\
&= \mathcal{U}(\tau_1,\ y\left(\tau_1\right),\ y(\tau_1 - \rho)) + \mathcal{V}(\tau_1,\ y\left(\tau_1\right),\ y(\tau_1 - \rho)) \frac{dW(\tau_1)}{d\tau_1}.
\end{aligned}$$

Moreover, from $y(\tau) \preceq y_{\varepsilon}^{\omega}(\tau),\ \tau \in [-\rho, 0]$ and (3.3), we can conclude that

$$y(\tau_1 + \eta) \prec y_{\varepsilon}^{\omega}(\tau_1 + \eta),\ \eta \in [-\rho, 0].$$

Given the nondecreasing property of the functions $\mathcal{U}(\tau,\ y\left(\tau\right),\ y(\tau - \rho))$, $\mathcal{V}(\tau,\ y\left(\tau\right),\ y(\tau - \rho))$ in y, it is easy to get

$$\begin{aligned}
&\mathcal{U}(\tau_1,\ y_{\varepsilon}^{\omega}\left(\tau_1\right),\ y_{\varepsilon}^{\omega}(\tau_1 - \rho)) + \mathcal{V}(\tau_1,\ y_{\varepsilon}^{\omega}\left(\tau_1\right),\ y_{\varepsilon}^{\omega}(\tau_1 - \rho)) \frac{dW(\tau_1)}{d\tau_1} \\
&\succeq \mathcal{U}(\tau_1,\ y\left(\tau_1\right),\ y(\tau_1 - \rho)) + \mathcal{V}(\tau_1,\ y\left(\tau_1\right),\ y(\tau_1 - \rho)) \frac{dW(\tau_1)}{d\tau_1},
\end{aligned}$$

which is a contradiction, and then we know that $y(\tau) \prec y_{\varepsilon}^{\omega}(\tau),\ \tau \in [0, T]$. Similarly, it can be proved that $y_{\varepsilon}^{\lambda}(\tau) \prec y(\tau),\ \tau \in [0, T]$. Then $y_{\varepsilon}^{\lambda}(\tau) \prec y(\tau) \prec y_{\varepsilon}^{\omega}(\tau),\ \tau \in [0, T]$ holds. Let $\varepsilon \to 0$, we deduce that $y_{\varepsilon}^{\lambda}(\tau) \preceq y(\tau) \preceq y_{\varepsilon}^{\omega}(\tau)$.

Step 3. We shall show the uniqueness of the solution. Suppose y and $\tilde{y}$ are two different solutions of (1.1). For all $\tau \in [-\rho, 0]$, $\tilde{y}(\tau) = \mathscr{P}(\tau)$. For all $\tau \in [0, T]$, by the same technique as Step 2, we have

$$\begin{aligned}
&\mathbb{E}d_\infty^2 \left[y(\tau), \tilde{y}(\tau)\right] \\
\leq &\frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2 \left[\mathcal{U}\left(\eta, y(\eta), y(\eta-\rho)\right), \mathcal{U}\left(\eta, \tilde{y}(\eta), \tilde{y}(\eta-\rho)\right)\right] d\eta \\
&+\frac{2T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2 \left[\mathcal{V}\left(\eta, y(\eta), y(\eta-\rho)\right), \mathcal{V}(\eta), \tilde{y}(\eta), \tilde{y}(\eta-\rho))\right] d\eta \\
\leq &\frac{4T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T \left(d_\infty^2 \left[y(\eta), y_{n-1}(\eta)\right] + d_\infty^2 \left[y(\eta-\rho), \tilde{y}(\eta-\rho)\right]\right) d\eta \\
\leq &\frac{4T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T \mathbb{E}\left(2d_\infty^2 \left[y(\eta), \tilde{y}(\eta)\right] + \sup_{0\leq\eta\leq\rho} d_\infty^2 \left[y(\eta-\rho), \tilde{y}(\eta-\rho)\right]\right) d\eta \\
= &\frac{8T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T \mathbb{E}d_\infty^2 \left[y(\eta), \tilde{y}(\eta)\right] d\eta,
\end{aligned}$$

which, with the aid of Grönwall-Bellman inequality, yields

$$\mathbb{E}d_\infty^2 \left[y(\tau), \tilde{y}(\tau)\right] \overset{P.1}{=} 0.$$

Therefore, the solution of (1.1) is unique. The proof is complete.

4. Ulam-Hyers stability results

Let $\varepsilon > 0$, we consider:

$$\begin{cases} \mathbb{E}d_\infty^2\left[{}^C\mathscr{D}_{0^+}^\beta \hbar(\tau),\ \mathcal{U}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho)) + \left\langle \mathcal{V}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho))\dfrac{dW(\tau)}{d\tau}\right\rangle\right] \leq \varepsilon,\ \tau \in [0,\ T], \\ \hbar(\tau) = \mathscr{P}(\tau),\ \tau \in [-\rho,\ 0]. \end{cases} \tag{4.1}$$

Definition 4.1. *([28]) If there is a real number $\delta > 0$, such that for $\forall\varepsilon > 0$ and for each solution $\hbar \in AC\left([-\rho,\ T],\ \mathscr{F}(\mathbb{R})\right)$ of the (4.1), there exists a solution $y \in AC\left([-\rho,\ T],\ \mathscr{F}(\mathbb{R})\right)$ of Eq. (1.1) with*

$$\mathbb{E}d_\infty^2[y(\tau),\ \hbar(\tau)] \leq \varepsilon\delta,$$

then the solution to Eq. (1.1) is U-H stable.

Remark 4.1. *An d-monotone function* $\hbar \in AC([-\rho,\ T],\ \mathscr{F}(\mathbb{R}))$ *is a solution of* (4.1) *if and only if there is* $g \in AC([-\rho,\ T],\ \mathscr{F}(\mathbb{R}))$ *such that*

(1) $\mathbb{E}d_\infty^2[g(\tau),\ \hat{0}] \le \varepsilon$;

(2) $$\begin{cases} {}^C\mathscr{D}_{0+}^{\beta}\hbar(\tau) = \mathcal{U}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho)) + \left\langle \mathcal{V}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho))\dfrac{dW(\tau)}{d\tau} \right\rangle + g(\tau),\ \tau \in [0,\ T], \\ \hbar(\tau) = \mathscr{P}(\tau),\ \tau \in [-\rho,\ 0]. \end{cases}$$

Theorem 4.1. *Under the* ***Assumption 1***(a_3)*, the solution to Eq.* (1.1) *is U-H stable.*

Proof. Taking remark 4.1, Definition 2.10 and remark 2.1 into account, we can derive

$$\begin{aligned} \hbar(\tau) =& \mathscr{P}_0\Theta\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}(\eta,\hbar(\eta),\hbar(\eta-\rho))d\eta \right. \\ &+ \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}g(\eta)d\eta \\ &\left. + \left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}(\eta,\hbar(\eta),\hbar(\eta-\rho))dW(\eta) \right\rangle \right), \end{aligned}$$

and $\tau \mapsto \Im_{0+}^{\beta}\tilde{\mathscr{W}}(\tau)$ is d-increasing on $[0,\ T]$, where $\tilde{\mathscr{W}}(\tau) = \mathcal{U}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho)) + \left\langle \mathcal{V}(\tau,\ \hbar(\tau),\ \hbar(\tau-\rho))\frac{dW(\tau)}{d\tau} \right\rangle + g(\tau)$ and $\Theta := \{+, \ominus(-1)\}$.

Then, employing inequality (3.1), Assumption 1, Hölder's inequality and Lemma 2.2, we attain

$$\begin{aligned} &\mathbb{E}d_\infty^2[y(\tau),\hbar(\tau)] \\ =&\mathbb{E}d_\infty^2\left[\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}(\eta,y(\eta),y(\eta-\rho))d\eta \right. \\ &+ \left\langle \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}(\eta,y(\eta),y(\eta-\rho))dW(\eta) \right\rangle, \\ &\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{U}(\eta,\hbar(\eta),\hbar(\eta-\rho))d\eta \\ &+ \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}g(\eta)d\eta \\ &\left. +\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\eta)^{\beta-1}\mathcal{V}(\eta,\hbar(\eta),\hbar(\eta-\rho))d\eta\right] \\ \le&\frac{3}{\Gamma^2(\beta)}\mathbb{E}d_\infty^2\left[\int_0^\tau (\tau-\eta)^{\beta-1}[\mathcal{U}(\eta,\hbar(\eta),\hbar(\eta-\rho)),\mathcal{U}(\eta,h(\eta),\hbar(\eta-\rho))]d\eta\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{\Gamma^2(\beta)}\mathbb{E}d_\infty^2\left[\int_0^\tau(\tau-\eta)^{\beta-1}[g(\eta),\langle 0\rangle]d\eta\right]\\
&+\frac{3}{\Gamma^2(\beta)}\mathbb{E}d_\infty^2\left\langle\int_0^\tau(\tau-\eta)^{\beta-1}[\mathcal{V}(\eta,y(\eta),y(\eta-\rho)),\mathcal{V}(\eta,\hbar(\eta),\hbar(\eta-\rho))]dW(\eta)\right\rangle\\
\leq&\frac{3}{\Gamma^2(\beta)}\int_0^\tau(\tau-\eta)^{2\beta-2}d\eta\mathbb{E}\int_0^\tau d_\infty^2[\mathcal{U}(\eta,y(\eta,y(\eta-\rho)),\mathcal{U}(\eta,\hbar(\eta),\hbar(\eta-\rho))]d\eta\\
&+\frac{3}{\Gamma^2(\beta)}\int_0^\tau(\tau-\eta)^{2\beta-2}d\eta\mathbb{E}\int_0^\tau d_\infty^2[g(\eta),\langle 0\rangle]d\eta\\
&+\frac{3}{\Gamma^2(\beta)}\mathbb{E}\sup_{0\leq\tau\leq T}d_\infty^2\left\langle\int_0^\tau(\tau-\eta)^{\beta-1}[\mathcal{V}(\eta,y(\eta),y(\eta-\rho)),\mathcal{V}(\eta,\hbar(\eta),\hbar(\eta-\rho))]dW(\eta)\right\rangle\\
\leq&\frac{3T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2[\mathcal{U}(\eta,y(\eta),y(\eta-\rho)),\mathcal{U}(\eta,\hbar(\eta),\hbar(\eta-\rho))]d\eta\\
&+\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\varepsilon\\
&+\frac{3T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T d_\infty^2[\mathcal{V}(\eta,y(\eta),y(\eta-\rho)),\mathcal{V}(\eta,\hbar(\eta),\hbar(\eta-\rho))]d\eta\\
\leq&\frac{6T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\mathbb{E}\int_0^T\left(d_\infty^2[y(\eta),\hbar(\eta)]+d_\infty^2[y(\eta-\rho),\hbar(\eta-\rho)]\right)d\eta\\
&+\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\varepsilon\\
\leq&\frac{6T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}\left(2d_\infty^2[y(\eta),\hbar(\eta)]+\sup_{0\leq\eta\leq\rho}d_\infty^2[y(\eta-\rho),\hbar(\eta-\rho)]\right)d\eta\\
&+\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\varepsilon\\
=&\frac{6T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}\left(2d_\infty^2[y(\eta),\hbar(\eta)]+\sup_{0\leq\eta\leq\rho}d_\infty^2[\mathscr{P}(\eta-\rho),\mathscr{P}(\eta-\rho)]\right)d\eta\\
&+\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\varepsilon\\
=&\frac{12T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}\int_0^T\mathbb{E}d_\infty^2[y(\eta),\hbar(\eta)]d\eta+\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\varepsilon.
\end{aligned}
$$

In terms of Grönwall-Bellman inequality, we get

$$
\begin{aligned}
\mathbb{E}d_\infty^2[y(\tau),\hbar(\tau)]&\leq\frac{3T^{2\beta}\varepsilon}{\Gamma^2(\beta)(2\beta-1)}\exp\left(\int_0^T\frac{12T^{2\beta-1}N_3}{\Gamma^2(\beta)(2\beta-1)}d\eta\right)\\
&=\frac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)}\exp\left(\frac{12T^{2\beta}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)\varepsilon.
\end{aligned}
$$

Let $\delta = \dfrac{3T^{2\beta}}{\Gamma^2(\beta)(2\beta-1)} \exp\left(\dfrac{12T^{2\beta}N_3}{\Gamma^2(\beta)(2\beta-1)}\right)$, we have

$$\mathbb{E}d_\infty^2[y(\tau), \hbar(\tau)] \le \varepsilon\delta.$$

Therefore, this Theorem is proved.

5. Examples

We present three examples to support the U-H stability theory, in the section.

Example 1 Consider the following FFSDEs on $[0, 8]$:

$$\begin{cases} {}^{C}\mathscr{D}_{0+}^{\frac{2}{3}}y(\tau) = \dfrac{1}{50}y(\tau) + \dfrac{1}{15}y(\tau-1) + (-4,\ 0,\ 4) \\ \qquad\qquad + \left\langle \dfrac{\sin(\tau+\tau^2)}{\pi^3} \dfrac{dW(\tau)}{d\tau} \right\rangle, \tau \in [0, 8], \\ y(\tau) = \left(-\tau^2 - 20,\ 0,\ 2\tau - \tau^2 + 20\right), \tau \in [-1, 0], \end{cases} \tag{5.1}$$

where

$$\mathcal{U}(\tau, y(\tau), y(\tau-1)) = \frac{1}{50}y(\tau) + \frac{1}{15}y(\tau-1) + (-4,\ 0,\ 4),$$
$$\mathcal{V}(\tau, y(\tau), y(\tau-1)) = \frac{\sin(\tau+\tau^2)}{\pi^3}.$$

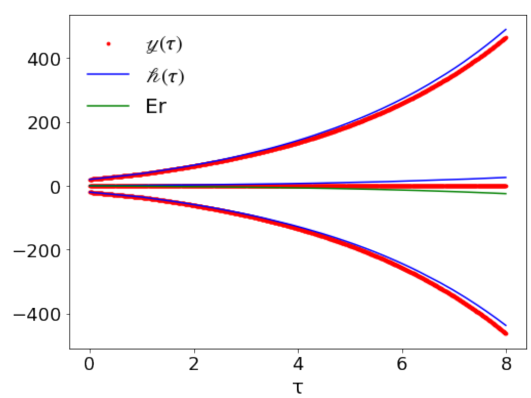

Figure 1: The U-H stability of the solution to Eq. (5.1) with $\varepsilon = 0.3$.

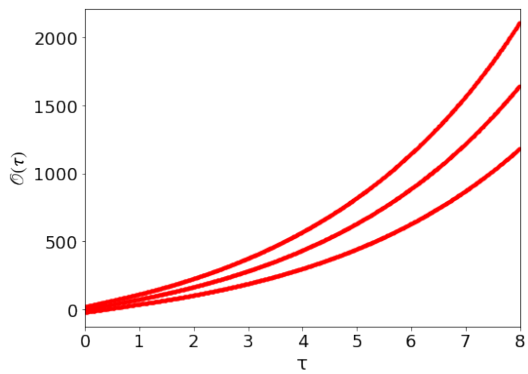

Figure 2: The d-monotone of the fuzzy function determined by Eq. (5.1).

From Figure 1, we can see that the solution y of Eq. (5.1) is d-increasing and from Figure 2 we know that the fuzzy function $\mathcal{O}$ is d-increasing, where the fuzzy function $\tau \mapsto \mathcal{O}(\tau) = \Im_{0+}^{\beta} \mathcal{W}(\tau)$, and $\mathcal{W}(\tau) = \mathcal{U}(\tau,\ y(\tau),\ y(\tau-1)) + \mathcal{V}(\tau,\ y(\tau),\ y(\tau-1))\dfrac{dW(\tau)}{d\tau}$.

Also, it is easy to verify that for $y,\ \hbar \in \mathcal{F}(\mathbb{R})$

$$d_\infty^2[\mathcal{U}(\tau, y(\tau), y(\tau-1)), \mathcal{U}(\tau, \hbar(\tau), \hbar(\tau-1))] \leq \frac{3}{15^2}\left(d_\infty^2[y(\tau), \hbar(\tau)] + d_\infty^2[y(\tau-1), \hbar(\tau-1)]\right),$$

$$d_\infty^2[\mathcal{V}(\tau, y(\tau), y(\tau-1)), \mathcal{V}(\tau, \hbar(\tau), \hbar(\tau-1))] \leq 0\left(d_\infty^2[y(\tau), \hbar(\tau)] + d_\infty^2[y(\tau-1), \hbar(\tau-1)]\right),$$

then

$$\begin{aligned} & d_\infty^2[\mathcal{U}(\tau, y(\tau), y(\tau-1)), \mathcal{U}(\tau, \hbar(\tau), \hbar(\tau-1))] \vee d_\infty^2[\mathcal{V}(\tau, y(\tau), y(\tau-1)), \mathcal{V}(\tau, \hbar(\tau), \hbar(\tau-1))] \\ \leq & \frac{3}{15^2}\left(d_\infty^2[y(\tau), \hbar(\tau)] + d_\infty^2[y(\tau-1), \hbar(\tau-1)]\right). \end{aligned}$$

Therefore, the condition of Theorem 4.1 holds, system (5.1) is U-H stable. And Figure 1 also shows that system (5.1) is U-H stable.

Example 2 Consider the following FFSDEs:

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\frac{5}{6}} y(\tau) = \dfrac{(-1)}{3.5\pi}\sin(y(\tau) + y(\tau-1)) + \dfrac{(-1)}{11} y(\tau) + (-1)\left(-\dfrac{1}{3}\tau, 0, \dfrac{1}{3}\tau\right) \\ \qquad + \left\langle \dfrac{(-1)}{20}\sin(2\tau)\dfrac{dW(\tau)}{d\tau} \right\rangle, \tau \in [0, 8], \\ y(\tau) = \left(\tau^{\frac{1}{2}} - 100,\ 2\tau^{\frac{1}{2}} + 30,\ 3\tau^{\frac{1}{2}} + 100\right),\ \tau \in [-1,\ 0], \end{cases} \tag{5.2}$$

where

$$\mathcal{U}(\tau, y(\tau), y(\tau-1)) = \frac{(-1)}{3.5\pi}\sin(y(\tau)+y(\tau-1)) + \frac{(-1)}{11}y(\tau) + (-1)\left(-\frac{1}{3}\tau, 0, \frac{1}{3}\tau\right),$$

$$\mathcal{V}(\tau, y(\tau), y(\tau-1)) = \frac{(-1)}{20}\sin(2\tau).$$

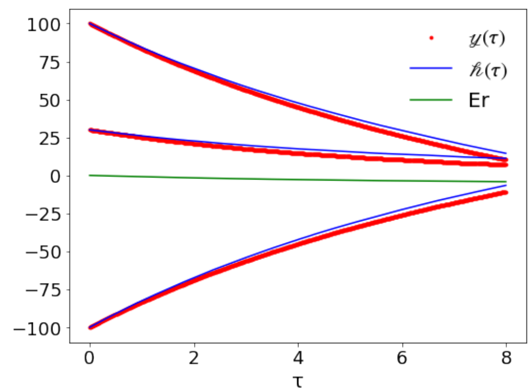

Figure 3: The U-H stability of the solution to Eq. (5.2) with $\varepsilon = 0.5$.

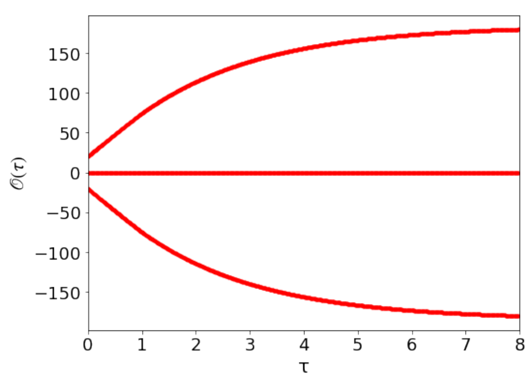

Figure 4: The d-monotone of the fuzzy function determined by Eq. (5.2).

It is easy to see from Figure 3 that the solution of system (5.2) is d-decreasing. And it is not difficult to find from Figure 4 that the fuzzy function $\mathcal{O}$ is d-increasing, where $\mathcal{O}(\tau) = \Im^{\beta}_{0+}\mathcal{W}(\tau)$, and $\mathcal{W}(\tau) = \mathcal{U}(\tau,\ y(\tau),\ y(\tau-1)) + \left\langle \mathcal{V}(\tau,\ y(\tau),\ y(\tau-1))\frac{dW(\tau)}{d\tau}\right\rangle$.

For $y,\ \hbar \in \mathscr{F}(\mathbb{R})$, we can easily verify that

$$
\begin{aligned}
& d_\infty^2[\mathcal{U}(\tau, y(\tau), y(\tau-1)), \mathcal{U}(\tau, \hbar(\tau), \hbar(\tau-1))] \vee d_\infty^2[\mathcal{V}(\tau, y(\tau), y(\tau-1)), \mathcal{V}(\tau, \hbar(\tau), \hbar(\tau-1))] \\
\leq & \frac{9}{(3.5\pi)^2}\left(d_\infty^2[y(\tau), \hbar(\tau)] + d_\infty^2[y(\tau-1), \hbar(\tau-1)]\right).
\end{aligned}
$$

Hence, from Theorem 4.1, Eq. (5.2) is U-H stable on [0, 8]. It is not difficult to see from Figure 3 that Eq. (5.2) is U-H stable.

Example 3 Consider the following FFSDEs:

$$
\begin{cases}
{}^C\mathscr{D}_{0^+}^{\frac{2}{3}} y(\tau) = & \frac{(-1)}{50} y(\tau) + \frac{(-1)}{20} y(\tau-1) + (-1)(-4, 0, 4) \\
& + \left\langle \frac{(-1)}{\pi^4} \sin(3\tau) \frac{dW(\tau)}{d\tau} \right\rangle, \tau \in [0, 8], \\
y(\tau) = \left(-\tau^2 - 20,\ 0,\ \tau^2 + 20\right), & \tau \in [-1, 0],
\end{cases}
\tag{5.3}
$$

where

$$
\begin{aligned}
\mathcal{U}(\tau, y(\tau), y(\tau-1)) &= \frac{(-1)}{50} y(\tau) + \frac{(-1)}{20} y(\tau-1) + (-1)(-4, 0, 4), \\
\mathcal{V}(\tau, y(\tau), y(\tau-1)) &= \frac{(-1)}{\pi^4} \sin(3\tau).
\end{aligned}
$$

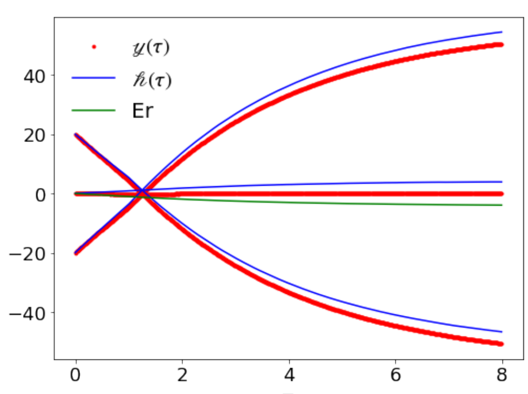

Figure 5: The U-H stability of the solution to Eq. (5.3) with $\varepsilon = 0.3$.

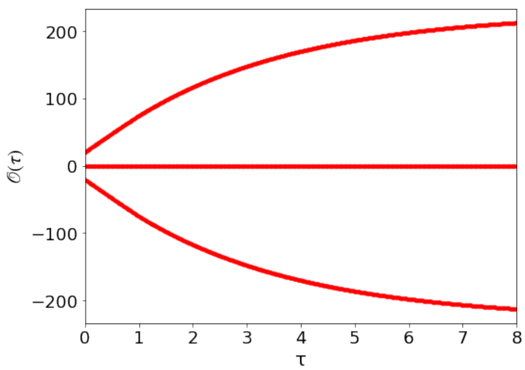

Figure 6: The d-monotone of the fuzzy function determined by Eq. (5.3).

Figure 5 shows that the solution of Eq. (5.3) is d-monotone, and Figure 6 implies that $\mathscr{O}$ is d-increasing, where $\mathscr{O}(\tau) = \Im_{0+}^{\beta}\mathscr{W}(\tau)$, and $\mathscr{W}(\tau) = \mathscr{U}(\tau,\ y(\tau),\ y(\tau-1)) + \left\langle \mathscr{V}(\tau,\ y(\tau),\ y(\tau-1))\dfrac{dW(\tau)}{d\tau} \right\rangle$.

Obviously, for $y,\ \hbar \in \mathscr{F}(\mathbb{R})$ we can verify that

$$
\begin{aligned}
&d_{\infty}^{2}[\mathscr{U}(\tau,y(\tau),y(\tau-1)),\mathscr{U}(\tau,\hbar(\tau),\hbar(\tau-1))] \vee d_{\infty}^{2}[\mathscr{V}(\tau,y(\tau),y(\tau-1)),\mathscr{V}(\tau,\hbar(\tau),\hbar(\tau-1))]\\
\leq& \frac{3}{20^2}\left(d_{\infty}^{2}[y(\tau),\hbar(\tau)] + d_{\infty}^{2}[y(\tau-1),\hbar(\tau-1)]\right),
\end{aligned}
$$

which means that Eq. (5.3) satisfies the conditions of Theorem 4.1, so Eq. (5.3) is U-H stable on [0, 8]. Similarly, we can also know from Figure 5 that Eq. (5.3) is U-H stable on [0, 8].

6. Conclusion

This article dealt with a FFSDEs with delays in the sense of generalized Hukuhara differentiability. And the existence of the solutions to FFSDEs was proved by the upper and lower solution method, and then its uniqueness was proved by means of contradictory method. Next, we explored the U-H stability of FFSDEs with the help of Hölder's inequality and Grönwall-Bellman inequality. Finally, we demonstrate the validity of the proposed theory through three examples with numerical simulations.

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