1ex plus .25ex minus .1ex 1ex plus .25ex minus .1ex

OPTIMIZATION AND CONVERGENCE OF NUMERICAL ATTRACTORS FOR DISCRETE-TIME QUASI-LINEAR LATTICE SYSTEM*

YANGRONG LI†, SHUANG YANG †, AND TOMÁS CARABALLO ‡

Abstract. Existence and connection of numerical attractors for discrete-time p-Laplace lattice systems via the implicit Euler scheme are proved. The numerical attractors are shown to have an optimized bound, which leads to the continuous convergence of the numerical attractors when the graph of the nonlinearity closes to the vertical axis or when the external force vanishes. A new type of Taylor expansion without Fréchet derivatives is established and applied to show the discretization error of order two, which is crucial to prove that the numerical attractors converge upper semi-continuously to the global attractor of the original continuous-time system as the step size of the time goes to zero. It is also proved that the truncated numerical attractors for finitely dimensional systems converge upper semi-continuously to the numerical attractor and the lower semi-continuity holds in special cases.

Key words. discrete-time equation, numerical attractor, p-Laplace lattice, finite-dimensional approximation; semi-continuity of attractors

AMS subject classifications. 65L20, 35B40, 37L60

1. Introduction.

We study the numerical scheme of attractors as well as solutions for the p-Laplace lattice dynamical system (LDS)

$$\frac{du_i(t)}{dt} = \nu(A_p u(t))_i + f(u_i(t)) + g_i, \; i \in \mathbb{Z}, \tag{1.1}$$

where $\nu > 0$, $p \geq 2$, $u = (u_i)_{i\in\mathbb{Z}}$, and the discrete p-Laplace operator is defined by

$$(A_p u)_i = |u_{i+1} - u_i|^{p-2}(u_{i+1} - u_i) - |u_i - u_{i-1}|^{p-2}(u_i - u_{i-1}), i \in \mathbb{Z}.$$

As one knows, a LDS has many applications in fluid dynamics, chemistry and neural networks, see [3, 5, 18, 32]. The p-Laplace LDS (1.1) is the space-discretization of the corresponding p-Laplace partial differential equation (defined on the real line), while the dynamics of the (deterministic or stochastic) p-Laplace PDE was studied in [8, 11, 12, 13, 23, 26, 27, 28, 29, 30, 33, 37].

As preliminaries, we show in Section 2 that the LDS (1.1) has a positively invariant ball $\mathcal{B}_{r^*}(0)$ and a global attractor $\mathcal{A}$ in ℓ^2, where the dissipative condition of $f \in C(\mathbb{R}, \mathbb{R})$ is different from those in [10, 14, 15, 36] and given by

$$\alpha := \inf_{s\neq 0} \frac{-f(s)}{s} > 0. \tag{1.2}$$

*Submitted to the editors DATE.

Funding: Li and Yang have been supported by National Natural Science Foundation of China grant 11571283. Yang has been partially supported by China Scholarship Council (No. 202006990054) for 1 year's study abroad at the University of Sevilla. Caraballo has been supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00.

†School of Mathematics and Statistics, Southwest University, Chongqing 400715, PR China (liyr@swu.edu.cn,ys1718@email.swu.edu.cn).

‡Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, C/Tarfia s/n, 41012-Sevilla, Spain (caraball@us.es).

In this paper, we mainly consider the numerical scheme in the discrete-time sense. Using the step size $\epsilon := t_{n+1} - t_n$ of the time to discretize the LDS (1.1), we obtain the p-Laplace implicit Euler scheme (IES)

$$u^\epsilon_{n,i} = u^\epsilon_{n-1,i} + \epsilon\nu(A_p u^\epsilon_n)_i + \epsilon f(u^\epsilon_{n,i}) + \epsilon g_i,\ \forall n \in \mathbb{N},\ i \in \mathbb{Z}. \tag{1.3}$$

As pointing out by Han, Kloeden, Sonner [17], the IES (1.3) with $p = 2$ is not globally solvable for a common step size (see Lemma 5.3 for the reason if $p > 2$). Instead the global solvability, we will prove in Theorem 3.1 that, for sufficiently small step sizes, the IES (1.3) is uniquely solvable when the initial datum belongs to the positively invariant ball $\mathcal{B}_{r^*}(0)$. In the recursive proof of Theorem 3.1, we use a new method of enlarging the radius to overcome the difficulty that the ball $\mathcal{B}_{r^*}(0)$ is no longer a positively invariant set under the operator defined by the right-hand side of (1.3). The proof is more careful and technical than in [17] even for the case of $p = 2$.

For the later purpose, we have to consider the numerical approximation of solutions as $\epsilon \to 0$. In this respect, Kloeden and Lorenz [25] (see also Jentzen et al. [4, 20, 21, 22]) have introduced the method of the Taylor expansion by using the Fréchet derivatives of the *linear* Laplacian A_p ($p = 2$) and f. However, the nonlinear operator A_p ($p > 2$) has not a Fréchet derivative.

To overcome the above difficulty, we establish a new type of Taylor expansions without Fréchet derivatives and give the continuous-time error of solutions for the LDS (1.1) (see Lemma 4.1). Using this continuous-time error, we can show the discretization error of order two for the solutions between LDS (1.1) and IES (1.3), see Theorem 4.2. Our method is suitable for a wider class of discrete-time equations even if the operators are not Fréchet differential.

From Section 5, our main purpose is to study the numerical scheme of attractors, which is a relative new subject (introduced by Han, Kloeden, Sonner [17], see also [38]) in both Numerical Analysis and Dynamical Systems [31]. More precisely, we study the discrete approximation of the global attractor $\mathcal{A}$ for LDS (1.1) in terms of numerical attractors for IES (1.3) and its finitely dimensional truncated system.

We prove in Theorem 5.2 that the discrete semigroup, generated from the IES (1.3), possesses a unique connected numerical attractor $\mathcal{A}_\epsilon$ for sufficiently small step sizes. In the proof, we need to recursively estimate the tail of solutions for all $n \in \mathbb{N}$, where the usual cut-off function technique (see [1, 2, 6, 7, 16, 19, 35, 39, 40]) is still valid in the discrete-time case.

Furthermore, we prove in Theorem 5.4 that $\mathcal{A}_\epsilon$ has an optimized bound given by $\|g\|/\alpha$. This bound is crucial to prove that the numerical attractor converges continuously (upper and lower) to zero as the graph of f closes enough to the y-axis and as $g \to 0$, respectively. This subject of optimization and convergence of numerical attractors is new in the literature.

In Theorem 6.1, we establish the upper semi-continuity from the numerical attractor $\mathcal{A}_\epsilon$ to the global attractor $\mathcal{A}$ as $\epsilon \to 0$, where the discretization error of solutions in Theorem 4.2 plays a crucial role in the proof.

In Section 7, we study the finitely dimensional approximation of the numerical attractor. For this end, we truncate the IES (1.3) on the $(2m+1)$-dimensional Euclid space to obtain the truncated numerical scheme with the periodic boundary condition, see the model (7.2). We then prove in Theorem 7.4 that the truncated IES (7.2) has an attractor denoted by $\mathcal{A}_{\epsilon,m}$, and that $\mathcal{A}_{\epsilon,m}$ converges upper semi-continuously to the numerical attractor $\mathcal{A}_\epsilon$ as $m \to \infty$. If the viscosity is zero, i.e. $\nu = 0$, the lower semi-continuity from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}_\epsilon$ still holds as proved in Theorem 7.6.

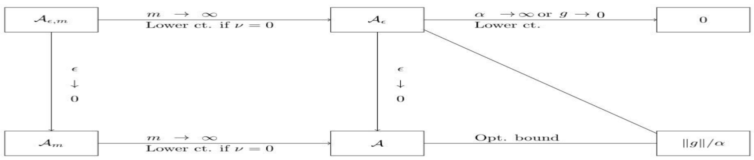

FIG. 1. *Convergence paths and bounds of attractors.*

In a word, we have established a convergence path from $\mathcal{A}_{\epsilon,m}$ to the global attractor $\mathcal{A}$ through $\mathcal{A}_\epsilon$. In fact, there is another convergence path from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}$ through $\mathcal{A}_m$, where $\mathcal{A}_m$ is the global attractor for the truncated system of the LDS (1.1) on the $(2m+1)$-dimensional Euclid space. All convergence paths and optimized bounds of attractors are displayed in FIG. 1.

2. Positively invariant ball and global attractor for p-Laplace lattice. The discrete p-Laplace operator A_p $(p \geq 2)$ can be formally written as

$$A_p u = -B^*(|Bu|^{p-2}Bu), \ (Bu)_i := u_{i+1} - u_i, \ (B^*u)_i := u_{i-1} - u_i,$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $|u|^q = (|u_i|^q)_{i\in\mathbb{Z}}$ and $uv = (u_i v_i)_{i\in\mathbb{Z}}$. By [14], we have $(A_p u, u) = -\|Bu\|_p^p$, where $\|\cdot\|_q$ (omitting the subscript if $q = 2$) denotes the norm in the Banach space

$$\ell^q := \{u = (u_i)_{i\in\mathbb{Z}} : \ \|u\|_q^q = \sum_{i\in\mathbb{Z}} |u_i|^q < \infty\}, \ q \geq 1.$$

We assume that $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$ and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschtz continuous, i.e. for each $r > 0$, there is $L_r \geq 0$ (increasingly in r) such that

$$|f(s_1) - f(s_2)| \leq L_r |s_1 - s_2|, \ \forall |s_1| \leq r, \ |s_2| \leq r, \tag{2.1}$$

and the dissipative condition (1.2) holds. Note that both (2.1) and (1.2) imply that

$$f(s)s \leq -\alpha s^2, \ \forall s \in \mathbb{R}, \quad f(0) = 0, \tag{2.2}$$

and the Nemytskii operator $F : \ell^2 \to \ell^2$, $F(u) = (f(u_i))_{i\in\mathbb{Z}}$ is bounded and locally Lipschtz continuous.

Now, the p-Laplace LDS (1.1) is rewritten as an abstract form

$$\frac{du(t)}{dt} = \nu A_p u(t) + F(u(t)) + g, \ t > 0, \quad u(0) = u_0 \in \ell^2, \tag{2.3}$$

where $\nu > 0$, $p > 2$. Although the dissipative condition ((1.2) or equivalently (2.2)) is different from those in [10, 14], one can similarly prove that the p-Laplace LDS (2.3) has a unique solution $u \in C([0,\infty),\ell^2)$, which generates a continuous semigroup (semi dynamical system) defined by

$$S(t) : \ell^2 \to \ell^2, \quad S(t)u_0 = u(t; u_0), \ \forall t \ge 0, \ u_0 \in \ell^2.$$

Lemma 2.1. *The semigroup $S(\cdot)$ has a positively invariant absorbing ball*

$$\mathcal{B}_{r^*}(0) := \{x \in \ell^2 : \|x\| \le r^* := \sqrt{1 + \|g\|^2/\alpha^2}\}.$$

Proof. By the inner product of (2.3) with $u(t)$, using (2.2) we obtain

$$\frac{d}{dt}\|u\|^2 = -2\nu\|Bu\|_p^p - 2\alpha \sum_{i\in\mathbb{Z}} f(u_i)u_i + 2(g,u) \le -\alpha\|u\|^2 + \frac{\|g\|^2}{\alpha}.$$

The Gronwall lemma yields

$$\|u(t)\|^2 \le e^{-\alpha t}\|u_0\|^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}), \ \forall t \ge 0. \tag{2.4}$$

For all $u_0 \in \mathcal{B}_r(0)$ with arbitrary radius $r > 0$, we have

$$\|u(t,u_0)\|^2 \le e^{-\alpha t}r^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}) \le 1 + \frac{\|g\|^2}{\alpha^2} = (r^*)^2$$

if $t \ge \frac{2}{\alpha}\log r$. Hence $\mathcal{B}_{r^*}(0)$ is an absorbing ball.

By (2.4) again, for all $u_0 \in \mathcal{B}_{r^*}(0)$ and $t \ge 0$,

$$\|u(t,u_0)\|^2 \le e^{-\alpha t}\Big(1 + \frac{\|g\|^2}{\alpha^2}\Big) + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}) = e^{-\alpha t} + \frac{\|g\|^2}{\alpha^2} \le 1 + \frac{\|g\|^2}{\alpha^2} = (r^*)^2.$$

Hence $\mathcal{B}_{r^*}(0)$ is also positively invariant under $S(\cdot)$. ∎

We remark here that the larger radius $\sqrt{1 + 2\|g\|^2/\alpha^2}$ was used in [17] for $p = 2$.

By the technique of a cut-off function (see e.g. [1, 39]), one can give the uniform estimates of the tail of the solution on the ball $\mathcal{B}_{r^*}(0)$, which leads to the existence of a global attractor. The proof is similar to the one given in [14].

Theorem 2.2. *The semigroup $S(\cdot)$, generated from the p-Laplace lattice, possesses a unique global attractor $\mathcal{A} \subset \mathcal{B}_{r^*}(0)$.*

3. Numerical solutions and discrete semigroup on a ball.

The implicit Euler scheme for the p-Laplace LDS (2.3) with the step size $\epsilon > 0$ can be read as

$$u_n^\epsilon = u_{n-1}^\epsilon + \epsilon\nu A_p u_n^\epsilon + \epsilon F(u_n^\epsilon) + \epsilon g, \qquad u_0^\epsilon = u_0 \in \ell^2. \tag{3.1}$$

Note that there does not exist a common step size such that (3.1) is solvable for all initial data (see Lemma 5.3 later or see [17, 31] in the case of $p = 2$). So, we will restrict (3.1) on the ball $\mathcal{B}_{r^*}(0)$ to ensure the existence of a discrete-time dynamical system for at least one step size.

We need to use the local Lipschitz continuity of the discrete p-Laplace operator

$$\|A_p u - A_p v\| \le L_{p,r}\|u - v\| \text{ and } \|A_p u\| \le L_{p,r}\|u\|, \ \forall u, v \in \mathcal{B}_r(0), \tag{3.2}$$

where $L_{p,r} := (p-1)2^{2p}r^{p-2}$ depends increasingly on $r \ge 0$, see [34] for the proof.

THEOREM 3.1. *There is $\epsilon^* > 0$ such that, for each $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the IES (3.1) has a unique solution such that*

$$u_n^\epsilon(u_0) \in \mathcal{B}_{r^*}(0),\ \forall n \in \mathbb{N}, \quad \text{where } r^* = \sqrt{1 + \|g\|^2/\alpha^2}.$$

Proof. We recursively prove the theorem in four steps.

Step 1. In the case of $n = 1$, we find an $\epsilon^* > 0$ such that the IES (3.1) has a solution

$$u_1^\epsilon(u_0) \in \mathcal{B}_{r^*+1}(0),\ \forall \epsilon \in (0, \epsilon^*],\ u_0 \in \mathcal{B}_{r^*}(0),$$

where the radius is temporarily enlarged (from r^* to $r^* + 1$).

For each $\epsilon > 0$ and $u_0 \in \mathcal{B}_{r^*}(0)$, we define an operator $\mathcal{M}_{u_0}^\epsilon : \ell^2 \to \ell^2$ by

$$\mathcal{M}_{u_0}^\epsilon(x) = u_0 + \epsilon\nu A_p x + \epsilon F(x) + \epsilon g, \quad \forall x \in \ell^2. \tag{3.3}$$

We prove that $\mathcal{M}_{u_0}^\epsilon$ maps $\mathcal{B}_{r^*+1}(0)$ into itself if ϵ is small enough. Since A_p is bounded, it follows from the second inequality of (3.2) that

$$\nu\|A_p x\| \le \nu L_{p,r^*+1}\|x\| \le (r^* + 1)\nu L_{p,r^*+1}, \quad \forall x \in \mathcal{B}_{r^*+1}(0).$$

By the local Lipschitz continuity (2.1) and $f(0) = 0$, we obtain

$$\|F(x)\| \le L_{r^*+1}\|x\| \le (r^* + 1)L_{r^*+1}, \quad \forall x \in \mathcal{B}_{r^*+1}(0).$$

Hence, for all $u_0 \in \mathcal{B}_{r^*}(0)$ and $x \in \mathcal{B}_{r^*+1}(0)$, we have

$$\begin{aligned}\|\mathcal{M}_{u_0}^\epsilon(x)\| &\le \|u_0\| + \epsilon(\nu\|A_p x\| + \|F(x)\| + \|g\|)\\ &\le r^* + \epsilon\Big((r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|\Big).\end{aligned}$$

We define an essential constant by

$$\epsilon^* := \frac{1}{(r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|}. \tag{3.4}$$

Then $\|\mathcal{M}_{u_0}^\epsilon(x)\| \le r^* + 1$ for all $\epsilon \in (0, \epsilon^*]$, $u_0 \in \mathcal{B}_{r^*}(0)$ and $x \in \mathcal{B}_{r^*+1}(0)$.

We then prove that, for each $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the mapping $\mathcal{M}_{u_0}^\epsilon : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ is contractive. Indeed, by the local Lipschitz continuity in (2.1) and (3.2), for all $x, y \in \mathcal{B}_{r^*+1}(0)$,

$$\begin{aligned}\|\mathcal{M}_{u_0}^\epsilon(x) - \mathcal{M}_{u_0}^\epsilon(y)\| &\le \epsilon(\nu\|A_p x - A_p y\| + \|F(x) - F(y)\|)\\ &\le \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})\|x - y\|.\end{aligned} \tag{3.5}$$

If $\epsilon \in (0, \epsilon^*]$, where ϵ^* is the constant defined by (3.4), then

$$\begin{aligned}\epsilon(\nu L_{p,r^*+1} + L_{r^*+1}) &\le \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1})\\ &= \frac{\nu L_{p,r^*+1} + L_{r^*+1}}{(r^* + 1)(\nu L_{p,r^*+1} + L_{r^*+1}) + \|g\|} \le \frac{1}{r^* + 1} < 1.\end{aligned}$$

By the contraction mapping principle, for each $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$, the mapping $\mathcal{M}_{u_0}^\epsilon : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ has a unique fixed point

$$u_1^\epsilon \in \mathcal{B}_{r^*+1}(0) \text{ such that } \mathcal{M}_{u_0}^\epsilon(u_1^\epsilon) = u_1^\epsilon,$$

which is the unique solution of the IES (3.1) for $n = 1$ in $\mathcal{B}_{r^*+1}(0)$.

Step 2. We further prove that the unique solution in **Step 1** satisfies

$$u_1^\epsilon(u_0) \in \mathcal{B}_{r^*}(0),\ \forall \epsilon \in (0, \epsilon^*],\quad u_0 \in \mathcal{B}_{r^*}(0).$$

For this purpose, we take the inner product of the equation (3.1) for $n = 1$ by u_1^ϵ to obtain

$$\|u_1^\epsilon\|^2 = (u_0, u_1^\epsilon) + \epsilon\nu(A_p u_1^\epsilon, u_1^\epsilon) + \epsilon(F(u_1^\epsilon), u_1^\epsilon) + \epsilon(g, u_1^\epsilon). \tag{3.6}$$

By (2.2),

$$\epsilon(F(u_1^\epsilon), u_1^\epsilon) = \epsilon \sum_{i\in\mathbb{Z}} f(u_{1,i}^\epsilon) u_{1,i}^\epsilon \le -\epsilon\alpha \|u_1^\epsilon\|^2.$$

The Young inequality implies

$$(u_0, u_1^\epsilon) \le \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u_1^\epsilon\|^2 \text{ and } \epsilon(g, u_1^\epsilon) \le \frac{\epsilon}{2\alpha}\|g\|^2 + \frac{\epsilon\alpha}{2}\|u_1^\epsilon\|^2.$$

Since $(A_p u_1^\epsilon, u_1^\epsilon) = -\|Bu_1^\epsilon\|_p^p \le 0$, it follows from (3.6) and the above estimates that

$$\|u_1^\epsilon\|^2 \le \frac{1}{1+\epsilon\alpha}\Big(\|u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big). \tag{3.7}$$

Since $u_0 \in \mathcal{B}_{r^*}(0)$, it follows from (3.7) that

$$\|u_1^\epsilon\|^2 \le \frac{1}{1+\epsilon\alpha}\Big(1 + \frac{\|g\|^2}{\alpha^2} + \frac{\epsilon}{\alpha}\|g\|^2\Big) \le 1 + \frac{\|g\|^2}{\alpha^2} = (r^*)^2,$$

which means that $u_1^\epsilon \in \mathcal{B}_{r^*}(0)$ as desired.

Step 3. We show that the solution is unique globally. Let $\epsilon \in (0, \epsilon^*]$ and $u_0 \in \mathcal{B}_{r^*}(0)$. By **Step 1**, the solution $u_1^\epsilon(u_0)$ is unique in $\mathcal{B}_{r^*+1}(0)$. By **Step 2**, there is not a solution outside $\mathcal{B}_{r^*}(0)$ and thus the solution $u_1^\epsilon(u_0)$ is unique in ℓ^2. So far, the theorem for $n = 1$ has been proved.

Step 4. Suppose the theorem holds for a certain n, that is, for each $\epsilon \in (0, \epsilon^*]$ (where ϵ^* is still the constant given by (3.4)) and $u_0 \in \mathcal{B}_{r^*}(0)$, the n-th IES (3.1) has a unique solution $u_n^\epsilon(u_0) \in \mathcal{B}_{r^*}(0)$. We then define a mapping by

$$\mathcal{M}_{u_n^\epsilon}^\epsilon(x) = u_n^\epsilon + \epsilon\nu A_p x + \epsilon F(x) + \epsilon g,\quad \forall x \in \mathcal{B}_{r^*+1}(0),$$

where $u_n^\epsilon \in \mathcal{B}_{r^*}(0)$ instead of $u_0 \in \mathcal{B}_{r^*}(0)$ in (3.3). Repeating the process in **Step 1**, we know that, for each $\epsilon \in (0, \epsilon^*]$, the mapping $\mathcal{M}_{u_n^\epsilon}^\epsilon : \mathcal{B}_{r^*+1}(0) \to \mathcal{B}_{r^*+1}(0)$ is well-defined and contractive, which implies the existence of a unique fixed point u_{n+1}^ϵ in $\mathcal{B}_{r^*+1}(0)$.

Repeating the estimates in **Step 2**, we obtain an analogue inequality of (3.7)

$$\|u_{n+1}^\epsilon\|^2 \le \frac{1}{1+\epsilon\alpha}\Big(\|u_n^\epsilon\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big). \tag{3.8}$$

By the recursive hypothesis $u_n^\epsilon \in \mathcal{B}_{r^*}(0)$, we infer from (3.8) that $u_{n+1}^\epsilon \in \mathcal{B}_{r^*}(0)$, which is the unique solution of the $(n+1)$-th IES (3.1). The recursive proof is complete. □

Remark 3.2. The above proof is more careful than the proof of [17, Lemma 2] even in the case of $p = 2$. In fact, $\mathcal{B}_{r^*}(0)$ may not be positively invariant under the operator $\mathcal{M}_{u_0}^\epsilon$ (although it is invariant under the solution mapping, see [17, Lemma 1]). To overcome this difficulty, we enlarge the radius r^* to $r^* + 1$ such that $\mathcal{B}_{r^*+1}(0)$ is positive invariant under $\mathcal{M}_{u_0}^\epsilon$ with a possible maximal size ϵ^*.

The following result shows the generation of a discrete-time dynamical system (see [24]), which has better properties than the continuous system. The proof is standard.

COROLLARY 3.3. *For each $\epsilon \in (0, \epsilon^*]$, where ϵ^* is given by (3.4), the unique solution of the IES (3.1) in $\mathcal{B}_{r^*}(0)$ generates a discrete semigroup given by*

$$S_\epsilon(n) : \mathcal{B}_{r^*}(0) \to \mathcal{B}_{r^*}(0),\ S_\epsilon(n)u_0 = u_n^\epsilon(u_0),\ \forall n \in \mathbb{N}_0,\ u_0 \in \mathcal{B}_{r^*}(0).$$

LEMMA 3.4. *For $\epsilon \in (0, \epsilon^*]$ and $n \in \mathbb{N}$, the operator $S_\epsilon(n)$ is Lipschitz continuous in $\mathcal{B}_{r^*}(0)$.*

Proof. Let $n = 1$ and $u_0, v_0 \in \mathcal{B}_{r^*}(0)$. By (3.5), the solutions $u_1^\epsilon = S_\epsilon(1)u_0$ and $v_1^\epsilon = S_\epsilon(1)v_0$ satisfy

$$\begin{aligned}\|u_1^\epsilon - v_1^\epsilon\| &\le \|u_0 - v_0\| + \epsilon(\nu\|A_p u_1^\epsilon - A_p v_1^\epsilon\| + \|F(u_1^\epsilon) - F(v_1^\epsilon)\|)\\ &\le \|u_0 - v_0\| + \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})\|u_1^\epsilon - v_1^\epsilon\|,\end{aligned}$$

which further implies that for all $\epsilon \in (0, \epsilon^*]$,

$$\|u_1^\epsilon - v_1^\epsilon\| \le \frac{\|u_0 - v_0\|}{1 - \epsilon(\nu L_{p,r^*+1} + L_{r^*+1})} \le \frac{\|u_0 - v_0\|}{1 - \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1})},$$

where $\epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1}) < 1$ in view of (3.4). By the semigroup property,

$$\|S_\epsilon(n)u_0 - S_\epsilon(n)v_0\| \le \frac{\|u_0 - v_0\|}{(1 - \epsilon^*(\nu L_{p,r^*+1} + L_{r^*+1}))^n}$$

for all $n \in \mathbb{N}$. The proof is complete. ∎

4. Generalized Taylor expansion and discretization error.

To study the convergence of attractors, we need to estimate the discretization error of solutions, for which we need to develop a *generalized* Taylor expansion.

4.1. Generalized Taylor expansion for continuous-time error.

According to the method in [17, 25], one must consider the Taylor expansion of LDS (2.3) starting from $u(t_{n+1}; u_0)$ and going back to $u(t_n; u_0)$ as follows:

$$u(t_n) = u(t_{n+1}) + (-\epsilon)\mathcal{H}_p(u(t_{n+1})) + \frac{1}{2}(-\epsilon)^2 D\mathcal{H}_p(u(\theta_\epsilon))$$

where $t_{n+1} - t_n = \epsilon$, $\theta_\epsilon \in (t_n, t_{n+1})$, the operator $\mathcal{H}_p : \ell^2 \to \ell^2$ is given by

$$\mathcal{H}_p(x) := \nu A_p x + F(x) + g,\ \forall x \in \ell^2 \tag{4.1}$$

and $D\mathcal{H}_p$ denotes the Fréchet derivative (perhaps formal) of $\mathcal{H}_p$. If $p = 2$, then $A := A_p$ is a bounded linear operator, which has a Fréchet derivative given by itself, and thus, by the method as in [21], one can clearly write the Fréchet derivative as $D\mathcal{H}(x) = (\nu A + \mathrm{diag}(f'(x_i))\mathcal{H}(x)$ for $x \in \ell^2$. However, if $p > 2$, then the nonlinear operator A_p has not a Fréchet derivative (even the original function $y = |s|^{p-2}s$ is not differential in $\mathbb{R}$).

To overcome the difficulty, we give an alternative for the second order Taylor expansion of LDS (2.3) without Fréchet derivatives, which will be useful for estimating the discretization error in the next subsection.

LEMMA 4.1. *Let $u(\cdot; u_0)$ be the solution of LDS (2.3), $t_{n+1} - t_n = \epsilon > 0$, $t_n \ge 0$. Then, for each $u_0 \in \mathcal{B}_r(0)$ with any radius $r > 0$, there is $\mathcal{M}_\epsilon(u_0) \in \ell^2$ such that*

$$u(t_n; u_0) = u(t_{n+1}; u_0) - \epsilon\mathcal{H}_p(u(t_{n+1}; u_0)) + \epsilon\mathcal{M}_\epsilon(u_0), \tag{4.2}$$

$$\|\mathcal{M}_\epsilon(u_0)\| \le \epsilon C_r, \quad \forall u_0 \in \mathcal{B}_r(0), \tag{4.3}$$

where C_r is increasing in r (but independent of ϵ) and the operator $\mathcal{H}_p : \ell^2 \to \ell^2$ is well-defined by (4.1).

Proof. The first order Taylor expansion of LDS (2.3) can be read as

$$\begin{aligned} u(t_n) &= u(t_{n+1}) - \epsilon \frac{du}{dt}(\theta) = u(t_{n+1}) - \epsilon \mathcal{H}_p(u(\theta)) \\ &= u(t_{n+1}) - \epsilon \mathcal{H}_p(u(t_{n+1})) + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u(\theta))), \end{aligned}$$

where $\theta \in (t_n, t_{n+1})$. Hence (4.2) follows if we put

$$\mathcal{M}_\epsilon(u_0) := \mathcal{H}_p(u(t_{n+1}; u_0)) - \mathcal{H}_p(u(\theta; u_0)).$$

To prove (4.3), we assume without loss of generality that $r > \|g\|/\alpha$ (otherwise, one can use $r + \|g\|/\alpha$ instead of r), and claim that $\mathcal{B}_r(0)$ is a positively invariant set for the semigroup $S(\cdot)$. Indeed, by (2.4), for all $t \geq 0$ and $u_0 \in \mathcal{B}_r(0)$,

$$\|u(t; u_0)\|^2 \leq e^{-\alpha t}\|u_0\|^2 + \frac{\|g\|^2}{\alpha^2}(1 - e^{-\alpha t}) \leq e^{-\alpha t}\Big(r^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{\|g\|^2}{\alpha^2} \leq r^2. \tag{4.4}$$

By the local Lipschitz continuity of A_p and F, we obtain

$$\begin{aligned} \|\mathcal{M}_\epsilon(u_0)\| &= \|\nu(A_p u(t_{n+1}) - A_p u(\theta)) + (F(u(t_{n+1})) - F(u(\theta)))\| \\ &\leq (\nu L_{p,r} + L_r)\|u(t_{n+1}) - u(\theta)\|. \end{aligned}$$

By the first order Taylor expansion again, we have

$$\begin{aligned} u(t_{n+1}) - u(\theta) &= (t_{n+1} - \theta)\frac{du}{dt}(\hat{\theta}) = (t_{n+1} - \theta)\mathcal{H}_p(u(\hat{\theta})) \\ &= (t_{n+1} - \theta)(\nu A_p u(\hat{\theta}) + F(u(\hat{\theta})) + g) \end{aligned}$$

for some $\hat{\theta} \in (\theta, t_{n+1})$. By the local Lipschitz continuity of A_p and F again, it follows from (4.4) that

$$\|u(t_{n+1}) - u(\theta)\| \leq |t_{n+1} - \theta|\Big((\nu L_{p,r} + L_r)\|u(\hat{\theta})\| + \|g\|\Big) \leq \epsilon(r(\nu L_{p,r} + L_r) + \|g\|),$$

which further implies that for all $u_0 \in \mathcal{B}_r(0)$,

$$\|\mathcal{M}_\epsilon(u_0)\| \leq \epsilon\Big(r(\nu L_{p,r} + L_r)^2 + \|g\|(\nu L_{p,r} + L_r)\Big) =: \epsilon C_r,$$

where C_r is obviously increasing in r. The proof is complete. ∎

4.2. Discretization error of order two. We now use the generalized Taylor expansion in Lemma 4.1 to estimate the discretisation error of solutions when the initial data are restricted on the ball $\mathcal{B}_{r^*}(0)$.

THEOREM 4.2. *Let $u(t; u_0)$ and $u_n^\epsilon(u_0)$ be the solutions of LDS (2.3) and IES (3.1) respectively, where $u_0 \in \mathcal{B}_{r^*}(0)$. We have the discretisation error of order 2:*

$$\|u(\epsilon; u_n^\epsilon(u_0)) - u_{n+1}^\epsilon(u_0)\| \leq \epsilon^2 C_{r^*}, \quad \forall \epsilon \in (0, \epsilon^*],\ n \in \mathbb{N}_0. \tag{4.5}$$

Furthermore, for each $T > 0$, there is a $C_{T,r^*} > 0$ such that

$$\|u(t_n; u_0) - u_n^\epsilon(u_0)\| \leq \epsilon C_{T,r^*}, \quad \forall t_n := \epsilon n \in [0, T],\ \epsilon \in (0, \epsilon^*]. \tag{4.6}$$

Proof. Both (4.2) and (3.1) can be rewritten as

$$u(t_{n+1}) = u(t_n) + \epsilon\mathcal{H}_p(u(t_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0) \text{ and } u^\epsilon_{n+1} = u^\epsilon_n + \epsilon\mathcal{H}(u^\epsilon_{n+1}),$$

where $\mathcal{H}_p = \nu A_p + F + gI$ as given in (4.1). From the difference between the above two equalities, we know that the discretisation error $\Delta^\epsilon_n(u_0) := u(t_n; u_0) - u^\epsilon_n(u_0)$ satisfies the following equation:

$$\Delta^\epsilon_{n+1} = \Delta^\epsilon_n + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0).$$

Taking the inner product with Δ^ϵ_{n+1} yields

$$\begin{aligned}\|\Delta^\epsilon_{n+1}\|^2 &= (\Delta^\epsilon_n + \epsilon(\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})) - \epsilon\mathcal{M}_\epsilon(u_0), \Delta^\epsilon_{n+1}) \\ &\le (\|\Delta^\epsilon_n\| + \epsilon\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| + \epsilon\|\mathcal{M}_\epsilon(u_0)\|)\|\Delta^\epsilon_{n+1}\|,\end{aligned}$$

which further implies

$$\|\Delta^\epsilon_{n+1}\| \le \|\Delta^\epsilon_n\| + \epsilon\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| + \epsilon\|\mathcal{M}_\epsilon(u_0)\|. \tag{4.7}$$

Since $\mathcal{B}_{r^*}(0)$ is positively invariant under both $S(t)$ and $S_\epsilon(n)$ (see Lemma 2.1 and Theorem 3.1), it follows that $u(t_{n+1}), u^\epsilon_n \in \mathcal{B}_{r^*}(0)$, and thus we see from the local Lipschitz continuity of A_p and F that

$$\begin{aligned}&\|\mathcal{H}_p(u(t_{n+1})) - \mathcal{H}_p(u^\epsilon_{n+1})\| \le \nu\|A_p u(t_{n+1}) - A_p u^\epsilon_{n+1}\| + \|F(u(t_{n+1})) - F(u^\epsilon_{n+1})\| \\ &\le (\nu L_{p,r^*} + L_{r^*})\|u(t_{n+1}) - u^\epsilon_{n+1}\| = (\nu L_{p,r^*} + L_{r^*})\|\Delta^\epsilon_{n+1}\|.\end{aligned}$$

By Lemma 4.1, $\|\mathcal{M}_\epsilon(u_0)\| \le \epsilon C_{r^*}$ for all $u_0 \in \mathbb{B}_{r^*}(0)$, and thus (4.7) yields

$$\|\Delta^\epsilon_{n+1}\| \le \|\Delta^\epsilon_n\| + \epsilon(\nu L_{p,r^*} + L_{r^*})\|\Delta^\epsilon_{n+1}\| + \epsilon^2 C_{r^*}.$$

Denote by $\hat{L}_{r^*} := \nu L_{p,r^*} + L_{r^*}$. By (3.4), for all $\epsilon \in (0, \epsilon^*]$, $\epsilon\hat{L}_{r^*} \le \epsilon^*\hat{L}_{r^*} < 1$ and thus

$$\|\Delta^\epsilon_{n+1}\| \le \frac{1}{1 - \epsilon\hat{L}_{r^*}}\|\Delta^\epsilon_n\| + \epsilon^2 C_{r^*}, \ \forall n \in \mathbb{N}_0, \ \epsilon \in (0, \epsilon^*], \tag{4.8}$$

where C_{r^*} is $1/(1 - \epsilon^*\hat{L}_{r^*})$-times bigger than the original constant. Since $\Delta^\epsilon_0 = 0$, we infer from (4.8) that $\|u(\epsilon; u_0) - u^\epsilon_1(u_0)\| = \|\Delta^\epsilon_1\| \le \epsilon^2 C_{r^*}$. Using u^ϵ_n as an initial datum in the above formula, we obtain the discretization error (4.5) of order 2.

On the other hand, for all $t_n = \epsilon n \in [0, T]$, by the recursive inequality (4.8) and $\Delta^\epsilon_0 = 0$, we have

$$\|\Delta^\epsilon_n\| \le \epsilon^2 C_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1 - \epsilon\hat{L}_{r^*})^j}. \tag{4.9}$$

Since $\epsilon\hat{L}_{r^*} < 1$ and $n \le T/\epsilon$, it follows that

$$\epsilon\hat{L}_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1 - \epsilon\hat{L}_{r^*})^j} = \frac{1 - (1 - \epsilon\hat{L}_{r^*})^n}{(1 - \epsilon\hat{L}_{r^*})^{n-1}} \le (1 - \epsilon\hat{L}_{r^*})^{-(n-1)} \le (1 - \epsilon\hat{L}_{r^*})^{-\frac{T}{\epsilon}} \uparrow e^{T\hat{L}_{r^*}}$$

as $\epsilon \downarrow 0$, where the last limit is deduced from the basic limit $(1 + 1/k)^k \uparrow e$ as $k \to \infty$. By (4.9),

$$\|\Delta^\epsilon_n\| \le \epsilon\frac{C_{r^*}}{\hat{L}_{r^*}}\epsilon\hat{L}_{r^*} \sum_{j=0}^{n-1} \frac{1}{(1 - \epsilon\hat{L}_{r^*})^j} \le \epsilon\frac{C_{r^*} e^{T\hat{L}_{r^*}}}{\hat{L}_{r^*}} =: \epsilon C_{T,r^*},$$

for all $\epsilon \in (0, \epsilon^*]$, $t_n = \epsilon n \in [0, T]$ and $u_0 \in \mathbb{B}_{r^*}(0)$. Hence (4.6) holds true. ∎

5. Numerical attractors: existence, optimized bound and continuity. In this section, we derive the existence, optimized bound and continuity of a numerical attractor for IES (3.1).

5.1. Estimates for tails of numerical solutions. We need to give the estimate of tails of the solutions in $\mathcal{B}_{r^*}(0)$.

LEMMA 5.1. *Let $\epsilon \in (0, \epsilon^*]$. Then, for each $\delta > 0$, there are $I(\delta) \in \mathbb{N}$ (independent of ϵ) and $N_\epsilon(\delta) \in \mathbb{N}$ such that the solution of IES* (3.1) *satisfies*

$$\|S_\epsilon(n)u_0\|^2_{\ell^2(|i|\geq I(\delta))} = \sum_{|i|\geq I(\delta)} |u^\epsilon_{n,i}|^2 < \delta,\ \forall n \geq N_\epsilon(\delta),\ u_0 \in \mathcal{B}_{r^*}(0). \tag{5.1}$$

Proof. As usual, we consider a cut-off function $\xi \in C^1(\mathbb{R}^+, [0, 1])$ such that $\xi(s) = 0$ for all $s \in [0, 1/2]$ and $\xi(s) = 1$ for all $s \in [1, +\infty)$. For each $k > 0$, we define

$$\xi_k := (\xi_{k,i})_{i\in\mathbb{Z}} \ \text{ where } \xi_{k,i} = \xi\Big(\frac{|i|}{k}\Big),\ \forall i \in \mathbb{Z}. \tag{5.2}$$

Since the IES (3.1) is well-defined in ℓ^2 and $\xi_k u^\epsilon_n = (\xi_{k,i}u^\epsilon_{n,i})_{i\in\mathbb{Z}} \in \ell^2$, we can use the inner product of (3.1) with $\xi_k u^\epsilon_n$ to obtain

$$\sum_{i\in\mathbb{Z}} \xi_{k,i}|u^\epsilon_{n,i}|^2 = (u^\epsilon_{n-1}, \xi_k u^\epsilon_n) + \epsilon(\nu A_p u^\epsilon_n + F(u^\epsilon_n) + g, \xi_k u^\epsilon_n). \tag{5.3}$$

We now estimate all terms on the right-hand side. First,

$$(u^\epsilon_{n-1}, \xi_k u^\epsilon_n) \leq \frac{1}{2}\sum_{i\in\mathbb{Z}} \xi_{k,i}|u^\epsilon_{n,i}|^2 + \frac{1}{2}\sum_{i\in\mathbb{Z}} \xi_{k,i}|u^\epsilon_{n-1,i}|^2.$$

Second, since $A_p x = -B^*(|Bx|^{p-2}Bx)$ and $B(xy) = xBy + yBx$, it follows that

$$\begin{aligned}
&(A_p u^\epsilon_n, \xi_k u^\epsilon_n) = -(|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, B(\xi_k u^\epsilon_n))\\
&= -(|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, \xi_k Bu^\epsilon_n) - (|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, u^\epsilon_n B\xi_k)\\
&= -\sum_{i\in\mathbb{Z}} \xi_{k,i}|(Bu^\epsilon_n)_i|^p - (|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, u^\epsilon_n B\xi_k) \leq |(|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, u^\epsilon_n B\xi_k)|.
\end{aligned}$$

Since $|\xi'(s)| \leq C$ for all $s \geq 0$, it follows from the mean-valued theorem that

$$|(B\xi_k)_i| = \Big|\xi\Big(\frac{|i+1|}{k}\Big) - \xi\Big(\frac{|i|}{k}\Big)\Big| \leq \frac{C}{k},\ \forall k \in \mathbb{N},\ i \in \mathbb{Z}.$$

By Theorem 3.1, $u^\epsilon_n \in \mathcal{B}_{r^*}(0)$ and thus $|(Bu^\epsilon_n)_i| \leq \|Bu^\epsilon_n\| \leq 2\|u^\epsilon_n\| \leq 2r^*$. Therefore,

$$\begin{aligned}
&\epsilon(\nu A_p u^\epsilon_n, \xi_k u^\epsilon_n) \leq \epsilon\nu|(|Bu^\epsilon_n|^{p-2}Bu^\epsilon_n, u^\epsilon_n B\xi_k)|\\
&\leq \epsilon\nu\sum_{i\in\mathbb{Z}} |(B\xi_k)_i||(Bu^\epsilon_n)_i|^{p-1}|u^\epsilon_{n,i}| \leq \epsilon\frac{C_p}{k}(r^*)^{p-1}\|u^\epsilon_{n,i}\| \leq \epsilon\frac{C_p}{k}(r^*)^p.
\end{aligned}$$

Third, by (2.2), we have

$$\epsilon(F(u^\epsilon_n) + g, \xi_k u^\epsilon_n) \leq -\frac{\epsilon\alpha}{2}\sum_{i\in\mathbb{Z}} \xi_{k,i}|u^\epsilon_{n,i}|^2 + \frac{\epsilon}{2\alpha}\sum_{i\in\mathbb{Z}} \xi_{k,i}g_i^2.$$

Substituting the three estimates into (5.3) we find

$$\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 \le \frac{1}{1+\epsilon\alpha}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n-1,i}|^2 + \frac{\epsilon}{1+\epsilon\alpha}\Big(\frac{C_p}{k}(r^*)^p + \frac{1}{\alpha}\sum_{|i|\ge -1+k/2} g_i^2\Big). \tag{5.4}$$

Given $\delta>0$, there is $I(\delta)\in\mathbb{N}$ (independent of ϵ) such that

$$\frac{C_p}{k}(r^*)^p + \frac{1}{\alpha}\sum_{|i|\ge -1+k/2} g_i^2 < \frac{\alpha}{2}\delta,\ \forall k\ge I(\delta),$$

which together with (5.4) implies that for all $k\ge I(\delta)$,

$$\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 \le \frac{1}{1+\epsilon\alpha}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n-1,i}|^2 + \frac{\epsilon}{1+\epsilon\alpha}\frac{\alpha}{2}\delta.$$

Iterating the above inequality yields

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 &\le \frac{1}{(1+\epsilon\alpha)^n}\sum_{i\in\mathbb{Z}}\xi_{k,i}|u_{0,i}|^2 + \frac{\delta}{2}\sum_{j=1}^{n}\frac{\epsilon\alpha}{(1+\epsilon\alpha)^j}\\ &\le \frac{\|u_0\|^2}{(1+\epsilon\alpha)^n} + \frac{\delta}{2} \le \frac{(r^*)^2}{(1+\epsilon\alpha)^n} + \frac{\delta}{2} \to \frac{\delta}{2} \text{ as } n\to\infty.\end{aligned}$$

Hence, there is $N_\epsilon(\delta)\in\mathbb{N}$ such that for all $n\ge N_\epsilon(\delta)$ and $k\ge I(\delta)$,

$$\sum_{|i|\ge k}|u^{\epsilon}_{n,i}|^2 \le \sum_{i\in\mathbb{Z}}\xi_{k,i}|u^{\epsilon}_{n,i}|^2 < \delta.$$

Setting $k=I(\delta)$ we obtain (5.1) as desired. ∎

5.2. Existence and connection of numerical attractors.

Recall that a compact subset $\mathcal{A}_\epsilon$ of $\mathcal{B}_{r^*}(0)$ is call a (numerical) attractor of the discrete-time dynamical system $\{S_\epsilon(n)\}_{n\in\mathbb{N}_0}$ for the IES (3.1) if $\mathcal{A}_\epsilon$ is invariant and attracting

$$S_\epsilon(n)\mathcal{A}_\epsilon = \mathcal{A}_\epsilon\ (\forall n\in\mathbb{N}),\ \text{and}\ \lim_{n\to\infty}\mathrm{dist}_{\ell^2}(S_\epsilon(n)\mathcal{B}_{r^*}(0),\mathcal{A}_\epsilon)=0.$$

THEOREM 5.2. *For each $\epsilon\in(0,\epsilon^*]$, the discrete semigroup $\{S_\epsilon(n)\}_{n\in\mathbb{N}_0}$ on $\mathcal{B}_{r^*}(0)$ has a unique numerical attractor $\mathcal{A}_\epsilon$ such that $\mathcal{A}_\epsilon$ is topologically connected in ℓ^2.*

Proof. We prove that the semigroup $S_\epsilon(\cdot)$ is asymptotically compact on $\mathcal{B}_{r^*}(0)$. It suffices to prove that the sequence $\{S_\epsilon(n)u_0^n : n\in\mathbb{N}\}$ is relative compact for any sequence $\{u_0^n : n\in\mathbb{N}\}$ in $\mathcal{B}_{r^*}(0)$.

Given $\delta>0$, we see from Lemma 5.1 that there are $N_\epsilon(\delta), I(\delta)\in\mathbb{N}$ such that

$$\|S_\epsilon(n)u_0^n\|^2_{\ell^2(|i|>I)} = \|u_n^\epsilon(u_0^n)\|^2_{\ell^2(|i|>I)} < \delta^2,\quad \forall\ n\ge N.$$

By Theorem 3.1, $\{S_\epsilon(n)u_0^n : n\in\mathbb{N}\}\subset\mathcal{B}_{r^*}(0)$, which is bounded in ℓ^2. In particular,

$$(S_\epsilon(n)u_0^n)_{|i|\le I} \text{ is bounded in } \ell^2(|i|\le I)\cong\mathbb{R}^{2I+1},$$

where the space is finitely dimensional. Then the sequence $\{(S_\epsilon(n)u_0^n)_{|i|\le I}\}_{n\ge N}$ has a finite δ-net with centers $x_1,x_2,\cdots,x_{k_0}\in\mathbb{R}^{2I+1}$. We define the null-expansion $\widetilde{y}$ of an element $y\in\mathbb{R}^{2I+1}$ by

$$\widetilde{y}_i = y_i,\forall|i|\le I \text{ and } \widetilde{y}_i = 0,\forall|i|>I.$$

Hence, for each $n \geq N$, there is $x_k \in \mathbb{R}^{2I+1}$, where $k \in \{1, 2, \cdots, k_0\}$, such that

$$\|S_\epsilon(n)u_0^n - \widetilde{x}_k\|^2 = \|S_\epsilon(n)u_0^n\|^2_{\ell^2(|i|>I)} + \|S_\epsilon(n)u_0^n - x_k\|^2_{\ell^2(|i|\leq I)} < 2\delta^2,$$

which means that the sequence $\{S_\epsilon(n)u_0^n : n \geq N\}$ has a finite $\sqrt{2}\delta$-net in ℓ^2. Since the finite set $\{S_\epsilon(n)u_0^n : n < N\}$ is compact, it follows that the whole sequence $\{S_\epsilon(n)u_0^n : n \in \mathbb{N}\}$ has a finite $\sqrt{2}\delta$-net too and thus relatively compact in ℓ^2.

Therefore, since the state space $\mathcal{B}_{r^*}(0)$ is bounded, it follows that the discrete semigroup $S_\epsilon(\cdot)$ has a unique numerical attractor denoted by $\mathcal{A}_\epsilon$.

Suppose that $\mathcal{A}_\epsilon$ is not topologically connected. Then there are two open sets $O_1, O_2 \subset \ell^2$ such that

$$O_1 \cup O_2 \supset \mathcal{A}_\epsilon \quad O_1 \cap \mathcal{A}_\epsilon \neq \emptyset, \; O_2 \cap \mathcal{A}_\epsilon \neq \emptyset.$$

Let $\mathcal{K}_\epsilon$ be the closed convex hull of $\mathcal{A}_\epsilon$ in ℓ^2. Then $\mathcal{K}_\epsilon$ is pathwise connected and thus topologically connected in ℓ^2. As the ball $\mathcal{B}_{r^*}(0)$ is closed and convex, we have $\mathcal{K}_\epsilon \subset \mathcal{B}_{r^*}(0)$ and thus the set $S_\epsilon(n)\mathcal{K}_\epsilon$ is well-defined. By the invariance of $\mathcal{A}_\epsilon$, we have $\mathcal{A}_\epsilon = S_\epsilon(n)\mathcal{A}_\epsilon \subset S_\epsilon(n)\mathcal{K}_\epsilon$ and thus

$$O_1 \cap S_\epsilon(n)\mathcal{K}_\epsilon \neq \emptyset, \; O_2 \cap S_\epsilon(n)\mathcal{K}_\epsilon \neq \emptyset, \; \forall n \in \mathbb{N}. \tag{5.5}$$

By Lemma 3.4, the operator $S_\epsilon(n) : \mathcal{B}_{r^*}(0) \to \mathcal{B}_{r^*}(0)$ is (Lipschitz) continuous. Since $\mathcal{K}_\epsilon$ is topologically connected, $S_\epsilon(n)\mathcal{K}_\epsilon$ is topologically connected too, which together with (5.5) implies that $O_1 \cup O_2$ cannot cover $S_\epsilon(n)\mathcal{K}_\epsilon$. In particular, for each $n \in \mathbb{N}$ there is $x_n \in S_\epsilon(n)\mathcal{K}_\epsilon$ so that $x_n \notin O_1 \cup O_2$. Since $\mathcal{A}_\epsilon$ attracts the bounded set $\mathcal{K}_\epsilon$, it follows that $d_{\ell^2}(x_n, \mathcal{A}_\epsilon) \to 0$. as $n \to \infty$. By the compactness of $\mathcal{A}_\epsilon$, passing to a subsequence, $x_n \to x$ for some $x \in \mathcal{A}_\epsilon$. Hence $x \in O_1 \cup O_2$, which contradicts $x_n \in \ell^2 \setminus (O_1 \cup O_2)$ (a closed set). ∎

5.3. Optimized bound and continuity of attractors on f, g. To give an optimized bound of the numerical attractors, we consider the restriction of the IES (3.1) on arbitrary balls.

LEMMA 5.3. *For each $r_0 > \|g\|/\alpha$, there is $\epsilon_{r_0} > 0$, given by*

$$\epsilon_{r_0} := \frac{1}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|},$$

such that, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $u_0 \in \mathcal{B}_{r_0}(0)$, the IES (3.1) has a unique solution $\{u_n^\epsilon\}_{n\in\mathbb{N}} \subset \mathcal{B}_{r_0}(0)$, which generates a discrete semigroup

$$S_{\epsilon,r_0}(n) : \mathcal{B}_{r_0}(0) \to \mathcal{B}_{r_0}(0), \quad S_{\epsilon,r_0}(n)u_0 = u_n^\epsilon(u_0), \; \forall \epsilon \in (0, \epsilon_{r_0}].$$

Proof. By the same method as in **Step 1** of Theorem 3.1, one can prove that, for each $u_0 \in \mathcal{B}_{r_0}(0)$ and $\epsilon \in (0, \epsilon_{r_0}]$, the operator $\mathcal{M}^\epsilon_{u_0} : \mathcal{B}_{r_0+1}(0) \to \mathcal{B}_{r_0+1}(0)$ is well-defined and contractive. Hence the IES (3.1) with $n = 1$ has a unique solution $u_1^\epsilon \in \mathcal{B}_{r_0+1}(0)$. By the method in **Step 2**, we have $u_1^\epsilon \in \mathcal{B}_{r_0}(0)$. Suppose the solution $u_n^\epsilon \in \mathcal{B}_{r_0}(0)$ for some $n \in \mathbb{N}$. Then we see from (3.8) in **Step 3** and $r_0 > \|g\|/\alpha$ that

$$\begin{aligned}
\|u_{n+1}^\epsilon\|^2 &\leq \frac{1}{1+\epsilon\alpha}\Big(\|u_n^\epsilon\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big) \leq \frac{1}{1+\epsilon\alpha}\Big(r_0^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big) \\
&= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{1}{1+\epsilon\alpha}\Big(\frac{\|g\|^2}{\alpha^2} + \epsilon\alpha\frac{\|g\|^2}{\alpha^2}\Big) \\
&\leq \Big(r_0^2 - \frac{\|g\|^2}{\alpha^2}\Big) + \frac{\|g\|^2}{\alpha^2} = r_0^2.
\end{aligned}$$

Hence the recursive proof is available. ∎

Note that $\epsilon_{r_0} \downarrow 0$ as $r_0 \to \infty$ and $\epsilon_{r^*} = \epsilon^*$, where ϵ^* is defined by (3.4).

THEOREM 5.4. *For each $r_0 > \|g\|/\alpha$, there is $\epsilon_{r_0} > 0$ such that, for each $\epsilon \in (0, \epsilon_{r_0}]$, the discrete semigroup $S_{\epsilon,r_0}(\cdot)$ has a unique attractor $\mathcal{A}_{\epsilon,r_0}$ in $\mathcal{B}_{r_0}(0)$. Moreover, the numerical attractor $\mathcal{A}_\epsilon$ in Theorem 5.2 fulfills*

$$\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r_0}, \ \forall \epsilon \in (0, \min\{\epsilon_{r_0}, \epsilon^*\}],$$

$$\|\mathcal{A}_\epsilon\| := \sup_{x \in \mathcal{A}_\epsilon} \|x\| \le \frac{\|g\|}{\alpha}, \ \forall \epsilon \in (0, \epsilon^*]. \tag{5.6}$$

Proof. By the same method as in Theorem 5.2, one can prove the existence of a unique attractor $\mathcal{A}_{\epsilon,r_0}$. To prove the equality between two attractors, we let $r_0 < \hat{r}_0$ and $r_0, \hat{r}_0 \in (\|g\|/\alpha, +\infty)$ (note that r^* belongs to this interval). Since $r_0 \to \epsilon_{r_0}$ is decreasing, we have $\min\{\epsilon_{r_0}, \epsilon_{\hat{r}_0}\} = \epsilon_{\hat{r}_0}$.

Next, we prove that $\mathcal{B}_{r_0}(0)$ is an absorbing set of the semigroup $S_{\epsilon,\hat{r}_0}(\cdot)$ on $\mathcal{B}_{\hat{r}_0}(0)$ for all $\epsilon \in (0, \epsilon_{\hat{r}_0}]$. Given any ball $\mathcal{B}_r(0)$ with the radius $r \in (0, \hat{r}_0]$. For each $u_0 \in \mathcal{B}_r(0) \subset \mathcal{B}_{\hat{r}_0}(0)$, it is similar to prove the recursive formula as in (3.8), given by

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \le \frac{1}{1+\epsilon\alpha}\Big(\|S_{\epsilon,\hat{r}_0}(n-1)u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2\Big), \ \forall n \in \mathbb{N}_0.$$

Iterating it yields

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \le \frac{1}{(1+\epsilon\alpha)^n}\|u_0\|^2 + \frac{\epsilon}{\alpha}\|g\|^2 \sum_{j=1}^{n} \frac{1}{(1+\epsilon\alpha)^j} \le \frac{r^2}{(1+\epsilon\alpha)^n} + \frac{\|g\|^2}{\alpha^2}.$$

Since $r^2/(1+\epsilon\alpha)^n \to 0$ as $n \to \infty$ and $r_0 > \|g\|/\alpha$, there is $N = N(r)$ such that for all $n \ge N$,

$$\|S_{\epsilon,\hat{r}_0}(n)u_0\|^2 \le \frac{r^2}{(1+\epsilon\alpha)^n} + \frac{\|g\|^2}{\alpha^2} \le (r_0^2 - \frac{\|g\|^2}{\alpha^2}) + \frac{\|g\|^2}{\alpha^2} = r_0^2.$$

Hence $\mathcal{B}_{r_0}(0)$ is a bounded absorbing set for $S_{\epsilon,\hat{r}_0}(\cdot)$.

Since an attractor is the omega-limit set of any bounded absorbing set, it follows that

$$\mathcal{A}_{\epsilon,\hat{r}_0} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\ge k} S_{\epsilon,\hat{r}_0}(n)\mathcal{B}_{r_0}(0)} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\ge k} S_{\epsilon,r_0}(n)\mathcal{B}_{r_0}(0)} = \mathcal{A}_{\epsilon,r_0},$$

where we have used the uniqueness of solutions to ensure $S_{\epsilon,\hat{r}_0}(n) = S_{\epsilon,r_0}(n)$ on $\mathcal{B}_{r_0}(0)$. In particular, since $\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r^*}$, it follows that

$$\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,r_0}, \ \forall 0 < \epsilon \le \min\{\epsilon_{r_0}, \epsilon^*\}, \ r_0 > \frac{\|g\|}{\alpha}.$$

If $r_0 \in (\|g\|/\alpha, r^*]$, then $\epsilon_{r_0} \ge \epsilon^*$. The above equality implies

$$\mathcal{A}_\epsilon \subset \mathcal{B}_{r_0}(0), \ \forall r_0 \in (\frac{\|g\|}{\alpha}, r^*], \ \epsilon \in (0, \epsilon^*].$$

Letting $r_0 \to \|g\|/\alpha$ we obtain $\mathcal{A}_\epsilon \subset \mathcal{B}_{\|g\|/\alpha}(0)$ for all $\epsilon \in (0, \epsilon^*]$. □

Example. The bound $\|g\|/\alpha$ of $\|\mathcal{A}_\epsilon\|$ in (5.6) seems to be optimized. Let $\nu = 0$ and $f(s) = -\alpha s$ (satisfying (2.2)). Then the IES (3.1) is read as

$$u_n = u_{n-1} - \epsilon\alpha u_n + \epsilon g.$$

It has an entire solution $u_n \equiv g/\alpha$ for all $n \in \mathbb{Z}$, which belongs to the attractor and $\|u_n\| = \|g\|/\alpha$.

To close this section, we deduce the continuity (upper and lower semi-continuity) of the numerical attractors depending on the nonlinearity f or the external force g. The Hausdorff metric between two subsets $X, Y \subset \ell^2$ is defined by

$$\mathrm{dist}_h(X,Y) = \max(d(X,Y), d(X,Y)),\ d(X,Y) := \sup_{x\in X} \inf_{y\in Y} \|x-y\|.$$

Corollary 5.5. *Denoting the numerical attractor $\mathcal{A}_\epsilon$ by $\mathcal{A}_\epsilon(\alpha, g)$, depending on the constant α in (1.2) and the force g, we have*

$$\lim_{\alpha\to\infty} dist_h(\mathcal{A}_\epsilon(\alpha,g), \{0\}) = 0 \text{ and } \lim_{g\to 0} dist_h(\mathcal{A}_\epsilon(\alpha,g), \{0\}) = 0.$$

In particular, if f_1 and f_2 satisfy (2.2) with the same constant α, then

$$\lim_{\alpha\to\infty} dist_h(\mathcal{A}_\epsilon(f_1), \mathcal{A}_\epsilon(f_2)) = 0.$$

Proof. By (5.6) we have

$$\mathrm{dist}_h(\mathcal{A}_\epsilon(\alpha,g), \{0\}) = \|\mathcal{A}_\epsilon(\alpha,g)\| \le \frac{\|g\|}{\alpha} \to 0$$

as $\alpha \to \infty$ or $g \to 0$. By (5.6) again,

$$\mathrm{dist}_h(\mathcal{A}_\epsilon(f_1), \mathcal{A}_\epsilon(f_2)) \le \|\mathcal{A}_\epsilon(f_1)\| + \|\mathcal{A}_\epsilon(f_2)\| \le 2\frac{\|g\|}{\alpha} \to 0$$

as $\alpha \to \infty$. □

Remark 5.6. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the dissipative condition (2.2) if and only if the curve $y = f(s)$ falls in the area surrounded by two straight lines $y = -\alpha s$ and $s = 0$, In particular, the graph of $y = f(s)$ closes to the vertical axis as $\alpha \to \infty$, see FIG. 2.

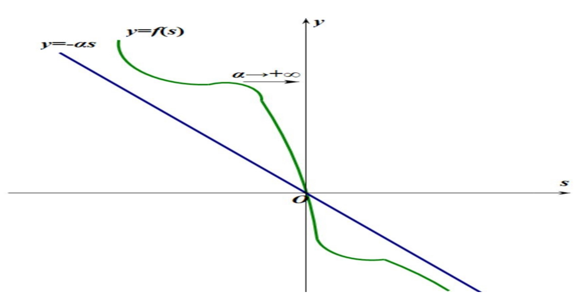

Fig. 2. *Graph and limit of f*

6. Convergence from numerical attractor to global attractor. We are in a position to establish the upper semi-continuity of the numerical attractors.

THEOREM 6.1. *Let $\mathcal{A}_\epsilon$ and $\mathcal{A}$ be the numerical attractor and the global attractor for IES (3.1) and Eq.(2.3) respectively. Then it holds the upper semi-continuity under the Hausdorff semi-metric*

$$\lim_{\epsilon\to 0^+} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}) = 0. \tag{6.1}$$

Proof. Suppose (6.1) is false, then there are $\epsilon_k \downarrow 0$ (as $k \to +\infty$), $x_k \in \mathcal{A}_{\epsilon_k}$ and $\delta_0 > 0$ such that

$$d_{\ell^2}(x_k, \mathcal{A}) \geq \delta_0,\ \forall k \in \mathbb{N}. \tag{6.2}$$

Since the global attractor $\mathcal{A}$ attracts $\mathbb{B}_{r^*}(0)$, we can find a $T > 0$ such that

$$d_{\ell^2}(u(t; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2},\ \forall t \geq T.$$

We can assume $\epsilon_k < 1$ for all $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\epsilon_k n_k \in [T, T + 1]$ and thus

$$d_{\ell^2}(u(\epsilon_k n_k; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2},\ \forall k \in \mathbb{N}.$$

By the invariance of each attractor $\mathcal{A}_{\epsilon_k}$, we have $x_k = S_{\epsilon_k}(n_k)y_k$ for some $y_k \in \mathcal{A}_{\epsilon_k} \subset \mathbb{B}_{r^*}(0)$. Now, the discretization error (4.6) in Lemma 4.5 implies

$$\|S_{\epsilon_k}(n_k)y_k - u(\epsilon_k n_k; y_k)\| \leq \epsilon_k C_{T+1,r^*},$$

where the constant depends on $T + 1$ in view of $\epsilon_k n_k \leq T + 1$. Since $\epsilon_k \downarrow 0$, there is an $k_0 \in \mathbb{N}$ such that

$$\|S_{\epsilon_k}(n_k)y_k - u(\epsilon_k n_k; y_k)\| < \frac{\delta_0}{2},\ \forall k \geq k_0.$$

Therefore, for all $k \geq k_0$,

$$\begin{aligned} d_{\ell^2}(x_k, \mathcal{A}) &= \mathrm{dist}_{\ell^2}(S_{\epsilon_k}(n_k)y_k, \mathcal{A}) \\ &\leq \|S_{\epsilon_k}(n_k)y_k - u(\epsilon_k n_k; y_k)\| + d_{\ell^2}(u(\epsilon_k n_k; \mathbb{B}_{r^*}(0)), \mathcal{A}) < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0, \end{aligned}$$

which gives a contradiction to (6.2). ∎

COROLLARY 6.2. *The union $\cup_{\epsilon\in(0,\epsilon^*]}\mathcal{A}_\epsilon$ is relatively compact in ℓ^2.*

Proof. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence taken from the union. Then there is $\{\epsilon_k\} \subset (0, \epsilon^*]$ such that $x_k \in \mathcal{A}_{\epsilon_k}$. We prove that $\{x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence in two cases.

Case 1: $\inf \epsilon_k > 0$. Then $\epsilon_k \in [\epsilon_0, \epsilon^*]$ for some $\epsilon_0 > 0$. By Lemma 5.1, the tail estimate of solutions is uniform for all ϵ_k. More precisely, for $\delta > 0$, there is $N(\delta), I(\delta) \in \mathbb{N}$ such that for all $n \geq N$,

$$\|S_{\epsilon_k}(n)u_0\|_{\ell^2(|i|>I)} < \delta,\quad \forall\ n \geq N,\ k \in \mathbb{N},\ u_0 \in \mathcal{B}_{r^*}(0).$$

The invariance implies $x_k = S_{\epsilon_k}(N)y_k$ for some $y_k \in \mathcal{B}_{r^*}(0)$ and thus

$$\|x_k\|_{\ell^2(|i|>I)} < \delta,\quad \forall k \in \mathbb{N}. \tag{6.3}$$

Since $\{x_k\}$ is bounded in ℓ^2, its truncation on $\ell^2(|i| \le I) = \mathbb{R}^{2I+1}$ is also bounded. Hence the truncated sequence of $\{x_k\}$ has a finite δ-net in $\mathbb{R}^{2I+1}$, which together with (6.3) implies that the sequence $\{x_k\}$ has a finite 2δ-net and thus it is relatively compact in ℓ^2.

Case 2: $\inf \epsilon_k = 0$. Passing to a subsequence, we assume $\epsilon_k \to 0$. By the upper semi-continuity as in Theorem 6.1, we have

$$d_{\ell^2}(x_k, \mathcal{A}) \le d_{\ell^2}(\mathcal{A}_{\epsilon_k}, \mathcal{A}) \to 0 \text{ as } k \to \infty.$$

Since $\mathcal{A}$ is compact, we see from [9, lemma 2.3] that the sequence $\{x_k\}$ has a convergent subsequence. □

Remark 6.3. The uniform compactness of attractors is usually applied to prove the upper semi-continuity, see [28, 37]. For the numerical attractors, the situation is reversed.

7. Finitely dimensional approximation of numerical attractors . How to truncate the IES (3.1) on a finite-dimensional space? For each $m \in \mathbb{N}$, the operator $F : \ell^2 \to \ell^2$ has a natural truncation given by

$$F_m : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1},\ F_m(x) = (f(x_i))_{|i|\le m},\ \forall x = (x_i)_{|i|\le m} \in \mathbb{R}^{2m+1}.$$

However, it is not easy to truncate the discrete p-Laplace operators A_p because $A_p x$ $(x = (x_i)_{|i|\le m})$ involves two unknown components x_{m+1} and x_{-m-1} outside $\mathbb{R}^{2m+1}$. To overcome it, we use the periodic boundary conditions (see [2, 17])

$$x_{m+1} = x_{-m} \text{ and } x_{-m-1} = x_m.$$

So, the truncation $A_{p,m} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ of A_p can be defined by

$$\begin{aligned}
&(A_{p,m}x)_{-m} = |x_{-m+1} - x_{-m}|^{p-2}(x_{-m+1} - x_{-m}) - |x_{-m} - x_m|^{p-2}(x_{-m} - x_m),\\
&(A_{p,m}x)_i = (A_p x)_i, \quad \forall |i| < m,\\
&(A_{p,m}x)_m = |x_{-m} - x_m|^{p-2}(x_{-m} - x_m) - |x_m - x_{m-1}|^{p-2}(x_m - x_{m-1})
\end{aligned}$$

for all $x = (x_i)_{|i|\le m} \in \mathbb{R}^{2m+1}$. For $p > 2$, the truncated operator $A_{p,m}$ is nonlinear and thus it is not a matrix. But $A_{p,m}$ is a function of matrixes

$$A_{p,m}x = -B_m^T(|B_m x|^{p-2}B_m x),\ \forall x \in \mathbb{R}^{2m+1},$$

where B_m^T is the transport matrix of B_m and

$$B_m = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} \in (\mathbb{R}^{2m+1})^2.$$

As in (3.2), $A_{p,m} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ is locally Lipschitz continuous:

$$\|A_{p,m}x - A_{p,m}y\| \le L_{p,r}\|x - y\| \text{ and } \|A_{p,m}x\| \le L_{p,r}\|x\|,\ \forall x, y \in \mathcal{B}_r^m(0), \tag{7.1}$$

where $\mathcal{B}_r^m(0)$ is the ball in $\mathbb{R}^{2m+1}$ and $L_{p,r} := (p-1)2^{2p}r^{p-2}$.

Then the IES (3.1) can be truncated as follows:

$$u_n^{\epsilon,m} = u_{n-1}^{\epsilon,m} + \epsilon\nu A_{p,m}u_n^{\epsilon,m} + \epsilon F_m(u_n^{\epsilon,m}) + \epsilon g|_m,\quad u_0^{\epsilon,m} = u_0^m \in \mathbb{R}^{2m+1}, \tag{7.2}$$

where $g|_m := (g_i)_{|i|\le m} \in \mathbb{R}^{2m+1}$ is the truncation of $g \in \ell^2$, and the unknown is denoted by $u_n^{\epsilon,m} = (u_{n,i}^{\epsilon,m})_{|i|\le m} \in \mathbb{R}^{2m+1}$.

7.1. Existence and bound of truncated numerical attractors. As in the infinite dimension case, we show that the truncated IES (7.2) with small step size is solvable when the initial datum belongs to some suitable balls.

LEMMA 7.1. *For each $r_0 > \|g|_m\|/\alpha$ and $m \in \mathbb{N}$, there is $\epsilon_{r_0} > 0$, such that, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $u_0^m \in \mathcal{B}_{r_0}^m(0)$, the truncated IES (7.2) has a unique solution $\{u_n^{\epsilon,m}\}_{n\in\mathbb{N}}$ which satisfies*

$$\|u_n^{\epsilon,m}\|^2 \leq \frac{1}{1+\epsilon\alpha}\Big(\|u_{n-1}^{\epsilon,m}\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big). \tag{7.3}$$

In particular, for all $n \in \mathbb{N}$, $u_n^{\epsilon,m} \in \mathcal{B}_{r_0}^m(0)$.

Proof. We recursively prove it as done in Theorem 3.1. Consider the case $n = 1$. For $\epsilon > 0$ and $u_0^m \in \mathcal{B}_{r_0}^m(0)$, we denote by

$$\mathcal{M}_{u_0^m}^\epsilon(x) = u_0^m + \epsilon\nu A_{p,m}x + \epsilon F_m(x) + \epsilon g|_m, \quad \forall x \in \mathcal{B}_{r_0+1}^m(0).$$

By the local Lipschitz continuity of $A_{p,m}$ and F_m, we have

$$\begin{aligned}\|\mathcal{M}_{u_0^m}^\epsilon(x)\| &\leq \|u_0^m\| + \epsilon(\nu\|A_{p,m}x\| + \|F_m(x)\| + \|g|_m\|)\\ &\leq r_0 + \epsilon\Big((r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g|_m\|\Big).\end{aligned}$$

Using $\|g\|$ instead of $\|g|_m\|$ we put

$$\epsilon_{r_0} := \frac{1}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|}, \tag{7.4}$$

which is independent of m. Since $\|g|_m\| \leq \|g\|$, it follows that for all $\epsilon \in (0, \epsilon_{r_0}]$,

$$\|\mathcal{M}_{u_0^m}^\epsilon(x)\| \leq r_0 + \frac{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g|_m\|}{(r_0+1)(\nu L_{p,r_0+1} + L_{r_0+1}) + \|g\|} \leq r_0 + 1,$$

which means that $\mathcal{M}_{u_0^m}^\epsilon : \mathcal{B}_{r_0+1}^m(0) \to \mathcal{B}_{r_0+1}^m(0)$ is well-defined. By the local Lipschitz continuity of $A_{p,m}$ and F_m again, for all $\epsilon \in (0, \epsilon_{r_0}]$ and $x, y \in \mathcal{B}_{r_0+1}^m(0)$,

$$\begin{aligned}&\|\mathcal{M}_{u_0^m}^\epsilon(x) - \mathcal{M}_{u_0^m}^\epsilon(y)\| \leq \epsilon(\nu\|A_{p,m}x - A_{p,m}y\| + \|F_m(x) - F_m(y)\|)\\ &\leq \epsilon(\nu L_{p,r_0+1} + L_{r_0+1})\|x-y\| \leq \frac{1}{r_0+1}\|x-y\|.\end{aligned}$$

Then the contraction mapping principle implies that the first equation of (7.2) has a unique solution $u_1^{\epsilon,m} \in \mathcal{B}_{r_0+1}^m(0)$ for all $\epsilon \in (0, \epsilon_{r_0}]$.

Now, we take the $\mathbb{R}^{2m+1}$-inner product of the truncted IES (7.2) with $u_1^{\epsilon,m}$, the result is

$$\|u_1^{\epsilon,m}\|^2 = \langle u_0^m, u_1^{\epsilon,m}\rangle + \epsilon\nu\langle A_{p,m}u_1^{\epsilon,m}, u_1^{\epsilon,m}\rangle + \epsilon\langle F_m(u_1^{\epsilon,m}), u_1^{\epsilon,m}\rangle + \epsilon\langle g|_m, u_1^{\epsilon,m}\rangle. \tag{7.5}$$

Since $A_{p,m}$ is the function of the matrix B_m, it follows that for all $x \in \mathbb{R}^{2m+1}$,

$$\langle A_{p,m}x, x\rangle = -\langle |B_m x|^{p-2}B_m x, B_m x\rangle = -\sum_{|i|\leq m}|(B_m x)_i|^p \leq 0.$$

Hence, by estimating other three terms in (7.5) and using the method in Theorem 3.1, we obtain

$$\|u_1^{\epsilon,m}\|^2 \leq \frac{1}{1+\epsilon\alpha}\Big(\|u_0^m\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big).$$

Since $r_0 > \|g|_m\|/\alpha$ and $u_0^m \in \mathcal{B}_{r_0}^m(0)$, it follows that

$$\begin{aligned}\|u_1^{\epsilon,m}\|^2 &\le \frac{1}{1+\epsilon\alpha}\Big(r_0^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)\\ &= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{1}{1+\epsilon\alpha}\Big(\frac{\|g|_m\|^2}{\alpha^2} + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)\\ &= \frac{1}{1+\epsilon\alpha}\Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{\|g|_m\|^2}{\alpha^2} \le r_0^2,\end{aligned}$$

which means $u_1^{\epsilon,m} \in \mathcal{B}_{r_0}^m(0)$ for all $\epsilon \in (0, \epsilon_{r_0}]$. Repeating the above process with $u_{n-1}^{\epsilon,m} \in \mathcal{B}_{r_0}^m(0)$ instead of $u_0^m \in \mathcal{B}_{r_0}^m(0)$, the recursive proof is available. ∎

Note that the radius r^* and step size ϵ^* in the previous sections satisfy

$$r^* = \sqrt{1 + \frac{\|g\|^2}{\alpha^2}} > \frac{\|g|_m\|}{\alpha}, \quad \text{and } \epsilon^* = \epsilon_{r^*},$$

where ϵ_{r^*} is defined as in (7.4). By Lemma 7.1, for each $\epsilon \in (0, \epsilon^*]$ and $m \in \mathbb{N}$, we can define a discrete semigroup by

$$S_{\epsilon,m}(n) : \mathcal{B}_{r^*}^m(0) \to \mathcal{B}_{r^*}^m(0), \; S_{\epsilon,m}(n)u_0^m = u_n^{\epsilon,m}(u_0^m), \; \forall n \in \mathbb{N}_0.$$

THEOREM 7.2. *For each $\epsilon \in (0, \epsilon^*]$ and $m \in \mathbb{N}$, the discrete semigroup $S_{\epsilon,m}(\cdot)$ has a (numerical) attractor $\mathcal{A}_{\epsilon,m}$ such that*

$$\mathcal{A}_{\epsilon,m} \subset \mathcal{B}_{\|g|_m\|/\alpha}^m(0) \;\; \textit{and } \mathcal{A}_{\epsilon,m} \textit{ is connected.} \tag{7.6}$$

Proof. The existence of a unique attractor $\mathcal{A}_{\epsilon,m}$ follows from the compactness of the state space $\mathcal{B}_{r^*}^m(0)$ immediately. The connection of $\mathcal{A}_{\epsilon,m}$ follows from the same method as in Theorem 5.2.

To prove the bound of the attractor, we put $r_0 \in (\|g|_m\|/\alpha, r^*)$ and prove that $\mathcal{B}_{r_0}^m(0)$ is an absorbing set for the semigroup $S_{\epsilon,m}(\cdot)$. It suffices to prove that $\mathcal{B}_{r_0}^m(0)$ absorbs the whole state space $\mathcal{B}_{r^*}^m(0)$. Iterating (7.3) in Lemma 7.1, we have for all $u_0^m \in \mathcal{B}_{r^*}^m(0)$,

$$\begin{aligned}\|u_n^{\epsilon,m}\|^2 &\le \frac{1}{1+\epsilon\alpha}\Big(\|u_{n-1}^{\epsilon,m}\|^2 + \frac{\epsilon}{\alpha}\|g|_m\|^2\Big)\\ &\le \frac{\|u_0^m\|^2}{(1+\epsilon\alpha)^n} + \frac{\epsilon}{\alpha}\|g|_m\|^2 \sum_{j=1}^{n} \frac{1}{(1+\epsilon\alpha)^j} \le \frac{(r^*)^2}{(1+\epsilon\alpha)^n} + \frac{\|g|_m\|^2}{\alpha^2}.\end{aligned}$$

Since $r_0 > \|g|_m\|/\alpha$, it follows that there is $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\|u_n^{\epsilon,m}\|^2 \le \Big(r_0^2 - \frac{\|g|_m\|^2}{\alpha^2}\Big) + \frac{\|g|_m\|^2}{\alpha^2} = r_0^2.$$

Since $r_0 < r^*$, we see from (7.4) that $\epsilon_{r_0} > \epsilon^*$. By Lemma 7.1, $\mathcal{B}_{r_0}^m(0)$ is positively invariant under $S_{\epsilon,m}(\cdot)$ for all $0 < \epsilon \le \epsilon^*$ (and thus $\epsilon \le \epsilon_{r_0}$). Therefore,

$$\mathcal{A}_{\epsilon,m} = \cap_{k\in\mathbb{N}} \overline{\cup_{n\ge k} S_{\epsilon,m}(n)\mathcal{B}_{r_0}^m(0)} \subset \mathcal{B}_{r_0}^m(0)$$

for all $\epsilon \in (0, \epsilon^*]$ and $r_0 \in (\|g|_m\|/\alpha, r^*)$. Taking the limit as $r_0 \to \|g|_m\|/\alpha$, we obtain the inclusion in (7.6). ∎

7.2. Convergence from truncated attractor to numerical attractor. The following result states that the tail of any element in $\mathcal{A}_{\epsilon,m}$ is uniformly small in $\epsilon \in (0, \epsilon^*]$ as $m \to \infty$.

LEMMA 7.3. *For each $\delta > 0$, there exists $I(\delta) \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $\epsilon \in (0, \epsilon^*]$,*

$$\sum_{I(\delta)\le|i|\le m} |x_i|^2 < \delta, \quad \forall (x_i)_{|i|\le m} \in \mathcal{A}_{\epsilon,m}, \tag{7.7}$$

where the sum is zero if $m < I(\delta)$.

Proof. Taking the inner product of the truncated IES (7.2) with $\xi_k^m u_n^{\epsilon,m}$ in $\mathbb{R}^{2m+1}$, where $\xi_k^m = (\xi_{k,i})_{|i|\le m}$, we obtain

$$\begin{aligned}\sum_{|i|\le m} \xi_{k,i}|u_{n,i}^{\epsilon}|^2 &= (u_{n-1}^{\epsilon,m}, \xi_k^m u_n^{\epsilon,m})\\ &\quad + \epsilon\nu(A_{p,m}u_n^{\epsilon,m}, \xi_k^m u_n^{\epsilon,m}) + \epsilon(F_m(u_n^{\epsilon,m}) + g|_m, \xi_k^m u_n^{\epsilon,m}).\end{aligned}$$

Since $A_{p,m}$ and A_p have the same local Lipschitz constants, we can similarly obtain

$$\nu(A_{p,m}u_n^{\epsilon,m}, \xi_k^m \diamond u_n^{\epsilon,m}) \le \frac{C_p}{k}(r^*)^p. \tag{7.8}$$

where the constant C_p is independent of m. Hence, by other same arguments as in the proof of Lemma 5.1, it follows that, for each $\delta > 0$ and $\epsilon \in (0, \epsilon^*]$, there are $N_\epsilon(\delta) \in \mathbb{N}$ and $I(\delta) \in \mathbb{N}$ such that

$$\sum_{I(\delta)\le|i|\le m} |u_{n,i}^{\epsilon,m}|^2 < \delta, \quad \forall n \ge N_\epsilon(\delta).$$

Given now $x \in \mathcal{A}_{\epsilon,m}$ with arbitrary $\epsilon \in (0, \epsilon^*]$ and $m \in \mathbb{N}$. The invariance implies that

$$x = S_{\epsilon,m}(N_\epsilon(\delta))u_0^m = u_{N_\epsilon(\delta)}^{\epsilon,m}(u_0^m), \text{ for some } u_0^m \in \mathcal{B}_{r^*}^m(0),$$

and thus

$$\sum_{I(\delta)\le|i|\le m} |x_i|^2 = \sum_{I(\delta)\le|i|\le m} |u_{N_\epsilon(\delta),i}^{\epsilon,m}|^2 < \delta,$$

which implies (7.7) as desired. ∎

Any $x \in \mathbb{R}^{2m+1}$ has a *null-extension* $\tilde{x} \in \ell^2$ defined by

$$\tilde{x}_i = 0, \ \forall |i| > m, \ \tilde{x}_i = x_i, \ \forall |i| \le m.$$

Then a set D in $\mathbb{R}^{2m+1}$ still has a *null-extension* set in ℓ^2 denoted by $\widetilde{D}$. In this viewpoint, both attractors $\mathcal{A}_{\epsilon,m}$ and $\mathcal{A}_\epsilon$ can be contained into the same ball $\mathcal{B}_{r^*}(0)$ of ℓ^2.

THEOREM 7.4. *For each $\epsilon \in (0, \epsilon^*]$, the numerical attractor $\mathcal{A}_{\epsilon,m}$ of the truncated IES (7.2) upper semi-converges to the attractor $\mathcal{A}_\epsilon$ of the IES (3.1), i.e.*

$$d_{\ell^2}(\mathcal{A}_{\epsilon,m}, \mathcal{A}_\epsilon) := d_{\ell^2}(\widetilde{\mathcal{A}}_{\epsilon,m}, \mathcal{A}_\epsilon) \to 0, \ \textit{as } m \to \infty. \tag{7.9}$$

Proof. Suppose (7.9) is false for a fixed $\epsilon \in (0, \epsilon^*]$. Then there are $\eta_0 > 0$, a subsequence $\{m_j\}$ of $\{m\}$ and $x^{m_j} \in \mathcal{A}_{\epsilon,m_j}$ such that the null-extension $\tilde{x}^{m_j}$ of x^{m_j} satisfies

$$d(\tilde{x}^{m_j}, \mathcal{A}_\epsilon) \geq \eta_0, \ \forall j \in \mathbb{N}. \tag{7.10}$$

The solution of (7.2) with the initial data x^{m_j} is given by

$$u_n^{\epsilon,m_j} = u_n^{\epsilon,m_j}(x^{m_j}) = S_{\epsilon,m_j}(n)x^{m_j}, \ \forall n \in \mathbb{N}_0, \ j \in \mathbb{N}.$$

Since $x^{m_j} \in \mathcal{A}_{\epsilon,m_j}$, it follows that the solution u_n^{ϵ,m_j} can be expanded as an entire solution, i.e. defined for all $n \in \mathbb{Z}$.

Since $u_n^{\epsilon,m_j} \in \mathcal{A}_{\epsilon,m_j}$ for all $n \in \mathbb{Z}$, it follows from Lemma 7.3 that, for each $\delta > 0$, there is $I(\delta) \in \mathbb{N}$ such that

$$\sum_{|i| \geq I(\delta)} |\tilde{u}_{n,i}^{\epsilon,m_j}|^2 = \sum_{I(\delta) \leq |i| \leq m_j} |u_{n,i}^{\epsilon,m_j}|^2 < \delta^2, \ \forall n \in \mathbb{Z}, \ j \in \mathbb{N}, \tag{7.11}$$

where $\tilde{u}$ is the null-extension of u. By the previous discussion, all attractors are contained in those balls of radius r^* and thus the double sequence

$$\{(\tilde{u}_{n,i}^{\epsilon,m_j})_{|i|<I(\delta)} : n \in \mathbb{Z}, \ j \in \mathbb{N}\}$$

is bounded in $\mathbb{R}^{2I(\delta)-1}$ and thus it has a finite δ-net, which together with (7.11) implies that the double sequence

$$\{\tilde{u}_n^{\epsilon,m_j} : n \in \mathbb{Z}, \ j \in \mathbb{N}\}$$

has a finite 2δ-net and thus it is relatively compact in ℓ^2. By a diagonal argument, there are $\{u_n^* : n \in \mathbb{Z}\} \subset \ell^2$ and an index subsequence (denoted by itself) of $\{j\}$ such that

$$\|\tilde{u}_n^{\epsilon,m_j} - u_n^*\| \to 0 \text{ as } j \to \infty, \ \forall n \in \mathbb{Z}. \tag{7.12}$$

We then prove that $\{u_n^* : n \in \mathbb{Z}\}$ is an entire solution of the IES (3.1). As an entire solution, $\{u_n^{\epsilon,m_j} : n \in \mathbb{Z}\}$ satisfies the truncated IES (7.2) for all $n \in \mathbb{Z}$:

$$u_n^{\epsilon,m_j} = u_{n-1}^{\epsilon,m_j} + \epsilon\nu A_{p,m_j} u_n^{\epsilon,m_j} + \epsilon F_m(u_n^{\epsilon,m_j}) + \epsilon g|_{m_j}.$$

We now fix $i \in \mathbb{Z}$, then there is $j_i \in \mathbb{N}$ such that for all $j \geq j_i$ we have $m_j \geq |i|+1$, and thus

$$u_{n,i}^{\epsilon,m_j} = \tilde{u}_{n,i}^{\epsilon,m_j}, \ \ (A_{p,m_j} u_n^{\epsilon,m_j})_i = (A_p \tilde{u}_n^{\epsilon,m_j})_i, \ (g|_{m_j})_i = g_i$$

for all $j \geq j_i$ and $n \in \mathbb{Z}$. Hence, the ith-component of the entire solution u_n^{ϵ,m_j} satisfies

$$\tilde{u}_{n,i}^{\epsilon,m_j} = \tilde{u}_{n-1,i}^{\epsilon,m_j} + \epsilon\nu(A_p \tilde{u}_n^{\epsilon,m_j})_i + \epsilon f(\tilde{u}_{n,i}^{\epsilon,m_j}) + \epsilon g_i \tag{7.13}$$

for all $n \in \mathbb{Z}$ and $j \geq j_i$. By the local Lipschitz continuity of A_p and F, we have

$$|(A_p \tilde{u}_n^{\epsilon,m_j})_i - (A_p u_n^*)_i| \leq \|A_p \tilde{u}_n^{\epsilon,m_j} - A_p u_n^*\| \leq L_{p,r^*} \|\tilde{u}_n^{\epsilon,m_j} - u_n^*\|,$$
$$|f(\tilde{u}_{n,i}^{\epsilon,m_j}) - f(u_{n,i}^*)| \leq \|F(\tilde{u}_n^{\epsilon,m}) - F(u_n^*)\| \leq L_{r^*} \|\tilde{u}_n^{\epsilon,m_j} - u_n^*\|.$$

Letting $j \to \infty$ (thus $m_j \to \infty$) in (7.13) and using (7.12), we obtain

$$u^*_{n,i} = u^*_{n-1,i} + \epsilon\nu(A_p u^*_n)_i + \epsilon f(u^*_{n,i}) + \epsilon g_i, \ \forall n \in \mathbb{Z}.$$

Since $i \in \mathbb{Z}$ is arbitrary, it follows that $\{u^*_n : n \in \mathbb{Z}\}$ is a (bounded) entire solution of the truncated IES (7.2). Hence, $u^*_0 \in \mathcal{A}_\epsilon$, which together with

$$\tilde{x}^{m_j} = \tilde{u}_0^{\epsilon,m_j} \to u^*_0 \text{ as } j \to \infty$$

gives a contradiction to (7.10). ∎

7.3. Lower semi-continuity of numerical attractors for viscosity zero. We denote the restrictions of an element $x \in \ell^2$ and a subset $D \subset \ell^2$ on $\mathbb{R}^{2m+1}$ by

$$x|_m = (x_i)_{|i|\le m} \text{ and } D|_m = \{y \in \mathbb{R}^{2m+1} : \exists x \in D, \text{ s.t. } y = x|_m\}.$$

PROPOSITION 7.5. *For each $\epsilon \in (0, \epsilon^*]$, the numerical attractor $\mathcal{A}_\epsilon$ of the IES (3.1) satisfies the following lower semi-continuity:*

$$\lim_{m\to\infty} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}_\epsilon|_m) = 0. \tag{7.14}$$

Proof. By Lemma 5.1, for each $\delta > 0$, there are $I(\delta) \in \mathbb{N}$ and $N_\epsilon(\delta) \in \mathbb{N}$ such that the solution of the IES (3.1) satisfies

$$\|S_\epsilon(n)u_0\|_{\ell^2(|i|\ge I(\delta))} < \delta, \ \forall n \ge N_\epsilon(\delta), \ u_0 \in \mathcal{B}_{r^*}(0).$$

Given any $x \in \mathcal{A}_\epsilon$. By the invariance, we have $x = S_\epsilon(N_\epsilon(\delta))y$ for some $y \in \mathcal{A}_\epsilon$. Hence, for all $m \ge I(\delta)$,

$$\|x - \widetilde{x|_m}\|^2 = \|x\|^2_{\ell^2(|i|\ge m)} = \|S_\epsilon(N_\epsilon(\delta))y\|^2_{\ell^2(|i|\ge m)} < \delta^2.$$

Since $\widetilde{x|_m} \in \widetilde{\mathcal{A}_\epsilon|_m}$, it follows that for all $m \ge I(\delta)$ and $x \in \mathcal{A}_\epsilon$,

$$d_{\ell^2}(x, \widetilde{\mathcal{A}_\epsilon|_m}) \le \|x - \widetilde{x|_m}\| < \delta,$$

which further implies

$$d_{\ell^2}(\mathcal{A}_\epsilon, \widetilde{\mathcal{A}_\epsilon|_m}) = \sup_{x\in\mathcal{A}_\epsilon} d_{\ell^2}(x, \widetilde{\mathcal{A}_\epsilon|_m}) < \delta,$$

for all $m \ge I(\delta)$. Hence the lower semi-continuity (7.14) holds as desired. ∎

However, $\mathcal{A}_{\epsilon,m} \neq \mathcal{A}_\epsilon|_m$ generally, where $\mathcal{A}_{\epsilon,m}$ is the truncated numerical attractor for the truncated IES (7.2). We only prove the lower semi-continuity in a special case of viscosity zero.

THEOREM 7.6. *Suppose $\nu = 0$ in both IES (3.1) and (7.2). Then, for each $\epsilon \in (0, \epsilon^*]$, we have the following lower semi-convergence:*

$$\lim_{m\to\infty} d_{\ell^2}(\mathcal{A}_\epsilon, \mathcal{A}_{\epsilon,m}) = 0. \tag{7.15}$$

Proof. Given $x \in \mathcal{A}_\epsilon$. We know that the solution $u_n := u^\epsilon_n(x) = S_\epsilon(n)x$ can be expanded into an entire solution defined for all $n \in \mathbb{Z}$. Hence, the entire solution $\{u_n : n \in \mathbb{Z}\}$ satisfies

$$u_n = u_{n-1} + \epsilon F(u_n) + \epsilon g, \ \forall n \in \mathbb{Z}, \quad u_0 = x. \tag{7.16}$$

The component form of (7.16) can be read as

$$u_{n,i} = u_{n-1,i} + \epsilon f(u_{n,i}) + \epsilon g_i, \ \forall n \in \mathbb{Z}, \ i \in \mathbb{Z}. \tag{7.17}$$

Considering the truncation of (7.17) for those components $|i| \leq m$, it follows that $u_n|_m$ satisfies

$$\begin{aligned} u_n|_m &= u_{n-1}|_m + \epsilon F_m(u_n|_m) + \epsilon g^m, \ \forall n \in \mathbb{Z}, \\ u_0|_m &= x|_m \in \mathbb{R}^{2m+1}, \end{aligned}$$

which means $\{u_n|_m : n \in \mathbb{Z}\}$ is an entire solution of the truncated IES (7.2) with $\nu = 0$. Due to the positively invariance, we know $u_n|_m \in \mathcal{B}^m_{r^*}(0)$ and thus the entire solution is bounded in $\mathbb{R}^{2m+1}$, which implies

$$x|_m = u_0|_m \in \mathcal{A}_{\epsilon,m}.$$

Denote by $\widetilde{x|_m}$ the null-expansion of $x|_m$, by $x \in \ell^2$, we have

$$\lim_{m\to\infty} \|\widetilde{x|_m} - x\| = \lim_{m\to\infty} \sum_{|i|>m} |x_i|^2 = 0. \tag{7.18}$$

Suppose now the lower semi-convergence (7.15) is false. Then there is a subsequence $\{m_j\}$ of $\{m\}$ and $\delta_0 > 0$ such that

$$d_{\ell^2}(\mathcal{A}_\epsilon, \widetilde{\mathcal{A}_{\epsilon,m_j}}) > \delta, \ \forall j \in \mathbb{N},$$

where the tilde denotes the null-expansion of the set. Furthermore, for each $j \in \mathbb{N}$, there is $y_j \in \mathcal{A}_\epsilon$ such that

$$d_{\ell^2}(y_j, \widetilde{\mathcal{A}_{\epsilon,m_j}}) > \delta, \ \forall j \in \mathbb{N}. \tag{7.19}$$

Since $\mathcal{A}_\epsilon$ is compact in ℓ^2, there is an index subsequence $\{j_k\}$ of $\{j\}$ such that $y_{j_k} \to x$ for some $x \in \mathcal{A}_\epsilon$.

By the previous proof, we know $x|_m \in \mathcal{A}_{\epsilon,m}$ such that (7.18) holds. In particular,

$$\lim_{k\to\infty} \|\widetilde{x|_{m_{j_k}}} - x\| = 0, \ \text{and} \ \widetilde{x|_{m_{j_k}}} \in \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}.$$

Hence,

$$\begin{aligned} d_{\ell^2}(y_{j_k}, \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}) &\leq \|y_{j_k} - x\| + \|x - \widetilde{x|_{m_{j_k}}}\| \\ &\quad + d_{\ell^2}(\widetilde{x|_{m_{j_k}}}, \widetilde{\mathcal{A}_{\epsilon,m_{j_k}}}) \to 0, \ \text{as} \ k \to \infty, \end{aligned}$$

which contradicts (7.19). ∎

7.4. Final Conclusions. As displaying in FIG. 1, we have established a path of upper semi-convergence from the truncated numerical attractor $\mathcal{A}_{\epsilon,m}$ to the global attractor $\mathcal{A}$ through the numerical attractor $\mathcal{A}_\epsilon$, see Theorems 7.4 and 6.1.

On the other hand, we can establish another path of upper semi-convergence from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}$ through $\mathcal{A}_m$, where $\mathcal{A}_m$ is the attractor of the following truncated LDS of LDS (2.3):

$$\frac{du(t)}{dt} = \nu A_{p,m} u(t) + F_m(u(t)) + g|_m, \ u(0) \in \mathbb{R}^{2m+1}.$$

In fact, by the similar method as in Theorem 6.1, one can prove the upper semi-convergence from $\mathcal{A}_{\epsilon,m}$ to $\mathcal{A}_m$, while the upper semi-convergence from $\mathcal{A}_m$ to $\mathcal{A}$ follows from the same method as in [2].

Only in the special case of $\nu = 0$, we can establish the two classes of lower semi-convergence as in FIG. 1. Lower semi-convergence in other cases remains open.

REFERENCES

[1] P. W. Bates, H. Lisei and K. Lu, *Attractors for stochastic lattice dynamical systems*, Stoch. Dyn. 1 (2006), pp. 1–21.

[2] P. W. Bates, K. Lu and B. Wang, *Attractors for lattice dynamical systems*, Internat. J. Bifur. Chaos 11 (2001), pp. 143–153.

[3] H. Bessaih, M.J. Garrido-Atienza, X. Han and B. Schmalfuss, *Stochastic lattice dynamical systems with fractional noise*, SIAM J. Math. Anal., 49 (2017), pp. 1495–1518.

[4] D. Blömker and A. Jentzen, *Galerkin approximations for the stochastic Burgers equation*, SIAM J. Numer. Anal. 51 (2013), no. 1, pp. 694–715.

[5] T. Caraballo, X. Han, P.E. Kloeden, *Nonautonomous chemostats with variable delays*, SIAM J. Math. Anal., 47 (2015), pp. 2178-2199.

[6] T. Caraballo, X. Han, B. Schmalfuss and J. Valero, *Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise*, Nonlinear Anal. 130 (2016), pp. 255–278.

[7] T. Caraballo, J.A. Langa and Z. Liu, *Gradient infinite-dimensional random dynamical systems*, SIAM J. Appl. Dyn. Syst., 11 (2012), pp. 1817–1847.

[8] A.N. Carvalho and C.B. Gentile, *Asymptotic behaviour of non-linear parabolic equations with monotone principal part* J. Math. Anal. Appl., 280 (2003), pp. 252–272.

[9] A.N. Carvalho, J.A. Langa and J.C. Robinson, *Attractors for Infinite-dimensional Nonautonomous Dynamical Systems*, Appl. Math. Sciences, Springer, New York, 2013.

[10] D. Cheban, C. Mammana and E. Michetti, *Global attractors of quasi-linear non-autonomous difference equations: A growth model with endogenous population growth*, Nonlinear Anal.-RWA 14 (2013), pp. 1716–1731.

[11] M.S. Efendiev and S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain*, Commun. Pure Appl. Math., 54 (2001), pp. 625-688.

[12] P. G. Geredeli, *On the existence of regular global attractor for p-Laplacian evolution equation*, Appl. Math. Optim., 71(2015), pp. 517–532.

[13] B. Gess, W. Liu and A. Schenke, *Random attractors for locally monotone stochastic partial differential equations*, J. Differential Equations, 269 (2020), pp. 3414-3455.

[14] A. Gu and P. E. Kloeden, *Asymptotic Behavior of a Nonautonomous p-Laplacian Lattice System*, Internat. J. Bifur. Chaos, 26 (2016), Paper No: 1650174.

[15] A. Gu and Y. Li, *Dynamic behavior of stochastic p-Laplacian-type lattice equations*, Stoch. Dyn., 17 (2017), Paper No: 1750040.

[16] X. Han and P.E. Kloeden, *Non-autonomous lattice systems with switching effects and delayed recovery*, J. Differential Equations, 261 (2016), pp. 2986–3009.

[17] X. Han, P. E. Kloeden and S. Sonner, Discretisation of global attractors for lattice dynamical systems, J. Dynam. Differ. Equ. 32 (2020), pp. 1457–1474.

[18] X. Han and H.N. Najm, *Dynamical Structures in Stochastic Chemical Reaction Systems*, SIAM J. Appl. Dyn. Syst. 13 (2014), pp. 1328–1351.

[19] X. Han, W. Shen and S. Zhou, *Random attractors for stochastic lattice dynamical systems in weighted spaces*, J. Differential Equations 250 (2011), pp. 1235–1266.

[20] A. Jentzen, *Higher order pathwise numerical approximations of SPDEs with additive noise*, SIAM J. Numer. Anal. 49 (2011), no. 2, pp. 642–667.

[21] A. Jentzen and P. E. Kloeden, *Taylor approximations of stochastic partial differential equations*, CBMS Lecture series. SIAM, Philadelphia, 2011.

[22] A. Jentzen and P. E. Kloeden, *Taylor expansions of solutions of stochastic partial differential equations with additive noise*, Ann. Probab. 38 (2010), no. 2, pp. 532–569.

[23] N. Ju, *Numerical analysis of parabolic p-Laplacian: Approximation of trajectories*, SIAM J. Numerical Anal. 37 (2000), pp. 1861-1884.

[24] C. Kiers, *Rate-Induced Tipping in Discrete-Time Dynamical Systems*, SIAM J. Appl. Dyn. Syst., 19 (2020),pp. 1200-1224.

[25] P. E. Kloeden and J. Lorenz,
Stable attracting sets in dynamical systems and in their one-step discretizations, SIAM J.

Numer. Anal. 23 (1986), pp. 986–995.

[26] P.E. Kloeden and J. Simsen, *Attractors of asymptotically autonomous quasi-linear parabolic equation with spatially variable exponents*, J. Math. Anal. Appl., 425 (2015), pp. 911–918.

[27] A. Krause, M. Lewis and B. Wang, *Dynamics of the non-autonomous stochastic p-Laplace equation driven by multiplicative noise*, Appl. Math. Comput. 246 (2014), pp. 365-376.

[28] Y. Li, A. Gu and J. Li, *Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations*, J. Differential Equations, 258 (2015), pp. 504–534.

[29] Y. Li, L. She and R. Wang, *Asymptotically autonomous dynamics for parabolic equation*, J. Math. Anal. Appl., 459 (2018), pp. 1106–1123.

[30] Y. Li, S. Yang and Q. Zhang, *Continuous Wong-Zakai approximations of random attractors for quasi-linear equations with nonlinear noise*, Qualitative Theory Dynam. Syst. 19(3)(2020), Paper No. 87, pp. 1–25.

[31] A.M. Stuart and A.R.Humphries, *Numerical Analysis and Dynamical Systems*, Cambridge University Press, Cambridge, 1996.

[32] M. Sui, Y. Wang, X. Han, and P.E. Kloeden, *Random recurrent neural networks with delays*, J. Differential Equations, 269 (2020), pp. 8597–8639.

[33] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.

[34] F. Wang and Y. Li, *Random attractors for multi-valued multi-stochastic delayed p-Laplace lattice equations*, J. Difference Equat. Appl., 27 (2021),pp. 1232–1258.

[35] R. Wang and Y. Li, *Regularity and backward compactness of attractors for non-autonomous lattice systems with random coefficients*, Appl. Math. Comput. 354 (2019), pp. 86–102.

[36] R. Wang and B. Wang, *Random dynamics of p-Laplacian lattice systems driven by infinite-dimensional nonlinear noise*, Stochastic Process Appl. 130 (2020), pp. 7431–7462.

[37] X. Wang, K. Lu and B. Wang, *Random attractors for delay parabolic equations with additive noise and deterministic nonautonomous forcing*, SIAM J. Appl. Dyn. Syst. 14 (2015), pp. 1018–1047.

[38] S. Yang and Y. Li, *Numerical attractors and approximations for stochastic or deterministic sine-Gordon lattice equations*, Appl. Math. Comput. 413 (2022), Paper No. 126640.

[39] W. Zhao, *Compactness and attracting of random attractors for non-autonomous stochastic lattice dynamical systems in weighted space*, Appl. Math. Comput. 291 (2016), pp. 226–243.

[40] S. Zhou and X. Han, *Pullback exponential attractors for non-autonomous lattice systems*, J. Dyn. Differ. Equ. 24 (2012), pp. 601–631.