NEW RESULTS FOR CONVERGENCE PROBLEM OF FRACTIONAL DIFFUSION EQUATIONS WHEN ORDER APPROACH TO 1^-

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ABSTRACT. This work studies the convergence problem for a class of fractional diffusion equations in which the time-derivative order approaches 1^-. Up to now, few works have investigated this topic. The purpose of the article consists of three main contents. The first result is related to the convergence of the Caputo derivative and the Mittag-Leffler operators when $\alpha \to 1^-$. The second is to investigate the convergence problem for a linear fractional diffusion equation on L^p spaces. And last result is concerned with the convergence problem for nonlinear fractional diffusion equations. The main analysis and techniques of the paper involve the evaluation related to Riemann-Liouville integration, Caputo derivative and Sobolev embeddings. Our analysis provides a complete and detailed answer to the convergence problem as fractional order tends to 1^-.
Keywords: Caputo derivative, fractional diffusion equations, Sobolev embeddings, Connections between time fractional and classical solutions.

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1. Introduction

1.1. **Setting of the problem.** In this paper, we are interested in the convergence problem for the following type of time-fractional parabolic equations:

$$\begin{cases} {}^C D_t^\alpha u + \mathcal{A}u = F(x,t,u(x,t)), & \text{in} \quad \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in} \quad \Omega, \\ u(x,t) = 0, & \text{in} \quad \partial\Omega. \end{cases} \tag{1}$$

where Ω be an open and bounded domain in $\mathbb{R}^N, (N \geq 1)$ with smooth boundary $\partial\Omega$. The notation ${}^C D_t^\alpha$ means the Caputo time-fractional derivative of order $\alpha \in (0,1)$ (see Definition 2.2). In the first equation of Problem (1), $\mathcal{A}$ is a symmetric and uniformly elliptic operator on Ω defined by

$$\mathcal{A}f(x) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^k a_{ij}(x)\frac{\partial}{\partial x_j} f(x)\right) - b(x)f(x), \quad x \in \overline{\Omega},$$

where $a_{ij} \in C^1(\overline{\Omega})$, $b \in C(\overline{\Omega}; [0,+\infty))$, and $a_{ij} = a_{ji}, 1 \leq i,j \leq N$. We assume that there exists a constant $c_0 > 0$ such that, for $x \in \overline{\Omega}$, $y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$, $\sum_{i,j=1}^N y_i a_{ij}(x) y_j \geq c_0|y|^2$. (see [16]). More detailed properties related to $\mathcal{A}$ can be found at the beginning of Section 2.

Fractional partial differential equations (FPDE) of the form (1) have many applications in various fields, for example, physics and probability theory. These equations model anomalous diffusion where time-fractional derivatives can be used to describe particles adhesion and trapping phenomena. Fractional derivatives have been used for more than 300 years, but, like the Pareto distribution with no mean, these derivatives are present in the physical sciences because of relatively recent observations of anomalous diffusion, see [13]. There are many new models related to fractional differential equations with different approaches, which have attracted the attention of many mathematicians, for example [6, 9, 16, 18, 20–24] and references therein. The well-posedness of problem (1) (including the existence and uniqueness of mild solution) with several different assumptions on the function F has been clarified in the interesting book [8].

In various research directions concerning FPDEs, researchers are interested in the convergence problem when the fractional order tends to 1^-. Our main purpose in this paper is to provide some answers to this open question: If the problem (1) has a solution, what asymptotic property does the solution possess when the fractional order α approaches 1^-?. The significance of the convergence problem for Problem (1) is given by the following question

(Q_1) *Does the solution of Problem* (1) *with* $0 < \alpha < 1$ *approach the solution of Problem* (1) *with* $\alpha = 1$ *as* α *approaches* 1^-*?*

This is an interesting and open question suggested by Carvalho-Neto and Planas [15]. This intriguing problem was also raised and investigated in the recent book [8] (see Section 3.5, p.127). In this book, Gal and Warma addressed the limiting behavior of the mild solution as $\alpha \to 1^-$ for a class of parabolic problems with polynomial nonlinearities. The relationship between the solutions of fractional PDEs and classical PDEs is a very interesting topic for mathematicians. while the structure of the solution of Problem (1) depends on the Caputo derivative and the Mittag-Leffler functions, the solution of Problem (1) ($\alpha = 1$) depends on the classical derivative with exponential functions. Noting that, if $F = 0$, then the solution to Problem (1) is given by

$$u_\alpha(t) = E_{\alpha,1}(-t^\alpha \mathcal{A})u_0, \tag{2}$$

when $0 < \alpha < 1$, and by

$$u^*(t) = e^{-t\mathcal{A}}u_0, \tag{3}$$

when $\alpha = 1$. As showed by [8] (see p. 28), the operator $e^{-t\mathcal{A}}$ has a semi-group structure, namely $e^{-(t+s)\mathcal{A}} = e^{-t\mathcal{A}}e^{-s\mathcal{A}}$, while the semigroup property does not hold for the operator $E_{\alpha,1}(-t^\alpha\mathcal{A})$, namely, $E_{\alpha,1}(-(t+s)^\alpha\mathcal{A}) \neq E_{\alpha,1}(-t^\alpha\mathcal{A})E_{\alpha,1}(-s^\alpha\mathcal{A})$. The great difference between the structure of $E_{\alpha,1}(-t^\alpha\mathcal{A})$ in (2) and $e^{-t\mathcal{A}}$ in (3) is the main challenge in finding an answer to question Q_1. Since the solution structure of (1), as indicated by (2), is related to the Mittag-Leffler function, the idea of question Q_1 in the simple case $F = 0$ is expressed simply as the following two questions:

(Q_2) *Does the function* $E_{\alpha,1}(-t^\alpha\mathcal{A})$*, with* $0 < \alpha < 1$*, approach the function* $e^{-t\mathcal{A}}$ *with* $\alpha = 1$*, in an appropriate sense, as* α *approaches* 1^-*?*

and

(Q_3) *Does the function* ${}^C D_t^\alpha v$ *(Caputo derivative of the function* v*) approach* $D_t v$ *(the classical first degree derivative of* v*) in appropriate sense as* α *approaches* 1^-*?*

It would be surprising if these issues related to questions Q_1, Q_2 or Q_3 would not have been thoroughly studied. Let us refer the reader to the interesting recent work of Chen-Stynes [5] where it can be found the following facts:

In the simple linear case $\mathcal{A}u = -p\frac{\partial^2 u}{\partial x^2}$ and $F(x,t,u) = -cu + f$, the question (Q_1) is first studied in [5] for a linear fractional diffusion equation

$${}^C D_t^\alpha u - pu_{xx} + cu = f. \tag{4}$$

When $\alpha \to 1^-$, they showed that the solution u_α of Problem (4) converges, uniformly to the solution of the classical parabolic initial-boundary value problem where D_t^α is replaced by D_t. They also provided an interesting result which describes the connection between the Mittag-Leffler function $E_{\alpha,\beta}$ and the exponential function as follows.

Lemma 1.1. *Let* $3/4 \leq \alpha < 1$ *and* $\alpha \leq \beta \leq 1$*. Then there exists a constant* C *which is independent of* α, β, z *such that, for any* $z < 0$*,*

$$\left| E_{\alpha,\beta}(z) - e^z \right| \leq \frac{C}{1+|z|}(1-\alpha). \tag{5}$$

The main analysis in the proof of Lemma 1.1 is based on the extension of the Mittag-Leffler function which is represented as a complex contour integral. Lemma 1.1 is very useful in investigating linear problems. However, we are not sure that Lemma 1.1 can be successfully applied to the nonlinear problem. In order to study the problem in the nonlinear case, and use embeddings in L^p, we need a different result. Thus, we need to state another theorem about the convergence of $E_{\alpha,1}$ when α approaches 1^-, using a new proof.

To the best of our knowledge, except for the article [5] (and [8] as mentioned above), there have been no studies concerning both proving global results and answering questions Q_1, Q_2, Q_3. From the previous motivations, the main objective of this paper is to provide accurate and detailed answers to these problems.

1.2. **MAIN RESULTS.** In order to study further on this topic, we need to establish a more detailed relationship between both functions $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha}(z)$ with e^z. In the following theorem, we will answer questions Q_2 and Q_3 as introduced before. Our main results are stated as follows.

Theorem 1.2. *a) Let $v : [0,T] \to \mathbb{R}$ such that $D_t v \in L^\infty(0,T)$ and further assume that $D_t v$ and ${}^C D_t^\alpha v$ are locally integrable functions. Then, for $0 < \alpha \le 1$,*

$$\left| {}^C D_t^\alpha v(t) - D_t v(t) \right| \le \frac{3}{\Gamma(1-\alpha)} t^{-\alpha} \widetilde{M}_\alpha(T,p) \big\| Dv \big\|_{L^p(0,T)}, \tag{6}$$

where

$$\big\| Dv \big\|_{L^p(0,T)} = \left(\int_0^T |D_s v(s)|^p ds \right)^{\frac{1}{p}} \tag{7}$$

and $p > \frac{1}{\alpha}$ and

$$\widetilde{M}_\alpha(T,p) = \left[\frac{p(1-\alpha)}{\alpha p - 1} + T - 1 - \frac{p-1}{\alpha p - 1} \Big(T^{\frac{\alpha p - 1}{p-1}} - 1 \Big) \right]^{\frac{p-1}{p}} + \left| \frac{\Gamma(\alpha) - 1}{\Gamma(\alpha)} \right| \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} T^{\frac{\alpha p - 1}{p}}. \tag{8}$$

b) Let $v : [0,T] \to L^2(\Omega)$ and and further assume that $D_t v$ and ${}^C D_t^\alpha v$ are locally integrable functions. Then

$$\Big\| {}^C D_t^\alpha v(t) - D_t v(t) \Big\|_{L^2(\Omega)} \le \frac{3}{\Gamma(1-\alpha)} t^{-\alpha} \widetilde{M}_\alpha(T,p) \big\| Dv \big\|_{L^p(0,T;L^2(\Omega))} \tag{9}$$

where

$$\big\| Dv \big\|_{L^p(0,T;L^2(\Omega))} = \left(\int_0^T \big\| D_s v(s) \big\|^p_{L^2(\Omega)} ds \right)^{1/p}.$$

c) Let $k > 0$ and $\alpha \in (0,1)$. Then for any $\mu > 0$, there exists two constants $\overline{C}(\mu,p,T)$ and $\widetilde{C}(\mu,p,T)$ such that

$$\left| E_{\alpha,1}(-kt^\alpha) - e^{-kt} \right| \le \frac{3\overline{C}(\mu,p,T)}{\Gamma(1-\alpha)} t^{-\alpha} \widetilde{M}_\alpha(T,p) k^{\frac{\mu-1}{p}}, \tag{10}$$

and

$$\Big| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-k(t-s)^\alpha) ds - \int_0^t e^{-k(t-s)} ds \Big| \le \frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)} t^{-\alpha} \widetilde{M}_\alpha(T,p) k^{\frac{\mu-1}{p}-1} \tag{11}$$

Remark 1.1. *Let us point out an interesting observation: if a measurable function is bounded on $(0,\infty)$, then it is locally integrable there. To satisfy the above assumptions, we can assume that $D_t v$ and ${}^C D_t^\alpha v$ are measurable function which are bounded on $(0,\infty)$.*

Theorem 1.2 is one of the basis to prove our main results. The main idea of the proof of this theorem is detailed in Section 3.

Based on the above theorem, we first investigate the convergence problem for the fractional diffusion equation (1) in the case $F = F(x,t)$. The following theorem shows that the mild solution of (1), when $0<\alpha<1$, converges to the mild solution of (12) when $\alpha \to 1^-$, where u^* solves

$$\begin{cases} D_t u^* + \mathcal{A}u^* = F(x,t), & \text{in} \quad \Omega \times (0,T], \\ u^*(x,0) = u_0(x), & \text{in} \quad \Omega, \\ u^*(x,t) = 0, & \text{in} \quad \partial\Omega. \end{cases} \tag{12}$$

Theorem 1.3. *Let $u_0 \in L^q(\Omega)$ and $F \in L^\infty(0,T; L^q(\Omega))$ for $\max\Big(1, \frac{2pN}{pN+4}\Big) < q \le 2$, $p > \frac{1}{\alpha}$. Then we have the following statement*

$$\Big\| u_\alpha - u^* \Big\|_{L^d(0,T; L^{\frac{2N}{N-4\theta}}(\Omega))} \lesssim \widetilde{M}_\alpha(T,p) \Big[\; \big\|u_0\big\|_{L^q(\Omega)} + \frac{3}{\Gamma(1-\alpha)} \big\|F\big\|_{L^\infty(0,T;L^q(\Omega))} \Big]. \tag{13}$$

where $d > \frac{1}{\alpha}$ and $\widetilde{M}_\alpha(T,p)$ is defined by (8). Here the hidden constant is independent of α and the constant θ satisfies

$$0 \le \theta < \min\Big(\frac{N}{4}, \frac{1 + \frac{pN}{4} - \frac{pN}{2q}}{p} \Big). \tag{14}$$

The idea to prove Theorem 1.3 is to apply Theorem 1.2 and some Sobolev embeddings between L^p and Hilbert scales (see Lemma 2.1).

Our last main result is concerned with the limit problem for the nonlinear fractional diffusion equation. Let us study the following nonlinear fractional diffusion equation with Caputo derivative

$${}^C D_t^\alpha u_\alpha + \mathcal{A}u_\alpha = F(u_\alpha(x,t)), \; u_\alpha(0) = u_0, \tag{15}$$

and the nonlinear parabolic diffusion equation with classical derivative

$$D_t u^* + \mathcal{A}u^* = F(u^*(x,t)), \; u^*(0) = u_0. \tag{16}$$

The following theorem shows that the mild solution of (15) converges to the mild solution of (16) when $\alpha \to 1^-$.

Theorem 1.4. *Let $F : L^r(\Omega) \to L^m(\Omega)$ such that $F(\mathbf{0}) = \mathbf{0}$ and*

$$\|F(\psi_1) - F(\psi_2)\|_{L^m(\Omega)} \le K_f \|\psi_1 - \psi_2\|_{L^r(\Omega)}, \tag{17}$$

for any $\psi_1, \psi_2 \in L^r(\Omega)$. Here $\mathcal{K}_f$ is a postive constant and two numbers m, r are chosen such that $1 \le m \le r$ and $\frac{1}{m} - \frac{1}{r} < \frac{2\alpha}{N}$. Let $u_0 \in \mathbb{X}^{\frac{N(r-2)}{4r} - \gamma}(\Omega)$ for any $0 < \gamma < 1$. Then there exists a postive $\beta_0 > 0$ such that Problem (15) has a unique mild solution in $\mathbf{Z}_{b,\beta_0}((0,T]; L^r(\Omega))$ where b satisfies that

$$\frac{N}{2}\Big(\frac{1}{m} - \frac{1}{r}\Big) - \alpha < b < \min\Big(1, \alpha - \frac{N}{2}\Big(\frac{1}{m} - \frac{1}{r}\Big)\Big).$$

Let us assume that m, r, p, N satisfy that $\frac{2}{m} - \frac{2}{r} + 1 < \frac{4}{pN} + \frac{4}{N}$. Let us assume that the initial condition $u_0 \in \mathbb{X}^{\frac{N(r-2)}{4r} + \frac{\mu_0 - 1}{p}}(\Omega) \cap L^r(\Omega)$, then Problem (15) has a unique mild solution $u_\alpha \in$

$\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$. *Furthermore, we obtain the following estimate*

$$\|u_\alpha - u^*\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))} \le \frac{6\overline{C}(\mu_0,p,T)T^{b-\alpha}}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)\|u_0\|_{\mathbb{X}^{\frac{N(r-2)}{4r}+\frac{\mu_0-1}{p}}(\Omega)} + \frac{6K_f T^{b-\alpha}\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)\|u^*\|_{L^\infty(0,T;L^r(\Omega))}, \tag{18}$$

where u^* *is the unique mild solution to Problem* (16)*. Here* $\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$ *is defined in* (21)*.*

Remark 1.2. *Using Proposition 4.1, under the initial condition* $u_0 \in L^r(\Omega)$*, by the same method as Theorem 1.4, we can show that Problem* (16) *has a unique solution* $u^* \in L^\infty(0,T;L^r(\Omega))$*. We omit the details of the proof.*

Remark 1.3. *The convergence behavior of solution to Problem* (15) *with a source of polynomial type is showed in [2, 8, 14] and references given there. Although we share the same topic with the above works, because of a different chosen case of source function, our results are novel. Indeed, the existence of global solution without any restriction on initial data when* F *satisfies* (17) *is not trivial. Our techniques used to derive the existence of global solution and convergence results in this paper are also different from those in [2, 8, 14].*

In Section 2, we introduce some notations on functional spaces, Riemann-Liouville integral and its properties. Section 3 will provide the proof of Theorem 1.2. The proof of Theorem 1.3 and Theorem 1.4 will be given in Section 4.

2. Notation and Preliminaries

Given $r \in (1,\infty)$ we denote by r^* its Hölder conjugate, i.e., the real number such that $\frac{1}{r}+\frac{1}{r^*}=1$. By $a \lesssim b$ we will denote that $a \le Cb$ for a constant C which does not depend on a, b and neither on the discretization parameters.

We consider the operator $\mathcal{A}$ acting on $W_*^{2,2}(\Omega) := W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \subset L^2(\Omega)$. Then, there exist sequences $\{\lambda_j\}_{j\ge 1}$ and $\{e_j\}_{j\ge1} \subset W_*^{2,2}(\Omega)$ which are the eigenvalues and eigenvectors of $-\mathcal{A}$ respectively. It is well known that $\{\lambda_j\}_{j\ge1}$ are positive, non-decreasing and $\lim_{j\to\infty}\lambda_j = \infty$ (see [19]). Moreover, $-\mathcal{A}e_j = \lambda_j e_j$. The sequence $\left\{e_j\right\}$, $j = 1,2,3...$ forms an orthonormal basis of $L^2(\Omega)$, see e.g. [12]. Let $\mathbb{X}^s(\Omega)$, $s \ge 0$, be the space which is defined in following form

$$\mathbb{X}^s(\Omega) = \left\{ v \in L^2(\Omega) \,\middle|\, \|v\|^2_{\mathbb{X}^s(\Omega)} := \sum_{j=1}^\infty \lambda_j^{2s}\left(\int_\Omega v(x)e_j(x)dx\right)^2 < \infty \right\}, \tag{19}$$

and we call $\mathbb{X}^s(\Omega)$ by a Hilbert scales space (see p. 1452, [16]). For $s=0$, we have $\mathbb{X}^0(\Omega) = L^2(\Omega)$ with the usual norm $\|\cdot\|$. We identify the dual space $\left[L^2(\Omega)\right]^*$ with $L^2(\Omega)$, and thus we can set $\mathbb{X}^{-s}(\Omega) := [\mathbb{X}^s(\Omega)]^*$ with

$$\|v\|^2_{\mathbb{X}^{-s}(\Omega)} := \sum_{j=1}^\infty \lambda_j^{-2s}\langle v, e_j\rangle^2_{-s,s}, \tag{20}$$

where $\langle\cdot,\cdot\rangle_{-s,s}$ is the duality bracket between $\mathbb{X}^s(\Omega)$ and $\mathbb{X}^{-s}(\Omega)$. More detailed information about these spaces can be found in [8, 16].

Let $\mathbf{Z}_{b,\beta}((0,T];X)$ denote the weighted space of all functions $v \in C((0,T];X)$ such that

$$\|v\|_{\mathbf{Z}_{b,\beta}((0,T];X)} := \sup_{t\in(0,T]} t^b e^{-\beta t}\|v(t,\cdot)\|_X < \infty,$$

where $b, \beta > 0$ (see [4]).

Lemma 2.1. *(see [16]) The following statements hold true:*
a) If $s = \frac{N}{4}$, $p \geq 1$ *or if* $0 \leq s < \frac{N}{4}$ *and* $1 \leq p \leq \frac{2N}{N-4s}$, *then we have the following Sobolev embedding*

$$\mathbb{X}^s(\Omega) \hookrightarrow W^{2s,2}(\Omega) \hookrightarrow L^p(\Omega) \tag{21}$$

b) If $-\dfrac{N}{4} < s \leq 0$ *and* $p \geq \dfrac{2N}{N-4s}$, *then we have the following Sobolev embedding*

$$L^p(\Omega) \hookrightarrow W^{2s,2}(\Omega) \hookrightarrow \mathbb{X}^s(\Omega) \tag{22}$$

Operators $E_{\alpha,1}(-t^\alpha \mathcal{A})$ and $E_{1,1}(-t^\alpha \mathcal{A})$, $0 < \alpha < 1$, $0 < \beta \leq 1$ are defined by

$$E_{\alpha,1}(-t^\alpha \mathcal{A})v = \sum_{j=1}^{\infty} E_{\alpha,1}(-t^\alpha \lambda_j)\langle v, e_j\rangle e_j, \qquad E_{1,1}(-t^\alpha \mathcal{A})v = \sum_{j=1}^{\infty} e^{-t\lambda_j}\langle v, e_j\rangle e_j \tag{23}$$

for any $v \in L^2(\Omega)$.

2.1. **Fractional integral and derivative.** We recall some definitions introduced in [8, 10].

Definition 2.2. If the function ψ is absolutely continuous in time, the Caputo derivative is the following function

$$^C D_t^\alpha \psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \psi'(s) ds, \tag{24}$$

where Γ is the Gamma function and $\psi'(s)$ is the first order integer derivative of function $\psi(s)$ with respect to its independent variable s.

Definition 2.3. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a locally integrable function $\psi : (0, \infty) \to \mathbb{R}$ is defined by

$$J^\alpha \psi(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \psi(s) ds.$$

Remark 2.1. *If the function* ψ *is absolutely continuous in time, then*

$$^c D_t^\alpha \psi(t) := J^{1-\alpha} \psi'(t), \quad t \geq 0,$$

2.2. **Wright function.**

Definition 2.4. The symbol $\mathcal{M}_\alpha$ denotes the Wright type function introduced by [10]

$$\mathcal{M}_\alpha(r) = \sum_{n=0}^{\infty} \frac{r^n}{n!\Gamma(1-\alpha(1+n))}, \quad r \in \mathbb{C}.$$

This function is an entire function on $\mathbb{C}$. The Mittag-Leffler function is expressed by $E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(\eta) e^{-z\eta} d\eta$, $z \in \mathbb{C}$.

3. Proof of Theorem (1.2)

3.1. **Proof of Part a).** In the following sentences, we present the challenge of assessing (1.2) as follows. In a first logical thought, we deal directly with the difference of $^C D_t^\alpha v$ and $D_t v$. However, we are stuck at some components. Therefore, the direction of thinking about direct evaluation is prevented.

Our novel ideas are discovered unexpectedly and interestingly as follows. First, we apply the Riemann–Liouville fractional integral operator J^α to the difference of $^C D_t^\alpha v$ and $D_t v$, denoted $\widetilde{W}_\alpha(t)$ which is given by (25). After several estimations, we process the upper bound of $\widetilde{W}_\alpha(t)$ as given in (39). In the next step, we compute the Caputo derivative of $\widetilde{W}_\alpha(t)$ in terms of itself (see (41)). The well-known formula $J^\alpha \left({}^C D_t^\alpha\right) f(t) = f(t) - f(0)$ (see [10]) allows us to deduce the desired result (1.2) since we have the following lucky fact that $\widetilde{W}_\alpha(0) = 0$.

In the sequel, we will present, in a detailed way, the proof of part a). Set the following function

$$\begin{aligned}\widetilde{W}_\alpha(t) &= J^\alpha \Big({}^C D_t^\alpha v(t) - D_t v(t)\Big) \\ &= J^\alpha \Big({}^C D_t^\alpha v(t) - \Gamma(\alpha) D_t v(t)\Big) + \Big(\Gamma(\alpha) - 1\Big) J^\alpha D_t v(t) = \widetilde{W}_\alpha^{(1)}(t) + \widetilde{W}_\alpha^{(2)}(t). \qquad (25)\end{aligned}$$

Since $D_t v$ and ${}^C D_t^\alpha v$ are locally integrable functions, it follows from Definition (2.3) that $\widetilde{W}_\alpha$ is well-defined. Our next aim is to estimate the quantity $\widetilde{W}_\alpha(t)$.

Step 1. Estimate of the term $\widetilde{W}_\alpha^{(1)}(t)$.

First, we need to find an upper bound for the term $\widetilde{W}_\alpha^{(1)}(t)$. Thanks to the formula $J^\alpha \left({}^C D_t^\alpha\right) v(t) = v(t) - v(0)$ (see [10]) and recalling that J^α is defined by Definition 2.3, we derive the following equality

$$\begin{aligned}\widetilde{W}_\alpha^{(1)}(t) &= J^\alpha \Big({}^C D_t^\alpha v(t) - \Gamma(\alpha) D_t v(t)\Big) = J^\alpha \big({}^C D_t^\alpha v(t)\big) - \Gamma(\alpha) J^\alpha \big(D_t v(t)\big) \\ &= \Big[v(t) - v(0) - \int_0^t (t-s)^{\alpha-1} D_s v(s) ds\Big] \\ &= \int_0^t \Big[1 - (t-s)^{\alpha-1}\Big] D_s v(s) ds. \qquad (26)\end{aligned}$$

To continute further, we consider two cases.

Case 1. $0 \le t \le 1$. For this case, it is easy to see that $1 - (t-s)^{\alpha-1} < 0$. Thus, we deduce that $|1 - (t-s)^{\alpha-1}| = -1 + (t-s)^{\alpha-1}$. Thanks to the Hölder inequality, we derive that

$$\begin{aligned}\left|\widetilde{W}_\alpha^{(1)}(t)\right| &= \left|J^\alpha \Big({}^C D_t^\alpha v(t) - \Gamma(\alpha) D_t v(t)\Big)\right| \\ &\le \int_0^t \Big|1 - (t-s)^{\alpha-1}\Big| |D_s v(s)| ds \\ &\le \left(\int_0^t \left((t-s)^{\alpha-1} - 1\right)^{p^*} ds\right)^{\frac{1}{p^*}} \left(\int_0^t |D_s v(s)|^p ds\right)^{\frac{1}{p}} \\ &\le \left(\int_0^t \left((t-s)^{\alpha-1} - 1\right)^{p^*} ds\right)^{\frac{1}{p^*}} \big\|Dv\big\|_{L^p(0,T)}, \qquad (27)\end{aligned}$$

where we remind (7) and we choose $0 < \beta \le \alpha_0$. Our next aim is to provide an upper bound for the integral term on the right hand side of (27). It is not difficult to see that if $0 \le a \le b$ then, for any $\mu \ge 1$, we have $(b-a)^\mu \le b^\mu - a^\mu$. Thus, we find the following estimate

$$\begin{aligned}\left(\int_0^t \left((t-s)^{\alpha-1} - 1\right)^{p^*} ds\right)^{\frac{1}{p^*}} &\le \left(\int_0^t \Big[(t-s)^{(\alpha-1)p^*} - 1\Big] ds\right)^{\frac{1}{p^*}} \\ &= \Big[\frac{p-1}{\alpha p - 1} t^{\frac{\alpha p - 1}{p-1}} - t\Big]^{\frac{p-1}{p}}. \qquad (28)\end{aligned}$$

It is easy to verify that the derivative of the function $\Phi(t) = \frac{p-1}{\alpha p-1} t^{\frac{\alpha p-1}{p-1}} - t$ is $\Phi'(t) = t^{\frac{p(\alpha-1)}{p-1}} - 1 > 0$ for any $0 < \alpha < 1$ and $0 < t \le 1$. This implies that the function Φ is increasing on $[0,1]$, so we arrive at $\Phi(t) \le \Phi(1) = \frac{p-1}{\alpha p-1} - 1 = \frac{p(1-\alpha)}{\alpha p-1}$. The latter inequality, together with (27) and (28), leads to

$$\left|\widetilde{W}_\alpha^{(1)}(t)\right| \le \Big[\frac{p(1-\alpha)}{\alpha p - 1}\Big]^{\frac{p-1}{p}} \big\|Dv\big\|_{L^p(0,T)}. \qquad (29)$$

Case 2. $t \geq 1$. In this case, we know that the function $1 - s^{1-\alpha}$ is negative for $s \in (0,1)$ and positive for $s > 1$. Hence, after changing variables for integration and (26), we have

$$\begin{aligned}\left|\widetilde{W}_\alpha^{(1)}(t)\right| &= \left|J^\alpha\Big({}^C D_t^\alpha v(t) - \Gamma(\alpha) D_t v(t)\Big)\right| = \Bigg|\int_0^t \Big[1-(t-s)^{\alpha-1}\Big] D_s v(s)\Bigg| ds \\ &\leq \Bigg(\int_0^t \Big|1-(t-s)^{\alpha-1}\Big|^{p^*} ds\Bigg)^{\frac{1}{p^*}} \Bigg(\int_0^t |D_s v(s)|^p ds\Bigg)^{\frac{1}{p}} \\ &\leq \Bigg(\int_0^t \Big|1-(t-s)^{\alpha-1}\Big|^{p^*} ds\Bigg)^{\frac{1}{p^*}} \big\|Dv\big\|_{L^p(0,T)}. \end{aligned} \tag{30}$$

It is easy to check that

$$\begin{aligned}\int_0^t \Big|1-(t-s)^{\alpha-1}\Big|^{p^*} ds &= \int_0^t \Big|1-s^{\alpha-1}\Big|^{p^*} ds \\ &= \int_0^1 \Big[s^{\alpha-1}-1\Big]^{p^*} ds + \int_1^t \Big[1-s^{\alpha-1}\Big]^{p^*} ds. \end{aligned} \tag{31}$$

Using $(b-a)^\mu \leq b^\mu - a^\mu$, for any $0 \leq a \leq b$ and $\mu \geq 1$, we arrive at

$$\int_0^1 \Big[s^{\alpha-1}-1\Big]^{p^*} ds \leq \int_0^1 \Big[s^{(\alpha-1)p^*}-1\Big] ds = \frac{p(1-\alpha)}{\alpha p - 1}. \tag{32}$$

and

$$\int_1^t \Big[1-s^{\alpha-1}\Big]^{p^*} ds \leq \int_1^t \Big[1-s^{(\alpha-1)p^*}\Big] ds = t - 1 - \frac{p-1}{\alpha p-1}\Big(t^{\frac{\alpha p-1}{p-1}} - 1\Big). \tag{33}$$

Set $\Psi(t) = t - 1 - \frac{p-1}{\alpha p-1}\Big(t^{\frac{\alpha p-1}{p-1}} - 1\Big)$ for $1 \leq t \leq T$. Its derivative is $\Psi'(t) = 1 - t^{\frac{\alpha p-p}{p-1}} > 0,\ t \geq 1,\ \frac{\alpha p - p}{p-1} < 0$. Thus, we obtain immediately that

$$\int_1^t \Big[1-s^{\alpha-1}\Big]^{p^*} ds \leq \Psi(t) \leq \Psi(T) = T - 1 - \frac{p-1}{\alpha p-1}\Big(T^{\frac{\alpha p-1}{p-1}} - 1\Big). \tag{34}$$

Combining (31), (32) and (34), we find that

$$\Bigg(\int_0^t \Big|1-(t-s)^{\alpha-1}\Big|^{p^*} ds\Bigg)^{\frac{1}{p^*}} \leq \Bigg[\frac{p(1-\alpha)}{\alpha p-1} + T - 1 - \frac{p-1}{\alpha p-1}\Big(T^{\frac{\alpha p-1}{p-1}} - 1\Big)\Bigg]^{\frac{p-1}{p}}. \tag{35}$$

It follows from (30) and (29) that

$$\begin{aligned}\left|\widetilde{W}_\alpha^{(1)}(t)\right| &= \left|J^\alpha\Big({}^C D_t^\alpha v(t) - \Gamma(\alpha) D_t v(t)\Big)\right| \\ &\leq \Bigg[\frac{p(1-\alpha)}{\alpha p-1} + T - 1 - \frac{p-1}{\alpha p-1}\Big(T^{\frac{\alpha p-1}{p-1}} - 1\Big)\Bigg]^{\frac{p-1}{p}} \big\|Dv\big\|_{L^p(0,T)}, \end{aligned} \tag{36}$$

for any $t > 0$.

Step 2. Estimate $\widetilde{W}_\alpha^{(2)}(t)$.

Now, Definition 2.3 and the definition of $\widetilde{W}_\alpha^{(2)}(t)$ in (25) imply

$$\widetilde{W}_\alpha^{(2)}(t) = \Big(\Gamma(\alpha) - 1\Big) J^\alpha D_t v(t) = \frac{\Gamma(\alpha)-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_s v(s) ds. \tag{37}$$

For this step, using the Hölder inequality and noting that $p > \frac{1}{\alpha}$, it is straightforward to see that

$$\begin{aligned} \left|\widetilde{W}_\alpha^{(2)}(t)\right| &\le \Big|\frac{\Gamma(\alpha)-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} D_s v(s) ds\Big| \\ &\le \Big|\frac{\Gamma(\alpha)-1}{\Gamma(\alpha)}\Big| \left(\int_0^t (t-s)^{(\alpha-1)p^*} ds\right)^{\frac{1}{p^*}} \left(\int_0^t |D_s v(s)|^p ds\right)^{\frac{1}{p}} \\ &\le \Big|\frac{\Gamma(\alpha)-1}{\Gamma(\alpha)}\Big| \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} T^{\frac{\alpha p-1}{p}} \big\|Dv\big\|_{L^p(0,T)}. \end{aligned} \tag{38}$$

Therefore, Step 2 is completed.
On account of (25), (36) and (38), we deduce an upper bound for the term $\widetilde{W}_\alpha(t)$ as follows

$$\left|\widetilde{W}_\alpha(t)\right| = \widetilde{W}_\alpha^{(1)}(t) + \widetilde{W}_\alpha^{(2)}(t) \le \widetilde{M}_\alpha(T,p)\big\|Dv\big\|_{L^p(0,T)}. \tag{39}$$

where we recall that $\widetilde{M}_\alpha(T,p)$ is defined in (8).
Notice that the following equality

$${}^C D_t^\alpha J^\alpha\Big({}^C D_t^\alpha v(t) - D_t v(t)\Big) = {}^C D_t^\alpha v(t) - D_t v(t) = {}^C D_t^\alpha \widetilde{W}_\alpha(t),$$

the fact that $\widetilde{W}_\alpha(0) = 0$, and the definition of ${}^C D_t^\alpha$ for the function $\widetilde{W}_\alpha(t)$, allow us to obtain the following equality

$$\begin{aligned} {}^C D_t^\alpha v(t) - D_t v(t) = {}^C D_t^\alpha \widetilde{W}_\alpha(t) &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\widetilde{W}_\alpha(t) \\ &\quad + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha-1}\Big[\widetilde{W}_\alpha(t) - \widetilde{W}_\alpha(s)\Big] ds. \end{aligned} \tag{40}$$

Due to the estimate (39), we bound the second term on the right hand side of the above expression as follows

$$\begin{aligned} \Big|\int_0^t (t-s)^{-\alpha-1}\Big[\widetilde{W}_\alpha(t) - \widetilde{W}_\alpha(s)\Big] ds\Big| &\le 2\widetilde{M}_\alpha(T,p)\big\|Dv\big\|_{L^p(0,T)}\int_0^t (t-s)^{-\alpha-1} ds \\ &= \frac{2\widetilde{M}_\alpha(T,p)\big\|Dv\big\|_{L^p(0,T)}}{\alpha} t^{-\alpha}. \end{aligned} \tag{41}$$

For any $t \ge 0$, this leads to

$$\begin{aligned} \Big|{}^C D_t^\alpha v(t) - D_t v(t)\Big| &\le \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\Big|\widetilde{W}_\alpha(t)\Big| + \Big|\int_0^t (t-s)^{-\alpha-1}\Big[\widetilde{W}_\alpha(t) - \widetilde{W}_\alpha(s)\Big] ds\Big| \\ &\le \frac{3}{\Gamma(1-\alpha)} t^{-\alpha}\widetilde{M}_\alpha(T,p)\big\|Dv\big\|_{L^p(0,T)}. \end{aligned} \tag{42}$$

3.2. **Proof of Part b).** In the case $v = v(x,t)$ and $v : [0,T] \to L^2(\Omega)$, we deduce that

$$\Big\|\widetilde{W}_\alpha^{(1)}(t)\Big\|_{L^2(\Omega)} = \Big\|\int_0^t \Big[1 - (t-s)^{\alpha-1}\Big] D_s v(s) ds\Big\|_{L^2(\Omega)} \le \int_0^t \Big|1 - (t-s)^{\alpha-1}\Big|\ \big\|D_s v(s)\big\|_{L^2(\Omega)} ds. \tag{43}$$

By a similar argument as in (36), we find that

$$\Big\|\widetilde{W}_\alpha^{(1)}(t)\Big\|_{L^2(\Omega)} \le \left[\frac{p(1-\alpha)}{\alpha p-1} + T - 1 - \frac{p-1}{\alpha p-1}\Big(T^{\frac{\alpha p-1}{p-1}} - 1\Big)\right]^{\frac{p-1}{p}} \big\|Dv\big\|_{L^p(0,T;L^2(\Omega))}, \tag{44}$$

for any $t > 0$, and we notice that

$$\|Dv\|_{L^p(0,T;L^2(\Omega))} = \left(\int_0^T \|D_s v(s)\|^p_{L^2(\Omega)} ds\right)^{1/p}.$$

By a similar argument as in (38), we find that

$$\left\|\widetilde{W}^{(1)}_\alpha(t)\right\|_{L^2(\Omega)} \le \left|\frac{\Gamma(\alpha)-1}{\Gamma(\alpha)}\right| \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} T^{\frac{\alpha p-1}{p}} \|Dv\|_{L^p(0,T;L^2(\Omega))}. \tag{45}$$

In the same spirit as in the proof of (42), we also obtain the desired result (9).

3.3. **Proof of Part c).** Set $\psi_\alpha(k,t) = E_{\alpha,1}(-kt^\alpha)$ and $\overline{\psi}(k,t) = E_{1,1}(-kt)$. It is obvious to see that

$${}^C D^\alpha_t \psi_\alpha(k,t) = -k\psi_\alpha(k,t),\ \ \psi_\alpha(k,0) = 1,$$

and $D_t\overline{\psi}(k,t) = -k\overline{\psi}(k,t)$, $\overline{\psi}(k,0) = 1$. These equations give that $\overline{\psi}(k,t) = e^{-kt}$ which implies $\frac{d}{dt}\overline{\psi}(k,t) = -ke^{-kt}$. Therefore, using the inequality $1 - e^{-z} \le C_\mu z^\mu$ for any $\mu > 0$, it is obvious to find that

$$\begin{aligned}\left\|\frac{d}{dt}\overline{\psi}(k,t)\right\|_{L^p(0,T)} &= \left(\int_0^T \left(ke^{-kt}\right)^p dt\right)^{1/p} = k^{1-\frac{1}{p}} p^{-1/p}\left(1-e^{-kpT}\right)^{1/p} \\ &\le C_\mu k^{1+\frac{\mu-1}{p}} p^{\frac{\mu-1}{p}} T^{\mu/p} = \overline{C}(\mu,p,T) k^{1+\frac{\mu-1}{p}},\end{aligned} \tag{46}$$

where in the above expression, we set $\overline{C}(\mu,p,T) = C_\mu p^{\frac{\mu+1}{p}} T^{\mu/p}$. Set $\varrho_\alpha(k,t) = \psi_\alpha(k,t) - \overline{\psi}(k,t)$. It is straightforward to check that

$${}^C D^\alpha_t \varrho_\alpha(k,t) = -k\varrho_\alpha(k,t) + {}^C D^\alpha_t \overline{\psi}(k,t) - D_t\overline{\psi}(k,t), \quad \varrho_\alpha(k,0) = 0. \tag{47}$$

Multiplying both sides of (47) by $\varrho_\alpha(k,t)$ and using the inequality

$${}^C D^\alpha_t \varrho_\alpha(k,t)\varrho_\alpha(k,t) \ge \frac{1}{2}{}^C D^\alpha_t |\varrho_\alpha(k,t)|^2,$$

we arrive at

$$\frac{1}{2}{}^C D^\alpha_t |\varrho_\alpha(k,t)|^2 + k|\varrho_\alpha(k,t)|^2 = \left\langle {}^C D^\alpha_t \overline{\psi}(k,t) - D_t\overline{\psi}(k,t), \varrho_\alpha(k,t)\right\rangle. \tag{48}$$

From the latter estimate, noting that

$$\left|\left\langle {}^C D^\alpha_t \overline{\psi}(k,t) - D_t\overline{\psi}(k,t), \varrho_\alpha(k,t)\right\rangle\right| \le \left|{}^C D^\alpha_t \overline{\psi}(k,t) - D_t\overline{\psi}(k,t)\right|\left|\varrho_\alpha(k,t)\right|,$$

we obtain that

$$\begin{aligned}k|\varrho_\alpha(k,t)| &\le \left|{}^C D^\alpha_t \overline{\psi}(k,t) - D_t\overline{\psi}(k,t)\right| \le \frac{3}{\Gamma(1-\alpha)} t^{-\alpha}\widetilde{M}_\alpha(T,p)\left\|\frac{d}{dt}\overline{\psi}(k,t)\right\|_{L^p(0,T)} \\ &\le \frac{3\overline{C}(\mu,p,T)}{\Gamma(1-\alpha)} t^{-\alpha}\widetilde{M}_\alpha(T,p) k^{1+\frac{\mu-1}{p}}.\end{aligned} \tag{49}$$

This implies the following estimate

$$\left|E_{\alpha,1}(-kt^\alpha) - e^{-kt}\right| = |\varrho_\alpha(k,t)| \le \frac{3\overline{C}(\mu,p,T)}{\Gamma(1-\alpha)} t^{-\alpha}\widetilde{M}_\alpha(T,p) k^{\frac{\mu-1}{p}}. \tag{50}$$

We continue to show the second part which is showed similarly as for the previous term. Set the following functions

$$w_\alpha(k,t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-k(t-s)^\alpha) F(s) ds, \quad \overline{w}(k,t) = \int_0^t e^{-k(t-s)} F(s) ds. \tag{51}$$

It is easy to verify that ${}^C D_t^\alpha w_\alpha(k,t) = -kw_\alpha(k,t) + F(t)$, $w_\alpha(k,0) = v_0$ and $D_t\overline{w}(k,t) = -k\overline{w} + F(t)$, $\overline{w}(k,0) = v_0$. By letting $z_\alpha(k,t) = w_\alpha(k,t) - \overline{w}(k,t)$, we find that

$$ {}^C D_t^\alpha z_\alpha(k,t) = -kz_\alpha(k,t) + {}^C D_t^\alpha \overline{w}(k,t) - D_t\overline{w}(k,t), \quad z_\alpha(k,0) = 0. \tag{52} $$

Multiplying both sides of (52) by $z_\alpha(k,t)$ and using the inequality

$$ {}^C D_t^\alpha z_\alpha(k,t) z_\alpha(k,t) \geq \frac{1}{2} {}^C D_t^\alpha |z_\alpha(k,t)|^2, $$

we derive that

$$ \frac{1}{2} {}^C D_t^\alpha |z_\alpha(k,t)|^2 + k|z_\alpha(k,t)|^2 = \Big\langle \Big({}^C D_t^\alpha \overline{w}(k,t) - D_t\overline{\psi}(k,t) \Big) z_\alpha(k,t) \Big\rangle, \tag{53} $$

which allows us to obtain

$$ k|z_\alpha(k,t)|^2 \leq \Big\langle \Big({}^C D_t^\alpha \overline{w}(k,t) - D_t\overline{\psi}(k,t) \Big) z_\alpha(k,t) \Big\rangle \leq \Big| {}^C D_t^\alpha \overline{w}(t) - D_t\overline{w}(t) \Big| |z_\alpha(k,t)|. $$

This inequality, together with part a) of Theorem 1.2, implies

$$ k|z_\alpha(k,t)| \leq \Big| {}^C D_t^\alpha \overline{w}(t) - D_t\overline{w}(t) \Big| \leq \frac{3}{\Gamma(1-\alpha)} t^{-\alpha} \widetilde{M}_\alpha(T,p) \Big\| D\overline{w} \Big\|_{L^p(0,T)}. \tag{54} $$

Let us continue to estimate the term $\Big\| D\overline{w} \Big\|_{L^p(0,T)}$ on the right hand side of the latter expression. Indeed, by a simple calculation and setting $F = 1$, we have

$$ D_t\overline{w}(t) = -k\overline{w} + F(t) = -k \int_0^t e^{-k(t-s)} ds + 1 = e^{-kt}. $$

In view of the inequality $1 - e^{-z} \leq C_\mu z^\mu$ for any $\mu > 0$, we obtain

$$ \Big\| D\overline{w} \Big\|_{L^p(0,T)} = \Big(\int_0^T e^{-ktp} dt \Big)^{1/p} \leq C_\mu T^{\mu/p} p^{\frac{\mu-1}{p}} k^{\frac{\mu-1}{p}} = \widetilde{C}(\mu,T,p) k^{\frac{\mu-1}{p}}. \tag{55} $$

This, together with (54), leads to the desired result.

4. Proof of Theorem 1.3 and Theorem 1.4

4.1. **Proof of Theorem 1.3.** From the paper [8], we have the following mild solution to Problem (1) in the case $F = F(x,t)$

$$ u_\alpha(t) = E_{\alpha,1}(-t^\alpha \mathcal{A}) u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) F(s) ds. \tag{56} $$

The mild solution to Problem (12) is defined by

$$ u^*(t) = E_{1,1}(-t^\alpha \mathcal{A}) u_0 + \int_0^t E_{1,1}(-(t-s)\mathcal{A}) F(s) ds. \tag{57} $$

By subtracting both sides of the latter equations, we have

$$ \begin{aligned} u_\alpha(t) - u^*(t) &= \Big(E_{\alpha,1}(-t^\alpha \mathcal{A}) u_0 - E_{1,1}(-t^\alpha \mathcal{A}) u_0 \Big) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) F(s) ds - \int_0^t E_{1,1}(-(t-s)\mathcal{A}) F(s) ds = \mathscr{J}_1(t) + \mathscr{J}_2(t). \end{aligned} \tag{58} $$

Take any $0 \le \theta < \frac{N}{4}$. By using Parseval's equality and the estimate (10) as in Theorem 1.2, we deduce

$$\begin{aligned}\Big\|E_{\alpha,1}(-t^\alpha \mathcal{A})u_0 - E_{1,1}(-t^\alpha \mathcal{A})u_0\Big\|_{\mathbb{X}^\theta(\Omega)} &= \sqrt{\sum_{j=1}^\infty \lambda_j^{2\theta}\Big|E_{\alpha,1}(-\lambda_j t^\alpha) - E_{1,1}(-\lambda_j t)\Big|^2\bigg(\int_\Omega u_0(x)e_j(x)dx\bigg)^2} \\ &\le \frac{3\overline{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\left[\sum_{j=1}^\infty \lambda_j^{2\theta+\frac{2\mu-2}{p}}\bigg(\int_\Omega u_0(x)e_j(x)dx\bigg)^2\right]^{1/2} \\ &= \frac{3\overline{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\big\|u_0\big\|_{\mathbb{X}^{\theta+\frac{\mu-1}{p}}(\Omega)}. \qquad (59)\end{aligned}$$

Since $\max\Big(1, \frac{2pN}{pN+4}\Big) < q \le 2$, we know that $1 + \frac{pN}{4} > \frac{pN}{2q}$. Let us recall θ as in (14). If we set $\mu = 1 + p\Big(\frac{Nq-2N}{4q} - \theta\Big)$, then by the above condition on θ, we immediately obtain that $\mu > 0$ and $\mathbb{X}^{\theta+\frac{\mu-1}{p}}(\Omega) = \mathbb{X}^{\frac{Nq-2N}{4q}}(\Omega)$. In view of Lemma 2.1, noticing that $1 < q \le 2$, we derive the Sobolev embedding $L^q(\Omega) \hookrightarrow \mathbb{X}^{\theta+\frac{\mu-1}{p}}(\Omega) = \mathbb{X}^{\frac{Nq-2N}{4q}}(\Omega)$. In addition, from (14) and using again Lemma 2.1, we have $\mathbb{X}^\theta(\Omega) \hookrightarrow L^{\frac{2N}{N-4\theta}}(\Omega)$. These observations, together with (59), yield to

$$\begin{aligned}\Big\|\mathscr{J}_1\Big\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} &= \Big\|E_{\alpha,1}(-t^\alpha \mathcal{A})u_0 - E_{1,1}(-t^\alpha \mathcal{A})u_0\Big\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \\ &\le C(N,\theta)\Big\|E_{\alpha,1}(-t^\alpha \mathcal{A})u_0 - E_{1,1}(-t^\alpha \mathcal{A})u_0\Big\|_{\mathbb{X}^\theta(\Omega)} \\ &\le \widetilde{C}_1\widetilde{M}_\alpha(T,p)t^{-\alpha}\big\|u_0\big\|_{L^q(\Omega)}, \qquad (60)\end{aligned}$$

where $\widetilde{C}_1$ depends on $p, q, N, \theta, T, \alpha$. Let us now estimate the second term on the right hand side of (58). Using Parseval's equality and estimate (11) as in Theorem (1.2), we have that, for any $\mu > 0$ and $p > \frac{1}{\alpha}$,

$$\begin{aligned}&\Big\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})F(s)ds - \int_0^t E_{1,1}(-(t-s)\mathcal{A})F(s)ds\Big\|_{\mathbb{X}^\theta(\Omega)} \\ &\le \left(\sum_{j=1}^\infty \lambda_j^{2\theta}\Big|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-s)^\alpha)ds - \int_0^t E_{1,1}(-\lambda_j(t-s))ds\Big|^2\bigg(\int_\Omega F(x,s)e_j(x)dx\bigg)^2\right)^{\frac{1}{2}} \\ &\qquad \le \frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\left[\sum_{j=1}^\infty \lambda_j^{2\theta+\frac{2\mu-2}{p}-2}\bigg(\int_\Omega F(x,s)e_j(x)dx\bigg)^2\right]^{1/2} \\ &\qquad = \frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\big\|F(\cdot,s)\big\|_{\mathbb{X}^{\theta+\frac{\mu-1}{p}-1}(\Omega)}. \qquad (61)\end{aligned}$$

By the same techniques as before, we know that the Sobolev embedding below is true

$$L^q(\Omega) \hookrightarrow \mathbb{X}^{\theta+\frac{\mu-1}{p}}(\Omega) \hookrightarrow \mathbb{X}^{\theta+\frac{\mu-1}{p}-1}(\Omega).$$

This implies that $\big\|F(\cdot,s)\big\|_{\mathbb{X}^{\theta+\frac{\mu-1}{p}-1}(\Omega)} \le \Big\|F\Big\|_{L^\infty(0,T;L^q(\Omega))}$, which yields

$$\Big\|\mathscr{J}_2\Big\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \le \frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\Big\|F\Big\|_{L^\infty(0,T;L^q(\Omega))}. \qquad (62)$$

Combining (58), (60) and (62), we derive that

$$\Big\|u_\alpha(t)-u^*(t)\Big\|_{L^{\frac{2N}{N-4\theta}}(\Omega)} \le t^{-\alpha}\widetilde{M}_\alpha(T,p)\Big[\widetilde{C}_1\ \|u_0\|_{L^q(\Omega)}+\frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\Big\|F\Big\|_{L^\infty(0,T;L^q(\Omega))}\Big]. \quad (63)$$

Therefore, we achieve the desired result (13). The proof is completed.

4.2. **Proof of Theorem 1.4.** In order to show the existence of the mild solution to Problem (15), we need to use next two results.

Proposition 4.1. *There exists a positive constant C such that, for any $1\le m\le r$,*

$$\big\|e^{-t\mathcal{A}}\varphi\big\|_{L^r(\Omega)} \le C(m,r)t^{-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}\|\varphi\|_{L^m(\Omega)},\ \ t>0, \varphi\in L^m(\Omega). \quad (64)$$

Proof. For the proof, see [11, Lemma 2.3]. □

First we state the following lemma which will be useful in our main results (this lemma can be found in [4], Lemma 8, page 9).

Lemma 4.1. *Let $a>-1$, $b>-1$ such that $a+b\ge-1$, $d>0$ and $t\in[0,T]$. For $d>0$, the following limit holds*

$$\lim_{\mu\to\infty}\left(\sup_{t\in[0,T]} t^d\int_0^1 s^a(1-s)^b e^{-\mu t(1-s)}\mathrm{d}s\right)=0.$$

In the first part, we show the existence and uniqueness of the mild solution to Problem (15). Define the mapping $\mathbf{Q}:\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))\to\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))$, $\beta>0$, by

$$\mathbf{Q}\psi(t):=E_{\alpha,1}(-t^\alpha\mathcal{A})u_0+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})F(\psi(s))ds. \quad (65)$$

In what follows, we shall prove the existence of a unique solution of Problem (65). This is based on the Banach principle argument. Thanks to the Sobolev embedding $\mathbb{X}^{\frac{N(p-2)}{4p}}(\Omega)\hookrightarrow L^r(\Omega)$, the Parseval equality and the boundedness of Mittag-Leffler functions, we find that

$$\begin{aligned}
\Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|^2_{L^r(\Omega)} &\lesssim \Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|^2_{\mathbb{X}^{\frac{N(r-2)}{4r}}(\Omega)}\\
&\le\sum_{j=1}^\infty\left(\int_\Omega u_0(x)e_j(x)dx\right)^2\big(E_{\alpha,1}\left(-\lambda_j^2t^\alpha\right)\big)^2\lambda_j^{\frac{N(r-2)}{2r}}\\
&=\sum_{j=1}^\infty\left(\int_\Omega u_0(x)e_j(x)dx\right)^2\frac{C^2}{(1+\lambda_jt^\alpha)^{2\gamma}}\lambda_j^{\frac{N(r-2)}{2r}}\\
&\le C^2t^{-2\alpha\gamma}\|u_0\|^2_{\mathbb{X}^{\frac{N(r-2)}{4r}-\gamma}(\Omega)}.
\end{aligned} \quad (66)$$

It follows from the assumption $b>\alpha\gamma$ that

$$t^be^{-\beta t}\Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|_{L^p(\Omega)}\le T^{b-\alpha\gamma}\|u_0\|_{\mathbb{X}^{\frac{N(r-2)}{4r}-\gamma}(\Omega)}.$$

This inequality also implies that if $\psi=0$ then $\mathbf{Q}\psi$ belongs to the space $\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))$. In the following, we need to estimate the upper bound for the term $\|\mathbf{Q}\psi_1(t)-\mathbf{Q}\psi_2(t)\|_{L^r(\Omega)}$ for any $\psi_1,\psi_2\in\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))$. It is noted that

$$E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})F(\psi(s))=\alpha\int_0^\infty\nu\mathcal{M}_\alpha(\nu)e^{-\nu(t-s)^\alpha\mathcal{A}}F(\psi(s)),$$

using the globally Lipschitz property of F and Proposition 4.1, we arrive at

$$\|\mathbf{Q}\psi_1(t)-\mathbf{Q}\psi_2(t)\|_{L^r(\Omega)} \le \alpha\int_0^t (t-s)^{\alpha-1}\int_0^\infty \nu\mathcal{M}_\alpha(\nu)\Big\|e^{-\nu(t-s)^\alpha\mathcal{A}}\Big[F(\psi_1(s))-F(\psi_2(s))\Big]\Big\|_{L^r(\Omega)} d\nu ds$$

$$\le \alpha C\int_0^t (t-s)^{\alpha-1}\int_0^\infty \nu^{-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}(t-s)^{-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}\mathcal{M}_\alpha(\nu)\Big\|F(\psi_1(s))-F(\psi_2(s))\Big\|_{L^m(\Omega)} d\nu ds$$

$$\le \frac{\alpha CK_f\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\int_0^t (t-s)^{\alpha-1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}\big\|\psi_1(s,.)-\psi_2(s,.)\big\|_{L^r(\Omega)} ds \tag{67}$$

where we have used the following identity $\int_0^\infty \nu^{-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}\mathcal{M}_\alpha(\nu)d\nu = \frac{\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}$, since condition $\frac{N}{2}(\frac{1}{m}-\frac{1}{r})<1$ holds. Multiplying both sides of the above expression by $t^be^{-\beta t}$ and by simple transformation, we obtain

$$\|\mathbf{Q}\psi_1-\mathbf{Q}\psi_2\|_{\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))} \le \frac{\alpha CK_f\mathbf{M}_\beta\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\|\psi_1-\psi_2\|_{\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))} \tag{68}$$

where

$$\mathbf{M}_\beta = \sup_{t\in(0,T]} t^b\int_0^t (t-s)^{\alpha-1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}s^{-b}e^{-\beta(t-s)}\mathrm{d}s$$
$$= t^{b+\alpha-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}\int_0^1 (1-z)^{\alpha-1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})}z^{-b}e^{-\beta t(1-z)}\mathrm{d}z. \tag{69}$$

It is easy to verify the following conditions $b+\alpha-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})>0,\ \alpha-1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})>-1,\ -b>-1,\ \alpha-\frac{N}{2}(\frac{1}{m}-\frac{1}{r})\ge b$. In view of Lemma 4.1, we infer that $\mathbf{M}_\beta\to 0$ when β tends to infinity. Hence, there exists a positive $\beta_0>0$ such that $\frac{\alpha CK_f\mathbf{M}_\beta\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\le\frac{1}{2}$, which allows us to deduce that $\mathbf{Q}$ is a contraction mapping on $\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$. This, together with (42), leads to $\mathbf{Q}\psi\in\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))$ if $\psi\in\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$. Hence, we conclude that $\mathbf{Q}$ has a fixed point u in $\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$, i.e, u is a unique mild solution of Problem (65) under the condition $u_0\in\mathbb{X}^{\frac{N(r-2)}{4r}-\gamma}(\Omega)$. If we let $\gamma=\frac{1-\mu_0}{p}\in(0,1)$ if $0<\mu_0<1$, then Problem (65) has a unique solution in $\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))$.
In the second part, we show that the mild solution to Problem (15) converges to the mild solution to Problem (16) when $\alpha\to1^-$.

Let u^* be a solution of the following classical parabolic problem (16). It follows from [7] that the mild solution u^* is given by the formula

$$u^*(t)=E_{1,1}(-t\mathcal{A})u_0+\int_0^t E_{1,1}(-(t-s)\mathcal{A})F(u^*(s))ds, \tag{70}$$

where we recall that $E_{1,1}(-t\mathcal{A})=e^{-t\mathcal{A}}$. In the next estimates, we compare both functions u_α and u^*. From the above identities, we can divide the differences of the mild solution of Problem (15) and the mild solution of (16) into the sum of three terms as follows

$$u_\alpha(t)-u^*(t)=\Big[E_{\alpha,1}(-t^\alpha\mathcal{A})-E_{1,1}(-t\mathcal{A})\Big]u_0$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\Big[F(u_\alpha(s))-F(u^*(s))\Big]ds$$
$$+\int_0^t\Big[(t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})-E_{1,1}(-(t-s)^\alpha\mathcal{A})\Big]F(u^*(s))ds=(I)+(II)+(III). \tag{71}$$

For the first term on the right hand side of (71), we apply the same techniques as in (59) and (66) to derive that

$$\begin{aligned}\Big\|\Big[E_{\alpha,1}(-t^\alpha\mathcal{A})-E_{1,1}(-t\mathcal{A})\Big]u_0\Big\|_{L^r(\Omega)} &\lesssim \Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0-E_{1,1}(-t\mathcal{A})u_0\Big\|_{\mathbb{X}^{\frac{N(r-2)}{4r}}(\Omega)} \\ &\le \sqrt{\sum_{j=1}^\infty\Bigg(\int_\Omega u_0(x)e_j(x)dx\Bigg)^2\Bigg((E_{\alpha,1}\left(-\lambda_j t^\alpha\right)-(E_{1,1}\left(-\lambda_j t\right)\Bigg)^2\lambda_j^{\frac{N(r-2)}{2r}}} \\ &\le 3\overline{C}(\mu_0,p,T)\left(\Gamma(1-\alpha)\right)^{-1}\widetilde{M}_\alpha(T,p)t^{-\alpha}\big\|u_0\big\|_{\mathbb{X}^{\frac{N(r-2)}{4r}+\frac{\mu_0-1}{p}}(\Omega)},\ \mu_0>0. \end{aligned} \tag{72}$$

It follows from the latter estimate and $b\ge\alpha$ that

$$\big\|(I)\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}\le 3\overline{C}(\mu_0,p,T)\left(\Gamma(1-\alpha)\right)^{-1}\widetilde{M}_\alpha(T,p)T^{b-\alpha}\big\|u_0\big\|_{\mathbb{X}^{\frac{N(r-2)}{4r}+\frac{\mu_0-1}{p}}(\Omega)}. \tag{73}$$

For the second term on the right hand side of (71), we observe that $(II)=\mathbf{Q}u_\alpha-\mathbf{Q}u^*$ and applied (67) to obtain that

$$\big\|(II)\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}\le\frac{\alpha CK_f\mathbf{M}_{\beta_0}\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\left\|u_\alpha-u^*\right\|_{\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))}. \tag{74}$$

As for the last term on the right hand side of (71), we use the similar techniques as in (61),

$$\big\|(III)(t)\big\|_{L^r(\Omega)}\le\frac{3\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)t^{-\alpha}\left[\sum_{j=1}^\infty\lambda_j^{\frac{N(r-2)}{2r}+\frac{2\mu-2}{p}-2}\Bigg(\int_\Omega F(u^*(x,s))e_j(x)dx\Bigg)^2\right]^{1/2}. \tag{75}$$

Notice that

$$\begin{aligned}&\sum_{j=1}^\infty\lambda_j^{\frac{N(r-2)}{2r}+\frac{2\mu-2}{p}-2}\Bigg(\int_\Omega F(u^*(x,s))e_j(x)dx\Bigg)^2 \\ &=\sum_{j=1}^\infty\lambda_j^{\frac{N(r-2)}{2r}+\frac{2\mu-2}{p}-2}\lambda_j^{\frac{N(m-2)}{2m}}\Bigg(\int_\Omega F(u^*(x,s))e_j(x)dx\Bigg)^2 \\ &\le\Bigg(\sum_{j=1}^\infty\lambda_j^{\frac{N}{m}-\frac{N}{r}+\frac{2\mu-2}{p}-2}\Bigg)\big\|F(u^*)\big\|_{L^\infty(0,T;\mathbb{X}^{\frac{N(m-2)}{4m}}(\Omega))} \\ &\le\Bigg(\sum_{j=1}^\infty\lambda_j^{\frac{N}{m}-\frac{N}{r}+\frac{2\mu-2}{p}-2}\Bigg)\big\|F(u^*)\big\|_{L^\infty(0,T;L^m(\Omega))},\end{aligned} \tag{76}$$

where we have used the Sobolev embedding $L^m(\Omega)\hookrightarrow\mathbb{X}^{\frac{Nm-2N}{4m}}(\Omega)$ for $1<m<2$. Since the condition $\frac{2}{m}-\frac{2}{r}+1<\frac{4}{pN}+\frac{4}{N}$ holds, we multiply both sides of this condition by $\frac{pN}{4}$ to deduce $p+\frac{pN}{2r}+1-\frac{pN}{2m}-\frac{pN}{4}>0$. By choosing μ such that $0<\mu<p+\frac{pN}{2r}+1-\frac{pN}{2m}-\frac{pN}{4}$, it is easy to verify that

$$\frac{N}{m}+\frac{2\mu}{p}<\frac{N}{m}+\frac{2}{p}\Big(p+\frac{pN}{2r}+1-\frac{pN}{2m}-\frac{pN}{4}\Big)=\frac{N}{r}+\frac{2}{p}+2-\frac{N}{2}. \tag{77}$$

From the constraint (77), we deduce $\frac{N}{m}-\frac{N}{r}+\frac{2\mu-2}{p}-2<0$. Thus, we arrive at the following observation

$$\sum_{j=1}^{\infty}\lambda_j^{\frac{N}{m}-\frac{N}{r}+\frac{2\mu-2}{p}-2}=\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{\frac{N}{r}+\frac{2}{p}+2-\frac{N}{m}-\frac{2\mu}{p}}}\le\frac{1}{C}\sum_{j=1}^{\infty}\frac{1}{j^{\left(\frac{N}{r}+\frac{2}{p}+2-\frac{N}{m}-\frac{2\mu}{p}\right)\frac{2}{N}}},$$

where we note that $\lambda_j\ge Cj^{2/N}$. Due to (77), it is easy to verify that the condition is true $\left(\frac{N}{r}+\frac{2}{p}+2-\frac{N}{m}-\frac{2\mu}{p}\right)\frac{2}{N}>1$ holds true. Therefore, the infinite sum $\sum_{j=1}^{\infty}\frac{1}{j^{\left(\frac{N}{r}+\frac{2}{p}+2-\frac{N}{m}-\frac{2\mu}{p}\right)\frac{2}{N}}}$ is convergent and it holds that the series on the right hand side of (76), $\sum_{j=1}^{\infty}\lambda_j^{\frac{N}{m}-\frac{N}{r}+\frac{2\mu-2}{p}-2}$, is convergent. In addition, the globally Lipschitz property of F as in (17) gives us

$$\big\|F(u^*)\big\|_{L^\infty(0,T;L^m(\Omega))}\le K_f\big\|u^*\big\|_{L^\infty(0,T;L^r(\Omega))}. \tag{78}$$

It follows from (75) that

$$\big\|(III)\big\|_{\mathbf{Z}_{b,\beta}((0,T];L^r(\Omega))}\lesssim\frac{3K_f\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)T^{b-\alpha}\big\|u^*\big\|_{L^\infty(0,T;L^r(\Omega))}. \tag{79}$$

Combining (73), (74) and (79), we deduce that

$$\begin{aligned}\big\|u_\alpha-u^*\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}&\le\big\|(I)\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}+\big\|(II)\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}+\big\|(III)\big\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}\\&\le 3\overline{C}(\mu_0,p,T)\left(\Gamma(1-\alpha)\right)^{-1}\widetilde{M}_\alpha(T,p)T^{b-\alpha}\big\|u_0\big\|_{\mathbb{X}^{\frac{N(r-2)}{4r}+\frac{\mu_0-1}{p}}(\Omega)}\\&\quad+\frac{\alpha CK_f\mathbf{M}_{\beta_0}\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\|u_\alpha-u^*\|_{\mathbf{Z}_{b,\beta_0}((0,T];L^r(\Omega))}\\&\quad+\frac{3K_f\widetilde{C}(\mu,p,T)}{\Gamma(1-\alpha)}\widetilde{M}_\alpha(T,p)T^{b-\alpha}\big\|u^*\big\|_{L^\infty(0,T;L^r(\Omega))}.\end{aligned} \tag{80}$$

Since $\frac{\alpha CK_f\mathbf{M}_{\beta_0}\Gamma(1-\frac{N}{2}(\frac{1}{m}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{m}-\frac{1}{r}))}\le\frac{1}{2}$, we obtain the desired result (18).

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