ASYMPTOTIC STABILITY OF EVOLUTION SYSTEMS OF PROBABILITY MEASURES FOR NONAUTONOMOUS STOCHASTIC SYSTEMS: THEORETICAL RESULTS AND APPLICATIONS

RENHAI WANG, TOMÁS CARABALLO, AND NGUYEN HUY TUAN*

Abstract. The limiting stability of invariant probability measures of time homogeneous transition semigroups for autonomous stochastic systems has been extensively discussed in the literature. In this paper we initially initiate a program to study the asymptotic stability of evolution systems of probability measures of time inhomogeneous transition operators for nonautonomous stochastic systems. Two general theoretical results on this topic are established in a Polish space by establishing some sufficient conditions which can be verified in applications. Our abstract results are applied to a stochastic lattice reaction-diffusion equation driven by a time-dependent nonlinear noise. A time-average argument and an extended Krylov-Bogolyubov method due to Da Prato and Röckner [11] are employed to prove the existence of evolution systems of probability measures. A mild condition on the time-dependent diffusion function is used to prove that the limit of every evolution system of probability measures must be an evolution system of probability measures of the limiting equation. The theoretical results are expected to be applied to various stochastic lattice systems/ODEs/PDEs in the future.

1. Introduction

1.1. **Statement of problems.** A basic approach to look at the asymptotic stability of stochastic systems with noise perturbations is to consider the limiting stability of their invariant probability measures with respect to the noise intensity. Recently, this kind of the limiting stability of invariant probability measures of time homogeneous transition semigroups was discussed in the literature for *autonomous* stochastic lattice systems/ODEs/PDEs, see e.g., [5, 6, 7, 14, 15]. As far as the authors can find, currently, there are no results reported in the literature on the limiting stability of an *evolution system of probability measures* (an extension of invariant measures from autonomous to nonautonomous developed by Da Prato and Röckner [10, 11]) of time inhomogeneous transition operators for *nonautonomous* stochastic systems.

1.2. **General framework and theoretical results.** The goal of the present work is to initiate a program of studying limiting stability of evolution systems of probability measures of time inhomogeneous transition operators. We will establish a general setting on the limiting stability of evolution system of probability measures for an abstract time inhomogeneous transition operator on a Polish space. More specifically, we will show, under certain conditions, that the limit of every evolution system of probability measures (if exists) must be an evolution system of probability measures of the limiting transition operator.

2010 *Mathematics Subject Classification.* 37L30,37L40,37L55, 35B40,35B41,60H10.

Key words and phrases. Invariant measure; tightness; Limit measure; nonlinear noise; lattice system.

*Corresponding author: Nguyen Huy Tuan (E-mail: nguyenhuytuan@vlu.edu.vn).

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Polish space, $\mathcal{P}(\mathbb{X})$ be the set of all probability measures on $\mathbb{X}$, $\mathscr{B}(\mathbb{X})$ be the Borel σ-algebra on $\mathbb{X}$, and $B_b(\mathbb{X})$ ($C_b(\mathbb{X})$) the space of all bounded Borel (continuous) functions on $\mathbb{X}$. Assume $\{X^\epsilon(t,\tau,x), t \geqslant \tau \in \mathbb{R}\}$ on $\mathbb{X}$ is a unique noise driven stochastic process with initial value $x \in \mathbb{X}$ and the noise intensity $\epsilon \in (0, \dot{\epsilon})$ for any $\dot{\epsilon} > 0$. For $\varphi \in B_b(\mathbb{X})$, $\Lambda \in \mathscr{B}(\mathbb{X})$ and $\eta \in \mathcal{P}(\mathbb{X})$, we define a transition operator $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ by $(P^\epsilon_{\tau,t}\varphi)(x) = \mathbb{E}[\varphi(X^\epsilon(t,\tau,x))]$, a transition probability function $P^\epsilon_{\tau,t}(x,\Lambda) = \mathbb{P}\{\omega \in \Omega : X^\epsilon(t,\tau,x) \in \Lambda\}$ and the adjoint operator $(Q^\epsilon_{\tau,t})_{t\geqslant\tau}$ of $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ by $Q^\epsilon_{\tau,t}\eta(\Lambda) = \int_{\mathbb{X}} P^\epsilon_{\tau,t}(x,\Lambda)\eta(dx)$. As in Da Prato and Röckner [10, 11], we say $\{\eta^\epsilon_t\}_{t\in\mathbb{R}} \subseteq \mathcal{P}(\mathbb{X})$ is an *evolution system of probability measures* of $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ if $Q^\epsilon_{\tau,t}\eta_\tau = \eta_t$ for all $t \geqslant \tau \in \mathbb{R}$.

For some technical reasons, the following assumption is needed when we discuss the limiting stability of evolution systems of probability measures of time inhomogeneous transition operators.

CIP(`Convergence in Probability`). For each compact set $\mathbf{K} \subseteq \mathbb{X}$, $\tau \in \mathbb{R}$, $t \geqslant \tau$, $\epsilon_0 \in (0, \dot{\epsilon}]$ and $\delta > 0$,

$$\lim_{\epsilon\to\epsilon_0} \sup_{x\in\mathbf{K}} \mathbb{P}\big(\big\{\omega \in \Omega : \|X^\epsilon(t,\tau,x) - X^{\epsilon_0}(t,\tau,x)\|_{\mathbb{X}} \geqslant \delta\big\}\big) = 0. \tag{1.1}$$

Theorem 1.1. *(**Theoretical result I**) Assume **CIP** holds and $(P^{\epsilon_0}_{\tau,t})_{t\geqslant\tau}$ is Feller. Let $\{\eta^{\epsilon_0}_t\}_{t\in\mathbb{R}}$ be a family of probability measures on $\mathbb{X}$ for $\epsilon_0 \in [0, \dot{\epsilon}]$, and $\{\eta^{\epsilon_n}_t\}_{t\in\mathbb{R}}$ be an evolution system of probability measures of $(P^{\epsilon_n}_{\tau,t})_{t\geqslant\tau}$ on $\mathbb{X}$ with $\epsilon_n \to \epsilon_0$. If $\eta^{\epsilon_n}_t \to \eta^{\epsilon_0}_t$ weakly for each $t \in \mathbb{R}$, then $\{\eta^{\epsilon_0}_t\}_{t\in\mathbb{R}}$ must be an evolution system of probability measures of $(P^{\epsilon_0}_{\tau,t})_{t\geqslant\tau}$.*

Remark 1.2. *(i) We only require $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ is Feller at $\epsilon = \epsilon_0$. (ii) The convergence in probability in* (1.1) *is not necessary to be uniform for t. But for most stochastic systems,* (1.1) *can be proved uniformly for t:* $\lim_{\epsilon\to\epsilon_0} \sup_{x\in\mathbf{K}} \mathbb{P}\big(\big\{\omega \in \Omega : \sup_{t\in[\tau,\tau+T]} \|X^\epsilon(t,\tau,x) - X^{\epsilon_0}(t,\tau,x)\|_{\mathbb{X}} \geqslant \delta\big\}\big) = 0$.

In applications, ϵ_n in Theorem 1.1 may depend on t, we also provide the following results.

Theorem 1.3. *(**Theoretical result II**) Assume **CIP** holds and $(P^{\epsilon_0}_{\tau,t})_{t\geqslant\tau}$ is Feller. Let $\{\eta^{\epsilon_0}_t\}_{t\in\mathbb{R}}$ be a family of probability measures on $\mathbb{X}$ for $\epsilon_0 \in [0, \dot{\epsilon}]$, and $\{\eta^{\epsilon_n}_t\}_{t\in\mathbb{R}}$ be an evolution system of probability measures of $(P^{\epsilon_n}_{\tau,t})_{t\geqslant\tau}$ on $\mathbb{X}$ with $\epsilon_n \to \epsilon_0$. For each $t \in \mathbb{R}$, if there exists a subsequence $\{\epsilon_{n_k}(t)\}_{k=1}^\infty$ of $\{\epsilon_n\}_{n=1}^\infty$ such that $\eta_t^{\epsilon_{n_k}(t)} \to \eta^{\epsilon_0}_t$ weakly, then $\{\eta^{\epsilon_0}_t\}_{t\in\mathbb{R}}$ must be an evolution system of probability measures of $(P^{\epsilon_0}_{\tau,t})_{t\geqslant\tau}$.*

Theorems 1.1 and 1.3 can be viewed as extension versions of limiting stability of invariant measures of time homogeneous transition operators from the autonomous framework to the nonautonomous setting.

1.3. **Application of theoretical results.** Our abstract results in Theorems 1.1 and 1.3 are expected to be applied to various stochastic ODEs/PDEs/lattice systems with noise perturbations. In particular, we apply these abstract results to a stochastic lattice reaction-diffusion equation driven by a *nonlinear* noise on $\mathbb{Z}$ for $t > \tau$ with $\tau \in \mathbb{R}$:

$$du^\epsilon_i(t) + \lambda u^\epsilon_i(t)dt - \nu\big(u^\epsilon_i(t) - 2u^\epsilon_{i-1}(t) + u^\epsilon_{i+1}(t)\big)dt + |u^\epsilon_i(t)|^{p-2}|u^\epsilon_i(t)|dt = \epsilon\sigma(t, u^\epsilon_i(t))dW(t), \tag{1.2}$$

with initial condition $u_i^\epsilon(\tau) = u_{0,i}$, where $\lambda, \nu > 0$, $p > 2$, $\epsilon \geqslant 0$, W is a two-sided, real-valued Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ the complete filtered probability space, and the nonlinear function $\sigma(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz such that $|\sigma(t,s)| \leqslant \delta|s| + g(t)$ for a constant $\delta > 0$ and a time-dependent deterministic function g satisfying certain conditions. We say the noise in (1.2) is a time-dependent nonlinear noise just because the diffusion function σ depends on time t, and is nonlinear in the unknown function u_i^ϵ.

Note that lattice systems can be regarded as space discretization versions of PDEs that have many applications in the real world such as electric circuits, pattern formation, propagation of nerve pulse, chemical reaction and others. The existence and limiting stability of invariant probability measures for *autonomous* stochastic lattice systems has been considered recently in [5, 6, 14, 15, 16]. The reader is referred to [1, 2, 3, 4, 12, 13, 18, 19] for other mathematical topics such as random attractors for stochastic lattice systems. In this paper we study existence and limiting stability of evolution systems of probability measures for the *nonautonomous* stochastic lattice system (1.2). To our knowledge, it seems that this is the first time to study evolution systems of probability measures for stochastic lattice systems.

Theorem 1.4. ***(Existence of evolution systems of probability measures)*** *If $\int_{-\infty}^0 e^{\lambda r}\|g(r)\|^2 dr < \infty$, then the transition operator $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ for (1.2) has an evolution system of probability measures $\{\mu_t^\epsilon\}_{t\in\mathbb{R}}$ on ℓ^2 for any $\epsilon \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$.*

Let $\mathcal{E}_t^\epsilon = \big\{\mu_t^\epsilon : \{\mu_t^\epsilon\}_{t\in\mathbb{R}}$ is an evolution system of probability measures of $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$ for (1.2)$\big\}$ for $t \in \mathbb{R}$ and $\epsilon \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$. Then we discuss the tightness of $\bigcup_{\epsilon\in[0,\frac{\sqrt{\lambda}}{2\delta}]} \mathcal{E}_t^\epsilon$ and limiting stability of any sequences of $\mathcal{E}_t^\epsilon$.

Theorem 1.5. ***(Application of theoretical results)*** *If $\int_{-\infty}^t e^{\lambda r}\|g(r)\|^2 dr < \infty$ for each $t \in \mathbb{R}$, then we have the following two results.*

(i) The union $\bigcup_{\epsilon\in[0,\frac{\sqrt{\lambda}}{2\delta}]} \mathcal{E}_t^\epsilon$ is tight on ℓ^2 for each $t \in \mathbb{R}$.

(ii)For each $t \in \mathbb{R}$, if $\mu_t^{\epsilon_n} \in \mathcal{E}_t^{\epsilon_n}$ with $\epsilon_n \to \epsilon_0 \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$, then there exists $\mu_t^{\epsilon_0} \in \mathcal{E}_t^{\epsilon_0}$ and a subsequence $\{\epsilon_{n_k}(t)\}_{k=1}^\infty$ of $\{\epsilon_n\}_{n=1}^\infty$ such that $\mu_t^{\epsilon_n} \to \mu_t^{\epsilon_0}$ weakly for each $t \in \mathbb{R}$.

2. Proof of Theorems 1.1 and 1.3

Proof. We only prove Theorem 1.3. Given $\tau \in \mathbb{R}$, $s \geqslant \tau$ and $\varphi \in C_b(\mathbb{X})$, it is sufficient to show

$$\int_{\mathbb{X}} \varphi(x)(Q^{\epsilon_0}_{\tau,s}\eta^{\epsilon_0}_\tau)(dx) = \int_{\mathbb{X}} (P^{\epsilon_0}_{\tau,s}\varphi)(x)\eta^{\epsilon_0}_\tau(dx) = \int_{\mathbb{X}} \varphi(x)\eta^{\epsilon_0}_s(dx). \tag{2.1}$$

The first equality is obvious. By the condition for $t = s$, there exists a subsequence $\{\epsilon_{n_k}(s)\}_{k=1}^\infty$ of $\{\epsilon_n\}_{n=1}^\infty$ such that $\eta_s^{\epsilon_{n_k}(s)} \to \eta_s^{\epsilon_0}$ weakly. Then, for every $\gamma > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_{n_k}(s)}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_0}(dx) \leqslant \gamma \quad \text{for all } k \geqslant N_1. \tag{2.2}$$

Since $\epsilon_{n_k}(s) \to \epsilon_0$ and $\Big\{\eta_t^{\epsilon_{n_k}(s)}\Big\}_{t\in\mathbb{R}}$ is an evolution system of probability measures of $\Big(P^{\epsilon_{n_k}(s)}_{\tau,t}\Big)_{t\geqslant\tau}$, by the condition again for $t = \tau$, there exists a subsequence $\{\epsilon_{n_{k_j}}(s,\tau)\}_{j=1}^\infty$ of $\{\epsilon_{n_k}(s)\}_{k=1}^\infty$ such that

$\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)} \to \eta_\tau^{\epsilon_0}$ weakly. This implies that for every $\gamma > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\int_{\mathbb{X}} (P^{\epsilon_0}_{\tau,s}\varphi)(x)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) - \int_{\mathbb{X}} (P^{\epsilon_0}_{\tau,s}\varphi)(x)\eta_\tau^{\epsilon_0}(dx) \leqslant \gamma \text{ for all } j \geqslant N_2. \tag{2.3}$$

where we have used the Feller property of $P^{\epsilon_0}_{\tau,s}$. By the assumption **CIP** we know that for every $\gamma > 0$ and $\delta > 0$, there exists $N_3 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbf{K}} \mathbb{P}\left(\left\{\omega \in \Omega : \|X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x) - X^{\epsilon_0}(s,\tau,x)\|_{\mathbb{X}} \geqslant \delta\right\}\right) \leqslant \gamma \text{ for all } j \geqslant N_3. \tag{2.4}$$

Since $\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)} \to \eta_\tau^{\epsilon_0}$ weakly, by Prohorov theorem (see [8, 9]) we know $\left\{\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}\right\}_{j=1}^{\infty}$ is tight on $\mathbb{X}$. This means that for every $\gamma > 0$, there exists a compact set $\mathbf{K}(\gamma, s, \tau) \subseteq \mathbb{X}$ such that

$$\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(\mathbb{X}/\mathbf{K}(\gamma, s, \tau)) < \gamma \text{ for all } j \in \mathbb{N}. \tag{2.5}$$

By $\varphi \in C_b(\mathbb{X})$ and the compactness of $\mathbf{K}(\gamma, s, \tau)$ we know that for every $\gamma > 0$, there exists $\eta > 0$ such that

$$|\varphi(y) - \varphi(z)| < \gamma \text{ for all } y, z \in \mathbf{K}(\gamma, s, \tau) \text{ with } \|y - z\|_{\mathbb{X}} \leqslant \eta. \tag{2.6}$$

Indeed, if (2.6) is false, then there exist $\gamma_0 > 0$, $y_n, z_n \in \mathbf{K}(\gamma, s, \tau)$ with $\|y_n - z_n\|_{\mathbb{X}} \leqslant 1/n$ such that $|\varphi(y_n) - \varphi(z_n)| \geqslant \gamma_0$. Since $\mathbf{K}(\gamma, s, \tau)$ is a compact subset of $\mathbb{X}$, there exist $x \in \mathbf{K}(\gamma, s, \tau)$ and a subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \{n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} \|y_{n_k} - x\|_{\mathbb{X}} = 0$. Then one can verify $\lim_{k\to\infty} \|z_{n_k} - x\|_{\mathbb{X}} = 0$. Since φ is continuous, letting $k \to \infty$ in $\gamma_0 \leqslant |\varphi(y_{n_k}) - \varphi(z_{n_k})| \leqslant |\varphi(y_{n_k}) - \varphi(x)| + |\varphi(z_{n_k}) - \varphi(x)|$, we find a contradiction $\gamma_0 \leqslant 0$.

Since $\left\{\eta_t^{\epsilon_{n_{k_j}}(s,\tau)}\right\}_{t\in\mathbb{R}}$ is an evolution system of probability measures of $\left(P_{\tau,t}^{\epsilon_{n_{k_j}}(s,\tau)}\right)_{t\geqslant\tau}$ on $\mathbb{X}$, by (2.4)-(2.6) we find that for all $j \geqslant N := \max\{N_1, N_2, N_3\}$,

$$\begin{aligned}
&\int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&= \int_{\mathbb{X}} \mathbb{E}\big[\varphi(X^{\epsilon_0}(s,\tau,x))\big]\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) - \int_{\mathbb{X}} \mathbb{E}\big[\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x))\big]\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\leqslant \int_{\mathbb{X}} \mathbb{E}\big[|\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)) - \varphi(X^{\epsilon_0}(s,\tau,x))|\big]\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\leqslant \int_{\mathbf{K}(\gamma,s,\tau)} \mathbb{E}\big[|\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)) - \varphi(X^{\epsilon_0}(s,\tau,x))|\big]\eta_\tau^{\epsilon_{n_{k_j}}(\tau)}(dx) + 2\sup_{x\in\mathbb{X}}|\varphi(x)|\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(\mathbb{X}/\mathbf{K}(\gamma,s,\tau)) \\
&\leqslant \int_{\mathbf{K}(\gamma,s,\tau)} \mathbb{E}\big[|\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)) - \varphi(X^{\epsilon_0}(s,\tau,x))|\big]\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) + 2\gamma\sup_{x\in\mathbb{X}}|\varphi(x)| \\
&\leqslant \int_{\mathbf{K}(\gamma,s,\tau)} \int_{\left\{\omega\in\Omega:\|X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)-X^{\epsilon_0}(s,\tau,x)\|_{\mathbb{X}}\geqslant\eta\right\}} \big|\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)) - \varphi(X^{\epsilon_0}(s,\tau,x))\big|\mathbb{P}(d\omega)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\quad + \int_{\mathbf{K}(\gamma,s,\tau)} \int_{\left\{\omega\in\Omega:\|X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)-X^{\epsilon_0}(s,\tau,x)\|_{\mathbb{X}}<\eta\right\}} \big|\varphi(X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x)) - \varphi(X^{\epsilon_0}(s,\tau,x))\big|\mathbb{P}(d\omega)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\quad + 2\gamma\sup_{x\in\mathbb{X}}|\varphi(x)| \\
&\leqslant 2\sup_{x\in\mathbb{X}}|\varphi(x)|\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(\mathbf{K}(\gamma,s,\tau))\mathbb{P}\bigg(\Big\{\omega\in\Omega : \|X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x) - X^{\epsilon_0}(s,\tau,x)\|_{\mathbb{X}} \geqslant \eta\Big\}\bigg) \\
&+ \gamma\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(\mathbf{K}(\gamma,s,\tau))\mathbb{P}\bigg(\Big\{\omega\in\Omega : \|X^{\epsilon_{n_{k_j}}(s,\tau)}(s,\tau,x) - X^{\epsilon_0}(s,\tau,x)\|_{\mathbb{X}} < \eta\Big\}\bigg) + 2\gamma\sup_{x\in\mathbb{X}}|\varphi(x)| \\
&\leqslant \gamma(1 + 4\sup_{x\in\mathbb{X}}|\varphi(x)|). \qquad (2.7)
\end{aligned}$$

In order to prove the second equality in (2.1), we consider the following equality:

$$\begin{aligned}
\int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_0}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_0}(dx) &= \int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_0}(dx) - \int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\quad + \int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_{n_{k_j}}(s,\tau)}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_{n_{k_j}}(s,\tau)}(dx) \\
&\quad + \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_{n_{k_j}}(s,\tau)}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_0}(dx). \qquad (2.8)
\end{aligned}$$

Thus, as a result of (2.2)-(2.3) and (2.7)-(2.8) we have

$$\int_{\mathbb{X}} (P_{\tau,s}^{\epsilon_0}\varphi)(x)\eta_\tau^{\epsilon_0}(dx) - \int_{\mathbb{X}} \varphi(x)\eta_s^{\epsilon_0}(dx) \leqslant \gamma(3 + 4\sup_{x\in\mathbb{X}}|\varphi(x)|).$$

Since $\gamma > 0$ is arbitrary, we thus complete the proof. □

3. Proof of Theorem 1.4

Let us consider the Banach space $\ell^q = \big\{u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |u_i|^q < +\infty\big\}$ endowed with the norm

$$\|u\|_q = \begin{cases} \big(\sum_{i\in\mathbb{Z}} |u_i|^q\big)^{1/q}, & \text{if } q \in [1,\infty), \\ \sup_{i\in\mathbb{Z}} |u_i|, & \text{if } q = \infty. \end{cases}$$

In particular, we write $\|\cdot\| = \|\cdot\|_2$. It is known from Wang [16] that for each $\tau \in \mathbb{R}$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$, equation (1.2) has a unique solution which is continuous ℓ^2-valued $\mathcal{F}_t$-adapted Markov process $u(\cdot, \tau, u_0) \in L^2(\Omega, C([\tau, \infty), \ell^2)) \cap L^p(\Omega, L^p(\tau, \infty); \ell^p))$. Then we can prove that the transition operator $(P^\epsilon_{\tau,t})_{t \geqslant \tau}$ for $u(t, \tau, u_0)$ with $u_0 \in \ell^2$ is Feller, and the process laws hold: $P^\epsilon_{\tau,t} = P^\epsilon_{\tau,r} P^\epsilon_{r,t}$ and $Q^\epsilon_{\tau,t} = Q^\epsilon_{r,t} Q^\epsilon_{\tau,r}$, $-\infty < \tau \leqslant r \leqslant t < +\infty$.

Next, we derive different uniform estimates of time average of the solutions.

Lemma 3.1. *If $\int_{-\infty}^t e^{\lambda r}\|g(r)\|^2 dr < \infty$ for each $t \in \mathbb{R}$, then we have the following results.*

(i) For any $t \geqslant \tau \in \mathbb{R}$ and $\big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$,

$$\mathbb{E}\big[\|u^\epsilon(t, \tau, u_0)\|^2\big] + \frac{\lambda}{2}\int_\tau^t e^{\lambda(r-t)}\mathbb{E}\left[\|u^\epsilon(r, \tau, u_0)\|^2\right] dr \leqslant e^{\lambda(\tau-t)}\mathbb{E}\big[\|u_0\|^2\big] + c\int_{-\infty}^t e^{\lambda(s-t)}\|g(s)\|^2 ds, \quad (3.1)$$

where $c > 0$ is a constant independent of ϵ, t and u_0.

(ii) For any $t \in \mathbb{R}$ and $\mathbb{N} \ni k > -t$ and $\epsilon \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$,

$$\begin{aligned} &\frac{1}{k+t}\int_{-k}^t \mathbb{E}[\|u^\epsilon(t, \tau, u_0)\|^2] d\tau + \frac{1}{k+t}\int_{-k}^t \int_\tau^t e^{\lambda(r-t)}\mathbb{E}\left[\|u^\epsilon(r, \tau, u_0)\|^2\right] dr d\tau \\ &\leqslant \frac{c}{(k+t)}\mathbb{E}\big[\|u_0\|^2\big] + c\int_{-\infty}^t e^{\lambda(r-t)}\|g(r)\|^2 dr, \end{aligned} \quad (3.2)$$

where $c > 0$ is a constant independent of ϵ, t, k and u_0.

(iii) For each $t \in \mathbb{R}$ and bounded set $\mathbf{B}$ of ℓ^2, the solutions satisfy

$$\lim_{n\to\infty} \lim_{\tau\to-\infty} \sup_{\epsilon \in \left[0, \frac{\sqrt{\lambda}}{2\delta}\right]} \sup_{u_0 \in \mathbf{B}} \sum_{|i| \geqslant n} \mathbb{E}[|u_i^\epsilon(t, \tau, u_0)|^2] = 0.$$

(iv) For each $t \in \mathbb{R}$ and compact set $\mathbf{K}$ of ℓ^2, the solutions satisfy

$$\lim_{n\to\infty} \sup_{\epsilon \in \left[0, \frac{\sqrt{\lambda}}{2\delta}\right]} \sup_{\mathbb{N} \ni k > -t} \sup_{u_0 \in \mathbf{K}} \frac{1}{k+t}\int_{-k}^t \sum_{|i| \geqslant n} \mathbb{E}[|u_i^\epsilon(t, \tau, u_0)|^2] d\tau = 0.$$

Proof. (i)-(ii) Applying Itô's formula to (1.2), we infer that for any $\epsilon \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$,

$$\frac{d}{dt}\mathbb{E}\big[\|u^\epsilon(t)\|^2\big] + \frac{3}{2}\lambda\mathbb{E}\big[\|u^\epsilon(t)\|^2\big] \leqslant c\|g(t)\|^2. \quad (3.3)$$

Multiplying (3.3) by $e^{\lambda t}$ and integrating over (τ, t), we deduce (3.1). Integrating (3.1) with respect to τ over $(-k, t)$, we obtain (3.2).

(iii)-(iv) By a cut-off technique as used by Wang [16, Lemma 4.2] (see also [17]) one can derive, for any $\epsilon \in \big[0, \frac{\sqrt{\lambda}}{2\delta}\big]$,

$$\begin{aligned} \mathbb{E}\Big[\sum_{|i| \geqslant 2n} |u_i^\epsilon(t, \tau, u_0)|^2\Big] &\leqslant e^{\lambda(\tau-t)}\sum_{|i| \geqslant n}|u_{0,i}|^2 + c\int_\tau^t e^{\lambda(r-t)}\sum_{|i| \geqslant n}|g_i(r)|^2 dr \\ &\quad + \frac{c}{n}\int_\tau^t e^{\lambda(r-t)}\mathbb{E}\left[\|u^\epsilon(r, \tau, u_0)\|^2\right] dr. \end{aligned} \quad (3.4)$$

This along with $\int_{-\infty}^t e^{\lambda r}\|g(r)\|^2 dr < \infty$ and (3.1) completes the proof of (iii).

Integrating (3.4) with respect to τ over $(-k,t)$, we have

$$\frac{1}{k+t}\int_{-k}^{t}\mathbb{E}\Big[\sum_{|i|\geqslant 2n}|u_i^\epsilon(t,\tau,u_0)|^2\Big]d\tau \leqslant \frac{1}{\lambda(k+t)}\sum_{|i|\geqslant n}|u_{0,i}|^2 + c\int_{-\infty}^{t}e^{\lambda(r-t)}\sum_{|i|\geqslant n}|g_i(r)|^2dr + \frac{c}{n(k+t)}\int_{-k}^{t}\int_{\tau}^{t}e^{\lambda(r-t)}\mathbb{E}\left[\|u^\epsilon(r,\tau,u_0)\|^2\right]drd\tau.$$

This along with $\int_{-\infty}^t e^{\lambda r}\|g(r)\|^2dr<\infty$ and (3.2) completes the proof of (iv). □

Proof. **Proof of Theorem 1.4**. The proof is completed by an extended Krylov-Bogolyubov method proposed by Da Prato and Röckner [11]. For $n\in\mathbb{N}$, we let $\chi_{[-n,n]}$ be the characteristic function of $[-n,n]$. Then $u^\epsilon(t,\tau,u_0)=\tilde{u}_n^\epsilon(t,\tau,u_0)+\hat{u}_n^\epsilon(t,\tau,u_0)$ with $\tilde{u}_n^\epsilon(t,\tau,u_0)=\big(\chi_{[-n,n]}(|i|)u_i^\epsilon(t,\tau,u_0)\big)_{i\in\mathbb{Z}}$ and $\hat{u}_n^\epsilon(t,\tau,u_0)=\big((1-\chi_{[-n,n]}(|i|))u_i^\epsilon(t,\tau,u_0)\big)_{i\in\mathbb{Z}}$. Given $\delta>0$, $l\in\mathbb{N}$ and $m,k\in\mathbb{N}$ with $m\leqslant k$, by (ii) and (iv) of Lemma 3.1 there exist $n_l^\delta(m)\in\mathbb{N}$ and $C(m)>0$ independent on ϵ and k such that

$$\sup_{\epsilon\in\left[0,\frac{\sqrt{\lambda}}{2\delta}\right]}\frac{1}{k-m}\int_{-k}^{-m}\mathbb{E}\big[\|u^\epsilon(-m,\tau,0)\|^2\big]d\tau\leqslant C(m),$$

and

$$\sup_{\epsilon\in\left[0,\frac{\sqrt{\lambda}}{2\delta}\right]}\frac{1}{k-m}\int_{-k}^{-m}\mathbb{E}\big[\|\hat{u}^\epsilon_{n_l^\delta(m)}(-m,\tau,0)\|^2\big]d\tau<\frac{\delta}{2^{4l}}.$$

Define

$$\mathcal{Y}_l^\delta(m)=\left\{u\in\ell^2: u_i=0 \text{ for } |i|>n_l^\delta(m) \text{ and } \|u\|\leqslant\frac{2^l\sqrt{C(m)}}{\sqrt{\delta}}\right\}$$

and

$$\mathcal{Z}_l^\delta(m)=\Big\{u\in\ell^2:\|u-v\|\leqslant\frac{1}{2^l}\text{ for some } v\in\mathcal{Y}_l^\delta(m)\Big\}.$$

Define a probability measure $\eta_{k,m}^\epsilon=\frac{1}{k-m}\int_{-k}^{-m}\mathbb{P}\Big\{\omega\in\Omega: u^\epsilon(-m,\tau,0)\in\cdot\Big\}d\tau$ on ℓ^2. Then by Chebychev's inequality one can verify

$$\eta_{k,m}^\epsilon(\ell^2\setminus\mathcal{Z}_l^\delta(m))\leqslant\frac{\delta}{2^{2l}C(m)}\frac{1}{k-m}\int_{-k}^{-m}\mathbb{E}\big(\|u^\epsilon(-m,\tau,0)\|^2\big)d\tau + \frac{2^{2l}}{k-m}\int_{-k}^{-m}\mathbb{E}\big[\|\hat{u}^\epsilon_{n_l^\delta(m)}(-m,\tau,0)\|^2\big]d\tau\leqslant\frac{\delta}{2^{2l-1}}. \tag{3.5}$$

Let $\mathcal{Z}^\delta(m)=\bigcap_{l=1}^\infty\mathcal{Z}_l^\delta(m)$. Since one can verify that $\mathcal{Z}^\delta(m)$ is closed and totally bounded in ℓ^2, it is compact in ℓ^2. By (3.5) we know $\eta_{k,m}^\epsilon(\ell^2\setminus\mathcal{Z}^\delta(m))\leqslant\sum_{l=1}^\infty\eta_{k,m}^\epsilon(\ell^2\setminus\mathcal{Z}_l^\delta(m))<\delta$. Then $\{\eta_{k,m}^\epsilon\}_{\mathbb{N}\ni k\geqslant m}$ is tight on ℓ^2 for each fixed $m\in\mathbb{N}$. Then there exists $\eta_m^\epsilon\in\mathcal{P}(\ell^2)$ and a subsequence (not relabeled) such that $\eta_{k,m}^\epsilon\to\eta_m^\epsilon$ weakly as $k\to\infty$. For each fixed $t\in\mathbb{R}$, we choose $m\in\mathbb{N}$ such that $-m\leqslant t$, and define $\mu_t^\epsilon:=Q_{-m,t}^\epsilon\eta_m^\epsilon$. By [11] one can verify that this definition is independent of the choice of m. Then for every fixed $t\in\mathbb{R}$ and for any $\mathbb{N}\ni m\geqslant-\tau\geqslant-t$, we have $Q_{\tau,t}^\epsilon\mu_\tau^\epsilon=(P_{-m,\tau}^\epsilon P_{\tau,t}^\epsilon)^*\eta_m^\epsilon=Q_{-m,t}^\epsilon\eta_m^\epsilon=\mu_t^\epsilon$. This complete the proof. □

4. Proof of Theorem 1.5

Proof. **Proof of Theorem 1.5**. (i) For every $\delta>0$, $t\in\mathbb{R}$, $\epsilon\in[0,\frac{\sqrt{\lambda}}{2\delta}]$ and $\mu_t^\epsilon\in\mathcal{E}_t^\epsilon$, we shall find a compact set $\mathcal{Z}^\delta(t)\subseteq\ell^2$ independent of ϵ such that $\mu_t^\epsilon(\mathcal{Z}^\delta(t))>1-\delta$. Given $t\in\mathbb{R}$ and $l\in\mathbb{N}$, by (i) and (iii) of Lemma 3.1 there exists $n_l^\delta(t)\in\mathbb{N}$, $\mathfrak{T}_l^\delta(t)\leqslant t$ and $C(t)>0$ independent of ϵ and u_0 such that

$$\sup_{\epsilon\in\left[0,\frac{\sqrt{\lambda}}{2\delta}\right]}\sup_{\tau\leqslant t}\sup_{u_0\in\ell^2}\mathbb{E}\big[\|u(t,\tau,u_0)\|^2\big]d\tau\leqslant C(t), \tag{4.1}$$

$$\sup_{\epsilon\in\left[0,\frac{\sqrt{\lambda}}{2\delta}\right]}\sup_{\tau\leqslant\mathfrak{T}_l^\delta(t)}\sup_{u_0\in\ell^2}\mathbb{E}\big[\|\hat{u}^\epsilon_{n_l^\delta(t)}(t,\tau,u_0)\|^2\big]<\frac{\delta}{2^{4l}}. \tag{4.2}$$

Define $\mathcal{Y}_l^\delta(t):=\Big\{u\in\ell^2: u_i=0 \text{ for } |i|>n_l^\delta(t) \text{ and } \|u\|\leqslant\frac{2^l\sqrt{C(t)}}{\sqrt{\delta}}\Big\}$, $\mathcal{Z}_l^\delta(t):=\Big\{u\in\ell^2: \|u-v\|\leqslant\frac{1}{2^l}$ for some $v\in\mathcal{Y}_l^\delta(t)\Big\}$ and $\mathcal{Z}^\delta(t):=\bigcap_{l=1}^\infty\mathcal{Z}_l^\delta(t)$. Note that $\mathcal{Z}^\delta(t)$ is compact in ℓ^2. In what follows we prove $\mu_t^\epsilon(\mathcal{Z}^\delta(t))>1-\delta$. Denote by $\mathcal{X}_n^\delta(t):=\bigcap_{l=1}^n\mathcal{Z}_l^\delta(t)$ for $n\in\mathbb{N}$. Then $\mu_t^\epsilon(\mathcal{Z}^\delta(t))=\mu_t^\epsilon\big(\bigcap_{n=1}^\infty\mathcal{X}_n^\delta(t)\big)=\lim_{n\to\infty}\mu_t^\epsilon\big(\mathcal{X}_n^\delta(t)\big)$, and hence there exists $N=N(\delta,t)\in\mathbb{N}$ such that $0\leqslant\mu_t^\epsilon(\mathcal{X}_n^\delta(t))-\mu_t^\epsilon(\mathcal{Z}^\delta(t))\leqslant\delta/3$ for all $n\geqslant N$.

Note that by (4.1)-(4.2) we have, for all $\tau\leqslant\mathfrak{T}_l^\delta(t)$,

$$\begin{aligned}
&\int_{\ell^2}\mathbb{P}\Big(\Big\{\omega\in\Omega: u^\epsilon(t,\tau,x)\notin\mathcal{X}_N^\delta(t)\Big\}\Big)\mu_\tau^\epsilon(dx)\\
&\leqslant\sum_{l=1}^N\frac{\delta}{2^{2l}C(t)}\int_{\ell^2}\mathbb{E}\big[\|\tilde{u}^\epsilon_{n_l^\delta(t)}(t,\tau,x)\|^2\big]\mu_\tau^\epsilon(dx)\\
&\quad+\sum_{l=1}^N 2^{2l}\int_{\ell^2}\mathbb{E}\big[\|\hat{u}^\epsilon_{n_l^\delta(t)}(t,\tau,x)\|^2\big]\mu_\tau^\epsilon(dx)\\
&\leqslant\sum_{l=1}^N\frac{\delta}{2^{2l-1}}\leqslant\frac{2\delta}{3}.
\end{aligned}\tag{4.3}$$

By (4.3) and the definition of $\{\mu_t^\epsilon\}_{t\in\mathbb{R}}$ of $(P^\epsilon_{\tau,t})_{t\geqslant\tau}$, we find that for all $\tau\leqslant\mathfrak{T}_l^\delta(t)$,

$$\begin{aligned}
\mu_t^\epsilon(\mathcal{X}_N^\delta(t))&=\int_{\ell^2}\mathbb{P}\Big(\Big\{\omega\in\Omega: u^\epsilon(t,\tau,x)\in\mathcal{X}_N^\delta(t)\Big\}\Big)\mu_\tau^\epsilon(dx)\\
&=1-\int_{\ell^2}\mathbb{P}\Big(\Big\{\omega\in\Omega: u^\epsilon(t,\tau,x)\notin\mathcal{X}_N^\delta(t)\Big\}\Big)\mu_\tau^\epsilon(dx)\\
&\geqslant 1-\frac{2\delta}{3}.
\end{aligned}\tag{4.4}$$

This concludes the proof of (i).

(ii) For every $\tau\in\mathbb{R}$, $t\geqslant\tau$ $\delta>0$, $\epsilon_0\in\big[0,\frac{\sqrt{\lambda}}{2\delta}\big]$ and compact set $\mathbf{K}$ of ℓ^2, since we can prove that all uniform estimates of the solutions are uniform for $\epsilon\in\big[0,\frac{\sqrt{\lambda}}{2\delta}\big]$, by a stoping time argument, we can prove

$$\lim_{\epsilon\to\epsilon_0}\sup_{u_0\in\mathbf{K}}\mathbb{P}\Big(\big\{\omega\in\Omega: \|u^\epsilon(t,\tau,u_0)-u^{\epsilon_0}(t,\tau,u_0)\|\geqslant\delta\big\}\Big)=0. \tag{4.5}$$

The proof of (4.5) is quite similar to the autonomous case as in [5, 6, 14, 15], the details are omitted here. Then by (i), (4.5), Theorem 1.3 and Prohorov's theorem, we complete the proof of (ii). □

5. Acknowledgements

Renhai Wang was supported by China Postdoctoral Science Foundation under grant numbers 2020TQ0053 and 2020M680456. Tomás Caraballo was supported by the Spanish Ministerio de Ciencia, Innovación y Universidades project PGC2018-096540-B-I00, and Junta de Andalucia (Spain) under projects US-1254251 and P18-FR-4509. Nguyen Huy Tuan was supported by Van Lang University. The authors would like to thank the referees for their valuable comments and suggestions which improve the quality of the manuscript

References

[1] T. Caraballo, X. Han, Applied nonautonomous and random dynamical systems: applied dynamical systems. Springer, 2017.
[2] T. Caraballo, X. Han, B. Schmalfuss, J. Valero, Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise, Nonlinear Anal. 130 (2016) 255-278.
[3] T. Caraballo, F.Morillas, J. Valerom, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, J. Differential Equations 253 (2012) 667-693.
[4] T. Caraballo, F. Morillas, J. Valero, On differential equations with delay in Banach spaces and attractors for retarded lattice dynamical systems, Discrete and Continuous Dynamical Systems, 34(1) (2014) 51-77.
[5] Z. Chen, B. Wang, Limit measures of stochastic Schrödinger lattice systems, Proceedings of the American Mathematical Society, 2022, https://doi.org/10.1090/proc/15769.
[6] Z. Chen, B. Wang, Weak mean attractors and invariant measures for stochastic Schrödinger delay lattice systems, Journal of Dynamics and Differential Equations, 2022, https://doi.org/10.1007/s10884-021-10085-3
[7] L. Chen, Z. Dong, J. Jiang, J. Zhai, On limiting behavior of stationary measures for stochastic evolution systems with small noisy intensity, Sci. China Math. 63 (2020) 1463-1504.
[8] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press,1992.
[9] G. Da Prato, J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes, 229, Cambridge University Press, 1996.
[10] G. Da Prato and M. Röckner, Dissipative stochastic equations in Hilbert space with time dependent coefficients, Rend. Lincei Mat. Appl. 17 (2006) 397-403.
[11] G. Da Prato and M. Röckner, A note on evolution systems of measures for time-dependent stochastic differential equations, Progress in Probability, 59 (2009) 115-122.
[12] X. Han, P. E. Kloeden, Sigmoidal approximations of Heaviside functions in neural lattice models, Journal of Differential Equations, 268(9) (2020) 5283-5300.
[13] X. Han, W. Shen, S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, J. Differential Equations 250 (2011) 1235-1266.
[14] D. Li, B. Wang, X. Wang, Periodic measures of stochastic delay lattice systems. J. Differ. Equ. 272 (2021) 74-104.
[15] D. Li, B. Wang, X. Wang, Limiting behavior of invariant measures of stochastic delay lattice systems, J. Dyn. Diff. Eqns. 2021, https://doi.org/10.1007/s10884-021-10011-7.
[16] B. Wang, Dynamics of stochastic reaction-diffusion lattice systems driven by nonlinear noise, J. Math. Anal. Appl. 477 (2019) 104-132.
[17] B. Wang, Attractors for reaction-diffusion equations in unbounded domains, Phys. D 128 (1999) 41-52.
[18] S. Zhou, X. Han, Pullback exponential attractors for non-autonomous lattice systems, J. Dyn. Differ. Equ. 24 (2012) 601-631.
[19] S. Zhou, Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise, J. Differential Equations 263 (2017) 2247-2279.

(Renhai Wang) School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China
E-mail address, R. Wang: rwang-math@outlook.com

(Tomás Caraballo) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas,, Universidad de Sevilla, C/ Tarfia s/n, 41012-Sevilla, Spain
E-mail address, T. Caraballo: caraball@us.es

(Nguyen Huy Tuan) Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City, Vietnam,, Faculty of Technology, Van Lang University, Ho Chi Minh City, Vietnam
E-mail address, N.H. Tuan: nguyenhuytuan@vlu.edu.vn