Existence, degenerate regularity and limit behavior of trajectory statistical solution for the 3D incompressible micropolar fluids flows with damping term*

Caidi Zhao[a]†, Binbin Miao[a]‡, Tomás Caraballo[b]§

[a] *Department of Mathematics, Wenzhou University, Wenzhou, Zhejiang Province, 325035, P. R. China*

[b] *Departmento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, c/Tarfia s/n, 41012-Sevilla, Spain*

April 26, 2023

Abstract

This article studies the existence, degenerate regularity and limit behavior of the trajectory statistical solution for a three-dimensional incompressible micropolar fluids flows with a damping term. The authors first prove the existence of the trajectory attractor and use it to construct the trajectory statistical solution. Then they establish that the trajectory statistical solution possesses partial regularity provided that the associated Grashof number is small enough. Finally, they investigate the limiting behavior of the trajectory statistical solution as the damping term vanishes.

Keywords: Micropolar fluids flows; Trajectory statistical solution; Trajectory attractor; Degenerate regularity; Grashof number.

1 Introduction

We study the following equations

$$u_t - (\nu + \kappa)\Delta u + (u \cdot \nabla)u + \sigma|u|^{\beta-1}u + \nabla p = 2\kappa\nabla \times \omega + f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$\omega_t - \gamma\Delta\omega + (u \cdot \nabla)\omega - \eta\nabla \mathrm{div}\omega + 4\kappa\omega = 2\kappa\nabla \times u + g, \tag{1.3}$$

*Supported by NSF of China with No.11971356, 11271290, by NSF of Zhejiang Province with No.LY17A010011 and Ministerio de Ciencia e Innovación (Spain) with No. PID2021-122991NB-C21.

†Corresponding author E-mail: zhaocaidi2013@163.com or zhaocaidi@wzu.edu.cn

‡E-mail:binbinmiao1997@163.com

§E-mail: caraball@us.es

with the initial and boundary conditions

$$u(x,t)\big|_{t=0}=u_0,\ \ \omega(x,t)\big|_{t=0}=\omega_0, \tag{1.4}$$

$$u(x,t)\big|_{\partial\Omega\times(0,\infty)}=0,\ \ \omega(x,t)\big|_{\partial\Omega\times(0,\infty)}=0, \tag{1.5}$$

where $(x,t)\in\Omega\times\mathbb{R}_+$, and Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In equations (1.1)-(1.3), the unknown functions

$$u=(u_1(x,t),u_2(x,t),u_3(x,t)),\ \ \omega=(\omega_1(x,t),\omega_2(x,t),\omega_3(x,t)),\ \ p=p(x,t),$$

denote, respectively, the velocity vector, the angular velocity of rotation of particles, and the pressure of the fluid, the functions $f=(f_1(x),f_2(x),f_3(x))$ and $g=(g_1(x),g_2(x),g_3(x))$ denote the external force and the angular momentum, respectively, and the positive constant σ is the damping coefficient. In addition, the parameters ν, γ, κ, η are positive constants.

Equations (1.1)-(1.3) describe the motion of micropolar fluids [24], which were first introduced to describe the micro-rotational motion and rotational inertia of fluids. In physics, micropolar fluids can represent fluids composed of rigid, randomly oriented (or spherical particles) suspended in a viscous medium. The theory of micropolar fluids was first introduced by Eringen in [13]. If $\kappa=\sigma=0$, $\omega=g=(0,0,0)$ in (1.1) and (1.3), then equations (1.1)-(1.3) turn to be the classical incompressible Navier-Stokes equations.

There are some references studying the incompressible micropolar fluids. For the case that there is no damping term, the global well-posedness and the existence of uniform attractor of two-dimensional (2D) micropolar fluids equations was proved in [7, 12]; initial and boundary-value problem for 2D micropolar equations with only angular velocity dissipation was investigated in [22]; the regular criteria of weak solutions for the 3D case was investigated in [10, 11, 20]; the existence and homogenization of trajectory statistical solution for the 3D case was established in [34, 42]. When equation (1.1) contains a damping term, the global existence of strong solution to the 3D case was verified in [35] for $\beta=3$ and $4\sigma(\nu+\kappa)>1$ or $\beta>3$, and the existence of global attractor was proved in [33] with the assumption $\beta\in(3,5)$.

The original motivation of this article is to investigate the existence of the trajectory statistical solution and its limiting behavior for equations (1.1)-(1.3) when $\beta\in(1,3]$. We are interested in the probability distribution of solutions within the temporal-spatial space. The invariant measure and statistical solution for evolution equations have been extensively studied. One can refer, for deterministic equations, to [4–6, 14–19, 23, 25–27, 31, 32, 38, 39, 43, 46, 49, 50] the well-posed systems and to [2, 21, 40, 42, 44, 45] the ill-posed systems, and to [48] the invariant sample measure and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations. For the impulsive differential equations, we can consult [28, 29, 47] for the existence of the statistical solutions. Especially, Zhao, Li and Caraballo [41] established, via the approach of trajectory attractor, some sufficient conditions ensuring the existence of trajectory statistical solution for general evolution equations, including those systems which possess global weak solutions but without a known result of global uniqueness. Recently, the abstract result of [41] was applied to equations (1.1)-(1.3) without damping term to construct the trajectory statistical solution [42].

The first result of this article is the existence of trajectory statistical solution for the 3D micropolar fluids with a damping term. We assume that $\beta \in (1,3]$. In this case we can obtain the existence of a weak solution to problem (1.1)-(1.5) corresponding to each initial value. However, it is not known whether the weak solution is unique or not. Therefore, here we cannot use the approach of classical semigroup [33] to investigate the asymptotic behavior of the solution for equations (1.1)-(1.3) because of the possible non-uniqueness of the weak solution. We will first prove the existence of the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ for equations (1.1)-(1.3) via the natural translation semigroup. Then we use the abstract results [41, Theorem2.1] to obtain the existence of the trajectory statistical solution μ_{σ,w_σ}, hereinafter w_σ is an element within $\mathcal{A}_\sigma^{\mathrm{tr}}$ and σ is the constant from equation (1.1).

The second goal of this article is to investigate the partial degenerate regularity of the trajectory statistical solution μ_{σ,w_σ}. The regularity of the trajectory statistical solution means that it is supported by a set in which all weak solutions are strong solutions. When constructing μ_{σ,w_σ}, we observe that it is supported by the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$. Thus, we naturally consider the degenerate regularity of the trajectory statistical solution via investigating the degenerate regularity of the trajectory attractor. Notice that μ_{σ,w_σ} is a probability measure defined on the trajectory space of equations (1.1)-(1.3) and $\mathcal{A}_\sigma^{\mathrm{tr}}$ itself consists of weak, bounded and complete trajectories of equations (1.1)-(1.3). According to these facts, we use the following form of generalized Grashof number

$$G := [\frac{\lambda+1}{\delta}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2)]^{1/2}, \tag{1.6}$$

to discuss the partial degenerate regularity of μ_{σ,w_σ}. We will prove that if the Grashof number G given by (1.6) is properly small, then the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ degenerates to a single complete trajectory, that is $\mathcal{A}_\sigma^{\mathrm{tr}} = \{\tilde{w}(t) \in \mathcal{K}_\sigma : t \geqslant 0\}$, where $\tilde{w}(t)$ is the unique complete and bounded weak solution of equations (1.1)-(1.3) and $\mathcal{K}_\sigma$ is the kernel of equations (1.1)-(1.3). Further, we will prove that $\mu_{\sigma,\tilde{w}_\sigma}$ possesses partial degenerate regularity of the Lusin type in the following sense: $\forall\, \epsilon > 0$, there is a subset $\mathbb{R}_+(\epsilon) \subset \mathbb{R}_+$ with Lebesgue measure mes$(\mathbb{R}_+(\epsilon) < \epsilon$, such that $\mu_{\sigma,\tilde{w}_\sigma}$ is regular on $\mathbb{R}_+ \setminus \mathbb{R}_+(\epsilon)$.

The third result of this article is to prove that the trajectory statistical solution μ_{σ,w_σ} converges to $\mu_{0,w}$ as $\sigma \to 0^+$, where $\mu_{0,w}$ is the trajectory statistical solution of the following classical micropolar fluids flows equations

$$u_t - (\nu + \kappa)\Delta u + (u \cdot \nabla)u + \nabla p = 2\kappa \nabla \times \omega + f, \tag{1.7}$$

$$\nabla \cdot u = 0, \tag{1.8}$$

$$\omega_t - \gamma \Delta \omega + (u \cdot \nabla)\omega - \eta \nabla \mathrm{div}\omega + 4\kappa\omega = 2\kappa \nabla \times u + g, \tag{1.9}$$

which have been constructed in [42]. To this end, we prove that the solution of equations (1.1)-(1.3) converges to that of equations (1.7)-(1.9), and that $\mathcal{A}_\sigma^{\mathrm{tr}}$ converges to $\mathcal{A}_0^{\mathrm{tr}}$ as $\sigma \to 0^+$, where $\mathcal{A}_0^{\mathrm{tr}}$ is the trajectory attractor of equations (1.7)-(1.9). In [3], Bronzi and Rosa proved the convergence of statistical solutions of the 3D Navier-Stokes-α model as α vanishes. The main tools used in [3] are topological analysis and measure theory. Here we investigate the convergence of trajectory statistical solutions of the 3D incompressible micropolar fluids flows with damping term as the damping term vanishes, via the approach of trajectory attractor. Our result reveals that the trajectory statistical information obtained from the 3D incompressible micropolar fluids flows with damping term are good approximations of the trajectory statistical information of the classical micropolar fluids flows.

The article is organized as follows. In the next section we prove the existence of the trajectory attractor and trajectory statistical solution. In Section 3 we prove the degenerate regularity of Lusin type for trajectory statistical solution. Section 4 is devoted to the convergence of the trajectory statistical solution μ_{σ,w_σ} to $\mu_{0,w}$ as $\sigma \to 0^+$.

2 Existence of trajectory attractor and trajectory statistical solution

In this section we first introduce the mathematical setting for problem (1.1)-(1.5) and then prove the existence of the trajectory attractor and trajectory statistical solution.

As usual, $\mathbb{L}^p(\Omega) = (L^p(\Omega))^3$ and $\mathbb{W}^{m,p}(\Omega) = (W^{m,p}(\Omega))^3$ stand for the 3D vector Lebesgue space and Sobolev space with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. We denote by $\mathbb{W}_0^{m,p}(\Omega)$ the closure of $\{\varphi : \varphi = (\varphi_1, \varphi_2, \varphi_3) \in (C_0^\infty(\Omega))^3\}$ in $\mathbb{W}^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$. We write $\mathbb{W}_0^{m,p}(\Omega) = \mathbb{H}_0^m$, $\mathbb{W}^m(\Omega) = \mathbb{H}^m$ and $\|\cdot\|_p = \|\cdot\|$ for $p = 2$. We also use the following spaces:

$\mathcal{V} = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (C_0^\infty(\Omega))^3 : \nabla \cdot \varphi = 0\}$,

$H =$ the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ with norm $\|\cdot\|_H = \|\cdot\|$ and inner product $(\cdot,\cdot)$,

$V =$ the closure of $\mathcal{V}$ in $\mathbb{H}^1(\Omega)$ with norm $\|\cdot\|_V = \|\cdot\|_{1,2}$,

$\widehat{H} = H \times \mathbb{L}^2(\Omega)$ with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_{\widehat{H}} = \|\cdot\|$ defined as

$$(\Phi, \Psi) = (\varphi, \psi) + (\phi, \xi), \ \ \Phi = (\varphi, \phi), \ \Psi = (\psi, \xi) \in \widehat{H},$$

$$\|\Phi\| = (\|\varphi\|^2 + \|\phi\|^2)^{1/2}, \ \ \Phi = (\varphi, \phi) \in \widehat{H},$$

$\widehat{V} = V \times \mathbb{H}_0^1(\Omega)$ with the norm $\|\cdot\|_{\widehat{V}}$ defined as

$$\|\Phi\|_{\widehat{V}} = (\|\varphi\|_V^2 + \|\phi\|_{1,2}^2)^{1/2}, \ \ \Phi = (\varphi, \phi) \in \widehat{V}.$$

In addition, we use H^*, V^*, $\widehat{H}^* = H^* \times \mathbb{L}^2(\Omega)$ and $\widehat{V}^* = V^* \times \mathbb{H}^{-1}(\Omega)$ to denote the dual spaces of H, V, $\widehat{H}$ and $\widehat{V}$, respectively, where $\mathbb{L}^2(\Omega)^* = \mathbb{L}^2(\Omega)$ and $\mathbb{H}^{-1}(\Omega)$ is the dual space of $\mathbb{H}_0^1(\Omega)$. Then we have $V \hookrightarrow H = H^* \hookrightarrow V^*$, $\widehat{V} \hookrightarrow \widehat{H} = \widehat{H}^* \hookrightarrow \widehat{V}^*$ and all the embedding is compact. Note that we have used the same notation $(\cdot, \cdot)$ to denote the inner product in the spaces $\mathbb{L}^2(\Omega)$, H and $\widehat{H}$. We will also use the same notation $\langle \cdot, \cdot \rangle$ to denote the dual pairing between the spaces V and V^*, $\widehat{V}$ and $\widehat{V}^*$, $\mathbb{H}_0^m(\Omega)$ and $\mathbb{H}^{-1}(\Omega)$ provided that there is no confusion.

Define the strong and weak distance in $\widehat{H}$ by

$$\mathrm{d_s}(w, \Phi) = \|w - \Phi\|, \quad \mathrm{d_w}(w, \Phi) = \sum_{\mathbf{j} \in \mathbb{Z}^3} \frac{1}{2^{|\mathbf{j}|}} \frac{|w_{\mathbf{j}} - \Phi_{\mathbf{j}}|}{1 + |w_{\mathbf{j}} - \Phi_{\mathbf{j}}|}, \quad w, \Phi \in \widehat{H},$$

where $w_{\mathbf{j}}$ and $\Phi_{\mathbf{j}}$ are Fourier coefficients of w and Φ, respectively. We next denote by $(\widehat{H}, \mathrm{d}_\bullet)$ ($\bullet = $ s or w) the strong or weak metric in $\widehat{H}$. Let $C([a,b]; \widehat{H}_\bullet)$ be the space of $\mathrm{d}_\bullet$ continuous $\widehat{H}$-valued functions on $[a, b]$ endowed with the metric

$$\mathrm{d}_{C([a,b];\widehat{H}_\bullet)}(w, \Phi) = \sup_{t \in [a,b]} \mathrm{d}_\bullet(w(t), \Phi(t)).$$

We also denote by $C([a, \infty); \widehat{H}_\bullet)$ the space of $\mathrm{d}_\bullet$ continuous $\widehat{H}$-valued functions on $[a, \infty)$ endowed with the metric

$$\mathrm{d}_{C([a,\infty);\widehat{H}_\bullet)}(w, \Phi) = \sum_{j \in \mathbb{N}} \frac{1}{2^j} \frac{\sup\{\mathrm{d}_\bullet(w(t), \Phi(t)) : a \leqslant t \leqslant a + j\}}{1 + \sup\{\mathrm{d}_\bullet(w(t), \Phi(t)) : a \leqslant t \leqslant a + j\}}. \tag{2.1}$$

Next, we introduce some operators. We first define the linear operators $A_1 : V \to V^*$ and $A_2 : \mathbb{H}_0^1(\Omega) \to \mathbb{H}^{-1}(\Omega)$ as

$$\langle A_1 u, \phi \rangle = (\nu + \kappa)(\nabla u, \nabla \phi), \ \ \forall\, u, \phi \in V,$$
$$\langle A_2 \omega, \varphi \rangle = \gamma(\nabla \omega, \nabla \varphi), \ \ \forall\, \omega, \varphi \in \mathbb{H}_0^1(\Omega).$$

It is not difficult to check that $A_1 = (\nu + \kappa)P\Delta$ and $A_2 = \gamma\Delta$ both with Dirichlet boundary condition, hereinafter P is the Leray projector from $\mathbb{L}^2(\Omega)$ into H. We can check that $D(A_1) = V \cap \mathbb{H}_0^2(\Omega)$ and $D(A_2) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}_0^2(\Omega)$.

Secondly, we define the following trilinear forms $b_1(\cdot, \cdot, \cdot)$ and $b_2(\cdot, \cdot, \cdot)$ as

$$b_1(u, v, \phi) = \sum_{j,k=1}^{3} \int_\Omega u_j \frac{\partial v_k}{\partial x_j} \phi_k \mathrm{d}x, \ \ \forall\, u,\, v,\, \phi \in V,$$
$$b_2(u, \omega, \varphi) = \sum_{j,k=1}^{3} \int_\Omega u_j \frac{\partial \omega_k}{\partial x_j} \varphi \mathrm{d}x, \ \ \forall\, u \in V,\, \omega,\, \varphi \in \mathbb{H}_0^1(\Omega).$$

We can check that the trilinear forms $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ are continuous on $V\times V\times V$ and $V\times \mathbb{H}_0^1(\Omega)\times \mathbb{H}_0^1(\Omega)$, respectively. By some simple computations, we have

$$\begin{cases} b_1(u,v,\phi)=-b_1(u,\phi,v),\ \ b_1(u,v,v)=0,\ \ \forall\, u,\, v,\, \phi\in V,\\ b_2(u,\omega,\varphi)=-b_2(u,\varphi,\omega),\ \ b_2(u,\omega,\omega)=0,\ \ \forall\, u\,\in V,\, \omega,\, \varphi\in \mathbb{H}_0^1(\Omega). \end{cases} \tag{2.2}$$

For every $u,v\in V$, the operator $B_1(u,v): V\times V\mapsto V^*$ defined as

$$\langle B_1(u,v),\phi\rangle=b_1(u,v,\phi),\ \ \forall\,\phi\in V,$$

is continuous. Similarly, for every $u\in V$ and $\omega\in\mathbb{H}_0^1(\Omega)$, the operator $B_2(u,\omega)$: $V\times\mathbb{H}_0^1(\Omega)\mapsto\mathbb{H}^{-1}(\Omega)$ defined via

$$\langle B_2(u,\omega),\varphi\rangle=b_2(u,\omega,\varphi),\ \ \forall\,\varphi\in\mathbb{H}_0^1(\Omega),$$

is continuous.

Using above operators, equations (1.1)-(1.3) can be written as

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}+A_1u+B_1(u,u)+\sigma|u|^{\beta-1}u=2\kappa\nabla\times\omega+Pf,\\ \dfrac{\mathrm{d}\omega}{\mathrm{d}t}+A_2\omega+B_2(u,\omega)-\eta\nabla\mathrm{div}\omega+4\kappa\omega=2\kappa\nabla\times u+g. \end{cases} \tag{2.3}$$

Further, we set for $w=(u,\omega)\in\widehat{V}$ that

$$\begin{cases} Aw=\big(A_1u,A_2\omega\big),\\ Bw=\big(B_1(u,u),B_2(u,\omega)\big),\\ N_\sigma w=\big(\sigma|u|^{\beta-1}u,\mathbf{0}\big),\\ Lw=\big(-2\kappa\nabla\times\omega,4\kappa\omega-2\kappa\nabla\times u-\eta\nabla\mathrm{div}\omega\big). \end{cases} \tag{2.4}$$

Using the above notations and setting $F=(Pf,g)$, we can write the weak form of problem (1.1)-(1.5) as

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t}+Aw(t)+Bw(t)+N_\sigma w(t)+Lw(t)=F,\ \text{ in }\ \mathcal{D}'((0,\infty),\widehat{V}^*), \tag{2.5}$$

$$w(0)=w_0=(u_0,\omega_0). \tag{2.6}$$

To handle with the nonlinear term $\sigma|u|^{\beta-1}u$, we denote by

$$\hat{\mathbb{L}}^{\beta+1}(\Omega)=\big\{(u,\mathbf{0})\in\mathbb{R}^3\times\mathbb{R}^3:\int_\Omega|u|^{\beta+1}\mathrm{d}x<+\infty\big\}.$$

Definition 2.1. *A weak solution of problem* (2.5)-(2.6) *on* $[0,\infty)$ *is an* $\widehat{H}$*-valued function* $w(t)=(u(t),\omega(t))$ *defined on* $[0,\infty)$ *with* $w(0)=w_0=(u_0,\omega_0)$*, such that*

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t}\in L^{4/3}_{\mathrm{loc}}([0,\infty);\hat{V}),\ w(\cdot)\in C([0,\infty);\widehat{H}_{\mathrm{w}})\cap L^2_{\mathrm{loc}}([0,\infty);\widehat{V})\cap L^{\beta+1}([0,\infty);\hat{\mathbb{L}}^{\beta+1}(\Omega))\\ (\dfrac{\mathrm{d}w}{\mathrm{d}t},\Phi)+\langle Aw,\Phi\rangle+\big\langle Bw,\Phi\big\rangle+\langle N_\sigma w,\Phi\rangle+\langle Lw,\Phi\rangle=(F,\Phi),\ \ \forall\,\Phi\in\widehat{V}, \end{cases}$$

hold in the distribution sense $\mathcal{D}'(0,\infty)$, and $w(t)$ satisfies the following energy inequality

$$\begin{aligned}&\frac{1}{2}\|w(t)\|^2+\int_0^t\langle Aw(s),w(s)\rangle\mathrm{d}s+\int_0^t\langle N_\sigma w(s),w(s)\rangle\mathrm{d}s+\int_0^t\langle Lw(s),w(s)\rangle\mathrm{d}s\\&\leqslant\frac{1}{2}\|w_0\|^2+\int_0^t\big(F,w(s)\big)\mathrm{d}s\end{aligned}$$

in the sense that

$$\begin{aligned}&-\frac{1}{2}\int_0^t\|w(s)\|^2\zeta'(s)\mathrm{d}s+\int_0^t\langle Aw(s),w(s)\rangle\zeta(s)\mathrm{d}s\\&+\int_0^t\langle N_\sigma w(s),w(s)\rangle\zeta(s)\mathrm{d}s+\int_0^t\langle Lw(s),w(s)\rangle\zeta(s)\mathrm{d}s\\&\leqslant\int_0^t\big(F,w(s)\big)\zeta(s)\mathrm{d}s,\ \ \forall\,\zeta(\cdot)\in C_0^\infty[0,t]\ \ \textit{with}\ \ \zeta(\cdot)\geqslant0,\,\forall\,t\geqslant0.\end{aligned}\tag{2.7}$$

If there exists some interval $I\subset\mathbb{R}_+=[0,\infty)$ such that $w(t)\in L^\infty(I;\widehat{V})$ for the weak solution $w(t)$, then we call that the weak solution $w(t)$ possesses "partial" regularity.

Lemma 2.1. *Suppose that $F\in\widehat{H}$. Then for each $w_0=(u_0,\omega_0)\in\widehat{H}$, problem* (2.5)-(2.6) *corresponds at least one weak solution $w(t)=w(t;w_0)$. Moreover, there exists a time $t_*\geqslant0$ such that*

$$w(t)\in X\triangleq\Big\{w\in\widehat{H}:\ \|w\|\leqslant R\Big\},\ \forall\,t\geqslant t_*,\tag{2.8}$$

where $R=\big(\frac{2\lambda}{\delta}[\frac{\lambda}{\nu}\|f\|^2+\frac{\lambda}{\gamma}\|g\|^2]\big)^{1/2}$, $\delta=\min\{\nu+\kappa,\gamma\}$ and λ is a positive constant depending only on Ω.

Proof. Let $w_0=(u_0,\omega_0)\in\widehat{H}$ be given. For the existence of a global weak solution $w=w(t;w_0)$, one can refer to [24, Theorem 1.6.1, P_{128}] or to [35]. Here we omit the details and establish (2.8). Taking the inner product $(\cdot,\cdot)$ of w with equation (2.5) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+(\nu+\kappa)\|\nabla u(t)\|^2+\gamma\|\nabla\omega(t)\|^2-2\kappa\int_\Omega\nabla\times\omega(t)\cdot u(t)\mathrm{d}x\\&-\eta\int_\Omega\nabla\mathrm{div}\omega(t)\cdot\omega(t)\mathrm{d}x-2\kappa\int_\Omega\nabla\times u(t)\cdot\omega(t)\mathrm{d}x+4\kappa\int_\Omega\omega^2(t)\mathrm{d}x+\sigma\|u(t)\|_{\mathbb{L}^{\beta+1}(\Omega)}^{\beta+1}\\=&\int_\Omega f\cdot u(t)\mathrm{d}x+\int_\Omega g\cdot\omega(t)\mathrm{d}x.\end{aligned}\tag{2.9}$$

Now using the following Poincaré's inequality

$$\|\varphi\|^2\leqslant\lambda\|\nabla\varphi\|^2,\,\forall\,\varphi\in\mathbb{H}_0^1(\Omega),\ \lambda\text{ is a constant depending only on}\,\Omega,\tag{2.10}$$

we have by some direct computations and estimates that

$$\begin{cases}\displaystyle\int_\Omega\nabla\times\omega(t)\cdot u(t)\mathrm{d}x=\int_\Omega\nabla\times u(t)\cdot\omega(t)\mathrm{d}x\leqslant\|\omega(t)\|^2+\|\nabla u(t)\|^2,\\ \displaystyle-\eta\int_\Omega\nabla\mathrm{div}\omega(t)\cdot\omega(t)\mathrm{d}x=\eta\int_\Omega|\mathrm{div}\omega(t)|^2\mathrm{d}x,\\ \displaystyle\int_\Omega\omega^2(t)\mathrm{d}x\leqslant\lambda\|\nabla\omega(t)\|^2,\\ \displaystyle\int_\Omega f\cdot u(t)\mathrm{d}x+\int_\Omega g\cdot\omega(t)\mathrm{d}x\leqslant\frac{\nu}{2}\|\nabla u(t)\|^2+\frac{\gamma}{2}\|\nabla\omega(t)\|^2+\frac{\lambda}{2\nu}\|f\|^2+\frac{\lambda}{2\gamma}\|g\|^2.\end{cases}$$

Substituting this into (2.9) gives

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + \frac{\delta}{\lambda}\|w(t)\|^2 + 2\sigma\|u(t)\|^{\beta+1}_{\mathbb{L}^{\beta+1}(\Omega)} \leqslant \frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2, \\ \frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + \delta\|\nabla w(t)\|^2 + 2\sigma\|u(t)\|^{\beta+1}_{\mathbb{L}^{\beta+1}(\Omega)} \leqslant \frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2, \end{cases} \tag{2.11}$$

hereinafter $\delta = \min\{\nu+\kappa, \gamma\}$. Applying Gronwall's inequality to $(2.11)_1$ yields

$$\|w(t)\|^2 \leqslant \|w(0)\|^2 e^{-\frac{\delta t}{\lambda}} + \frac{\lambda}{\delta}[\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2]. \tag{2.12}$$

We end the proof by setting $R = \sqrt{\frac{2\lambda}{\delta}[\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2]}$. □

We next select some definitions concerning the trajectory attractor. Define the trajectory space $\mathcal{T}^+_\sigma$ and kernel $\mathcal{K}_\sigma$ of equation (2.5) respectively as

$$\mathcal{T}^+_\sigma = \{w(\cdot) : w(\cdot) \text{ is a weak solution of (2.5) and } w(t) \in X \text{ for all } t \in \mathbb{R}_+\},$$
$$\mathcal{K}_\sigma = \{w(\cdot) : w(\cdot) \text{ is a weak solution of (2.5) and } w(t) \in X \text{ for all } t \in \mathbb{R}\}.$$

We also set

$$\mathcal{F}^+ = C_{\mathrm{loc}}([0,\infty); X_{\mathrm{w}})$$

be the space of continuous functions from $[0,\infty)$ to X_{w}, where $X_{\mathrm{w}} = (X, \mathrm{d}_{\mathrm{w}})$ denotes the space X endowed with the weak distance. Define the weak distance in $\mathcal{F}^+$ as

$$\mathrm{d}_{\mathcal{F}^+}(w,\Phi) = \mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(w,\Phi) = \sum_{j\in\mathbb{N}} \frac{1}{2^j}\frac{\sup\{\mathrm{d}_{X_{\mathrm{w}}}(w(t),\Phi(t)) : 0 \leqslant t \leqslant j\}}{1+\sup\{\mathrm{d}_{X_{\mathrm{w}}}(w(t),\Phi(t)) : 0 \leqslant t \leqslant j\}},$$

which is compatible with the compact-open topology (denoted by Θ^+_{loc}) of $\mathcal{F}^+$. Note that $(\mathcal{F}^+, \Theta^+_{\mathrm{loc}})$ is a Hausdorff topological space. Define the natural translation operator $\{T(s)\}_{s\geqslant 0}$ on $\mathcal{F}^+$ as

$$T(s)w(t) = w(t+s)\big|_{[0,\infty)}, \quad w \in \mathcal{F}^+. \tag{2.13}$$

Because that the trajectory space $\mathcal{T}^+_\sigma$ includes all bounded weak solutions of (2.5), we have $T(s)\mathcal{T}^+_\sigma \subset \mathcal{T}^+_\sigma$ for any $s \geqslant 0$. For a set $\mathcal{P} \subset \mathcal{F}^+$ and some $r > 0$, we set

$$\mathcal{B}(\mathcal{P}, r) = \{u \in \mathcal{F}^+ \big|\ \mathrm{d}_{\mathcal{F}^+}(w,\mathcal{P}) = \inf_{\Phi\in P} \mathrm{d}_{\mathcal{F}^+}(w,\Phi) < r\}.$$

Definition 2.2. *A set $\mathcal{P} \subset \mathcal{F}^+$ is said to uniformly attract a set $Q \subset \mathcal{T}^+$ if for any $\epsilon > 0$ there is a $t_\epsilon > 0$ such that $T(t)Q \subset \mathcal{B}(P,\epsilon)$, $\forall\, t \geqslant t_\epsilon$. A set $\mathcal{P} \subset \mathcal{F}^+$ is said to be a trajectory attracting set if it uniformly attracts $\mathcal{T}^+$. A set $\mathcal{U} \subset \mathcal{F}^+$ is called a trajectory attractor if $\mathcal{U}$ is the minimal compact trajectory attracting set and $T(t)\mathcal{U} = \mathcal{U}$ for all $t \geqslant 0$.*

To prove the existence of a trajectory attractor for $\{T(s)\}_{s\geqslant 0}$ in $\mathcal{F}^+$, we next establish two lemmas. In the sequel, we will use the notation $a \lesssim b$ (also $a \gtrsim b$) to mean that $a \leqslant cb$ (also $a \geqslant cb$) for a universal constant $c > 0$ that only depends on the parameters coming from the problem.

Lemma 2.2. *Let $w = (u, \omega) \in L^\infty([0,T]; \widehat{H}) \cap L^2([0,T]; \widehat{V}) \cap L^{\beta+1}([0,T]; \hat{\mathbb{L}}^{\beta+1}(\Omega))$ for all $T > 0$, then*

$$t \longmapsto Aw(t) \in L^2([0,T]; \widehat{V}^*), \tag{2.14}$$

$$t \longmapsto Bw(t) \in L^{4/3}([0,T]; \widehat{V}^*), \tag{2.15}$$

$$t \longmapsto Lw(t) \in L^2([0,T]; \widehat{V}^*), \tag{2.16}$$

$$t \longmapsto N_\sigma w(t) \in L^{(\beta+1)^*}([0,T]; \hat{\mathbb{L}}^{(\beta+1)^*}(\Omega)), \tag{2.17}$$

hereinafter $(\beta+1)^* = (\beta+1)/\beta$ is the conjugate exponent of $\beta+1$.

Proof. Consider given $T > 0$. For a.e. $t \in [0,T]$, we see from the definitions of operators A, B, N_σ and L that $Aw(t)$, $B(u(t), w(t))$, $Lw(t)$ and $N_\sigma w(t)$ belong to $\widehat{V}^*$ for $w(t) \in L^\infty([0,T]; \widehat{H}) \cap L^2([0,T]; \widehat{V}) \cap L^{\beta+1}([0,T]; \hat{\mathbb{L}}^{\beta+1}(\Omega))$. The measurability of the functions $t \longmapsto Aw(t)$, $t \longmapsto Bw(t)$, $t \longmapsto Lw(t)$, $t \longmapsto N_\sigma w(t)$ is not difficult to check.

Now, for any $\Phi = (\phi, \varphi) \in \widehat{V}$ we have, by using Cauchy's inequality,

$$\begin{aligned} |\langle Aw(t), \Phi\rangle| &\lesssim |(\nabla u(t), \nabla\phi)| + |(\nabla\omega(t), \nabla\varphi)| \lesssim \|\nabla u(t)\|\|\nabla\phi\| + \|\nabla\omega(t)\|\|\nabla\varphi\| \\ &\lesssim \|u(t)\|_V \|\phi\|_V + \|\omega(t)\|_{1,2}\|\varphi\|_{1,2} \\ &\lesssim (\|u(t)\|_V^2 + \|\omega(t)\|_{1,2}^2)^{1/2}(\|\phi\|_V^2 + \|\varphi\|_{1,2}^2)^{1/2} \lesssim \|w(t)\|_{\widehat{V}}\|\Phi\|_{\widehat{V}}. \end{aligned} \tag{2.18}$$

At the same time, by using Gagliardo-Nirenberg's inequality and the embedding $\widehat{V} \hookrightarrow \mathbb{L}^6(\Omega) \times \mathbb{L}^6(\Omega)$, we have

$$\begin{aligned} |\langle Bw(t), \Phi\rangle| &\lesssim \|u(t)\|^{1/2}\|\nabla u(t)\|^{1/2}(\|\nabla u(t)\| + \|\nabla\omega(t)\|)\|\Phi\|_{\widehat{V}} \\ &\lesssim \|w(t)\|^{1/2}\|w(t)\|_{\widehat{V}}^{3/2}\|\Phi\|_{\widehat{V}}, \ \forall\, \Phi \in \widehat{V}. \end{aligned} \tag{2.19}$$

It is also clear that

$$|\langle Lw(t), \Phi\rangle| \lesssim \|w(t)\|_{\widehat{V}}\|\Phi\|_{\widehat{V}}, \ \forall\, \Phi \in \widehat{V}, \tag{2.20}$$

$$\begin{aligned} |\langle N_\sigma w(t), \Phi\rangle| &\leqslant \sigma\int_\Omega |u|^\beta|\phi|\mathrm{d}x \leqslant \sigma\||u|^\beta\|_{\mathbb{L}^{(\beta+1)^*}(\Omega)}\|\phi\|_{\mathbb{L}^{\beta+1}(\Omega)} \\ &\lesssim \|u(t)\|_{\mathbb{L}^{(\beta+1)}(\Omega)}^\beta\|\phi\|_{\mathbb{L}^{\beta+1}(\Omega)} \\ &\lesssim \|w(t)\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}^\beta\|\Phi\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}, \ \forall\, \Phi = (\phi,\varphi) \in \hat{\mathbb{L}}^{\beta+1}(\Omega). \end{aligned} \tag{2.21}$$

Then inequalities (2.18)-(2.21) give, respectively,

$$\|Aw(t)\|_{\widehat{V}^*} \lesssim \|w(t)\|_{\widehat{V}}, \tag{2.22}$$

$$\|Bw(t)\|_{\widehat{V}^*} \lesssim \|w(t)\|_{\widehat{V}}^{3/2}\|w(t)\|^{1/2}, \tag{2.23}$$

$$\|Lw(t)\|_{\widehat{V}^*} \lesssim \|w(t)\|_{\widehat{V}}, \tag{2.24}$$

$$\|N_\sigma w(t)\|_{\hat{\mathbb{L}}^{(\beta+1)^*}(\Omega)} \lesssim \|w(t)\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}^{\beta}. \tag{2.25}$$

Therefore,

$$\int_0^T \|Aw(t)\|_{\widehat{V}^*}^2 \mathrm{d}t \lesssim \int_0^T \|w(t)\|_{\widehat{V}}^2 \mathrm{d}t, \tag{2.26}$$

$$\begin{aligned}\int_0^T \|Bw(t)\|_{\widehat{V}^*}^{4/3} \mathrm{d}t &\lesssim \int_0^T \|w(t)\|^{2/3}\|w(t)\|_{\widehat{V}}^2 \mathrm{d}t \\ &\lesssim \|w(t)\|_{L^\infty([0,T];\widehat{H})}^{2/3} \int_0^T \|w(t)\|_{\widehat{V}}^2 \mathrm{d}t,\end{aligned} \tag{2.27}$$

$$\int_0^T \|Lw(t)\|_{\widehat{V}^*}^2 \mathrm{d}t \lesssim \int_0^T \|w(t)\|_{\widehat{V}}^2 \mathrm{d}t, \tag{2.28}$$

$$\int_0^T \|N_\sigma w(t)\|_{\hat{\mathbb{L}}^{(\beta+1)^*}(\Omega)}^{(\beta+1)^*} \mathrm{d}t \lesssim \int_0^T \|w(t)\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}^{\beta+1} \mathrm{d}t. \tag{2.29}$$

The proof is complete. □

Lemma 2.3. *Let $\{w_n(t)\}_{n\geqslant 1}$ be a sequence of weak solutions of equation* (2.5) *such that $w_n(t) \in X$ for all $t \geqslant 0$. Then*

(1) *For all $T > 0$,*

$$\begin{cases} \{w_n\}_{n\geqslant 1} \ \text{is bounded in} \ L^2([0,T];\widehat{V}) \cap L^\infty([0,T];\widehat{H}) \cap L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega)), \\ \{\frac{\mathrm{d}}{\mathrm{d}t} w_n\}_{n\geqslant 1} \ \text{is bounded in} \ L^{4/3}([0,T];\widehat{V}^*). \end{cases} \tag{2.30}$$

(2) *There exists a subsequence $\{w_{n_j}(t)\}_{n_j\geqslant 1}$ which converges in $C([0,T];\widehat{H}_{\mathrm{w}})$ to some weak solution $w(t)$ of* (2.5)*, that is,*

$$(w_{n_j}(t), \Phi) \longrightarrow (w(t), \Phi) \ \text{uniformly on} [0,T], \ \ n_j \to \infty, \ \ \forall\, \Phi \in \widehat{H}. \tag{2.31}$$

Proof. Let $\{w_n(t)\}_{n\geqslant 1}$ be a sequence of weak solutions of (2.5) such that $w_n(t) \in X$ for all $t \geqslant 0$. Similar to $(2.11)_2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_n(t)\|^2 + \delta\|\nabla w_n(t)\|^2 + 2\sigma\|u(t)\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}^{\beta+1} \lesssim \|f\|^2 + \|g\|^2, \ \ t > 0. \tag{2.32}$$

Consider any $T > 0$. Integrating (2.32) over $[0,T]$ implies

$$\begin{aligned}&\|w_n(t)\|^2 + \delta \int_0^T \|\nabla w_n(t)\|^2 \mathrm{d}t + 2\sigma \int_0^T \|u(t)\|_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}^{\beta+1} \mathrm{d}t \\ \lesssim &\|w_n(0)\|^2 + T(\|f\|^2 + \|g\|^2), \ \ t \in [0,T],\end{aligned} \tag{2.33}$$

which implies that

$$\{w_n\}_{n\geqslant 1} \text{ is bounded in } L^2([0,T];\widehat{V})\cap L^\infty([0,T];\widehat{H})\cap L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega)). \quad (2.34)$$

Note that $\beta\in(1,3]$. We have $(\beta+1)^*=\frac{\beta+1}{\beta}\in[\frac{4}{3},2)$ and thus

$$L^{(\beta+1)^*}([0,T];\hat{\mathbb{L}}^{(\beta+1)^*}(\Omega))\hookrightarrow L^{4/3}([0,T];\widehat{V}^*).$$

We then conclude from equation (2.5), Lemma 2.2 and the embedding $L^2([0,T];\widehat{V}^*)\hookrightarrow L^{4/3}([0,T];\widehat{V}^*)$ that

$$\begin{aligned}
&\|\frac{\mathrm{d}w_n(t)}{\mathrm{d}t}\|_{L^{4/3}([0,T];\widehat{V}^*)}\\
\leqslant&\|Aw_n(t)\|_{L^{4/3}([0,T];\widehat{V}^*)}+\|Bw_n(t)\|_{L^{4/3}([0,T];\widehat{V}^*)}\\
&+\|N_\sigma w_n(t)\|_{L^{4/3}([0,T];\widehat{V}^*)}+\|Lw_n(t)\|_{L^{4/3}([0,T];\widehat{V}^*)}+\|F\|_{L^{4/3}([0,T];\widehat{V}^*)}\\
\lesssim&\|w_n(t)\|_{L^2([0,T];\widehat{V})}+\|w_n(t)\|^{1/2}_{L^\infty([0,T];\widehat{H})}\|w_n(t)\|^{3/2}_{L^2([0,T];\widehat{V})}\\
&+\|w_n(t)\|_{L^2([0,T];\widehat{V})}+\|w_n(t)\|_{L^{(\beta+1)^*}([0,T];\hat{\mathbb{L}}^{(\beta+1)^*}(\Omega))}+T\|F\|. \qquad (2.35)
\end{aligned}$$

Note that $F\in\widehat{H}$. We conclude from (2.34) that the right hand side of (2.35) is bounded by a constant independent of n. (2.30) is proved.

We next prove (2.31). By (2.30) and the diagonal procedure, we claim that there exist some $w(t)\in L^\infty([0,T];\widehat{H})\cap L^2([0,T];\widehat{V})\cap L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega))$ with $\frac{\mathrm{d}w(t)}{\mathrm{d}t}\in L^{4/3}([0,T];\widehat{V}^*)$ and a subsequence $\{w_{n_j}(t)\}_{n_j\geqslant 1}$ of $\{w_n(t)\}_{n\geqslant 1}$ such that

$$\begin{cases}
w_{n_j}(t)\rightharpoonup w(t) \text{ weakly star in } L^\infty([0,T];\widehat{H}) \text{ as } n_j\to\infty,\\
w_{n_j}(t)\rightharpoonup w(t) \text{ weakly in } L^2([0,T];\widehat{V}) \text{ as } n_j\to\infty,\\
w_{n_j}(t)\rightharpoonup w(t) \text{ weakly in } L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega)) \text{ as } n_j\to\infty,\\
\frac{\mathrm{d}w_{n_j}(t)}{\mathrm{d}t}\rightharpoonup\frac{\mathrm{d}w(t)}{\mathrm{d}t} \text{ weakly in } L^{4/3}([0,T];\widehat{V}^*) \text{ as } n_j\to\infty.
\end{cases} \qquad (2.36)$$

We now denote by

$$\mathcal{Y}=\left\{\Phi(t):\Phi(t)\in L^2([0,T];\widehat{V})\cap L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega)),\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}\in L^{4/3}([0,T];\widehat{V}^*)\right\}$$

and endow it with the norm

$$\|\Phi\|_{\mathcal{Y}}=\|\Phi\|_{L^2([0,T];\widehat{V})}+\|\Phi\|_{L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega))}+\|\Phi\|_{L^{4/3}([0,T];\widehat{V}^*)}.$$

Then [24, Lemma 2.3.1, P_{55}] has proved that

$$\mathcal{Y}\subset C([0,T];\widehat{H}_{\mathrm{w}}) \text{ and } \mathcal{Y}\hookrightarrow L^2([0,T];\widehat{H}) \text{ with compact embedding.} \qquad (2.37)$$

Since $\beta\in(1,3]$ and the embedding $\widehat{V}\hookrightarrow\mathbb{L}^6(\Omega)$ is compact, we can check by interpolation that

$$\mathcal{Y}\hookrightarrow L^{\beta+1}([0,T];\hat{\mathbb{L}}^{\beta+1}(\Omega)) \text{ with compact embedding.} \qquad (2.38)$$

It then follows from (2.36)-(2.37) that

$$w_{n_j}(t) \longrightarrow w(t) \ \text{ in } \ C([0,T];\widehat{H}_{\mathrm{w}}) \ \text{ as } \ n_j \to \infty. \tag{2.39}$$

We shall prove that $w(t)$ satisfies (2.5) and (2.7). Firstly, from (2.36) and the definitions of the operators A, N_σ and L, we derive that

$$\begin{cases} Aw_{n_j}(t) \rightharpoonup Aw(t) \ \text{ weakly in } \ L^2([0,T];\widehat{V}^*) \quad \text{as } \ n_j \to \infty, \\ Lw_{n_j}(t) \rightharpoonup Lw(t) \ \text{ weakly in } \ L^2([0,T];\widehat{V}^*) \quad \text{as } \ n_j \to \infty, \\ N_\sigma w_{n_j}(t) \rightharpoonup N_\sigma w(t) \ \text{ weakly in } \ L^{(\beta+1)^*}([0,T];\hat{\mathbb{L}}^{(\beta+1)^*}(\Omega)) \ \text{ as } \ n_j \to \infty. \end{cases} \tag{2.40}$$

For the nonlinear term $Bw_{n_j}(t)$, we can use the derivations similar to those of [30] to deduce

$$Bw_{n_j}(t) \rightharpoonup Bw(t) \ \text{ weakly in } \ L^{4/3}([0,T];\widehat{V}^*) \quad \text{as } \ n_j \to \infty. \tag{2.41}$$

Now, for a.e. $t > 0$ we have in $L^{4/3}([0,T];\widehat{V}^*)$ that

$$\frac{\mathrm{d}w_{n_j}(t)}{\mathrm{d}t} + Aw_{n_j}(t) + Bw_{n_j}(t) + N_\sigma w_{n_j}(t) + Lw_{n_j}(t) = F, \ \ n = 1, 2, \cdots.$$

Taking (2.37) and (2.39)-(2.40) into account, we have in $L^{4/3}([0,T];\widehat{V}^*)$ that

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} + Aw(t) + Bw(t) + N_\sigma w(t) + Lw(t) = F, \quad \text{a.e.} \quad t > 0.$$

Notice that $L^{4/3}[0,T] \subset \mathcal{D}'[0,T]$. Thus, $w(t)$ satisfies (2.5) in the sense of distribution in $\mathcal{D}'([0,T];\widehat{V}^*)$. Secondly, we check that $w(t)$ fulfills (2.7). By (2.31), (2.37) and (2.38) we have that

$$w_{n_j}(t) \longrightarrow w(t) \quad \text{strongly} \quad \text{in } \ L^2([0,T];\widehat{H}) \quad \text{as } \ n_j \to \infty, \tag{2.42}$$

$$\|w_{n_j}(t)\|^2 \longrightarrow \|w(t)\|^2 \ \text{ for a.e. } \ t \in [0,T] \quad \text{as } \ n_j \to \infty, \tag{2.43}$$

$$\|w_{n_j}(t)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)} \longrightarrow \|w(t)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)} \ \text{ for a.e. } \ t \in [0,T] \quad \text{as } \ n_j \to \infty. \tag{2.44}$$

For given $\xi(\cdot) \in \mathcal{C}_0^\infty([0,T])$ with $\xi(t) \geqslant 0$. It is clear that $\|w_{n_j}(\cdot)\|^2\xi'(\cdot) \in L^1([0,T])$. $(2.36)_1$ implies that $\|w_{n_j}(t)\|^2\xi'(t)$ has essential upper bound and thus possesses an integrable dominated function. Using Lebesgue's Dominated Convergence Theorem and (2.42), we obtain that

$$\lim_{n_j\to\infty} \int_0^T \|w_{n_j}(t)\|^2\xi'(t)\mathrm{d}t = \int_0^T \|w(t)\|^2\xi'(t)\mathrm{d}t. \tag{2.45}$$

Similarly, we have that $\|w_{n_j}(\cdot)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}\xi'(\cdot) \in L^1([0,T])$. $(2.36)_3$ implies that $\|w_{n_j}(t)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}\xi'(t)$ has essential upper bound and thus possesses an integrable dominated function. Again, using Lebesgue's Dominated Convergence Theorem and (2.42), we arrive at

$$\lim_{n_j\to\infty} \int_0^T \|w_{n_j}(t)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}\xi'(t)\mathrm{d}t = \int_0^T \|w(t)\|^{\beta+1}_{\hat{\mathbb{L}}^{\beta+1}(\Omega)}\xi'(t)\mathrm{d}t. \tag{2.46}$$

At the same time, from $(2.36)_1$, $(2.36)_2$ and the lower semicontinuity of the norm we infer that

$$\begin{cases} \displaystyle\int_0^T \|w(t)\|_{\widehat{V}}^2 \xi'(t)\mathrm{d}t \leqslant \lim_{n_j\to\infty} \int_0^T \|w_{n_j}(t)\|_{\widehat{V}}^2 \xi'(t)\mathrm{d}t, \\ \displaystyle\int_0^T \langle Lw(t), w(t)\rangle \xi'(t)\mathrm{d}t \leqslant \lim_{n_j\to\infty} \int_0^T \langle Lw_{n_j}(t), w_{n_j}(t)\rangle \xi'(t)\mathrm{d}t. \end{cases} \tag{2.47}$$

Notice that (cf. [37, Lemma 2.1]) $\sqrt{\langle A\Phi, \Phi\rangle + \langle L\Phi, \Phi\rangle}$ induces a norm which is equivalent to $\|\Phi\|_{\widehat{V}}$. Thus,

$$\begin{aligned} &\int_0^T \langle Aw(t), w(t)\rangle \xi'(t)\mathrm{d}t + \int_0^T \langle Lw(t), w(t)\rangle \xi'(t)\mathrm{d}t \\ &\leqslant \liminf_{n_j\to\infty} \int_0^T \langle Aw_{n_j}(t), w_{n_j}(t)\rangle \xi'(t)\mathrm{d}t + \liminf_{n_j\to\infty} \int_0^T \langle Lw_{n_j}(t), w_{n_j}(t)\rangle \xi'(t)\mathrm{d}t. \end{aligned} \tag{2.48}$$

Now, $\{w_{n_j}(t)\}_{n_j\geqslant 1}$ is a sequence of weak solutions satisfying

$$\begin{aligned} &-\frac{1}{2}\int_0^T \|w_{n_j}(t)\|^2 \xi'(t)\mathrm{d}t + \int_0^T \langle Aw_{n_j}(t), w_{n_j}(t)\rangle \xi(t)\mathrm{d}t \\ &\quad + \int_0^T \langle N_\sigma w_{n_j}(t), w_{n_j}(t)\rangle \xi(t)\mathrm{d}t + \int_0^T \langle Lw_{n_j}(t), w_{n_j}(t)\rangle \xi(t)\mathrm{d}t \\ &\leqslant \int_0^T \big(F, w_{n_j}(t)\big)\xi(t)\mathrm{d}t, \ \ \forall\, \xi(t) \in C_0^\infty([0,T]) \quad \text{with } \ \xi(t) \geqslant 0, \end{aligned} \tag{2.49}$$

we pass to the limit in (2.49), using (2.45)-(2.46) and (2.48), and demonstrate that $w(t)$ satisfies (2.7). This ends the proof. □

Next, we use Π_+ to denote the restriction operator (with respect to time variable) to the semi-infinite interval $\mathbb{R}_+$. At this stage, we can prove the existence of the trajectory attractor.

Theorem 2.1. *The natural translation semigroup $\{T(t)\}_{t\geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ in $\mathcal{F}^+$ satisfying*

$$\mathcal{A}_\sigma^{\mathrm{tr}} = \Pi_+ \mathcal{K}_\sigma = \{w(\cdot)|_{[0,\infty)} \big| \ \ u \in \mathcal{K}_\sigma\} \subset \mathcal{T}^+. \tag{2.50}$$

Proof. According to [9, Theorem 7.4], it is sufficient to prove that $\mathcal{T}_\sigma^+$ is compact in $\mathcal{F}^+$. Factually, $\mathcal{T}_\sigma^+ \subset \mathcal{F}^+$ is obvious. Now, for any sequence $\{w_n\}_{n\geqslant 1}$ in $\mathcal{T}_\sigma^+$, we see from Lemma 2.3 that there is a subsequence (still denote by $\{w_n\}_{n\geqslant 1}$) that converges to some $w^{(1)} \in \mathcal{T}_\sigma^+$ in $C([0,1]; \widehat{H}_{\mathrm{w}})$ as $n \to \infty$. Passing to a subsequence and dropping the subindex again, we find that there exists $w^{(2)} \in \mathcal{T}_\sigma^+$ yielding

$$w_n \longrightarrow w^{(2)} \quad \text{in} \quad C([0,2]; \widehat{H}_{\mathrm{w}}) \quad \text{as} \quad n \to \infty.$$

Obviously, $w^{(1)}(t) = w^{(2)}(t)$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence $\{w_{n_k}\}_{n_k\geqslant 1}$ of $\{w_n\}_{n\geqslant 1}$ and some $w \in \mathcal{T}_\sigma^+$ such that

$$w_{n_k} \longrightarrow w \ \text{ in } \ \mathcal{F}^+ \quad \text{as } \ n_k \to \infty.$$

The proof is complete. □

We next introduce the definition of trajectory statistical solution for equation (2.5).

Definition 2.3. *A Borel probability measure μ on $\mathcal{F}^+$ is a $\mathcal{T}_\sigma^+$-trajectory statistical solution over* $[0,\infty)$ *(or simply a trajectory statistical solution) for equation* (2.5) *if*

(1) *μ is tight in the sense that for any $\mathcal{B} \in \mathcal{B}(\mathcal{F}^+)$ (the collection of Borel sets of $\mathcal{F}^+$),*

$$\mu(\mathcal{B}) = \sup\left\{\mu(E)\middle|E \in \mathcal{B}(\mathcal{F}^+) \text{ and } E \subset \mathcal{B}\right\};$$

(2) *μ is supported by a Borel subset of $\mathcal{F}^+$ contained in $\mathcal{T}_\sigma^+$.*

To state the existence of the trajectory statistical solution, we first recall the definition of generalized Banach limit.

Definition 2.4. ([27]) *A generalized Banach limit, which we denote by* $\mathrm{LIM}_{t\to+\infty}$*, is any linear continuous functional defined on the space of all bounded real-valued functions on* $[0,\infty)$ *that satisfies*

(1) $\mathrm{LIM}_{t\to+\infty}\rho(t) \geqslant 0$ *for nonnegative functions $\rho(\cdot)$ on* $[0,\infty)$;

(2) $\mathrm{LIM}_{t\to+\infty}\rho(t) = \lim\limits_{t\to+\infty}\rho(t)$ *if the usual limit* $\lim\limits_{t\to+\infty}\rho(t)$ *exists.*

The main result of this section reads as follows.

Theorem 2.2. *Let* $\mathrm{LIM}_{t\to+\infty}$ *be a given generalized Banach limit. Then for any $w_\sigma \in \mathcal{T}_\sigma^+$, there corresponds a unique Borel probability measure μ_{σ,w_σ} on $\mathcal{F}^+$ such that*

$$\int_{\mathcal{F}^+}\zeta(z)\mathrm{d}\mu_{\sigma,w_\sigma}(z) = \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\zeta(T(s)w)\mathrm{d}s, \quad \forall\zeta \in C(\mathcal{F}^+). \tag{2.51}$$

Moreover, μ_{σ,w_σ} is supported by $\mathcal{A}_\sigma^{\mathrm{tr}}$ and is a trajectory statistical solution for equation (2.5)*, and μ_{σ,w_σ} is invariant under the acting of $\{T(t)\}_{t\geqslant 0}$ in the following sense*

$$\int_{\mathcal{F}^+}\zeta(T(t)z)\mathrm{d}\mu_{\sigma,w_\sigma}(z) = \int_{\mathcal{F}^+}\zeta(z)\mathrm{d}\mu_{\sigma,w_\sigma}(z), \quad \forall\, t \geqslant 0,\ \ \forall\zeta \in C(\mathcal{F}^+). \tag{2.52}$$

Proof. Let $\mathcal{M}_{\mathrm{loc}}^+ = C_{\mathrm{loc}}([0,+\infty); X_{\mathrm{w}})$ be the space of continuous functions from $[0,+\infty)$ to X_{w}, where we have denoted by X_{w} the space X endowed with the weak topology inherited from $\hat{H}_{\mathrm{w}}$. We endow $\mathcal{M}_{\mathrm{loc}}^+$ also with the topology Θ_{loc}^+. Since X is a fixed bounded subset of the Banach space $\hat{H}$, the topology Θ_{loc}^+ in $\mathcal{M}_{\mathrm{loc}}^+$ is metrizable (cf. [1]). Then, by Theorem 2.1 and the abstract result [41, Theorem 2.1], we obtain the results of Theorem 2.2. The proof is complete. □

3 Lusin type degenerate regularity of the trajectory statistical solution

In this section, we will prove that if the Grashof number G given by (1.6) is sufficiently small, then the trajectory statistical solution μ_{σ,w_σ} obtained in Theorem 2.2 possesses Lusin type degenerate regularity.

For brevity, we set

$$\vartheta = \delta^{-1}\lambda^{-1},\ \ M(k) = \frac{k(2\lambda + \frac{1}{\lambda})}{\delta^2}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2),\ k \in \mathbb{N}.$$

Lemma 3.1. *Let $F \in \widehat{H}$. Then, for any $\varsigma \geqslant 0$ and any $w \in \mathcal{A}_\sigma^{\mathrm{tr}}$, the Lebesgue measure of the set $\{\tau \in [\varsigma, \varsigma + \vartheta : \|\nabla w(\tau)\|^2 \leqslant M(k)\}$ satisfies*

$$\mathrm{mes}(\{\tau \in [\varsigma, \varsigma + \vartheta] : \|\nabla w(\tau)\|^2 \leqslant M(k)\}) \geqslant \frac{k-1}{k}\vartheta. \tag{3.1}$$

Proof. For any given $\varsigma \geqslant 0$, let $w \in \mathcal{A}_\sigma^{\mathrm{tr}}$ with initial value $w(\varsigma) = w_\varsigma$ at time ς. Then we can derive from (2.8) and (2.50) that

$$\|w(\varsigma)\|^2 = \|w_0\|^2 \leqslant \frac{2\lambda}{\delta}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2), \tag{3.2}$$

which, together with $(2.11)_2$, gives for any $t \geqslant \varsigma$ that

$$\|w(t)\|^2 + \delta\int_\varsigma^t \|\nabla w(\tau)\|^2 \mathrm{d}\tau \leqslant \frac{2\lambda}{\delta}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2) + (\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2)(t - \varsigma). \tag{3.3}$$

Consequently,

$$\delta\int_\varsigma^t \|\nabla w(\tau)\|^2 \mathrm{d}\tau \leqslant \frac{2\lambda}{\delta}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2) + (\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2)(t - \varsigma). \tag{3.4}$$

Picking $t = \varsigma + \vartheta$, we arrive at

$$\int_\varsigma^{\varsigma+\vartheta} \|\nabla w(\tau)\|^2 \mathrm{d}\tau \leqslant \frac{(2\lambda + \frac{1}{\lambda})}{\delta^2}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2). \tag{3.5}$$

Inequality (3.5) shows that the Lebesgue measure of the set $\{\tau \in [\varsigma, \varsigma+\vartheta] :\ \|\nabla w(\tau)\|^2 \geqslant M(k)\}$ satisfies

$$\begin{aligned} &\mathrm{mes}(\{\tau \in [\varsigma, \varsigma + \vartheta] :\ \|\nabla w(\tau)\|^2 \geqslant M(k)\}) \\ \leqslant &\frac{1}{M(k)}\frac{(2\lambda + \frac{1}{\lambda})}{\delta^2}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2) = \frac{\vartheta}{k}. \end{aligned} \tag{3.6}$$

Otherwise, if

$$\mathrm{mes}(\{\tau \in [\varsigma, \varsigma + \vartheta] :\ \|\nabla w(\tau)\|^2 \geqslant M(k)\}) > \frac{\vartheta}{k},$$

then

$$\int_\varsigma^{\varsigma+\vartheta} \|\nabla w(\tau)\|^2 \mathrm{d}\tau > \frac{(2\lambda + \frac{1}{\lambda})}{\delta^2}(\frac{\lambda}{\nu}\|f\|^2 + \frac{\lambda}{\gamma}\|g\|^2).$$

This contradicts equation (3.5). The proof of Lemma 3.1 is therefore complete. $\square$

From (3.1) we can conclude that each weak solution $w(\cdot)$ within $\mathcal{A}_\sigma^{\mathrm{tr}}$ is "nearly" regular on $[\varsigma, \varsigma+\vartheta]$ for each $\varsigma \in \mathbb{R}_+$. But we cannot say that each $w(\cdot)$ within $\mathcal{A}_\sigma^{\mathrm{tr}}$ is a.e. regular on $[\varsigma, \varsigma+\vartheta]$. This is much like Lusin's Theorem describing the relations between measurable functions and continuous functions: each measurable function on an interval I is "nearly" continuous on I. This "nearly" regular result of the elements within $\mathcal{A}_\sigma^{\mathrm{tr}}$ is stated as follows.

Corollary 3.1. *For any $\varsigma \geqslant 0$, denote $E(\varsigma) = [\varsigma, \varsigma+\vartheta]$. Then, for any $\epsilon > 0$ and each $w \in \mathcal{A}_\sigma^{tr}$, there corresponds a subset $E_{\epsilon,w,\varsigma} \subseteq E(\varsigma)$ and a positive constant C_ϵ such that*

(1) *The Lebesgue measure of $E_{\epsilon,w,\varsigma}$ satisfies $\mathrm{mes}(E_{\epsilon,w,\varsigma}) < \epsilon$;*

(2) $\|w(\tau)\|_{\widehat{V}}^2 \leqslant C_\epsilon, \quad \forall\, \tau \in E(s) \setminus E_{\epsilon,w,\varsigma}$.

Proof. Note that the norms $\|\nabla w\|$ and $\|w\|_{\widehat{V}}$ are equivalent. For any given $\epsilon > 0$, there is some $k_\epsilon \in \mathbb{N}$ such that $\vartheta/k_\epsilon < \epsilon$. For each $w \in \mathcal{A}^{tr}$ and above ϵ, we set

$$E_{\epsilon,w,\varsigma} = \{\theta \in [\varsigma, \varsigma+\vartheta] \,:\, \|w(\tau)\|_{\widehat{V}}^2 \geqslant M(k_\epsilon)\}.$$

Then from (3.1) we have that

$$\mathrm{mes}(E_{\epsilon,w,\varsigma}) = \vartheta - \mathrm{mes}(\{\theta \in [\varsigma, \varsigma+\vartheta] \,:\, \|w(\theta)\|_{\widehat{V}}^2 \leqslant M(k_\epsilon)\}) \leqslant \vartheta - \frac{k_\epsilon - 1}{k_\epsilon}\vartheta < \epsilon.$$

Item (2) obviously holds true. The proof is complete. □

We next prove that if the Grashof number G given by (1.6) is sufficiently small, then the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ degenerates to a single bounded complete trajectory.

Lemma 3.2. *Let that the Grashof number G given by (1.6) satisfy*

$$\frac{7^7 c_1^8}{2\delta_1^7} G^8 + \frac{c_2^2 2^{\beta+2}\sigma^2}{\delta_1}\left(1+\frac{1}{\lambda}\right)^{\frac{\beta+1}{2}} G^{\beta+1} < \frac{\delta_1}{\lambda}, \tag{3.7}$$

where c_1 and c_3 are positive constants depending only on Ω. Then, the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ given by (2.50) degenerates to a single bounded complete trajectory:

$$\mathcal{A}_\sigma^{\mathrm{tr}} = \{\tilde{w}(t) \in \mathcal{K}_\sigma : t \geqslant 0\}. \tag{3.8}$$

Proof. Let $w(t) = (u(t), \omega(t))$ and $\Phi(t) = (\phi(t), \varphi(t))$ lie in $\mathcal{A}_\sigma^{\mathrm{tr}}$. Then (2.50) shows that both $w(t)$ and $\Phi(t)$ belong to $\mathcal{K}_\sigma$. Set $z(t) = w(t) - \Phi(t)$. Then $z(t)$ satisfies

$$\frac{\partial z}{\partial t} + Az + Bw - B\Phi + N_\sigma w - N_\sigma \Phi + Lz = 0, \quad t \in \mathbb{R}. \tag{3.9}$$

Taking the dual pairing $\langle \cdot, \cdot \rangle$ between z and equation (3.9) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + \langle Az, z\rangle + \langle Lz, z\rangle + \langle Bw - B\Phi, z\rangle + \langle N_\sigma w - N_\sigma \Phi, z\rangle = 0. \tag{3.10}$$

Similar to [37, (2.7)], we have

$$\langle Az, z\rangle + \langle Lz, z\rangle \geqslant \delta_1 \|z\|_{\hat{V}}, \text{ where } \delta_1 = \min\{\gamma, \nu\}. \tag{3.11}$$

We shall handle the nonlinear terms $\langle Bw - B\Phi, z\rangle$ and $\langle N_\sigma w - N_\sigma \Phi, z\rangle$. Firstly, by (2.2) and $(2.4)_2$, we obtain

$$\begin{aligned}\langle Bw - B\Phi, z\rangle =& \langle B_1(u,u) - B_1(\phi,\phi), u-\phi\rangle + \langle B_2(u,\omega) - B_2(\phi,\varphi), \omega - \varphi\rangle \\ =& b_1(u-\phi, \phi, u-\phi) + b_2(u-\phi, \varphi, \omega-\varphi).\end{aligned} \tag{3.12}$$

At the same time, from the classical estimation (see [14, (A.26f)]) we see that there is a positive constant c_1 depending only on Ω such that

$$\begin{cases} |b_1(u-\phi,\phi,u-\phi)| \leqslant c_1\|u-\phi\|^{1/4}\|u-\phi\|_V^{7/4}\|\phi\|^{1/4}\|\phi\|_V^{3/4} \\ \qquad\qquad\qquad\qquad \leqslant \dfrac{7^7 c_1^8}{8\delta_1^7}\|z\|^2\|\Phi\|_{\hat{V}}^2\|\nabla\Phi\|^6 + \dfrac{\delta_1}{8}\|z\|_{\hat{V}}^2, \\ |b_2(u-\phi,\varphi,\omega-\varphi)| \leqslant \dfrac{7^7 c_1^8}{8\delta_1^7}\|z\|^2\|\Phi\|_{\hat{V}}^2\|\nabla\Phi\|^6 + \dfrac{\delta_1}{8}\|z\|_{\hat{V}}^2. \end{cases} \tag{3.13}$$

It then follows from (3.12) and (3.13) that

$$|\langle Bw - B\Phi, z\rangle| \leqslant \frac{7^7 c_1^8}{4\delta_1^7}\|z\|^2\|\Phi\|_{\hat{V}}^2 + \frac{\delta_1}{4}\|z\|_{\hat{V}}^2. \tag{3.14}$$

Secondly,

$$\begin{aligned}\langle N_\sigma w - N_\sigma\Phi, z\rangle &= \langle \sigma(|u|^{\beta-1}u - |\phi|^{\beta-1}\phi), u-\phi\rangle \\ =&\sigma\||u|^{\frac{\beta-1}{2}}|u-\phi|\|^2 + \sigma\int_\Omega \big(|u|^{\beta-1} - |\phi|^{\beta-1}\big)(u-\phi)\phi \mathrm{d}x.\end{aligned} \tag{3.15}$$

Set $\varrho(y) = |y|^{\beta-1}$, $y = (y_1,y_2,y_3) \in \mathbb{R}^3$. Then $\varrho(y)$ is differentiable for $y \neq \mathbf{0}$, and

$$\nabla\varrho(y) = (|y|^{\frac{\beta-3}{2}}y_1, |y|^{\frac{\beta-3}{2}}y_2, |y|^{\frac{\beta-3}{2}}y_3), \ \ |\nabla\varrho(y)| \leqslant |y|^{\frac{\beta-1}{2}}.$$

By the differential mean value theorem we see that there exists $\theta \in (0,1)$ such that

$$\begin{aligned}&\big||u|^{\beta-1} - |\phi|^{\beta-1}\big| = |\varrho(u) - \varrho(\phi)| \leqslant |\nabla\varrho(\phi + \theta(u-\phi))||u-\phi| \\ \leqslant& |(1-\theta)\phi + \theta u|^{\frac{\beta-1}{2}}|u-\phi| \leqslant 2^{\frac{\beta-1}{2}}|\phi|^{\frac{\beta-1}{2}}|u-\phi| + (2\theta)^{\frac{\beta-1}{2}}|u-\phi||u|^{\frac{\beta-1}{2}}.\end{aligned} \tag{3.16}$$

Taking (3.15)-(3.16) into account and using Hölder's and Cauchy's inequalities, we arrive at

$$
\begin{aligned}
&\sigma\int_\Omega\big(|u|^{\beta-1}-|\phi|^{\beta-1}\big)(u-\phi)\phi\mathrm{d}x\\
\geqslant&-2^{\frac{\beta-1}{2}}\sigma\int_\Omega|\phi|^{\frac{\beta+1}{2}}|u-\phi|^2\mathrm{d}x-2^{\frac{\beta-1}{2}}\sigma\int_\Omega|\phi||u|^{\frac{\beta-1}{2}}|u-\phi|^2\mathrm{d}x\\
\geqslant&-2^{\frac{\beta-1}{2}}\sigma\|\phi\|^{\frac{\beta+1}{2}}_{\mathbb{L}^{\frac{3(\beta+1)}{2}}(\Omega)}\|u-\phi\|\|u-\phi\|_{\mathbb{L}^6(\Omega)}\\
&-2^{\frac{\beta-1}{2}}\sigma\|\phi\|_{\mathbb{L}^6(\Omega)}\|u\|^{\frac{\beta-1}{2}}_{\mathbb{L}^{3(\beta-1)}(\Omega)}\|u-\phi\|_{\mathbb{L}^6(\Omega)}\|u-\phi\|\\
\geqslant&-2^{\frac{\beta-1}{2}}\sigma\|\phi\|^{\frac{\beta+1}{2}}_{\mathbb{L}^{\frac{3(\beta+1)}{2}}(\Omega)}\|u-\phi\|\|u-\phi\|_V-2^{\frac{\beta-1}{2}}c_2\sigma\|\Phi\|_{\widehat{V}}\|w\|^{\frac{\beta-1}{2}}_{\widehat{V}}\|z\|\|z\|_{\widehat{V}}\\
\geqslant&-\frac{c_2^22^\beta\sigma^2}{\delta_1}\big(\|\Phi\|^{\beta+1}_{\widehat{V}}+\|\Phi\|^2_{\widehat{V}}\|w\|^{\beta-1}_{\widehat{V}}\big)\|z\|^2-\frac{\delta_1}{4}\|z\|^2_{\widehat{V}},
\end{aligned}\tag{3.17}
$$

where we have also used Cauchy's inequality, the embedding $V\hookrightarrow\mathbb{L}^6(\Omega)$, $V\hookrightarrow\mathbb{L}^{3(\beta-1)}(\Omega)$ and $V\hookrightarrow\mathbb{L}^{\frac{3(\beta+1)}{2}}(\Omega)$, and c_2 is a positive embedding constant depending only on Ω. It then follows from (3.10), (3.11), (3.14) and (3.17) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2+\delta_1\|z\|^2_{\widehat{V}}\leqslant\|z\|^2\big(\frac{7^7c_1^8}{2\delta_1^7}\|\Phi\|^2\|\nabla\Phi\|^6+\frac{c_2^22^{\beta+1}\sigma^2}{\delta_1}\big(\|\Phi\|^{\beta+1}_{\widehat{V}}+\|\Phi\|^2_{\widehat{V}}\|w\|^{\beta-1}_{\widehat{V}}\big)\big).\tag{3.18}
$$

For any $\tau\in\mathbb{R}$ with $\tau<t$, we integrate (3.18) over $[\tau,t]$ and use (2.10) to obtain

$$
\begin{aligned}
&\|z(t)\|^2-\|z(\tau)\|^2\\
\leqslant&\int_\tau^t\big(\frac{7^7c_1^8}{2\delta_1^7}\|\Phi\|^2\|\nabla\Phi\|^6+\frac{c_2^22^{\beta+1}\sigma^2}{\delta_1}\big(\|\Phi\|^{\beta+1}_{\widehat{V}}+\|\Phi\|^2_{\widehat{V}}\|w\|^{\beta-1}_{\widehat{V}}\big)-\frac{\delta_1}{\lambda}\big)\|z(\theta)\|^2\mathrm{d}\xi.
\end{aligned}\tag{3.19}
$$

Since $w(t)$ and $\Phi(t)$ are bounded complete trajectories of equation (2.5), we obtain from (2.11) that

$$
\begin{cases}
\|\Phi(t)\|^2+\dfrac{\delta_1}{\lambda}\displaystyle\int_\tau^t\|\Phi(\theta)\|^2\mathrm{d}\theta\leqslant\|\Phi(\tau)\|^2+(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2)(t-\tau),\\
\|w(t)\|^2+\dfrac{\delta_1}{\lambda}\displaystyle\int_\tau^t\|\nabla w(\theta)\|^2\mathrm{d}\theta\leqslant\|w(\tau)\|^2+(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2)(t-\tau),\\
\|\Phi(t)\|^2+\delta_1\displaystyle\int_\tau^t\|\nabla\Phi(\theta)\|^2\mathrm{d}\theta\leqslant\|\Phi(\tau)\|^2+(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2)(t-\tau).
\end{cases}\tag{3.20}
$$

Therefore,

$$
\begin{cases}
\limsup\limits_{\tau\to-\infty}\dfrac{1}{t-\tau}\displaystyle\int_\tau^t\|\Phi(\theta)\|^2\mathrm{d}\theta\leqslant\dfrac{\lambda}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2),\\
\limsup\limits_{\tau\to-\infty}\dfrac{1}{t-\tau}\displaystyle\int_\tau^t\|\nabla w(\theta)\|^2\mathrm{d}\theta\leqslant\dfrac{\lambda}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2),\\
\limsup\limits_{\tau\to-\infty}\dfrac{1}{t-\tau}\displaystyle\int_\tau^t\|\nabla\Phi(\theta)\|^2\mathrm{d}\theta\leqslant\dfrac{1}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2+\dfrac{\lambda}{\gamma}\|g\|^2),
\end{cases}
$$

which implies for $s \in (-\infty, t]$ that

$$
\begin{cases}
\|\Phi(s)\|^2 \leqslant \limsup\limits_{\tau\to-\infty} \dfrac{1}{t-\tau}\displaystyle\int_\tau^t \|\Phi(\theta)\|^2 \mathrm{d}\theta \leqslant \dfrac{\lambda}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2 + \dfrac{\lambda}{\gamma}\|g\|^2),\\
\|\nabla w(s)\|^2 \leqslant \limsup\limits_{\tau\to-\infty} \dfrac{1}{t-\tau}\displaystyle\int_\tau^t \|\nabla w(\theta)\|^2 \mathrm{d}\theta \leqslant \dfrac{\lambda}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2 + \dfrac{\lambda}{\gamma}\|g\|^2),\\
\|\nabla\Phi(s)\|^2 \leqslant \limsup\limits_{\tau\to-\infty} \dfrac{1}{t-\tau}\displaystyle\int_\tau^t \|\nabla\Phi(\theta)\|^2 \mathrm{d}\theta \leqslant \dfrac{1}{\delta_1}(\dfrac{\lambda}{\nu}\|f\|^2 + \dfrac{\lambda}{\gamma}\|g\|^2).
\end{cases}
\tag{3.21}
$$

Now if (3.7) holds true, we infer from (3.21) for $\theta \in [\tau, t]$ that

$$
\begin{aligned}
&\frac{7^7 c_1^8}{2\delta_1^7}\|\Phi\|^2\|\nabla\Phi\|^6 + \frac{c_2^2 2^{\beta+1}\sigma^2}{\delta_1}\big(\|\Phi\|_{\widehat{V}}^{\beta+1} + \|\Phi\|_{\widehat{V}}^2\|w\|_{\widehat{V}}^{\beta-1}\big) - \frac{\delta_1}{\lambda}\\
\leqslant&\frac{7^7 c_1^8}{2\delta_1^7}\|\Phi(\theta)\|\|\nabla\Phi(\theta)\|^6 + \frac{c_2^2 2^{\beta+1}\sigma^2}{\delta_1}(1+\frac{1}{\lambda})^{\frac{\beta+1}{2}}\|\nabla\Phi(\theta)\|^{\beta+1}\\
&+\frac{c_2^2 2^{\beta+1}\sigma^2}{\delta_1}(1+\frac{1}{\lambda})^{\frac{\beta+1}{2}}\|\nabla\Phi(\theta)\|^2\|\nabla w\|^{\beta-1} - \frac{\delta_1}{\lambda}\\
\leqslant&\frac{7^7 c_1^8}{2\delta_1^7}G^8 + \frac{c_2^2 2^{\beta+2}\sigma^2}{\delta_1}(1+\frac{1}{\lambda})^{\frac{\beta+1}{2}}G^{\beta+1} - \frac{\delta_1}{\lambda} := \alpha < 0.
\end{aligned}
\tag{3.22}
$$

Inserting (3.22) into (3.19) and then using Gronwall's inequality, we have

$$\|z(t)\|^2 \leqslant \|z(\tau)\|^2 e^{\alpha(t-\tau)},\ t > \tau.$$

Letting $\tau \to -\infty$ gives $w(\theta) = \Phi(\theta)$ for $\theta \in (-\infty, t]$. Particularly, $w(\theta) = \Phi(\theta)$ for $\theta \in [0, t]$. The proof of Lemma 3.2 is complete. □

Recall that the partial regularity of the trajectory statistical solution means that it is carried by a set in which all weak solutions are "partially" regular. Combining Corollary 3.1 and Lemma 3.2, we conclude that the statistical solution $\mu_{\sigma,\tilde{w}}$ is "nearly" regular (but not a.e. regular) on each interval $\mathbb{R}_+$. This Lusin type degenerate regularity result of the statistical solution is stated as follows.

Theorem 3.1. *Let $F \in \widehat{H}$ and assumption (3.7) hold. Then, the trajectory statistical solution $\mu_{\sigma,\tilde{w}}$ guaranteed by Theorem 2.2 and Lemma 3.2 possesses the following Lusin's type degenerate regularity: for any $\epsilon > 0$, there exists a subset $\mathbb{R}_+(\epsilon) \subset \mathbb{R}_+$ such that mes $(\mathbb{R}_+(\epsilon)) < \epsilon$ and $\mu_{\sigma,\tilde{w}}$ is regular on $\mathbb{R}_+ \setminus \mathbb{R}_+(\epsilon)$.*

Proof. Let $\tilde{w}(t)$ be the unique complete and bounded weak solution from Lemma 3.2. For each $j \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and for any $\epsilon > 0$, we see from Corollary 3.1 and Lemma 3.2 that there exists an interval $E(\epsilon, j) \subset I_j = [j\vartheta, (j+1)\vartheta]$ and a positive constant $C_{\epsilon,j}$ such that mes $(E(\epsilon, j)) < \frac{\epsilon}{2^{2^j}}$ and

$$\|\tilde{w}(\theta)\|_{\widehat{V}}^2 \leqslant C_{\epsilon,j},\ \forall \theta \in I_j \setminus E(\epsilon, j). \tag{3.23}$$

Set $\mathbb{R}_+(\epsilon) = \bigcup_{j\in\mathbb{Z}_+} E(\epsilon, j)$. We obviously have

$$\text{mes}\,(\mathbb{R}_+(\epsilon)) \leqslant \sum_{j\in\mathbb{Z}_+} \text{mes}(E(\epsilon, j)) \leqslant \sum_{j\in\mathbb{Z}_+} \frac{\epsilon}{2^{2^j}} \leqslant \epsilon.$$

At the same time, we have $\mathbb{R}_+ = \bigcup_{j\in\mathbb{Z}_+} I_j$, and by (3.23),

$$\|\tilde{w}(\theta)\|^2_{\widehat{V}} < +\infty, \ \forall \theta \in \mathbb{R}_+ \setminus \mathbb{R}_+(\epsilon).$$

This ends the proof. $\square$

4 Convergence of the trajectory statistical solution to that of the 3D incompressible micropolar fluids flows

Let $w_\sigma \in \mathcal{A}^{\text{tr}}_\sigma$ and $w_\sigma \to w$ in the topology Θ^+_{loc} as $\sigma \to 0^+$, we will prove that w is a bounded complete trajectory of the classical 3D incompressible micropolar fluids flows (see equation (4.1) below), and that the trajectory statistical solution μ_{σ,w_σ} converges to the trajectory statistical solution $\mu_{0,w}$ of equation (4.1) as $\sigma \to 0^+$.

Firstly, we select some known results concerning the following 3D incompressible micropolar flows (see e.g. [42]):

$$\begin{cases} u_t - (\nu+\kappa)\Delta u + (u\cdot\nabla)u + \nabla p = 2\kappa\nabla\times\omega + f, \\ \nabla\cdot u = 0, \\ \omega_t - \gamma\Delta\omega + (u\cdot\nabla)\omega - \eta\nabla\text{div}\omega + 4\kappa\omega = 2\kappa\nabla\times u + g, \\ u(x,t)\big|_{t=0} = u_0, \ \ u(x,t)\big|_{\partial\Omega\times(0,\infty)} = 0, \\ \omega(x,t)\big|_{t=0} = \omega_0, \ \ \omega(x,t)\big|_{\partial\Omega\times(0,\infty)} = 0. \end{cases} \tag{4.1}$$

Excluding the pressure p, we can write, using the notations as in Section 2, the weak form of problem (4.1) as

$$\frac{\partial w}{\partial t} + Aw + Bw + Lw = F, \quad \text{in } \mathcal{D}'((0,\infty); \widehat{V}^*), \tag{4.2}$$

$$w(0) = w_0 = (u_0, \omega_0). \tag{4.3}$$

Definition 4.1. *([42]) A weak solution to problem* (4.2)-(4.3) *on* $[0,\infty)$ *is an* $\widehat{H}$*-valued function* $w(t) = (u(t), \omega(t))$ *defined on* $[0,\infty)$ *with* $w(0) = w_0 = (u_0, \omega_0)$*, such that*

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} \in L^{4/3}_{\text{loc}}([0,\infty); V^*), \quad w(\cdot) \in C([0,\infty); \widehat{H}_{\mathrm{w}}) \cap L^2_{\text{loc}}([0,\infty); \widehat{V}) \\ \\ (\dfrac{\mathrm{d}w}{\mathrm{d}t}, \Phi) + \langle Aw, \Phi\rangle + \big\langle Bw, \Phi\big\rangle + \langle Lw, \Phi\rangle = (F, \Phi), \ \ \forall\, \Phi \in \widehat{V}, \end{cases}$$

holds in the distribution sense $\mathcal{D}'(0,\infty)$*, and* $w(t)$ *satisfies the following energy inequality*

$$\frac{1}{2}\|w(t)\|^2 + \int_0^t \langle Aw(s), w(s)\rangle \mathrm{d}s + \int_0^t \big\langle Lw(s), w(s)\big\rangle \mathrm{d}s \leqslant \frac{1}{2}\|w_0\|^2 + \int_0^t \big(F, w(s)\big)\mathrm{d}s$$

in the sense that

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t \|w(s)\|^2\xi'(s)\mathrm{d}s+\int_0^t \langle Aw(s),w(s)\rangle\xi(s)\mathrm{d}s+\int_0^t \langle Lw(s),w(s)\rangle\xi(s)\mathrm{d}s\\
&\leqslant \int_0^t \big(F,w(s)\big)\xi(s)\mathrm{d}s,\ \ \forall\,\xi(\cdot)\in C_0^\infty([0,t])\quad \textit{with}\ \xi(t)\geqslant 0,\,\forall\,t\geqslant 0.
\end{aligned}\tag{4.4}
$$

Lemma 4.1. *([42]) Let $w_0=(u_0,\omega_0)\in\widehat{H}$ and suppose $F\in\widehat{H}$. Then, problem* (4.2)*-*(4.3) *possesses at least one weak solution $w(t)=w(t;w_0)$ corresponding to the initial value w_0. Moreover, there exists a time $t'_*\geqslant 0$ yielding*

$$
w(t)\in X,\,\forall\,t\geqslant t'_*,\tag{4.5}
$$

where X is given by (2.8)*.*

The trajectory space $\mathcal{T}_0^+$ and kernel $\mathcal{K}_0$ of equation (4.2) is defined respectively as

$$
\begin{aligned}
&\mathcal{T}_0^+=\{w(\cdot):w(\cdot)\ \text{is a weak solution of (4.2) and}\ w(t)\in X\ \text{for all}\ t\in\mathbb{R}_+\},\\
&\mathcal{K}_0=\{w(\cdot):w(\cdot)\ \text{is a weak solution of (4.2) and}\ w(t)\in X\ \text{for all}\ t\in\mathbb{R}\}.
\end{aligned}
$$

Lemma 4.2. *([42]) Let $F\in\widehat{H}$.*

(a) *The translation semigroup $\{T(t)\}_{t\geqslant 0}$ defined by* (2.13) *possesses a trajectory attractor $\mathcal{A}_0^{\mathrm{tr}}\subset\mathcal{T}_0^+$ with respect to the topology Θ^+_{loc}:*

$$
\mathcal{A}_0^{\mathrm{tr}}=\Pi_+\mathcal{K}_0=\{u(\cdot)|_{[0,\infty)}\big|\ \ u\in\mathcal{K}_0\}\subset\mathcal{T}_0^+.\tag{4.6}
$$

(b) *For a given generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$ and for each $w\in\mathcal{T}_0^+$, there corresponds a unique Borel probability measure $\mu_{0,w}$ on $\mathcal{F}^+$ such that*

$$
\int_{\mathcal{F}^+}\zeta(z)\mathrm{d}\mu_{0,w}(z)=\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t\zeta(T(s)w)\mathrm{d}s,\quad \forall\zeta\in C(\mathcal{F}^+).\tag{4.7}
$$

Moreover, $\mu_{0,w}$ is supported by $\mathcal{A}_0^{\mathrm{tr}}$ and is a trajectory statistical solution for equation (4.2)*, and $\mu_{0,w}$ is invariant under the acting of $\{T(t)\}_{t\geqslant 0}$ in the following sense*

$$
\int_{\mathcal{F}^+}\zeta(T(t)z)\mathrm{d}\mu_{0,w}(z)=\int_{\mathcal{F}^+}\zeta(z)\mathrm{d}\mu_{0,w}(z),\quad \forall\,t\geqslant 0,\ \forall\zeta\in C(\mathcal{F}^+).\tag{4.8}
$$

We now prove the convergence of solution of equation (2.5) to that of equation (4.2) as $\sigma\to 0^+$.

Lemma 4.3. *Let a sequence $w_{\sigma_n}(\cdot)\in\mathcal{T}_{\sigma_n}^+$, $n\in\mathbb{N}$, satisfy the following conditions:*
(I) $\sigma_n\to 0^+$ *as $n\to\infty$;* (II) $w_{\sigma_n}(\cdot)\to w(\cdot)$ *in the topology Θ^+_{loc} as $n\to\infty$.*
Then, $w\in\mathcal{T}_0^+$.

Proof. We first prove that w satisfies (4.2). In fact, since $w_{\sigma_n}(\cdot) \in \mathcal{T}_{\sigma_n}^+$, $w_{\sigma_n}(\cdot)$ satisfies for each $T > 0$ that

$$\frac{\partial w_{\sigma_n}}{\partial t} + Aw_{\sigma_n} + Bw_{\sigma_n} + N_{\sigma_n} w_{\sigma_n} + Lw_{\sigma_n} = F, \text{ in } \mathcal{D}'(0,T;\widehat{V}^*). \tag{4.9}$$

From (2.35) we infer that both the nonlinear terms Bw_{σ_n} and $N_{\sigma_n} w_{\sigma_n}$ are uniformly (with respect to $n \in \mathbb{N}$) bounded in $L^{4/3}([0,T];\widehat{V}^*)$. Hence, by (2.36) and condition (II), we have as $\sigma_n \to 0^+$ that

$$\begin{cases} \dfrac{\partial w_{\sigma_n}}{\partial t} \rightharpoonup \dfrac{\partial w}{\partial t} \text{ weakly in } L^{4/3}(0,T;\widehat{V}^*), \\ Aw_{\sigma_n} \rightharpoonup Aw \text{ weakly in } L^2(0,T;\widehat{V}^*), \\ Bw_{\sigma_n} \rightharpoonup Bw \text{ weakly in } L^{4/3}(0,T;\widehat{V}^*), \\ N_{\sigma_n} w_{\sigma_n} \rightharpoonup 0 \text{ weakly in } L^2(0,T;\widehat{V}^*), \\ Lw_{\sigma_n} \rightharpoonup Lw \text{ weakly in } L^{4/3}(0,T;\widehat{V}^*). \end{cases} \tag{4.10}$$

(4.9) and (4.10) imply

$$\frac{\partial w}{\partial t} + Aw + Bw + Lw = F, \text{ in } \mathcal{D}'(0,T;\widehat{V}^*). \tag{4.11}$$

The fact that $w(\cdot)$ meets (4.4) can be proved by using (2.49) and (4.10). By (4.11) and the same derivations of Lemma 2.1, we conclude that $w(\cdot)$ satisfies (4.5). The proof is complete. $\square$

Next we set $\mathfrak{B}_\sigma = \{w_\sigma \mid w_\sigma \in \mathcal{T}_\sigma^+, \ \|w_\sigma\|_{\mathfrak{B}_\sigma} \leqslant c(R), \text{ for all } w_\sigma \in \mathcal{B}_\sigma\}$, where $c(R)$ is a positive constant which depends only on R and can be chosen for our purpose, and

$$\|w_\sigma\|_{\mathfrak{B}_\sigma} = \sup_{t\geqslant 0}\|w_\sigma(t)\| + \sup_{t\geqslant 0}(\int_t^{t+1}\|w_\sigma(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta)^{1/2} + \sup_{t\geqslant 0}(\int_t^{t+1}\|\partial_t w_\sigma(\theta)\|_{\widehat{V}^*}^{3/4}\mathrm{d}\theta)^{3/4}.$$

Lemma 4.4. *Let $\sigma \in (0,1]$. Then*

$$T(t)\mathfrak{B}_\sigma \to \mathcal{A}_0^{\mathrm{tr}} \textit{ in the topology } \Theta_{\mathrm{loc}}^+ \textit{ as } t \to +\infty \textit{ and } \sigma \to 0^+, \tag{4.12}$$

where (4.12) is interpreted in following sense: for $\{w_\sigma\}_{0<\sigma\leqslant 1}$ with $w_\sigma \in \mathfrak{B}_\sigma$, there is some $w \in \mathcal{A}_0^{\mathrm{tr}}$ such that $T(t)w_\sigma \to w$ in the topology Θ_{loc}^+ as $t \to +\infty$ and $\sigma \to 0^+$.

Proof. We prove (4.14) by contradiction. Assume that there is a neighbourhood $\mathcal{O}(\mathcal{A}_0^{\mathrm{tr}})$ of $\mathcal{A}_0^{\mathrm{tr}}$ in the topology Θ_{loc}^+ and two sequences $\sigma_n \to 0^+$, $t_n \to +\infty$ as $n \to \infty$, such that

$$T(t_n)\mathfrak{B}_{\sigma_n} \not\subset \mathcal{O}(\mathcal{A}_0^{\mathrm{tr}}). \tag{4.13}$$

Then there is a solution $w_{\sigma_n} \in \mathfrak{B}_{\sigma_n}$ such that

$$z_{\sigma_n}(t) := T(t_n)w_{\sigma_n}(t) \notin \mathcal{O}(\mathcal{A}_0^{\mathrm{tr}}). \tag{4.14}$$

Notice that $z_{\sigma_n}(t)$ is a solution of equation (2.5) with $\sigma = \sigma_n$ on the interval $[-t_n, +\infty)$ and $z_{\sigma_n}(t)$ is the backward time shift of the solution $w_{\sigma_n}(t)$ by t_n. By the definition of $\mathfrak{B}_\sigma$, we have that

$$\sup_{t \geqslant -t_n} \|z_{\sigma_n}(t)\| + \sup_{t \geqslant -t_n} (\int_t^{t+1} \|z_{\sigma_n}(\theta)\|^2_{\widehat{V}} \mathrm{d}\theta)^{1/2} \\ + \sup_{t \geqslant -t_n} (\int_t^{t+1} \|\partial_\theta z_{\sigma_n}(\theta)\|^{3/4}_{\widehat{V}^*} \mathrm{d}\theta)^{3/4} \leqslant c(R). \tag{4.15}$$

For every $T > 0$, we consider σ_n with the index n such that $t_n \geqslant T$. Then (4.15) implies that we can extract a subsequence (still denote by) $\{z_{\sigma_n}(\cdot)\}$ and a function $z(t)$, $t \in (-T, T)$, such that

$$\begin{cases} z_{\sigma_n} \rightharpoonup z \text{ weakly star in } L^\infty([0,T]; \widehat{H}) \text{ as } n \to \infty, \\ z_{\sigma_n} \rightharpoonup z \text{ weakly in } L^2([0,T]; \widehat{V}) \text{ as } n \to \infty, \\ z_{\sigma_n} \rightharpoonup \frac{\mathrm{d}z}{\mathrm{d}t} \text{ weakly in } L^{4/3}([0,T]; \widehat{V}^*) \text{ as } n \to \infty. \end{cases} \tag{4.16}$$

Using the standard diagonal procedure, we can construct a function $z(t)$, $t \in \mathbb{R}$, such that

$$z_{\sigma_n}(\cdot) \to z(\cdot) \text{ in the topology } \Theta^+_{\mathrm{loc}} \text{ as } n \to \infty. \tag{4.17}$$

By Lemma 4.3 we see that $z(\cdot) \in \mathcal{T}_0^{\mathrm{tr}}$. This fact, together with (4.15) and (4.17), implies that $z \in \mathcal{K}_0$. Now, by (4.6) we have $\Pi_+ z \in \Pi_+ \mathcal{K}_0 = \mathcal{A}_0^{\mathrm{tr}}$. Eq. (4.17) shows that

$$\Pi_+ z_{\sigma_n} \to \Pi_+ z \text{ in the topology } \Theta^+_{\mathrm{loc}} \text{ as } n \to \infty.$$

Therefore, we have for large enough n that

$$\Pi_+ z_{\sigma_n} \in \mathcal{O}(\Pi_+ z) \subset \mathcal{O}(\mathcal{A}_0^{tr}),$$

which contradicts (4.13). The proof is complete. □

Lemma 4.4 implies the convergence of the trajectory attractor $\mathcal{A}_\sigma^{\mathrm{tr}}$ to $\mathcal{A}_0^{\mathrm{tr}}$ as $\sigma \to 0^+$. The convergence of this type was investigated in [36] for the convective Brinkman-Forchheimer equations.

At this stage, we can state and prove the main results of this section.

Theorem 4.1. *Suppose that $F \in \widehat{H}$. Let $\mathcal{A}_\sigma^{\mathrm{tr}}$ and $\mathcal{A}_0^{\mathrm{tr}}$ be the trajectory attractors guaranteed by Theorem 2.1 and Lemma 2.2(I), respectively. Let that $w_\sigma \in \mathcal{A}_\sigma^{\mathrm{tr}}$ and $w_\sigma \to w$ in the topology Θ^+_{loc} as $\sigma \to 0^+$, and that μ_{σ, w_σ} and $\mu_{0,w}$ be the trajectory statistical solutions guaranteed by Theorem 2.2 and Lemma 4.2(b), respectively. Then*

$$\mathcal{A}_\sigma^{\mathrm{tr}} \to \mathcal{A}_0^{\mathrm{tr}} \text{ in the topology } \Theta^+_{\mathrm{loc}} \text{ as } \sigma \to 0^+, \tag{4.18}$$

$$\lim_{\sigma \to 0^+} \int_{\mathcal{F}^+} \zeta(z) \mathrm{d}\mu_{\sigma, w_\sigma}(z) = \int_{\mathcal{F}^+} \zeta(z) \mathrm{d}\mu_{0,w}(z), \forall \zeta \in C(\mathcal{F}^+). \tag{4.19}$$

Proof. From (2.8), (2.11), (2.35) and the definition $\|\cdot\|_{\mathfrak{B}_\sigma}$, we see that the family $\{\mathcal{A}_\sigma^{\mathrm{tr}}\}_{0<\sigma\leqslant 1}$ is uniformly (with respect to $\sigma \in (0,1]$) bounded when we choose $c(R)$ large enough. Applying Lemma 4.4 with $\mathfrak{B}_\sigma = \mathcal{A}_\sigma^{\mathrm{tr}}$ and the invariant property of $\mathcal{A}_\sigma^{\mathrm{tr}}$, we obtain by letting $t \to +\infty$ that

$$\mathcal{A}_\sigma^{\mathrm{tr}} = T(t)\mathcal{A}_\sigma^{\mathrm{tr}} \to \mathcal{A}_0^{\mathrm{tr}} \text{ in the topology } \Theta_{\mathrm{loc}}^+ \text{ as } \sigma \to 0^+,$$

that is, (4.18) is proved.

Now, let $w_\sigma \in \mathcal{A}_\sigma^{\mathrm{tr}}$ such that $w_\sigma \to w$ in the topology Θ_{loc}^+ as $\sigma \to 0^+$. By Lemma 4.3 we see that $w \in \mathcal{A}_0^{\mathrm{tr}}$. Then, Theorem 2.2 and Lemma 4.2(b) show that there exist trajectory statistical solutions μ_{σ,w_σ} and $\mu_{0,w}$ corresponding respectively to w_σ and w, and that for a given generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$ and for any $\zeta \in C(\mathcal{F}^+)$ there holds

$$\begin{cases} \displaystyle\int_{\mathcal{F}^+} \zeta(z)\mathrm{d}\mu_{\sigma,w_\sigma}(z) = \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \zeta(T(s)w_\sigma)\mathrm{d}s, \\ \displaystyle\int_{\mathcal{F}^+} \zeta(z)\mathrm{d}\mu_{0,w}(z) = \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \zeta(T(s)w)\mathrm{d}s. \end{cases} \tag{4.20}$$

Since that $\mathrm{LIM}_{t\to+\infty}$ is a linear continuous functional, $\zeta \in C(\mathcal{F}^+)$ and that $w_\sigma \to w$ in the topology Θ_{loc}^+ as $\sigma \to 0^+$, by (4.20) we have

$$\begin{aligned} \lim_{\sigma\to 0^+}\int_{\mathcal{F}^+} \zeta(z)\mathrm{d}\mu_{\sigma,w_\sigma}(z) &= \lim_{\sigma\to 0^+}\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \zeta(T(s)w_\sigma)\mathrm{d}s \\ &=\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \zeta(T(s)\lim_{\sigma\to 0^+} w_\sigma)\mathrm{d}s \\ &=\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \zeta(T(s)w)\mathrm{d}s \\ &= \int_{\mathcal{F}^+} \zeta(z)\mathrm{d}\mu_{0,w}(z), \end{aligned}$$

where we have also used the continuity of the translation semigroup $T(\cdot)$ on $\mathcal{F}^+$ with respect to the topology Θ_{loc}^+ (cf. [8]). The proof is complete. □

Acknowledgements

The authors warmly thank the anonymous referee for his/her careful reading of the article and many pertinent remarks that lead to various improvements to this article.

References

[1] C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis*, *A Hithhiker's Guide*, third editon, Springer-Verlag, 2006.

[2] A. Bronzi, C. F. Mondaini, R. Rosa, Trajectory statistical solutions for three-dimensional Navier-Stokes-like systems, *SIAM J. Math. Anal.*, **46**(2014), 1893-1921.

[3] A. Bronzi, R. Rosa, On the convergence of statistical solutions of the 3D Navier-Stokes-α model as α vanishes, *Discrete Cont. Dyn. Syst.*, **34**(2014), 19-49.

[4] A. Bronzi, C. F. Mondaini, R. Rosa, Abstract framework for the theory of statistical solutions, *J. Differential Equations*, **260**(2016), 8428-8484.

[5] T. Caraballo, P. E. Kloeden, J. Real, Invariant measures and statistical solutions of the globally modified Navier-Stokes equations, *Discrete Cont. Dyn. Syst.-B*, **10**(2008), 761-781.

[6] M. Chekroun, N. E. Glatt-Holtz, Invariant measures for dissipative dynamical systems: Abstract results and applications, *Comm. Math. Phys.*, **316**(2012), 723-761.

[7] J. Chen, B. Dong, Z. Chen, Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains, *Nonlinearity*, **20**(2007), 1619-1635.

[8] V. V. Chepyzhov, M. I. Vishik, *Attractors for Equations of Mathematical Physics*, AMS Colloquium Publications, Vol. 49, AMS, Providence, R.I., 2002.

[9] A. Cheskidov, Global attractors of evolutionary systems, *J. Dyn. Differential Equations*, **21**(2009), 249-268.

[10] B. Dong, Z. Chen, Regularity criteria of weak solutions to the three-dimensional micropolar flows, *J. Math. Phy.,* **50**(2009), 103525.

[11] B. Dong, Y. Jia, Z. Chen, Pressure regularity criteria of the three-dimensional micropolar fluid flows, *Math. Meth. Appl. Sci.*, **34**(2011), 595-606.

[12] B. Dong, J. Li, J. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, *J. Differential Equations*, **262**(2017), 3488-3523.

[13] A. C. Eringen, Theory of micropolar fluids, *J. Math. Mech.*, **16**(1966), 1-18.

[14] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes equations and turbulence*, Cambridge University Press, Cambridge, 2001.

[15] C. Foias, R. Rosa, R. Temam, A note on statistical solutions of the three-dimensional Navier-Stokes equations: the stationary case, *C. R. Math.*, **348**(2010), 235-240.

[16] C. Foias, R. Rosa, R. Temam, A note on statistical solutions of the three-dimensional Navier-Stokes equations: the time-dependent case, *C. R. Math.*, **348**(2010), 347-353.

[17] C. Foias, R. Rosa, R. Temam, Properties of time-dependent statistical solutions of the three-dimensional Navier-Stokes equations, *Ann. L'Inst. Fourier*, **63**(2013), 2515-2573.

[18] C. Foias, R. Rosa, R. Temam, Convergence of time averages of weak solutions of the three-dimensional Navier-Stokes equations, *J. Stat. Phys.*, 160(2015), 519-531.

[19] C. Foias, R. Rosa, R. Temam, Properties of stationary statistical solutions of the three-dimensional Navier-Stokes equations, *J. Dyn. Differential Equations*, **31**(2019), 1689-1741.

[20] X. He, J. Fan, A regularity criterion for 3D micropolar fluid flows, *Appl. Math. Lett.*, **25**(2012), 47-51.

[21] H. Jiang, C. Zhao, Trajectory statistical solutions and Liouville type theorem for nonlinear wave equations with polynomial growth, *Adv. Differential Equations*, **3-4**(2021), 107-132.

[22] Q. Jiu, J. Liu, J. Wu, H. Yu, On the initial and boundary-value problem for 2D micropolar equations with only angular velocity dissipation, *Z. Angew. Math. Phys.*, **68**(2017), 1-24.

[23] P. E. Kloeden, P. Marín-Rubio, J. Real, Equivalence of invariant measures and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations, *Comm. Pure Appl. Anal.*, **8**(2009), 785-802.

[24] G. Łukaszewicz, *Micropolar Fluids-Theory and Applications*, Birkhäuser, Boston, 1999.

[25] G. Łukaszewicz, Pullback attractors and statistical solutions for 2-D Navier-Stokes equations, *Discrete Cont. Dyn. Syst.-B*, **9**(2008), 643-659.

[26] G. Łukaszewicz, J. Real, J. C. Robinson, Invariant measures for dissipative dynamical systems and generalised Banach limits, *J. Dyn. Differential Equations*, **23**(2011), 225-250.

[27] G. Łukaszewicz, J. C. Robinson, Invariant measures for non-autonomous dissipative dynamical systems, *Discrete Cont. Dyn. Syst.*, **34**(2014), 4211-4222.

[28] Z. Lin, *et al.* Statistical solution and Kolmogorov entropy for the impulsive discrete Klein-Gordon-Schrödinger type equations, *Discrete Cont. Dyn. Syst.-B*, **28**(2023), 20-49.

[29] B. Miao, C. Xu, C. Zhao, Statistical solution and piecewise Liouville theorem for the impulsive discrete Zakharov equations, *AIMS Math.*, **7(5)**(2022): 9089-9116.

[30] R. Temam, *Navier-Stokes Equations* (Theory and Numerical Analysis), North-Holland: Amsterdam, 1984.

[31] J. Wang, C. Zhao, T. Caraballo, Invariant measures for the 3D globally modified Navier-Stokes equations with unbounded variable delays, *Comm. Nonl. Sci. Numer. Simu.*, **91**(2020), 105459.

[32] X. Wang, Upper-semicontinuity of stationary statistical properties of dissipative systems, *Discrete Cont. Dyn. Syst.*, **23**(2009), 521-540.

[33] X. Yang, H. Liu, C. Sun, Global attractors of the 3D mircopolar equations with damping term, *Math Foundations Comp.*, **4**(2021), 117-130.

[34] H. Yang, X. Han, C. Zhao, Homogenization of trajectory statistical solutions for the 3D incompressible micropolar fluids with rapidly oscillating terms, *Math.*, **10**(2022), 1-15.

[35] Z. Ye, Global existence of strong solution to the 3D micropolar equations with a damping term, *Appl. Math. Lett.*, **83**(2018), 188-193.

[36] C. Zhao, L. Kong, G. Liu, M. Zhao, The trajectory attractor and its limiting behavior for the convective Brinkman-Forchheimer equations, *Topological Meth. Nonl. Anal.*, **44**(2014), 413-433.

[37] C. Zhao, W. Sun, C. Hsu, Pullback dynamical behaviors of the non-autonomous

micropolar fluid flows, *Dynamics PDE*, **12**(2015),265-288.

[38] C. Zhao, L. Yang, Pullback attractor and invariant measure for the globally modified Navier-Stokes equations, *Comm. Math. Sci.*, **15**(2017), 1565-1580.

[39] C. Zhao, G. Xue, G. Łukaszewicz, Pullback attractors and invariant measures for discrete Klein-Gordon-Schrödinger equations, *Discrete Cont. Dyn. Syst.-B*, **23**(2018), 4021-4044.

[40] C. Zhao, T. Caraballo, Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations, *J. Differential Equations*, **266**(2019), 7205-7229.

[41] C. Zhao, Y. Li, T. Caraballo, Trajectory statistical solutions and Liouville type equations for evolution equations: Abstract results and applications, *J. Differential Equations*, **269**(2020), 467-494.

[42] C. Zhao, Y. Li, Y. Sang, Using trajectory attractor to construct trajectory statistical solution for the 3D incompressible micropolar flows. *Z. Angew. Math. Mech.*, **100**(2020), e201800197.

[43] C. Zhao, Y. Li, G. Łukaszewicz, Statistical solution and partial degenerate regularity for the 2D non-autonomous magneto-micropolar fluids, *Z. Angew. Math. Phys.*, **71**(2020), 141-1-24.

[44] C. Zhao, Z. Song, T. Caraballo, Strong trajectory statistical solutions and Liouville type equations for dissipative Euler equations, *Appl. Math. Lett.*, **99**(2020), 105981.

[45] C. Zhao, Y. Li, Z. Song, Trajectory statistical solutions for the 3D Navier-Stokes equations: The trajectory attractor approach, *Nonlinear Anal.-RWA*, **53**(2020), 103077.

[46] C. Zhao, T. Caraballo, G. Łukaszewicz, Statistical solution and Liouville type theorem for the Klein-Gordon-Schrödinger equations, *J. Differential Equations*, **281**(2021), 1-32.

[47] C. Zhao, H. Jiang, T. Caraballo, Statistical solutions and piecewise Liouville theorem for the impulsive reaction-diffusion equations on infinite lattices, *Appl. Math. Comp.*, **404**(2021), 126103.

[48] C. Zhao, J. Wang, T. Caraballo, Invariant sample measures and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations, *J. Differential Equations*, **317** (2022), 474–494.

[49] Z. Zhu, C. Zhao, Pullback attractor and invariant measures for the three-dimensional regularized MHD equations, *Discrete Cont. Dyn. Syst.*, **38**(2018), 1461-1477.

[50] Z. Zhu, Y. Sang, C. Zhao, Pullback attractor and invariant measures for the discrete Zakharov equations, *J. Appl. Anal. Comp.*, **9**(2019), 2333-2357.