Dynamics of stochastic nonlocal reaction-diffusion equations driven by multiplicative noise

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Abstract This paper deals with fractional stochastic nonlocal partial differential equations driven by multiplicative noise. We first prove the existence and uniqueness of solution to this kind of equations with white noise by applying the Galerkin method. Then, the existence and uniqueness of tempered pullback random attractor for the equation are ensured in an appropriate Hilbert space. When the fractional nonlocal partial differential equations are driven by colored noise, which indeed are approximations of the previous ones, we show the convergence of solutions of Wong-Zakai approximations and the upper semicontinuity of random attractors of the approximate random system as $\delta \to 0$.

Keywords: Fractional stochastic nonlocal PDEs, Multiplicative noise, Random attractors, Colored noise, Upper semicontinuity, Wong-Zakai approximations.
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1 Introduction

In this paper, we mainly consider the following stochastic fractional nonlocal reaction-diffusion equation driven by multiplicative white noise,

$$\begin{cases} \frac{\partial u}{\partial t} + a(l(u))(-\Delta)^{\gamma} u = f(u) + h(t) + \alpha u \circ \frac{dW}{dt}, & \text{in } \mathcal{O} \times (\tau, \infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x), & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $(-\Delta)^{\gamma}$, $\gamma \in (0,1)$, stands for the fractional Laplacian operator, $\mathcal{O}$ is a smooth bounded domain of $\mathbb{R}^n$, $\tau \in \mathbb{R}$, α is a positive constant, $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{O}))$. The symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration. Throughout this paper, the function $a \in C(\mathbb{R}; \mathbb{R}^+)$ and there exist two positive constants m and M, such that

$$m \le a(s) \le M, \qquad \forall s \in \mathbb{R}. \tag{1.2}$$

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Moreover, the function $f \in C(\mathbb{R})$ and there exist positive constants C_f, κ, β_1 and $\beta_2 > 1$, such that for some $p \geq 2$,

$$(f(s) - f(r))(s - r) \leq C_f (s - r)^2, \qquad \forall s, r \in \mathbb{R}. \tag{1.3}$$

$$-\kappa - \beta_1 |s|^p \leq f(s)s \leq \kappa - \beta_2 |s|^p, \qquad \forall s \in \mathbb{R}. \tag{1.4}$$

From (1.4), we deduce there exists a constant $\beta_3 > 1$, such that

$$|f(s)| \leq \beta_3 (|s|^{p-1} + 1), \qquad \forall s \in \mathbb{R}. \tag{1.5}$$

The identification $l(u)$ in (1.1) is in fact (l, u), however, we keep the usual notation in the existing previous literature $l(u)$ instead of (l, u) for the operator l acting on u. At last, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ satisfying the usual condition, that is, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all $\mathbb{P}$-null sets. W of (1.1) is a two-sided cylindrical Wiener process in a Hilbert space defined on this complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

The operator $(-\Delta)^\gamma$ with $\gamma \in (0, 1)$ denotes the fractional Laplacian, which has properties that $\lim_{\gamma\to 1^-}(-\Delta)^\gamma u = -\Delta u$ and $\lim_{\gamma\to 0^+}(-\Delta)^\gamma u = u$ (see [28, Proposition 4.4]). Recently in the literature, great attention has been devoted to the study of fractional partial differential equations, not only for pure academic interest, but also for various applications in physics, finance, probability and materials science, see, e.g., [1, 19, 24, 25] and the references therein. In fact, there are different definitions about fractional Laplacian operator [28]. Two well-known and widely studied fractional Laplacian operators are integral one (which reduces to the classical fractional Laplacian, see, for example, [6, 8, 30, 31] and the references therein) and the spectral one (sometimes called the local, fractional Laplacian; see, for example, [7, 9, 32, 33] and the references therein), respectively. Note that these two fractional operators are different, indeed, the spectral operator depends on the domain $\mathcal{O}$ considered (since its eigenfunctions and eigenvalues depend on $\mathcal{O}$), while the integral one $(-\Delta)^\gamma$ evaluated at some point is independent of the domain in which the equation is set, for more details, see [31] and the references therein.

In our paper, we focus on the integral fractional format (see Section 2) since the solutions of (1.1) with integral operator are closely related to the solutions of the equation defined on the entire domain $\mathbb{R}^n$ (see [37]). In general, defining this operator based on the spectral decomposition of the Dirichlet Laplacian is also frequently used, as adopted in [29].

On the one hand, in the real world, the different stochastic perturbations originate from many natural sources. Sometimes, they cannot be ignored and we need to incorporate them into the corresponding deterministic models, in this way, the stochastic differential equations are produced. In more recent decades, the random attractors for stochastic equations related to the standard Laplacian have been extensively studies in the literature, see, e.g., [2, 4, 5, 10, 11, 12, 13, 17, 18, 20, 23, 27, 41] and the references therein. However, as far as the authors are aware, the attractors of the fractional stochastic PDEs are not well studied, until recently B. X. Wang and his collaborators have been intensively working on a class of non-autonomous fractional PDEs, see [34, 35, 36, 37, 38, 39, 40]. On the other hand, since 1990, Chipot et al. [14, 15, 16] studied the behavior of a population of bacteria with nonlocal term in a container and extended the nonlocal effects from a constant a into a general nonlocal operator $a(l(\cdot))$ which is adopted in our model. Motivated by these previous works, it is interesting to study the dynamics of (1.1).

To prove the existence and uniqueness of problem (1.1), one advantage is fractional Laplacian operator $(-\Delta)^\gamma$, $\gamma \in (0,1)$, shares the same property of classical Laplacian operator, namely, $H^\gamma(\mathcal{O})$ is compactly embedded in $L^2(\mathcal{O})$ for a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ (e.g., [28]), which allows us to use the parallel ideas as classical reaction-diffusion equations to analyse the dynamics of problem (1.1). The difficulty appearing in our problem is the "double" nonlocal terms: one is the well-known fractional Laplacian $(-\Delta)^\gamma$ (see (2.1)), that cannot provide us enough dissipativity when directly doing energy estimates; $a(l(\cdot))$ is another nonlocal term accompanying fractional Laplacian operator. However, an a priori estimates shows the regularity of our solutions could arrive at $L^2(\tau, \tau+T; H^\gamma(\mathbb{R}^n))$ that is a subspace of $L^2(\tau, \tau+T; L^2(\mathbb{R}^n))$. Therefore, a Nemytskii operator including nonlocal term $a(l(\cdot))$ to problem (2.11) is introduced so that problem (1.1) can be solved smoothly.

This paper is structured as follows. The next section is devoted to introduce the definitions of fractional Laplacian operator and Ornstein-Uhlenbeck process, set the fractional stochastic nonlocal problem into a random one in a proper way. In Section 3, we mainly prove the existence and uniqueness of solution to fractional stochastic nonlocal PDEs (2.4) using the Galerkin method, thereupon a continuous cocycle is defined based on this solution operator. Section 4 is fully dedicated to prove the existence and uniqueness of tempered pullback random attractor for our problem in an appropriate Hilbert space. In Section 5, we discuss the Wong-Zakai approximated equation of our model, construct another cocycle and study the dynamics of this approximated equation with multiplicative colored noise in comparison to the dynamics established in the previous sections. Finally, the last section deals with the convergence of solutions and attractors of problem (2.4) as $\delta \to 0$.

2 Preliminaries

In this section, we will recall the concept of integral fractional Laplacian operator and introduce the definition of solutions of the stochastic nonlocal reaction-diffusion equations with fractional Laplacian (1.1).

2.1 Fractional setting

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^n$. For any fixed $0 < \gamma < 1$, for every $u \in \mathcal{S}$, the fractional Laplacian operator $(-\Delta)^\gamma$ at the point x is defined by,

$$(-\Delta)^\gamma u(x) = -\frac{1}{2}C(n,\gamma)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2\gamma}}dy, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where $C(n,\gamma)$ is a positive constant given by

$$C(n,\gamma) = \frac{\gamma 4^{\gamma \Gamma(\frac{n+2\gamma}{2})}}{\pi^{\frac{n}{2}}\Gamma(1-\gamma)}, \tag{2.2}$$

for more details on the integral fractional Laplacian operators, see [28] and the references therein. It can be also defined using Fourier transform by

$$\mathcal{F}((-\Delta)^\gamma u)(\xi) = |\xi|^{2\gamma}(\mathcal{F}u), \qquad \xi \in \mathbb{R}^n,$$

where $\mathcal{F}$ is the Fourier transform defined by

$$(\mathcal{F}u)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx, \qquad u \in \mathcal{S}.$$

For any real $0 < \gamma < 1$, the fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^n) := H^\gamma(\mathbb{R}^n)$ is defined by:

$$H^\gamma(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy < \infty\right\},$$

endowed with the norm

$$\|u\|_{H^\gamma(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy\right)^{\frac{1}{2}}.$$

From now on, we denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}^n)$ for some $p \geq 2$. Especially, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. Moreover, the Gagliardo semi-norm of $H^\gamma(\mathbb{R}^n)$ denoted by $\|\cdot\|_{\dot{H}^\gamma(\mathbb{R}^n)}$ is written as:

$$\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy, \quad u \in H^\gamma(\mathbb{R}^n).$$

Thus, $\|u\|^2_{H^\gamma(\mathbb{R}^n)} = \|u\|^2 + \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}$ for all $u \in H^\gamma(\mathbb{R}^n)$. Note that $H^\gamma(\mathbb{R}^n)$ is a Hilbert space with inner product

$$(u,v)_{H^\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2\gamma}}dxdy, \qquad \forall u,v \in H^\gamma(\mathbb{R}^n).$$

By [28], the norm $\|u\|_{H^\gamma(\mathbb{R}^n)}$ is equivalent to $\left(\|u\|^2 + \|(-\Delta)^{\frac{\gamma}{2}}u\|^2\right)^{\frac{1}{2}}$ for $u \in H^\gamma(\mathbb{R}^n)$. More precisely, we have

$$\|u\|^2_{H^\gamma(\mathbb{R}^n)} = \|u\|^2 + \frac{2}{C(n,\gamma)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2, \qquad \forall u \in H^\gamma(\mathbb{R}^n). \tag{2.3}$$

We will relate the integral fractional Laplacian with solutions of problem (1.1). Since $(-\Delta)^\gamma$ of (2.1) is obvious a nonlocal operator, we here interpret the homogeneous Dirichlet boundary (1.1) as $u=0$ on $\mathbb{R}^n\backslash\mathcal{O}$ instead of $u=0$ only on $\partial\mathcal{O}$ (consistent with the nonlocal character of $(-\Delta)^\gamma$), then problem (1.1) becomes

$$\begin{cases} \frac{\partial u}{\partial t}+a(l(u))(-\Delta)^\gamma u=f(u)+h(t)+\alpha u\circ\frac{dW}{dt}, & x\in\mathcal{O},\ t>\tau,\\ u=0, & x\in\mathbb{R}^n\backslash\mathcal{O},\ t>\tau,\\ u(x,\tau)=u_\tau(x), & x\in\mathcal{O}. \end{cases} \tag{2.4}$$

Based on this interpretation, we define two spaces $V=\{u\in H^\gamma(\mathbb{R}^n):u=0\ \ \text{a.e. on } \mathbb{R}^n\backslash\mathcal{O}\}$ and $H=\{u\in L^2(\mathbb{R}^n):u=0\ \ \text{a.e. on } \mathbb{R}^n\backslash\mathcal{O}\}$, their dual spaces are V^* and H, respectively. Furthermore, let $b:V\times V\to\mathbb{R}$ be a bilinear form given by, for $v_1,\ v_2\in V$,

$$b(v_1,v_2)=\mu(v_1,v_2)+\frac{1}{2}C(n,\gamma)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v_1(x)-v_1(y))(v_2(x)-v_2(y))}{|x-y|^{n+2\gamma}}dxdy, \tag{2.5}$$

where $C(n,\gamma)$ is the same as in (2.2) and μ is also a constant given later. For convenience, we associate an operator $A:V\to V^*$ with b such that

$$<A(v_1),v_2>_{(V^*,V)}=b(v_1,v_2),\qquad \text{for all } v_1,v_2\in V, \tag{2.6}$$

where $<\cdot,\cdot>_{(V^*,V)}$ is the duality paring of V^* and V.

In order to prove our results in the fractional framework, the properties of operator A introduced in [37] are considered: On the one hand, since A is injective and surjective, the inverse $A^{-1}:V^*\to V$ is well-defined; On the other hand, two facts that $H=H^*\subset V^*$ and the embedding $V\hookrightarrow H$ is compact (see, e.g., [28]) yield that $A^{-1}:H\to V\subset H$ is a symmetric compact operator. Then, by means of the Hilbert-Schmidt theorem, A has a family of eigenfunctions $\{e_j\}_{j=1}^\infty$ which forms an orthonormal basis of H. Moreover, if λ_j denotes the eigenvalue of operator A corresponding to e_j, i.e.,

$$Ae_j=\lambda_j e_j,\quad j=1,2,\cdots,$$

then λ_j satisfies

$$0<\mu<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to\infty\quad\text{as}\quad j\to\infty.$$

The following lemma is crucial to deal with the nonlinearity f in (2.4) with arbitrary growth order.

Lemma 2.1 *[37, Lemma 2.1] Suppose* $0<\gamma<1$, $2\le p<\infty$ *and* $r>\frac{n}{4\gamma}(1-\frac{2}{p})$. *Then* $D(A^r)$ *is continuously embedded into* $L^p(\mathcal{O})$.

2.2 Ornstein-Uhlenbeck process

The standard probability space $(\Omega,\mathcal{F},\mathbb{P})$ presented in previous section is used throughout this paper, where $\Omega=\{\omega\in C(\mathbb{R};\mathbb{R}):\omega(0)=0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega,\mathcal{F})$. For any given $t\in\mathbb{R}$, we identify $W(t,\omega):=\omega(t)$ and define the time shift $\theta_t:\Omega\to\Omega$ by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\qquad \omega\in\Omega,$$

then $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system. Furthermore, let $z:\Omega\to\mathbb{R}$ be a random variable given by $z(\omega):=-\int_{-\infty}^{0}e^{\tau}\omega(\tau)d\tau$ for $\omega\in\Omega$, then $z(t,\omega):=z(\theta_t\omega)$ is the unique stationary solution of the following stochastic equation:

$$dz=-zdt+dW$$

with initial value z_0 (for more details, see [3, 42] and the references therein). In addition, it follows from [3] that there exists a θ_t-invariant set of full measure (still denoted by Ω), such that the random variable $z(\theta_t\omega)$ satisfies the following properties: for all $\omega\in\Omega$,

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0\qquad\text{and}\qquad\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds=0. \tag{2.7}$$

2.3 Setting of the problem

For convenience, we fix a number $0<\mu<\min\left\{\frac{\beta_1}{m},\frac{\beta_2}{M},C(n,\gamma)\right\}$ and define the following Nemytskii operator $F:L^2(\mathbb{R}^n)\to L^q(\mathbb{R}^n)$ by

$$F(u)(x):=F(u(x))=a(l(u))\mu u(x)+f(u(x)), \tag{2.8}$$

where $q\in(1,2]$ is the conjugate number of p. By (1.4) and the Young inequality, we have:
Case 1. $p=q=2$

$$-\kappa-(\beta_1-m\mu)|u(x)|^2\le F(u(x))u(x)\le\kappa-(\beta_2-M\mu)|u(x)|^2.$$

Case 2. $p>2$

$$\begin{aligned}-\left(\frac{p-2}{p}|m\mu|^{\frac{p}{p-2}}+\kappa\right)-\left(\beta_1+\frac{2}{p}\right)|u(x)|^p&\le F(u(x))u(x)\\&\le\left(\kappa+\frac{p-2}{p}|M\mu|^{\frac{p}{p-2}}\right)-\left(\beta_2-\frac{2}{p}\right)|u(x)|^p.\end{aligned}$$

In conclusion, since $\beta_2>1$, there exist positive constants α_1, α_2 and κ_1, such that for all $x\in\mathbb{R}^n$, some $p\ge2$, F satisfies

$$-\kappa_1-\alpha_1|u(x)|^p\le F(u(x))u(x)\le\kappa_1-\alpha_2|u(x)|^p. \tag{2.9}$$

From (2.9), we can deduce that there exists $\alpha_3 > 0$, such that

$$|F(u(x))| \leq \alpha_3(|u(x)|^{p-1} + 1). \tag{2.10}$$

Then problem (2.4) can be written as

$$\begin{cases} \frac{\partial u}{\partial t} + a(l(u))(-\Delta)^\gamma u + a(l(u))\mu u = F(u) + h(t) + \alpha u \circ \frac{dW}{dt}, & x \in \mathcal{O},\ t > \tau, \\ u = 0, & x \in \mathbb{R}^n \backslash \mathcal{O},\ t > \tau, \\ u(x,\tau) = u_\tau(x), & x \in \mathcal{O}. \end{cases} \tag{2.11}$$

To study the dynamics of problem (2.11), we first transform the stochastic fractional nonlocal differential equation (2.11) into a pathwise deterministic one by doing a change of variable. Given $\tau \in \mathbb{R}$, $t \geq \tau$, $\omega \in \Omega$ and $u_\tau \in H$, if $u = u(t,\tau,\omega,u_\tau)$ is a solution of (2.11), then we define a new variable $v = v(t,\tau,\omega,v_\tau)$ by

$$v(t,\tau,\omega,v_\tau) = e^{-\alpha z(\theta_t\omega)} u(t,\tau,\omega,u_\tau) \quad \text{with} \quad v_\tau = e^{-\alpha z(\theta_\tau\omega)} u_\tau. \tag{2.12}$$

In terms of (2.11)-(2.12), for $t > \tau$, we obtain

$$\frac{\partial v}{\partial t} + a(l(e^{\alpha z(\theta_t\omega)}v))\left((-\Delta)^\gamma v + \mu v\right) = \alpha z(\theta_t\omega)v + e^{-\alpha z(\theta_t\omega)}F(e^{\alpha z(\theta_t\omega)}v) + e^{-\alpha z(\theta_t\omega)}h(t), \quad x \in \mathcal{O}, \tag{2.13}$$

with boundary condition

$$v(t,x) = 0, \qquad x \in \mathbb{R}^n \backslash \mathcal{O} \ \text{ and } \ t > \tau, \tag{2.14}$$

and initial condition

$$v(x,\tau) = v_\tau(x) = e^{-\alpha z(\theta_\tau\omega)} u_\tau, \quad x \in \mathcal{O}. \tag{2.15}$$

3 Main result

First of all, the notion of weak solutions to problem (2.13)-(2.15) is stated before proving the existence and uniqueness of solution to this equation. Then, the solution of (2.11) is obtained via the transform (2.12).

Definition 3.1 *Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in H$, a weak solution to* (2.13)-(2.15) *is a continuous function $v(\cdot,\tau,\omega,v_\tau) : [\tau,\infty) \to H$ with $v(\tau,\tau,\omega,v_\tau) = v_\tau$, fulfilling*

$$v \in L^2_{loc}(\tau,\infty;V) \cap L^p_{loc}(\tau,\infty;L^p(\mathbb{R}^n)), \qquad \frac{dv}{dt} \in L^2_{loc}(\tau,\infty;V^*) + L^q_{loc}(\tau,\infty;L^q(\mathbb{R}^n)).$$

Moreover, for every $\zeta \in V \cap L^p(\mathbb{R}^n)$, v satisfies

$$\begin{aligned} &\frac{d}{dt}(v,\zeta) + a(l(e^{\alpha z(\theta_t\omega)}v))\left(\frac{1}{2}C(n,\gamma)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v(x)-v(y))(\zeta(x)-\zeta(y))}{|x-y|^{n+2\gamma}}dxdy + \mu(v,\zeta)\right) \\ &\quad = \alpha z(\theta_t\omega)(v,\zeta) + e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha z(\theta_t\omega)}v)\zeta(x)dx + e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} h(t)\zeta(x)dx, \end{aligned} \tag{3.1}$$

where the previous equation must be understood in the sense of distribution on (τ,∞).

We are now ready to achieve our goal by using Galerkin method, to this end, the basis $\{e_j\}_{j=1}^{\infty}$ will be used based on the properties of operator A and the results of Lemma 2.1.

Theorem 3.2 *Assume $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz and satisfies* (1.2), *$f \in C(\mathbb{R})$ fulfills* (1.3)-(1.4), *which implies $F : L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ satisfies* (2.9)-(2.10). *In addition, $h \in L^2_{loc}(\mathbb{R}; H)$ and $l \in \mathcal{L}(L^2(\mathcal{O});\mathbb{R})$. Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and each initial datum $v_\tau \in H$, there exists a unique weak solution $v(t,\tau,\omega,v_\tau)$ to problem* (2.13)-(2.15) *in the sense of Definition 3.1. Also, this solution is $(\mathcal{F},\mathcal{B}(H))$-measurable in ω and continuous in initial data v_τ in H. Moreover, the following energy equality holds:*

$$\begin{aligned}&\frac{d}{dt}\|v(t)\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v))\left(C(n,\gamma)\|v\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + 2\mu\|v\|^2\right)\\&= 2\alpha z(\theta_t\omega)\|v\|^2 + 2e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha z(\theta_t\omega)}v)vdx + 2e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} h(t)vdx,\end{aligned} \tag{3.2}$$

for almost all $t \geq \tau$.

Proof. We will first construct a sequence of approximate solutions by Galerkin method, then derive uniform estimates using energy equality and take the limit of these approximate solutions, finally this limit is proved to be the desired solution.

Step 1: Approximate solutions. Given $n \in \mathbb{N}$, let $X_n := \text{span}[e_1, e_2, \cdots, e_n]$ and $P_n : H \to X_n$ be the projection given by

$$P_n v := v_n(t,\tau,\omega,v_\tau) = \sum_{j=1}^{n}(v,e_j)e_j, \qquad \forall v \in H.$$

Meanwhile, the projection operator P_n can be extended to V^* and $(L^p(\mathbb{R}^n))^*$ by

$$P_n\varphi = \sum_{j=1}^{n}(\varphi,e_j)e_j := \sum_{j=1}^{n}(\varphi(e_j))e_j, \qquad \forall \varphi \in V^* \text{ or } \varphi \in (L^p(\mathbb{R}^n))^*.$$

Now, for each $v_n \in X_n$ and $t \geq \tau$, we consider the following system:

$$\frac{dv_n}{dt} + a(l(e^{\alpha z(\theta_t\omega)}v_n))P_nA(v_n) = \alpha z(\theta_t\omega)v_n + e^{-\alpha z(\theta_t\omega)}P_nF(e^{\alpha z(\theta_t\omega)}v_n) + e^{-\alpha z(\theta_t\omega)}P_nh(t), \tag{3.3}$$

with initial condition

$$v_n(\tau) = P_nv_\tau. \tag{3.4}$$

With the help of (1.2) and (2.9)-(2.10), we know that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in H$, system (3.3)-(3.4) has a maximal solution $v_n(\cdot,\tau,\omega,v_n(\tau)) \in C([\tau,\tau+T);X_n)$ for some $T > 0$, that is also measurable in $\omega \in \Omega$. In fact, we can show this $T = \infty$, which implies the solutions are globally defined.

Step 2. Uniform estimates. Since $v_n(x,t)=0$ for all $x\notin\mathcal{O}$, it follows from (3.3) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_n\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v_n))b(v_n,v_n) &= \alpha z(\theta_t\omega)\|v_n\|^2 \\ &+ e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha z(\theta_t\omega)}v_n)v_n dx + e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} h(t)v_n dx.\end{aligned} \tag{3.5}$$

By (2.9), we obtain

$$\begin{aligned}e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha z(\theta_t\omega)}v_n)v_n dx &\le e^{-2\alpha z(\theta_t\omega)}\int_{\mathcal{O}}\left(\kappa_1-\alpha_2|e^{\alpha z(\theta_t\omega)}v_n|^p\right)dx \\ &\le \kappa_1|\mathcal{O}|e^{-2\alpha z(\theta_t\omega)} - \alpha_2 e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathcal{O}}|v_n|^p dx.\end{aligned} \tag{3.6}$$

Since $h\in L^2_{loc}(\mathbb{R};H)$, by the Young inequality, we have

$$e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} h(t)v_n dx \le \frac{1}{2}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2 + \frac{1}{2}\|v_n\|^2. \tag{3.7}$$

Making use of (1.2), (2.5) and (3.5)-(3.7), we derive

$$\begin{aligned}&\frac{d}{dt}\|v_n\|^2 + m\left(C(n,\gamma)\|v_n\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + 2\mu\|v_n\|^2\right) + 2\alpha_2 e^{(p-2)\alpha z(\theta_t\omega)}\|v_n\|_p^p \\ &\le (2\alpha z(\theta_t\omega)+1)\|v_n\|^2 + 2\kappa_1|\mathcal{O}|e^{-2\alpha z(\theta_t\omega)} + e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2.\end{aligned} \tag{3.8}$$

Given $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$, by the continuity of $z(\theta_t\omega)$ in t, it follows from (3.8) that

$$\{v_n\}_{n=1}^\infty \text{ is bounded in } L^\infty(\tau,\tau+T;H)\cap L^2(\tau,\tau+T;V)\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n)), \tag{3.9}$$

which along with (2.6) implies

$$\{A(v_n)\}_{n=1}^\infty \text{ is bounded in } L^2(\tau,\tau+T;V^*). \tag{3.10}$$

Furthermore, by (2.10), we derive there exists a constant $C:=C(\omega,\alpha,T,p)$, such that

$$\begin{aligned}\int_\tau^{\tau+T}\int_{\mathcal{O}}|F(e^{\alpha z(\theta_t\omega)}v_n)|^q dxdt &\le \alpha_3^q\int_\tau^{\tau+T}\int_{\mathcal{O}}\left(|e^{\alpha z(\theta_t\omega)}v_n|^{p-1}+1\right)^q dxdt \\ &\le 2^{q-1}\alpha_3^q\int_\tau^{\tau+T}\int_{\mathcal{O}}\left(|e^{\alpha z(\theta_t\omega)}v_n|^p+1\right)dxdt \\ &\le 2^{q-1}\alpha_3^q C\|v_n\|^p_{L^p(\tau,\tau+T;L^p(\mathbb{R}^n))} + 2^{q-1}\alpha_3^q T|\mathcal{O}|.\end{aligned}$$

Consequently, by (3.9), we have

$$\{F(e^{\alpha z(\theta_t\omega)}v_n)\}_{n=1}^\infty \text{ is bounded in } L^q(\tau,\tau+T;L^q(\mathbb{R}^n)). \tag{3.11}$$

Subsequently, it follows from (3.3) and (3.10)-(3.11) that

$$\left\{\frac{dv_n}{dt}\right\}_{n=1}^\infty \text{ is bounded in } L^2(\tau,\tau+T;V^*)+L^q(\tau,\tau+T;L^q(\mathbb{R}^n)). \tag{3.12}$$

Next, we consider the limiting process of (3.3)-(3.4) as $n \to \infty$.

Step 3. Existence of solutions. By (3.9)-(3.12), from compactness arguments and the Aubin-Lions lemma, there exist $\tilde{v} \in H$, $v \in L^\infty(\tau, \tau+T; H) \cap L^2(\tau, \tau+T; V) \cap L^p(\tau, \tau+T; L^p(\mathbb{R}^n))$, $\chi \in L^q(\tau, \tau+T; L^q(\mathbb{R}^n))$ and a subsequence of $\{v_n\}_{n=1}^\infty$ (which is denoted the same) such that

$$\begin{cases} v_n \to v \quad \text{weak-star in} \;\; L^\infty(\tau, \tau+T; H); \\ v_n \to v \quad \text{weakly in} \;\; L^2(\tau, \tau+T; V); \\ v_n \to v \quad \text{weakly in} \;\; L^p(\tau, \tau+T; L^p(\mathbb{R}^n)); \\ F(e^{\alpha z(\theta_t \omega)} v_n) \to \chi \quad \text{weakly in} \;\; L^q(\tau, \tau+T; L^q(\mathbb{R}^n)); \\ \frac{dv_n}{dt} \to \frac{dv}{dt} \quad \text{weakly in} \;\; L^q(\tau, \tau+T; L^q(\mathbb{R}^n)) + L^2(\tau, \tau+T; V^*); \\ v_n(\tau+T, \tau, \omega) \to \tilde{v} \quad \text{weakly in} \;\; H. \end{cases} \tag{3.13}$$

Actually, $\{\frac{dv_n}{dt}\}_{n=1}^\infty$ is bounded in $L^q(\tau, \tau+T; (V \cap L^p(\mathbb{R}^n))^*)$ since q, the conjugate number of p, belongs to $(1, 2]$. Notice that, the embedding $V \hookrightarrow H$ is compact and $H \hookrightarrow (V \cap L^p(\mathbb{R}^n))^*$ is continuous, together with (3.9) and (3.12), we infer from [26] that there exists a subsequence of $\{v_n\}_{n=1}^\infty$ (which is relabeled the same) such that

$$v_n \to v \quad \text{strongly in} \quad L^2(\tau, \tau+T; H). \tag{3.14}$$

Eq. (3.14) implies there exists a further subsequence of $\{v_n\}_{n=1}^\infty$ (relabeled the same again) such that

$$v_n \to v \quad \text{for almost every} \;\; (t, x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.15}$$

Now we will check $\chi = F(e^{\alpha z(\theta_t \omega)} v)$. Indeed, by (3.15) and the continuity of F, we deduce

$$F(e^{\alpha z(\theta_t \omega)} v_n) \to F(e^{\alpha z(\theta_t \omega)} v) \quad \text{for almost every} \;\; (t, x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.16}$$

By applying [28, Lemma 1.3], we infer from (3.13) and (3.16) that

$$F(e^{\alpha z(\theta_t \omega)} v_n) \to F(e^{\alpha z(\theta_t \omega)} v) \quad \text{weakly in} \;\; L^q(\tau, \tau+T; L^q(\mathbb{R}^n)). \tag{3.17}$$

Combing (3.11) with (3.17), we have

$$\chi = F(e^{\alpha z(\theta_t \omega)} v). \tag{3.18}$$

Furthermore, by means of the fact that $a \in C(\mathbb{R}; \mathbb{R}^+)$ and (3.15), we obtain

$$a(l(e^{\alpha z(\theta_t \omega)} v_n)) = a((l, e^{\alpha z(\theta_t \omega)} v_n)) \xrightarrow{n \to \infty} a((l, e^{\alpha z(\theta_t \omega)} v)) = a(l(e^{\alpha z(\theta_t \omega)} v)). \tag{3.19}$$

Next, we prove the energy equality (3.2) holds. Consider $j \in \mathbb{N}$ and $\phi \in C_0^\infty(\tau, \tau+T)$, multiplying by ϕe_j in (3.3), integrating the resulting identity and taking the limit when $n \to \infty$, with the help of (3.13)-(3.19), we derive

$$
\begin{aligned}
&-\int_\tau^{\tau+T}(v,e_j)\phi' dt + a(l(e^{\alpha z(\theta_t\omega)}v))\int_\tau^{\tau+T} b(v,e_j)\phi dt\\
&= \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,e_j)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v),e_j)_{(L^q,L^p)}\phi dt\\
&\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t),e_j)\phi dt.
\end{aligned}
\tag{3.20}
$$

Based on the result of Lemma 2.1, we pick up a positive integer k satisfying $k > \frac{n}{4\gamma}(1-\frac{2}{p})$, such that $D(A^k)$ is continuously embedded into $L^p(\mathcal{O})$. Since equality (3.20) holds for every e_j ($j\in\mathbb{N}$) and the linear combinations of $\{e_j, j\in\mathbb{N}\}$ are dense in $D(A^k)$, by a limiting process, for every $\vartheta\in D(A^k)$, we find

$$
\begin{aligned}
&-\int_\tau^{\tau+T}(v,\vartheta)\phi' dt + a(l(e^{\alpha z(\theta_t\omega)}v))\int_\tau^{\tau+T} b(v,\vartheta)\phi dt\\
&= \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,\vartheta)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v),\vartheta)_{(L^q,L^p)}\phi dt\\
&\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t),\vartheta)\phi dt,
\end{aligned}
\tag{3.21}
$$

which implies the following equality

$$
\frac{dv}{dt} = -a(l(e^{\alpha z(\theta_t\omega)}v))Av + \alpha z(\theta_t\omega)v + e^{-\alpha z(\theta_t\omega)}F(e^{\alpha z(\theta_t\omega)}v) + e^{-\alpha z(\theta_t\omega)}h(t), \tag{3.22}
$$

holds in $L^q(\tau,\tau+T;(D(A^k))^*)$. Indeed, (3.13) shows the identity (3.22) actually holds in $L^2(\tau,\tau+T;V^*)+L^q(\tau,\tau+T;L^q(\mathbb{R}^n))$ and $L^q(\tau,\tau+T;(V\cap L^p(\mathbb{R}^n))^*)$. This means, for all $\vartheta\in V\cap L^p(\mathbb{R}^n)$,

$$
\begin{aligned}
\frac{d}{dt}(v,\vartheta) + a(l(e^{\alpha z(\theta_t\omega)}v))b(v,\vartheta) &= \alpha z(\theta_t\omega)(v,\vartheta)\\
&\quad + e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v),\vartheta)_{(L^q,L^p)} + e^{-\alpha z(\theta_t\omega)}(h(t),\vartheta),
\end{aligned}
\tag{3.23}
$$

holds true in the sense of distribution on $(\tau,\tau+T)$. Thus, the required (3.1) is proved.

Step 4. Continuity of solutions v. We now show $v:[\tau,\infty)\to H$ is continuous. Since $v\in L^2(\tau,\tau+T;V)\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n))$ and $\frac{dv}{dt}\in L^2(\tau,\tau+T;V^*)+L^q(\tau,\tau+T;L^q(\mathbb{R}^n))$ (cf. (3.13)), we deduce from [28] that $v\in C([\tau,\tau+T];H)$ and

$$
\frac{1}{2}\frac{d}{dt}\|v\|^2 = \left(\frac{dv}{dt}, v\right)_{(V^*+L^q, V\cap L^p)} \quad \text{for almost every} \quad t\in(\tau,\tau+T). \tag{3.24}
$$

Subsequently, by (3.23)-(3.24), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v))b(v,v) &= \alpha z(\theta_t\omega)(v,v)\\
&\quad + e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v),v)_{(L^q,L^p)} + e^{-\alpha z(\theta_t\omega)}(h(t),v),
\end{aligned}
\tag{3.25}
$$

which implies the energy equation (3.2).

We next consider two endpoints of (2.13)-(2.15), prove $v(\tau) = v_\tau$ and $v(\tau + T) = \tilde{v}$. For any $\phi \in C^1([\tau, \tau + T])$, multiplying by ϕe_j in (3.3) and taking the limit as $n \to \infty$, it follows from (3.2), (3.13) and (3.18) that

$$\begin{aligned} &(\tilde{v}, e_j)\phi(\tau + T) - (v_\tau, e_j)\phi(\tau) \\ &= \int_\tau^{\tau+T} (v, e_j)\phi' dt - a(l(e^{\alpha z(\theta_t\omega)}v)) \int_\tau^{\tau+T} b(v, e_j)\phi dt + \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, e_j)\phi dt \\ &\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v), e_j)_{(L^q, L^p)}\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t), e_j)\phi dt. \end{aligned} \tag{3.26}$$

By (3.23), we also have

$$\begin{aligned} &(v(\tau + T), e_j)\phi(\tau + T) - (v(\tau), e_j)\phi(\tau) \\ &= \int_\tau^{\tau+T} (v, e_j)\phi' dt - a(l(e^{\alpha z(\theta_t\omega)}v)) \int_\tau^{\tau+T} b(v, e_j)\phi dt + \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, e_j)\phi dt \\ &\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v), e_j)_{(L^q, L^p)}\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t), e_j)\phi dt. \end{aligned}$$

Comparing (3.26) with the above expression, we obtain

$$(v(\tau + T), e_j)\phi(\tau + T) - (v(\tau), e_j)\phi(\tau) = (\tilde{v}, e_j)\phi(\tau + T) - (v_\tau, e_j)\phi(\tau).$$

As $\phi \in C^1([\tau, \tau + T])$ is arbitrary and $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of H, we obtain from the above equality that

$$v(\tau) = v_\tau \quad \text{and} \quad v(\tau + T) = \tilde{v} \ \text{ in } \ H. \tag{3.27}$$

Moreover, making use of (3.13) and (3.27), we arrive at

$$v_n(\tau + T, \tau, \omega, v_n(\tau)) \to v(\tau + T, \tau, \omega, v_\tau) \quad \text{weakly in} \ \ H. \tag{3.28}$$

By similar arguments as (3.28), we obtain the weak convergence of v_n for all $t \geq \tau$, that is,

$$v_n(t, \tau, \omega, v_n(\tau)) \to v(t, \tau, \omega, v_\tau) \quad \text{weakly in} \ \ H, \ \ \forall t \geq \tau. \tag{3.29}$$

At last, (3.23) and (3.27) indicate v is a solution of problem (2.13)-(2.15) in the sense of Definition 3.1.

Step 5. Uniqueness of solution. Suppose v_1 and v_2 are two solutions of problem (2.13)-(2.15), let $w = v_1 - v_2$, then we have

$$\begin{aligned} &\frac{dw}{dt} + a(l(e^{\alpha z(\theta_t\omega)}v_1))\left((-\Delta)^\gamma v_1 + \mu v_1\right) - a(l(e^{\alpha z(\theta_t\omega)}v_2))\left((-\Delta)^\gamma v_2 + \mu v_2\right) \\ &\quad = \alpha z(\theta_t\omega)w + e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v_1) - F(e^{\alpha z(\theta_t\omega)}v_2)). \end{aligned}$$

By energy equation we infer that,

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v_1))\,((-\Delta)^\gamma v_1 + \mu v_1, w) - a(l(e^{\alpha z(\theta_t\omega)}v_1))\,((-\Delta)^\gamma v_2 + \mu v_2, w)$$

$$= a(l(e^{\alpha z(\theta_t\omega)}v_2))\,((-\Delta)^\gamma v_2 + \mu v_2, w) - a(l(e^{\alpha z(\theta_t\omega)}v_1))\,((-\Delta)^\gamma v_2 + \mu v_2, w)$$

$$+ \alpha z(\theta_t\omega)\|w\|^2 + e^{-\alpha z(\theta_t\omega)}(F(e^{\alpha z(\theta_t\omega)}v_1) - F(e^{\alpha z(\theta_t\omega)}v_2), w).$$

Since $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz, we denote this Lipschitz constant by L_a, together with (1.2)-(1.3), (2.8) and the Young inequality, we obtain

$$\frac{d}{dt}\|w\|^2 + 2mb(w,w) \le 2L_a\|l\|e^{\alpha z(\theta_t\omega)}\|w\|((-\Delta)^\gamma v_2 + \mu v_2, w) + 2|\alpha z(\theta_t\omega)|\|w\|^2$$

$$+ 2M\mu\|w\|^2 + 2\mu L_a\|l\|e^{\alpha z(\theta_t\omega)}\|v_1\|\|v_2\|\|w\| + 2C_f\|w\|^2$$

$$\le \Big(2C(n,\gamma)(\mu+1)L_a\|l\|e^{\alpha z(\theta_t\omega)}\|v_2\|_{H^\gamma(\mathcal{O})} + 2\alpha|z(\theta_t\omega)| + 2C_f$$

$$+ 2M\mu + \mu^2L_a^2\|l\|^2e^{2\alpha z(\theta_t\omega)}\|v_1\|^2\|v_2\|^2\Big)\|w\|^2,$$

which along with the Gronwall Lemma implies the uniqueness and continuity of solution in initial data in H.

Step 6. Measurability of solutions in ω. Note that (3.29) and the uniqueness of solution indicate for any $t \ge \tau$ and $\omega \in \Omega$, the whole sequence $v_n(t,\tau,\omega) \to v(t,\tau,\omega)$ weakly in H. Since $v_n(t,\tau,\omega)$ is measurable in $\omega \in \Omega$, we know the weak limit $v(t,\tau,\omega)$ is also measurable in ω. The proof of this theorem is finished. □

The following lemma establishes the compactness of the solution operators in H so that the existence of random attractors for system (2.13)-(2.15) can be proved later.

Lemma 3.3 *Suppose the conditions of Theorem 3.2 hold true. Then for any $\tau \in \mathbb{R}$, $t > \tau$ and $\omega \in \Omega$, the solution operator $v(t,\tau,\omega,\cdot) : H \to H$ of problem* (2.13)-(2.15) *is compact. That is, for every bounded sequence $\{v_{0,n}\}_{n=1}^\infty$ in H, the sequence $\{v(t,\tau,\omega,v_{0,n})\}_{n=1}^\infty$ has a convergent subsequence in H.*

Proof. Choose $T > 0$ such that $t \in [\tau, \tau+T]$. Thanks to (3.14), there exists $\tilde{v} \in L^2(\tau,\tau+T;H)$ and a subsequence of $\{v_{0,n}\}_{n=1}^\infty$ (which is relabeled the same), such that

$$v(\cdot,\tau,\omega,v_{0,n}) \to \tilde{v} \quad \text{in} \quad L^2(\tau,\tau+T;H).$$

We deduce from the above that there exists a set I of measure zero with $I \subset [\tau,\tau+T]$ and a subsequence of $\{v_{0,n}\}_{n=1}^\infty$ (which is relabeled the same), such that

$$v(r,\tau,\omega,v_{0,n}) \to \tilde{v}(r) \ \text{ in } \ H, \ \ \forall r \in [\tau,\tau+T] \setminus I. \tag{3.30}$$

Since $t > \tau$, $(\tau, t) \subset [\tau, \tau + T]$ and I has measure zero, by (3.30), we find that there exists $r_0 \in (\tau, t) \setminus I$, such that

$$v(r_0, \tau, \omega, v_{0,n}) \to \tilde{v}(r_0) \qquad \text{in} \ \ H. \tag{3.31}$$

By means of (3.31) and the continuity of solutions in initial data, we obtain

$$v(t, \tau, \omega, v_{0,n}) = v(t, r_0, \omega, v(r_0, \tau, \omega, v_{0,n})) \to v(t, r_0, \omega, \tilde{v}(r_0)) \quad \text{in} \ \ H,$$

which concludes the proof. □

4 Existence of random attractors to problem (2.11)

This section is devoted to uniform estimates of solutions for the fractional stochastic nonlocal reaction-diffusion equations (2.11). Before doing this, we first construct a cocycle generated by (2.11). Then, by means of the solution v of (2.13)-(2.15) and the transform (2.12), we obtain a solution u of stochastic fractional nonlocal PDE (2.11), which is

$$u(t, \tau, \omega, u_\tau) = e^{\alpha z(\theta_t \omega)} v(t, \tau, \omega, v_\tau),$$

with $u_\tau = e^{\alpha z(\theta_\tau \omega)} v_\tau$. We deduce from Theorem 3.2 that $u(t, \tau, \omega, u_\tau)$ is both continuous in $t \in [\tau, \infty)$ and in $u_\tau \in H$. Moreover, $u(t, \tau, \cdot, u_\tau) : \Omega \to H$ is measurable. Thus, we can define a continuous cocycle in H for problem (2.11). Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ be a mapping given by, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = e^{\alpha z(\theta_t \omega)} v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau), \tag{4.1}$$

where $v_\tau = e^{-\alpha z(\omega)} u_\tau$.

In the sequel, we will present the existence and upper semicontinuity of tempered random attractors for Φ in H. To this end, it is necessary to introduce some notation and assumptions. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of H, we say D is tempered if for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{ct} \|D(\tau + t, \theta_t \omega)\| = 0,$$

where the norm $\|D\|$ of a set D in H is given by $\|D\| = \sup_{u \in D} \|u\|$. From now on, we use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets in H:

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \ D \text{ is tempered in } H\}.$$

Furthermore, we assume that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{0} e^{m\mu s} \|h(s + \tau)\|^2 ds < \infty. \tag{4.2}$$

Also, let h be tempered in the following sense: for every $c > 0$,

$$\lim_{r\to\infty} e^{-cr}\int_{-\infty}^{0} e^{m\mu s}\|h(s-r)\|^2 ds = 0. \tag{4.3}$$

It is easy to see these two conditions do not require h is bounded in H when $t \to \infty$.

Lemma 4.1 *Suppose the conditions of Theorem 3.2 and (4.2) hold. Then, for every $\alpha_0 > 0$, $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\sigma,\alpha_0) > 0$ such that for all $t \geq T$ and $0 < \alpha \leq \alpha_0$, the solution v of problem (2.13)-(2.15) satisfies*

$$\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{1}{2}m\mu\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_V^2 ds\\
&\quad + 2\alpha_2\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr} e^{(p-2)\alpha z(\theta_s\omega)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_p^p ds\\
&\leq 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} e^{\frac{1}{4}m\mu(\tau-\sigma)} e^{-2\alpha\int_{\sigma-\tau}^{0} z(\theta_r\omega)dr}\\
&\quad + \frac{4}{m\mu}\int_{-\infty}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds,
\end{aligned}$$

where $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$.

Proof. We mainly use the energy equation (3.2) to complete this proof. First of all, for the last term of (3.2), by the Young inequality, we have

$$2e^{-\alpha z(\theta_t\omega)}(h(t),v_n) \leq \frac{m\mu}{4}\|v_n\|^2 + \frac{4}{m\mu}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2. \tag{4.4}$$

Next, with the help of (2.9), we obtain

$$\begin{aligned}
2e^{-\alpha z(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha z(\theta_t\omega)}v)v dx &\leq 2e^{-2\alpha z(\theta_t\omega)}\int_{\mathcal{O}}\left(\kappa_1 - \alpha_2|e^{\alpha z(\theta_t\omega)}v|^p\right)dx\\
&\leq 2\kappa_1|\mathcal{O}|e^{-2\alpha z(\theta_t\omega)} - 2\alpha_2 e^{(p-2)\alpha z(\theta_t\omega)}\|v\|_p^p.
\end{aligned} \tag{4.5}$$

It follows from (1.2), (3.2) and (4.4)-(4.5) that

$$\begin{aligned}
&\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2 + \left(\frac{5}{4}m\mu - 2\alpha z(\theta_t\omega)\right)\|v\|^2 + mC(n,\gamma)\|v\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}\\
&\quad + \frac{1}{2}m\mu\|v\|^2 + 2\alpha_2 e^{(p-2)\alpha z(\theta_t\omega)}\|v\|_p^p \leq 2\kappa_1|\mathcal{O}|e^{-2\alpha z(\theta_t\omega)} + \frac{4}{m\mu}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2.
\end{aligned} \tag{4.6}$$

Multiplying (4.6) by $e^{\frac{5}{4}m\mu t-2\alpha\int_0^t z(\theta_r\omega)dr}$ and then integrating the inequality on $(\tau-t,\sigma)$ with

$\sigma > \tau - t$, we derive

$$\begin{aligned}
&\|v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2 + \frac{1}{2}m\mu\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_r\omega)dr}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2 ds\\
&\quad + mC(n,\gamma)\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_r\omega)dr}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{\dot H^\gamma(\mathbb{R}^n)}ds\\
&\quad + 2\alpha_2\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_r\omega)dr}e^{(p-2)\alpha z(\theta_s\omega)}\|v(s,\tau-t,\omega,v_{\tau-t})\|_p^p ds\\
&\le e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t} z(\theta_r\omega)dr}\|v_{\tau-t}\|^2 + 2\kappa_1|\mathcal{O}|\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}ds\\
&\quad + \frac{4}{m\mu}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s)\|^2 ds.
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above inequality, we have

$$\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{1}{2}m\mu\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds\\
&\quad + mC(n,\gamma)\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot H^\gamma(\mathbb{R}^n)}ds\\
&\quad + 2\alpha_2\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{(p-2)\alpha z(\theta_{s-\tau}\omega)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_p^p ds\\
&\le e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t} z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t}\|^2 + 2\kappa_1|\mathcal{O}|\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}ds\\
&\quad + \frac{4}{m\mu}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}m\mu(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|h(s)\|^2 ds.
\end{aligned}$$

After doing change of variables, we arrive at

$$\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\\
&\quad + \frac{1}{2}m\mu\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^s z(\theta_r\omega)dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds\\
&\quad + mC(n,\gamma)\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^s z(\theta_r\omega)dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot H^\gamma(\mathbb{R}^n)}ds\\
&\quad + 2\alpha_2\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^s z(\theta_r\omega)dr}e^{(p-2)\alpha z(\theta_s\omega)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_p^p ds\\
&\le e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_{\sigma-\tau}^{-t} z(\theta_r\omega)dr}\|v_{\tau-t}\|^2 + 2\kappa_1|\mathcal{O}|\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}ds\\
&\quad + \frac{4}{m\mu}\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds.
\end{aligned} \tag{4.7}$$

On the one hand, the first term of the right hand side of (4.7) is equivalent to,

$$
\begin{aligned}
&e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_{\sigma-\tau}^{-t}z(\theta_r\omega)dr}\|v_{\tau-t}\|^2\\
&=e^{\frac{5}{4}m\mu(\tau-\sigma)}e^{-\frac{5}{4}m\mu t}e^{-2\alpha\int_{\sigma-\tau}^{0}z(\theta_r\omega)dr}e^{-2\alpha\int_{0}^{-t}z(\theta_r\omega)dr}e^{-2\alpha z(\theta_{-t}\omega)}e^{2\alpha z(\theta_{-t}\omega)}\|v_{\tau-t}\|^2.
\end{aligned}
$$

Due to $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and $0<\alpha\le\alpha_0$, we obtain from the above equality that

$$
\begin{aligned}
&e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_{\sigma-\tau}^{-t}z(\theta_r\omega)dr}\|v_{\tau-t}\|^2\\
&\le e^{\frac{5}{4}m\mu(\tau-\sigma)}e^{-\frac{5}{4}m\mu t}e^{2\alpha_0\left|\int_{\sigma-\tau}^{0}z(\theta_r\omega)dr\right|}e^{2\alpha_0\left|\int_{0}^{-t}z(\theta_r\omega)dr\right|}e^{2\alpha_0|z(\theta_{-\tau}\omega)|}\|D(\tau-t,\theta_{-t}\omega)\|^2.
\end{aligned}
\tag{4.8}
$$

On the other hand, (2.7) indicates that there exists $T_1=T_1(\omega,\alpha_0,m,\mu)$, such that for all $t\ge T_1$,

$$
|z(\theta_{-t}\omega)|\le\frac{1}{16}\frac{\mu m}{\alpha_0}t\qquad\text{and}\qquad\left|\int_0^{-t}z(\theta_r\omega)dr\right|\le\frac{1}{16}\frac{\mu m}{\alpha_0}t.\tag{4.9}
$$

Hence, it follows from (4.8)-(4.9) that for all $t\ge T_1$,

$$
e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_{\sigma-\tau}^{-t}z(\theta_r\omega)dr}\|v_{\tau-t}\|^2\le e^{\frac{5}{4}m\mu(\tau-t)}e^{2\alpha_0\left|\int_{\sigma-\tau}^{0}z(\theta_r\omega)dr\right|}e^{-\mu mt}\|D(\tau-t,\theta_{-t}\omega)\|^2.\tag{4.10}
$$

In fact, $e^{-\mu mt}\|D(\tau-t,\theta_{-t}\omega)\|\to0$ as $t\to\infty$ since $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is tempered. Therefore, by (4.10), we find that there exists $T_2=T_2(\tau,\omega,\sigma,D,\alpha_0,m,\mu)\ge T_1$, such that for all $t\ge T_2$,

$$
e^{\frac{5}{4}m\mu(\tau-t-\sigma)-2\alpha\int_{\sigma-\tau}^{-t}z(\theta_r\omega)dr}\|v_{\tau-t}\|^2\le1.\tag{4.11}
$$

For the last term on the right hand side of (4.7), we have

$$
\begin{aligned}
&\frac{4}{m\mu}\int_{-t}^{\sigma-\tau}e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s}z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&=\frac{4}{m\mu}e^{\frac{5}{4}m\mu(\tau-\sigma)}e^{-2\alpha\int_{\sigma-\tau}^{0}z(\theta_r\omega)dr}\int_{-t}^{\sigma-\tau}e^{\frac{5}{4}m\mu s}e^{-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds.
\end{aligned}
\tag{4.12}
$$

(4.9) implies for all $t\ge T_1$,

$$
\begin{aligned}
&\int_{-t}^{-T_1}e^{\frac{5}{4}m\mu s}e^{-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&\le\int_{-t}^{-T_1}e^{\frac{5}{4}m\mu s}e^{-2\alpha_0\left|\int_0^s z(\theta_r\omega)dr\right|}e^{-2\alpha_0|z(\theta_s\omega)|}\|h(s+\tau)\|^2ds\\
&\le\int_{-t}^{-T_1}e^{\mu ms}\|h(s+\tau)\|^2ds\\
&\le\int_{-\infty}^{-T_1}e^{\mu ms}\|h(s+\tau)\|^2ds,
\end{aligned}
\tag{4.13}
$$

the last integral of (4.13) is convergent thanks to (4.2). It follows from (4.12)-(4.13) that

$$\begin{aligned}&\frac{4}{m\mu}\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds\\ &\le \frac{4}{m\mu}\int_{-\infty}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds,\end{aligned} \tag{4.14}$$

where the integral is convergent because of (4.13). By the same arguments as for (4.14), we deduce that

$$\begin{aligned}&2\kappa_1|\mathcal{O}|\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} ds\\ &\le 2\kappa_1|\mathcal{O}|e^{\frac{5}{4}m\mu(\tau-\sigma)}e^{-2\alpha\int_{\sigma-\tau}^{0} z(\theta_r\omega)dr}\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu s}e^{-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}ds\\ &\le 2\kappa_1|\mathcal{O}|e^{\frac{5}{4}m\mu(\tau-\sigma)}e^{-2\alpha\int_{\sigma-\tau}^{0} z(\theta_r\omega)dr}\int_{-\infty}^{\sigma-\tau} e^{m\mu s}ds \le \frac{2\kappa_1|\mathcal{O}|}{m\mu}e^{\frac{1}{4}m\mu(\tau-\sigma)}e^{-2\alpha\int_{\sigma-\tau}^{0} z(\theta_r\omega)dr}.\end{aligned} \tag{4.15}$$

By (4.7) and (4.14)-(4.15), for all $t \ge T_2$, we obtain

$$\begin{aligned}&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{1}{2}m\mu\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds\\ &\quad + mC(n,\gamma)\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} ds\\ &\quad + 2\alpha_2\int_{-t}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr}e^{(p-2)\alpha z(\theta_s\omega)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_p^p ds\\ &\le 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu}e^{\frac{1}{4}m\mu(\tau-\sigma)}e^{-2\alpha\int_{\sigma-\tau}^{0} z(\theta_r\omega)dr}\\ &\quad + \frac{4}{m\mu}\int_{-\infty}^{\sigma-\tau} e^{\frac{5}{4}m\mu(s-\sigma+\tau)-2\alpha\int_{\sigma-\tau}^{s} z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds.\end{aligned}$$

Thus, the proof of this lemma is complete. □

According to the results of Lemma 4.1, we claim the solution operator of problem (2.13)-(2.15) has a random pullback absorbing set.

Lemma 4.2 *Suppose the conditions of Theorem 3.2 and* (4.3) *hold. For each* $\alpha > 0$, *let* $B_\alpha = \{B_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ *be a random set given by*

$$B_\alpha(\tau,\omega) = \{v \in H : \|v\|^2 \le R_\alpha(\tau,\omega)\},$$

where $R_\alpha(\tau,\omega)$ *is defined by*

$$R_\alpha(\tau,\omega) = 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^{0} e^{\frac{5}{4}m\mu s-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2 ds. \tag{4.16}$$

Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\alpha,m,\mu) > 0$ such that the solution v of (2.13)-(2.15) *with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ satisfies, for all $t \geq T$,*

$$v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \in B_\alpha(\tau,\omega). \tag{4.17}$$

In addition, the random variable R_α defined by (4.16) *is tempered, i.e., for any $c > 0$,*

$$\lim_{t\to\infty} e^{-ct} R_\alpha(\tau-t,\theta_{-t}\omega) = 0. \tag{4.18}$$

Proof. (4.16) is obtained directly from Lemma 4.1 with $\sigma = \tau$. It remains to check (4.18), by (4.16) we infer that,

$$\begin{aligned} R_\alpha(\tau-t,\theta_{-t}\omega) &= 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s z(\theta_{r-t}\omega)dr} e^{-2\alpha z(\theta_{s-t}\omega)} \|h(s+\tau-t)\|^2 ds \\ &= 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_{-t}^0 z(\theta_r\omega)dr - 2\alpha\int_0^{s-t} z(\theta_r\omega)dr} e^{-2\alpha z(\theta_{s-t}\omega)} \|h(s+\tau-t)\|^2 ds. \end{aligned} \tag{4.19}$$

Given $c > 0$, let $c_1 = \min\left\{\frac{1}{16}\frac{m\mu}{\alpha}, \frac{1}{12}\frac{c}{\alpha}\right\}$, by (2.11), we find there exists $T_0 = T_0(c_1) > 0$, such that for all $t \geq T_0$,

$$|z(\theta_{-t}\omega)| \leq c_1 t \qquad \text{and} \qquad \left|\int_{-t}^0 z(\theta_r\omega)dr\right| \leq c_1 t. \tag{4.20}$$

Notice that if $t \geq T_0$ and $s \leq 0$, then we have $t - s \geq t \geq T_0$. Therefore, for all $t \geq T_0$ and $s \leq 0$, we obtain from (4.20) that

$$\left|-2\alpha\int_{-t}^0 z(\theta_r\omega)dr\right| \leq 2\alpha c_1 t, \quad \left|-2\alpha\int_0^{s-t} z(\theta_r\omega)dr\right| \leq 2\alpha c_1(t-s), \quad |-2\alpha z(\theta_{s-t}\omega)| \leq 2\alpha c_1(t-s). \tag{4.21}$$

Thus, by (4.19) and (4.21), we deduce, for all $t \geq T_0$,

$$R_\alpha(\tau-t,\theta_{-t}\omega) \leq 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}e^{\frac{1}{2}ct}\int_{-\infty}^0 e^{m\mu s}\|h(s+\tau-t)\|^2 ds.$$

Therefore, by (4.3), we derive that

$$\begin{aligned} \limsup_{t\to\infty} e^{-ct} R_\alpha(\tau-t,\theta_{-t}\omega) &\leq \limsup_{t\to\infty} \frac{4}{m\mu} e^{-\frac{1}{2}ct}\int_{-\infty}^0 e^{m\mu s}\|h(s+\tau-t)\|^2 ds \\ &= \limsup_{r\to\infty} \frac{4}{m\mu} e^{-\frac{1}{2}cr} e^{-\frac{1}{2}c\tau}\int_{-\infty}^0 e^{m\mu s}\|h(s-r)\|^2 ds = 0, \end{aligned}$$

which means R_α is tempered, as desired in (4.18). The proof of this lemma is complete. □

Next, we prove the asymptotic compactness of solution operator for problem (2.13)-(2.15).

Lemma 4.3 *Assume the conditions of Lemma 4.2 hold. Then, the sequence $v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})$ of solutions of* (2.13)-(2.15) *has a convergent subsequence in H as $t_n \to \infty$ when $e^{\alpha z(\theta_{-t_n}\omega)} v_{0,n} \in D(\tau-t_n,\theta_{-t_n}\omega)$ with $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Proof. By Lemma 4.1 with $\sigma = \tau - 1$, we find there exists $T = T(\tau, \omega, D, \alpha, m, \mu) > 0$ and $C = C(\tau, \omega, \alpha, m) > 0$, such that for all $t \geq T$,

$$\|v(\tau - 1, \tau - t, \theta_{-\tau}\omega, v_0)\| \leq C, \tag{4.22}$$

for any $v_0 \in H$ with $e^{\alpha z(\theta_{-t}\omega)}v_0 \in D(\tau - t, \theta_{-t}\omega)$. Since $t_n \to \infty$, there is $N = N(\tau, \omega, D, \alpha, m, \mu) \geq 1$, such that for all $t_n \geq T$ with $n \geq N$, we have

$$\|v(\tau - 1, \tau - t, \theta_{-\tau}\omega, v_{0,n})\| \leq C. \tag{4.23}$$

In addition,

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) = v(\tau, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})),$$

together with (4.23) and Lemma 3.3, we find the sequence $v(\tau, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}))$ is precompact in H. That is, the sequence $v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})$ has a convergent subsequence in H, which concludes the proof. $\square$

In what follows, we will present the existence of tempered random pullback attractors for the fractional nonlocal stochastic equation (2.11). Based on the uniform estimates on the solutions of (2.13)-(2.15), we first prove the existence of tempered pullback absorbing set and the asymptotic compactness of (2.11).

Lemma 4.4 *Assume the conditions of Lemma 4.2 are true. Given $\alpha > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let*

$$K_\alpha(\tau, \omega) = \{u \in H : \|u\|^2 \leq e^{2\alpha z(\omega)} R_\alpha(\tau, \omega)\},$$

where $R_\alpha(\tau, \omega)$ is the same as in (4.16). Then $K_\alpha = \{K_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a closed measurable tempered pullback absorbing set of the cocycle Φ.

Proof. We first show that K_α absorbs every member D of $\mathcal{D}$. By (2.12), we have

$$u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau - t}) = e^{\alpha z(\omega)} v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau - t}) \quad \text{with} \quad u_{\tau - t} = e^{\alpha z(\theta_{-t}\omega)} v_{\tau - t}. \tag{4.24}$$

If $u_{\tau - t} \in D(\tau - t, \theta_{-t}\omega)$, then by (4.24), we obtain $e^{\alpha z(\theta_{-t}\omega)} v_{\tau - t} \in D(\tau - t, \theta_{-t}\omega)$, which together with Lemma 4.2 implies that there exists $T = T(\tau, \omega, D, \alpha, m, \mu) > 0$, such that

$$v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau - t}) \in B_\alpha(\tau, \omega), \tag{4.25}$$

where $B_\alpha(\tau, \omega)$ is the same as in (4.17). It follows from (4.24)-(4.25) and (4.16)-(4.17) that for all $t \geq T$,

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau - t})\|^2 \leq e^{2\alpha z(\omega)} R_\alpha(\tau, \omega). \tag{4.26}$$

On the other hand, by (4.1), we have

$$\Phi(t, \tau - t, \theta_{-t}\omega, u_\tau) = u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau - t}), \tag{4.27}$$

which along with (4.26) shows $\Phi(t,\tau-t,\theta_{-t}\omega,v_{\tau-t})\in K_\alpha(\tau,\omega)$ for all $t\ge T$, hence K_α absorbs all elements of $\mathcal{D}$. We now prove K_α is tempered, namely, $K_\alpha\in\mathcal{D}$. By (2.7) and (4.18), for any $c>0$, we obtain

$$\lim_{t\to\infty} e^{-ct}\|K_\alpha(\tau-t,\theta_{-t}\omega)\| = \lim_{t\to\infty} e^{-\frac{1}{2}ct+\alpha z(\theta_{-t}\omega)}\left(e^{-ct}R_\alpha(\tau-t,\theta_{-t}\omega)\right)^{\frac{1}{2}} = 0,$$

which implies $K_\alpha\in\mathcal{D}$. Note that $R_\alpha(\tau,\omega)$ is measurable in $\omega\in\Omega$, and so is $K_\alpha(\tau,\omega)$, which completes this proof. □

The $\mathcal{D}$-pullback asymptotic compactness of Φ is presented below.

Lemma 4.5 *Assume the conditions of Lemma 4.2 hold. Then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$, the sequence $\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,u_{0,n})$ has a convergent subsequence in H provided $t_n\to\infty$ and $u_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$.*

Proof. It follows from (4.24) and (4.27) that

$$\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,u_{0,n}) = u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) = e^{\alpha z(\omega)}v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}),\quad \forall n\in\mathbb{N}, \tag{4.28}$$

where $v_{0,n}=e^{-\alpha z(\theta_{-t_n}\omega)}u_{0,n}$. Thanks to $u_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$, we find that $e^{\alpha z(\theta_{-t_n}\omega)}v_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$, thus by Lemma 4.3 we infer that the sequence $v(\tau,\tau-t_n,\theta_{-t_n}\omega,v_{0,n})$ has a convergent subsequence in H, which along with (4.28) completes the proof. □

The main result of this section is the existence and uniqueness of tempered pullback attractor of Φ in H as stated below.

Theorem 4.6 *Assume $a\in C(\mathbb{R};\mathbb{R}^+)$ satisfies* (1.2)*, $f\in C(\mathbb{R})$ fulfills* (1.3)-(1.4)*, which implies $F:L^2(\mathbb{R}^n)\to L^q(\mathbb{R}^n)$ satisfies* (2.9)-(2.10)*. In addition, suppose $h\in L^2_{loc}(\mathbb{R};H)$, $l\in\mathcal{L}(L^2(\mathcal{O});\mathbb{R})$ and* (4.2)-(4.3) *hold. Then the cocycle Φ generated by* (2.11) *has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\alpha=\{\mathcal{A}_\alpha(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in H.*

Proof. The existence and uniqueness of the $\mathcal{D}$-pullback attractor $\mathcal{A}_\alpha$ follows from [34, 35] immediately based on lemmas 4.4 and 4.5. □

5 Existence of attractors to fractional random nonlocal PDEs driven by colored noise

In this section, we discuss the approximations of fractional stochastic nonlocal differential equation (1.1), namely, the following pathwise Wong-Zakai approximated equation,

$$\begin{cases} \frac{\partial u_\delta}{\partial t} + a(l(u_\delta))(-\Delta)^\gamma u_\delta = f(u_\delta)+h(t)+\alpha u_\delta\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty),\\ u_\delta = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty),\\ u_\delta(x,\tau)=u_{\delta,\tau}(x), & \text{in } \mathcal{O}, \end{cases} \tag{5.1}$$

where functions a and f fulfill the same conditions (1.2)-(1.5) as in the previous sections and $h \in L^2_{loc}(\mathbb{R}; H)$, ζ_δ is the colored noise with correlation time $\delta > 0$. To study (5.1), we proceed similarly as in Section 2: define an operator $F(u_\delta) : L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ as (2.8) satisfying (2.9)-(2.10) and consider the integral fractional format of Laplacian operator. More precisely, for $\gamma \in (0, 1)$, we investigate the problem

$$\begin{cases} \frac{\partial u_\delta}{\partial t} + a(l(u_\delta))(-\Delta)^\gamma u_\delta + a(l(u_\delta))\mu u_\delta = F(u_\delta) + h(t) + \alpha u_\delta \zeta_\delta(\theta_t\omega), & x \in \mathcal{O},\ t > \tau, \\ u_\delta = 0, & x \in \mathbb{R}^n \backslash \mathcal{O},\ t > \tau, \\ u_\delta(x, \tau) = u_{\delta,\tau}(x), & x \in \mathcal{O}. \end{cases} \tag{5.2}$$

To better understand the relations between solutions of (2.11) and (5.2), we define a new variable

$$v_\delta(t) = u_\delta(t)e^{-\alpha y_\delta(\theta_t\omega)}. \tag{5.3}$$

Recall that y_δ satisfies

$$\frac{dy_\delta}{dt} = -\eta y_\delta + \zeta_\delta(\theta_t\omega). \tag{5.4}$$

For almost all $\omega \in \Omega$, one special solution of (5.4) can be represented by

$$Y_\delta(t, \omega) = e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \zeta(\theta_s\omega) ds,$$

which, in fact, can be rewritten as $Y_\delta(t, \omega) = y_\delta(\theta_t\omega)$. Where $y_\delta : \Omega \to \mathbb{R}$ is a well-defined random variable given by $y_\delta(\omega) := \int_{-\infty}^{0} e^{\eta s}\zeta_\delta(\theta_s\omega)ds$ and has the following properties:

Lemma 5.1 *([21, Lemma 3.2]) Let y_δ be the random variable defined as above. Then the mapping*

$$(t, \omega) \to y_\delta(\theta_t\omega) = e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \zeta_\delta(\theta_s\omega) ds, \tag{5.5}$$

is a stationary solution of (5.4) with continuous trajectories. In addition, $\mathbb{E}(y_\delta) = 0$ and for every $\omega \in \Omega$,

$$\lim_{\delta \to 0} y_\delta(\theta_t\omega) = z(\theta_t\omega) \quad \textit{uniformly on } [\tau, \tau + T] \textit{ with } \tau \in \mathbb{R},\ T > 0; \tag{5.6}$$

$$\lim_{t \to \pm\infty} \frac{|y_\delta(\theta_t\omega)|}{|t|} = 0 \quad \textit{uniformly for } \ 0 < \delta \le \tilde{\eta}; \tag{5.7}$$

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t y_\delta(\theta_r\omega)dr = 0 \quad \textit{uniformly for } \ 0 < \delta \le \tilde{\eta}, \tag{5.8}$$

where $\tilde{\eta} = \min\{1, \frac{1}{2\eta}\}$ and $z(\theta_t\omega)$ is given in Subsection 2.2.

Remark 5.2 *In this manuscript, in order to simplify the computations, we take $\eta = 1$ in equation (5.4), then the results of Lemma 5.1 are true for $\eta = 1$.*

Thus, it follows from (5.2)-(5.4) that, for $t > \tau$,

$$\begin{aligned}\frac{\partial v_\delta}{\partial t} + a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta))(-\Delta)^\gamma v_\delta + a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta))\mu v_\delta = \alpha y_\delta(\theta_t\omega)v_\delta \\ + e^{-\alpha y_\delta(\theta_t\omega)}F(e^{\alpha y_\delta(\theta_t\omega)}v_\delta) + e^{-\alpha y_\delta(\theta_t\omega)}h(t), \qquad x \in \mathcal{O},\end{aligned} \tag{5.9}$$

with boundary condition

$$v_\delta(t,x) = 0, \qquad x \in \mathbb{R}^n \backslash \mathcal{O} \text{ and } t > \tau, \tag{5.10}$$

and initial condition

$$v_\delta(x,\tau) = v_{\delta,\tau} = e^{-\alpha y_\delta(\theta_\tau\omega)}u_{\delta,\tau}, \qquad x \in \mathcal{O}. \tag{5.11}$$

Since (5.9)-(5.11) can be viewed as a deterministic equation parameterized by $\omega \in \Omega$, by the same procedures as in Theorem 3.2, we can prove that under conditions (1.2) and (2.9)-(2.10), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_{\delta,\tau} \in H$, (5.9)-(5.11) has a unique solution,

$$v_\delta(\cdot,\tau,\omega,v_{\delta,\tau}) \in L^2_{loc}(\tau,\infty;V) \cap L^p_{loc}(\tau,\infty;L^p(\mathbb{R}^n)).$$

Moreover, this solution v_δ satisfies the following energy equation,

$$\begin{aligned}&\frac{d}{dt}\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|^2 + a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta))\left(C(n,\gamma)\|v_\delta\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + 2\mu\|v_\delta\|^2\right) \\ &= 2\alpha y_\delta(\theta_t\omega)\|v_\delta\|^2 + 2e^{-\alpha y_\delta(\theta_t\omega)}\int_{\mathcal{O}} F(e^{\alpha y_\delta(\theta_t\omega)}v_\delta)v_\delta dx + 2e^{-\alpha y_\delta(\theta_t\omega)}\int_{\mathcal{O}} h(t)v_\delta dx,\end{aligned} \tag{5.12}$$

for almost all $t \geq \tau$. At this point, thanks to the transform (5.11), there exists a unique solution $u_\delta(\cdot,\tau,\omega,u_{\delta,\tau}) \in L^2_{loc}(\tau,\infty;V) \cap L^p_{loc}(\tau,\infty;L^p(\mathbb{R}^n))$ to problem (5.2). In addition, this solution is $(\mathcal{F},\mathcal{B}(H))$-measurable in ω and behaves continuously in H with respect to the initial value. Next, we define a cocycle $\Xi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ such that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\delta,\tau} \in H$,

$$\Xi(t,\tau,\omega,u_{\delta,\tau}) = u_\delta(t+\tau,\tau,\theta_{-\tau}\omega,u_{\delta,\tau}) = e^{\alpha y_\delta(\theta_t\omega)}v_\delta(t+\tau,\tau,\theta_{-\tau}\omega,v_{\delta,\tau}) \tag{5.13}$$

with $v_{\delta,\tau} = e^{-\alpha y(\omega)}u_{\delta,\tau}$.

In the following results, we establish the existence of tempered absorbing sets for equation (5.2) in H as well as the pullback asymptotic compactness of solutions. Finally, we prove the existence of tempered random attractors for this equation.

Lemma 5.3 *Assume $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz and satisfies* (1.2), *$f \in C(\mathbb{R})$ fulfills* (1.3)-(1.4), *which implies $F : L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ satisfies* (2.9)-(2.10). *In addition, let $h \in L^2_{loc}(\mathbb{R};H)$, $l \in \mathcal{L}(L^2(\mathcal{O});\mathbb{R})$ and* (4.2)-(4.3) *hold. Then, given $\alpha > 0$, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the cocyle Ξ associated with equation* (5.2) *possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K^\delta_\alpha = \{K^\delta_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in H given by*

$$K^\delta_\alpha(\tau,\omega) = \{u_\delta \in H : \|u_\delta\|^2 \leq e^{2\alpha y_\delta(\omega)}R^\delta_\alpha(\tau,\omega)\}, \tag{5.14}$$

where $R^\delta_\alpha(\tau,\omega)$ *is defined by*

$$R^\delta_\alpha(\tau,\omega) = 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr} e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2 ds. \tag{5.15}$$

Moreover, for every $\alpha > 0$, $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\delta\to 0} R^\delta_\alpha(\tau,\omega) = R_\alpha(\tau,\omega), \tag{5.16}$$

where $R_\alpha(\tau,\omega)$ *is defined in* (4.16).

Proof. It follows from (1.2), (2.9), (5.12) and the Young inequality that,

$$\frac{d}{dt}\|v_\delta(t,\tau,\omega,v_\tau)\|^2 + \left(\frac{5}{4}m\mu - 2\alpha y_\delta(\theta_t\omega)\right)\|v_\delta\|^2 + mC(n,\gamma)\|v_\delta\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + \frac{1}{2}m\mu\|v_\delta\|^2$$
$$+ 2\alpha_2 e^{(p-2)\alpha y_\delta(\theta_t\omega)}\|v_\delta\|_p^p \le 2\kappa_1|\mathcal{O}|e^{-2\alpha y_\delta(\theta_t\omega)} + \frac{4}{m\mu}e^{-2\alpha y_\delta(\theta_t\omega)}\|h(t)\|^2.$$

Multiplying the above inequality by $e^{\frac{5}{4}m\mu t - 2\alpha\int_0^t y_\delta(\theta_r\omega)dr}$, integrating the inequality on $(\tau - t, \tau)$ with $t > 0$ and replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}
&\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 + \frac{1}{2}m\mu\int_{-t}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}\|v_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 ds\\
&\quad + mC(n,\gamma)\int_{-t}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}\|v_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}ds\\
&\quad + 2\alpha_2\int_{-t}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{(p-2)\alpha y_\delta(\theta_s\omega)}\|v_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|_p^p ds\\
&\le e^{-\frac{5}{4}m\mu t - 2\alpha\int_0^{-t} y_\delta(\theta_r\omega)dr}\|v_{\delta,\tau-t}\|^2 + 2\kappa_1|\mathcal{O}|\int_{-t}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}ds\\
&\quad + \frac{4}{m\mu}\int_{-t}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2 ds,
\end{aligned} \tag{5.17}$$

where $e^{\alpha y_\delta(\theta_{-t}\omega)}v_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. Together with (5.7)-(5.8), the same reasoning as (4.11)-(4.15) yields that

$$\begin{aligned}
&\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2\\
&\quad \le 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2 ds.
\end{aligned} \tag{5.18}$$

(2.7) and (4.2) imply (5.18) is well-defined. Furthermore, it follows from (5.3) and (5.18) that

$$\begin{aligned}
\Xi(t,\tau-t,\theta_{-t}\omega,v_{\delta,\tau-t}) &= u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\\
&= v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})e^{\alpha y_\delta(\omega)} \in K^\delta_\alpha(\tau,\omega),
\end{aligned} \tag{5.19}$$

with $u_{\delta,\tau-t} = v_{\delta,\tau-t}e^{\alpha y_\delta(\theta_{-t}\omega)}$. Now, by (5.19), along with (4.3) and (5.7), we can easily verify $K^\delta_\alpha(\tau,\omega)$ is tempered.

It remains to show (5.16). On the one hand, by (5.7)-(5.8), we find there exists $T_3 = T_3(\alpha,\omega,\mu,m) > 0$, such that for all $t \geq T_3$, we have

$$|y_\delta(\theta_t\omega)| \leq \frac{1}{16}\frac{m\mu}{\alpha}t \quad \text{and} \quad \left|\int_0^t y_\delta(\theta_r\omega)dr\right| \leq \frac{1}{16}\frac{m\mu}{\alpha}t. \tag{5.20}$$

On the other hand, notice that

$$\begin{aligned}
&\int_{-\infty}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&=\int_{-\infty}^{-T_3} e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&\quad+\int_{-T_3}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds.
\end{aligned} \tag{5.21}$$

We now deal with the first term on the right hand side of (5.21). By (5.20) and (4.2), we have

$$\int_{-\infty}^{-T_3} e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds \leq \int_{-\infty}^{-T_3} e^{m\mu s}\|h(s+\tau)\|^2ds < \infty.$$

Thus, (5.6) and the Lebesgue Dominated Theorem imply that,

$$\begin{aligned}
&\lim_{\delta\to0}\int_{-\infty}^{-T_3} e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&=\int_{-\infty}^{-T_3} e^{\frac{5}{4}m\mu s-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds.
\end{aligned} \tag{5.22}$$

The same reasoning as above shows

$$\begin{aligned}
&\lim_{\delta\to0}\int_{-T_3}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&=\int_{-T_3}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds.
\end{aligned} \tag{5.23}$$

Consequently, it follows from (5.21)-(5.23) that

$$\begin{aligned}
&\lim_{\delta\to0}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s y_\delta(\theta_r\omega)dr}e^{-2\alpha y_\delta(\theta_s\omega)}\|h(s+\tau)\|^2ds\\
&=\int_{-\infty}^0 e^{\frac{5}{4}m\mu s-2\alpha\int_0^s z(\theta_r\omega)dr}e^{-2\alpha z(\theta_s\omega)}\|h(s+\tau)\|^2ds.
\end{aligned} \tag{5.24}$$

By the definitions of $R_\alpha(\tau,\omega)$ (cf. (4.16)) and $R^\delta_\alpha(\tau,\omega)$ (cf. (5.15)), we derive (5.16) immediately. The proof of this lemma is complete. □

Theorem 5.4 *Under the conditions of Lemma 5.3, the cocycle* Ξ *of problem* (5.2) *possesses a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}^\delta_\alpha = \{\mathcal{A}^\delta_\alpha(\tau,\omega) : \tau\in\mathbb{R}, \omega\in\Omega\} \in \mathcal{D}$ *in* H.

Proof. In order to prove this result, we need to establish analogous results to lemmas 4.4 and 4.5 for the cocycle generated by (5.2). As the proof is similar as the one in Theorem 4.6, we prefer to omit it here. □

6 Upper semicontinuity of attractors

In this section, we consider the limiting behavior of random pullback attractors $\mathcal{A}_\alpha^\delta$ of problem (5.2) as $\delta \to 0$, for which the following condition is needed: there exists a constant $C'_f > 0$, such that

$$\left|\frac{df(s)}{ds}\right| \le C'_f \left(1 + |s|^{p-2}\right), \qquad \forall s \in \mathbb{R}. \tag{6.1}$$

Lemma 6.1 *Assume $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz and satisfies (1.2), $f \in C(\mathbb{R})$ fulfills (1.3)-(1.4) and (6.1), which implies $F : L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ satisfies (2.9)-(2.10). In addition, $h \in L^2_{loc}(\mathbb{R}; H)$ and $l \in \mathcal{L}(L^2(\mathcal{O});\mathbb{R})$. Let u and u_δ be the solutions of (2.11) and (5.2), respectively. Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0,1)$, there exist $\delta_0 = \delta_0(\tau,\omega,T,\varepsilon)$ and $c = c(\tau,\omega,T,\alpha,v_\delta,M,C_f,C'_f,\mu) > 0$, such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau,\tau+T]$,*

$$\|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\|^2 \le c\|u_{\delta,\tau} - u_\tau\|^2 + c\varepsilon\left(1 + \int_\tau^t \|h(s)\|^2 ds\right). \tag{6.2}$$

Proof. Before showing the details, we emphasize that in this proof, c is a positive constant which is different from line to line even in the same line. Let $\bar{v} = v_\delta - v$. By (5.9) and (2.13), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\bar{v}\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v))\left((-\Delta)^\gamma \bar{v} + \mu\bar{v}, \bar{v}\right) \\
&= \left|\left(a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta))((-\Delta)^\gamma v_\delta + \mu v_\delta) - a(l(e^{\alpha z(\theta_t\omega)}v))((-\Delta)^\gamma v_\delta + \mu v_\delta), \bar{v}\right)\right| \\
&\quad + \alpha(y_\delta(\theta_t\omega)v_\delta - z(\theta_t\omega)v, \bar{v}) + \int_{\mathcal{O}}\left(e^{-\alpha y_\delta(\theta_t\omega)}F(e^{\alpha y_\delta(\theta_t\omega)}v_\delta) - e^{-\alpha z(\theta_t\omega)}F(e^{\alpha z(\theta_t\omega)}v)\right)\bar{v}dx \\
&\quad + \left(e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right)(h(t), \bar{v}).
\end{aligned} \tag{6.3}$$

Since v_δ, $v \in C(\tau,\tau+T;H)$, there exists a bounded set $\mathcal{S} \in H$ such that, v_δ and v both belong to $\mathcal{S}$. Besides, taking into account that $l \in L^2(\mathcal{O})$, there exists a constant $R > 0$ such that $l(v_\delta) \in [-R,R]$ and $l(v) \in [-R,R]$. Then, by means of the locally Lipschitz continuity of function a, we obtain

$$\begin{aligned}
&\left|\left(a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta))((-\Delta)^\gamma v_\delta + \mu v_\delta) - a(l(e^{\alpha z(\theta_t\omega)}v))((-\Delta)^\gamma v_\delta + \mu v_\delta), \bar{v}\right)\right| \\
&\le \left|a(l(e^{\alpha y_\delta(\theta_t\omega)}v_\delta)) - a(l(e^{\alpha z(\theta_t\omega)}v))\right|((-\Delta)^\gamma v_\delta + \mu v_\delta, \bar{v}) \\
&\le L_a\|l\|\left(\|e^{\alpha y_\delta(\theta_t\omega)}v_\delta - e^{\alpha z(\theta_t\omega)}v_\delta\| + e^{\alpha z(\theta_t\omega)}\|\bar{v}\|\right)\left(\frac{C(n,\gamma)}{2}\|v_\delta\|_{\dot{H}^\gamma(\mathbb{R}^n)}\|\bar{v}\| + \mu\|v_\delta\|\|\bar{v}\|\right) \\
&= \frac{C(n,\gamma)}{2}L_a\|l\|\left|e^{\alpha y_\delta(\theta_t\omega)} - e^{\alpha z(\theta_t\omega)}\right|\|v_\delta\|\|v_\delta\|_{\dot{H}^\gamma(\mathbb{R}^n)}\|\bar{v}\| + L_a\|l\|\mu e^{\alpha z(\theta_t\omega)}\|v_\delta\|\|\bar{v}\|^2.
\end{aligned} \tag{6.4}$$

Observe that

$$
\begin{aligned}
&\int_{\mathcal{O}} \left(e^{-\alpha y_\delta(\theta_t\omega)} F(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - e^{-\alpha z(\theta_t\omega)} F(e^{\alpha z(\theta_t\omega)} v)\right) \bar{v} dx \\
&= \int_{\mathcal{O}} e^{-\alpha y_\delta(\theta_t\omega)} \left(F(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - F(e^{\alpha y_\delta(\theta_t\omega)} v)\right) \bar{v} dx \\
&\quad + \int_{\mathcal{O}} \left(e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right) F(e^{\alpha y_\delta(\theta_t\omega)} v) \bar{v} dx \\
&\quad + \int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} \left(F(e^{\alpha y_\delta(\theta_t\omega)} v) - F(e^{\alpha z(\theta_t\omega)} v)\right) \bar{v} dx := I_1 + I_2 + I_3.
\end{aligned} \tag{6.5}
$$

For I_1, in terms of (1.2), (1.3), (2.8) and the locally Lipschitz continuity of function a, we have

$$
\begin{aligned}
I_1 &= \int_{\mathcal{O}} e^{-\alpha y_\delta(\theta_t\omega)} \left(F(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - F(e^{\alpha y_\delta(\theta_t\omega)} v)\right) \bar{v} dx \\
&= e^{-\alpha y_\delta(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha y_\delta(\theta_t\omega)} v_\delta)) \mu e^{\alpha y_\delta(\theta_t\omega)} v_\delta - a(l(e^{\alpha y_\delta(\theta_t\omega)} v)) \mu e^{\alpha y_\delta(\theta_t\omega)} v\right) \bar{v} dx \\
&\quad + e^{-\alpha y_\delta(\theta_t\omega)} \int_{\mathcal{O}} \left(f(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - f(e^{\alpha y_\delta(\theta_t\omega)} v)\right) \bar{v} dx \\
&= e^{-\alpha y_\delta(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha y_\delta(\theta_t\omega)} v_\delta)) \mu e^{\alpha y_\delta(\theta_t\omega)} v_\delta - a(l(e^{\alpha y_\delta(\theta_t\omega)} v)) \mu e^{\alpha y_\delta(\theta_t\omega)} v_\delta\right) \bar{v} dx \\
&\quad + e^{-\alpha y_\delta(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha y_\delta(\theta_t\omega)} v)) \mu e^{\alpha y_\delta(\theta_t\omega)} v_\delta - a(l(e^{\alpha y_\delta(\theta_t\omega)} v)) \mu e^{\alpha y_\delta(\theta_t\omega)} v\right) \bar{v} dx \\
&\quad + e^{-\alpha y_\delta(\theta_t\omega)} \int_{\mathcal{O}} \left(f(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - f(e^{\alpha y_\delta(\theta_t\omega)} v)\right) \bar{v} dx \\
&\leq L_a \|l\| \mu e^{\alpha y_\delta(\theta_t\omega)} \|v_\delta\| \|\bar{v}\|^2 + M\mu \|\bar{v}\|^2 + C_f \|\bar{v}\|^2.
\end{aligned} \tag{6.6}
$$

For I_2, it follows from (2.10) and the Young inequality that

$$
\begin{aligned}
I_2 &= \int_{\mathcal{O}} \left(e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right) F(e^{\alpha y_\delta(\theta_t\omega)} v) \bar{v} dx \\
&\leq \left|e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right| \int_{\mathcal{O}} F(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) \bar{v} dx \\
&\leq \left|e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right| \left(\alpha_3 e^{\alpha(p-1) y_\delta(\theta_t\omega)} \int_{\mathcal{O}} |v_\delta|^{p-1} \bar{v} dx + \alpha_3 \int_{\mathcal{O}} \bar{v} dx\right) \\
&\leq c \left|e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)}\right| \left(\|v_\delta\|_p^p + \|\bar{v}\|_p^p + 1\right).
\end{aligned} \tag{6.7}
$$

For I_3, making use of the locally Lipschitz continuity of function a, (1.2), (2.8) and (6.1), we arrive at

$$
\begin{aligned}
I_3 &= \int_{\mathcal{O}} e^{-\alpha z(\theta_t\omega)} \left(F(e^{\alpha y_\delta(\theta_t\omega)}v) - F(e^{\alpha z(\theta_t\omega)}v) \right) \bar{v} dx \\
&= e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha y_\delta(\theta_t\omega)}v))\mu e^{\alpha y_\delta(\theta_t\omega)}v - a(l(e^{\alpha z(\theta_t\omega)}v))\mu e^{\alpha z(\theta_t\omega)}v \right) \bar{v} dx \\
&\quad + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} f(e^{\alpha y_\delta(\theta_t\omega)}v) - f(e^{\alpha z(\theta_t\omega)}v)\bar{v} dx \\
&\le e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha y_\delta(\theta_t\omega)}v))\mu e^{\alpha y_\delta(\theta_t\omega)}v - a(l(e^{\alpha z(\theta_t\omega)}v))\mu e^{\alpha y_\delta(\theta_t\omega)}v \right) \bar{v} dx \\
&\quad + e^{-\alpha z(\theta_t\omega)} \int_{\mathcal{O}} \left(a(l(e^{\alpha z(\theta_t\omega)}v))\mu e^{\alpha y_\delta(\theta_t\omega)}v - a(l(e^{\alpha z(\theta_t\omega)}v))\mu e^{\alpha z(\theta_t\omega)}v \right) \bar{v} dx \qquad (6.8) \\
&\quad + \left| e^{\alpha y_\delta(\theta_t\omega)-\alpha z(\theta_t\omega)} - 1 \right| \int_{\mathcal{O}} \left| \frac{df}{ds} \right| v\bar{v} dx \\
&\le \left| e^{\alpha y_\delta(\theta_t\omega)} - e^{\alpha z(\theta_t\omega)} \right| \left(e^{\alpha y_\delta(\theta_t\omega)-\alpha z(\theta_t\omega)} \right) \mu \|v\|_{C(\tau,\tau+T;H)} L_a \|l\| \|v\| \|\bar{v}\| \\
&\quad + \left| e^{\alpha y_\delta(\theta_t\omega)-\alpha z(\theta_t\omega)} - 1 \right| \left(2\mu M \|v\| \|\bar{v}\| + C_f' \int_{\mathcal{O}} \left(e^{\alpha y_\delta(\theta_t\omega)} + e^{\alpha z(\theta_t\omega)} \right)^{p-2} |v|^{p-1} |\bar{v}|) dx \right) \\
&\le c \left| e^{\alpha y_\delta(\theta_t\omega)-\alpha z(\theta_t\omega)} - 1 \right| (\|v\|_p^p + \|\bar{v}\|_p^p) + c \left| e^{\alpha y_\delta(\theta_t\omega)} - e^{\alpha z(\theta_t\omega)} \right| (\|v\|^2 + \|\bar{v}\|^2).
\end{aligned}
$$

With the help of (5.6), we find for every $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon, \tau, \omega, T) > 0$, such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,

$$
\left| e^{-\alpha y_\delta(\theta_t\omega)} - e^{-\alpha z(\theta_t\omega)} \right| \le \varepsilon \qquad \text{and} \qquad \left| e^{\alpha y_\delta(\theta_t\omega)-\alpha z(\theta_t\omega)} - 1 \right| \le \varepsilon. \tag{6.9}
$$

It follows from (6.5)-(6.9) that there exists a constant $c = c(\tau, \omega, T, \alpha, v_\delta, M, C_f, C_f', \mu) > 0$, such that for every $\varepsilon > 0$, $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,

$$
\int_{\mathcal{O}} \left(e^{-\alpha y_\delta(\theta_t\omega)} F(e^{\alpha y_\delta(\theta_t\omega)} v_\delta) - e^{-\alpha z(\theta_t\omega)} F(e^{\alpha z(\theta_t\omega)} v) \right) \bar{v} dx \le c\|\bar{v}\|^2 + c\varepsilon + c\varepsilon \|v_\delta\|_p^p + c\varepsilon \|v\|_p^p. \tag{6.10}
$$

By (6.9) and the Young inequality, for every $\varepsilon > 0$, for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$, we obtain

$$
\left| e^{-\alpha z(\theta_t\omega)} - e^{-\alpha y_\delta(\theta_t\omega)} \right| (h(t), \bar{v}) \le \frac{1}{2}\varepsilon \|\bar{v}\|^2 + \frac{1}{2}\varepsilon \|h(t)\|^2, \tag{6.11}
$$

and

$$
\begin{aligned}
\alpha(y_\delta(\theta_t\omega)v_\delta - z(\theta_t\omega)v, \bar{v}) &\le \alpha|(y_\delta(\theta_t\omega)v_\delta - z(\theta_t\omega)v_\delta, \bar{v})| + \alpha|(z(\theta_t\omega)v_\delta - z(\theta_t\omega)v, \bar{v})| \\
&\le c\|\bar{v}\|^2 + \frac{1}{2}\varepsilon \|\bar{v}\|^2 + \frac{1}{2}\varepsilon \|v_\delta\|^2.
\end{aligned} \tag{6.12}
$$

Taking into account (6.3)-(6.12) and (1.2), together with the fact that v and v_δ both belong to $C(\tau,\tau+T;H)\cap L^2(\tau,\tau+T;V)$, we derive

$$\frac{d}{dt}\|\bar v\|^2 \le c\|\bar v\|^2 + c\varepsilon\left(1+\|v\|_p^p+\|v_\delta\|_p^p+\|h(t)\|^2\right). \tag{6.13}$$

Solving (6.13), we find for all $0<|\delta|<\delta_1$ and $t\in[\tau,\tau+T]$,

$$\|\bar v\|^2 \le e^{c(t-\tau)}\|\bar v(\tau)\|^2 + c\varepsilon e^{c(t-\tau)}\int_\tau^t \left(1+\|v\|_p^p+\|v_\delta\|_p^p+\|h(s)\|^2\right)ds. \tag{6.14}$$

Since we have proved that v and v_δ both belong to $L^p_{loc}(\tau,\infty;L^p(\mathbb{R}^n))$, there exists a positive constant $C_1:=C_1(T,\omega)$ such that

$$\int_\tau^t \left(\|v\|_p^p+\|v_\delta\|_p^p\right)ds < C_1.$$

Therefore, (6.14) is equivalent to

$$\|v_\delta(t,\tau,\omega,v_{\delta,\tau})-v(t,\tau,\omega,v_\tau)\|^2 \le e^{c(t-\tau)}\|v_{\delta,\tau}-v_\tau\|^2 + c\varepsilon e^{c(t-\tau)}\left(1+\int_\tau^t \|h(s)\|^2 ds\right). \tag{6.15}$$

Note that

$$\begin{aligned} u_\delta(t,\tau,\omega,u_{\delta,\tau})-u(t,\tau,\omega,u_\tau) &= e^{\alpha y_\delta(\theta_t\omega)}v_\delta(t,\tau,\omega,v_{\delta,\tau}) - e^{\alpha z(\theta_t\omega)}v(t,\tau,\omega,v_\tau)\\ &= e^{\alpha y_\delta(\theta_t\omega)}(v_\delta(t,\tau,\omega,v_{\delta,\tau})-v(t,\tau,\omega,v_\tau)) + \left(e^{\alpha y_\delta(\theta_t\omega)}-e^{\alpha z(\theta_t\omega)}\right)v(t,\tau,\omega,v_\tau),\end{aligned}$$

where $v_{\delta,\tau}=e^{-\alpha y_\delta(\theta_\tau\omega)}u_{\delta,\tau}$ and $v_\tau=e^{-\alpha z(\theta_\tau\omega)}u_\tau$. Then, by the continuity of $y_\delta(\theta_t\omega)$ in t and (6.15), we obtain that there exists $\delta_2\in(0,\delta_1)$, such that for all $0<|\delta|<\delta_2$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}\|u_\delta(t,\tau,\omega,u_{\delta,\tau})-u(t,\tau,\omega,u_\tau)\|^2 &\le c\|v_\delta(t,\tau,\omega,v_{\delta,\tau})-v(t,\tau,\omega,v_\tau)\|^2\\ &\quad + c\left|e^{\alpha y_\delta(\theta_t\omega)}-e^{\alpha z(\theta_t\omega)}\right|\|v(t,\tau,\omega,v_\tau)\|^2,\end{aligned}$$

which along with (5.6) and (6.15) implies (6.2). □

An immediate result of Lemma 6.1 is:

Corollary 6.2 *Assume the conditions of Lemma 6.1 are true and $\delta_n\to 0$ as $n\to\infty$. Let u_{δ_n} and u be the solutions of* (5.2) *and* (2.11) *with initial data $u_{\delta_n,\tau}$ and u_τ, respectively. If $u_{\delta_n,\tau}\to u_\tau$ in H as $n\to\infty$, then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t>\tau$,*

$$u_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau})\to u(t,\tau,\omega,u_\tau) \quad in \quad H \quad as \quad n\to\infty.$$

We have stated that for each $\delta\in(0,1)$, $\mathcal{A}^\delta_\alpha$ is the unique $\mathcal{D}$-pullback attractor of Ξ_δ in H. To establish the upper semicontinuity of these attractors as $\delta\to 0$, we need the following compactness result.

Lemma 6.3 *Assume the conditions of Lemma 6.1 and* (4.2)-(4.3) *hold. Let* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$ *be fixed, if* $\delta_n \to 0$ *as* $n \to \infty$ *and* $u_n \in \mathcal{A}_\alpha^{\delta_n}(\tau, \omega)$, *then the sequence* $\{u_n\}_{n=1}^\infty$ *is precompact in* H.

Proof. The proof is standard based on the arguments of Lemma 3.3. Since the attractor $\mathcal{A}_\alpha^{\delta_n}$ is invariant and $u_n \in \mathcal{A}_\alpha^{\delta_n}(\tau, \omega)$, we see that for each $n \in \mathbb{N}$, there exists $\tilde{u}_n \in \mathcal{A}_\alpha^{\delta_n}(\tau - 1, \theta_{-1}\omega)$ such that

$$u_n = \Xi_{\delta_n}(1, \tau - 1, \theta_{-1}\omega, \tilde{u}_n) = u_{\delta_n}(\tau, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n). \tag{6.16}$$

By (5.16), there exists $N = N(\tau, \omega, \alpha) > 1$, such that for all $n \geq N$,

$$R_\alpha^{\delta_n}(\tau - 1, \theta_{-1}\omega) \leq 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s z(\theta_{r-1}\omega)dr - e^{-2\alpha z(\theta_{s-1}\omega)}}\|h(s + \tau - 1)\|^2 ds. \tag{6.17}$$

Since $\tilde{u}_n \in \mathcal{A}_\alpha^{\delta_n}(\tau - 1, \theta_{-\tau}\omega) \subset K_\alpha^{\delta_n}(\tau - 1, \theta_{-1}\omega)$, by (5.14) and (6.17), for all $n \geq N$, we obtain

$$\begin{aligned}\|\tilde{u}_n\|^2 \leq e^{2\alpha z(\omega)}\bigg(1 + &\frac{2\kappa_1|\mathcal{O}|}{m\mu} \\ &+ \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s z(\theta_{r-1}\omega)dr - e^{-2\alpha z(\theta_{s-1}\omega)}}\|h(s + \tau - 1)\|^2 ds\bigg).\end{aligned} \tag{6.18}$$

It follows from (5.3) that for all $t \geq \tau - 1$,

$$v_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{v}_n) = u_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n)e^{-\alpha y_{\delta_n}(\theta_{t-\tau}\omega)}, \tag{6.19}$$

where $\tilde{v}_n = e^{-\alpha y_{\delta_n}(\theta_{-1}\omega)}\tilde{u}_n$. In terms of (5.6), we know that

$$\lim_{n\to\infty} e^{-\alpha y_{\delta_n}(\theta_{-1}\omega)} = e^{-\alpha z(\theta_{-1}\omega)},$$

which along with (6.18)-(6.19) shows the sequence $\{\tilde{v}_n\}_{n=1}^\infty$ is bounded in H. Then, similar arguments to those in Lemma 3.3 infer that $\{v_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{v}_n)\}_{n=1}^\infty$ is precompact in H, that is, there exists $v' \in H$, such that for all $t \in (\tau - 1, \tau)$,

$$v_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{v}_n) \to v'(t) \quad \text{in} \quad H. \tag{6.20}$$

By (5.3), (6.19)-(6.20), for all $n \geq N$, we have

$$u_{\delta_n}(t, \tau, \theta_{-\tau}\omega, \tilde{u}_n) \to e^{-\alpha z(\theta_{t-\tau}\omega)}v'(t) \quad \text{in} \quad H \quad \text{for almost all} \quad t \in (\tau - 1, \tau). \tag{6.21}$$

Since $\delta_n \to 0$, it follows from Corollary 6.2 and (6.21) that

$$u_{\delta_n}(\tau, t, \theta_{-\tau}\omega, u_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n)) \to u(\tau, t, \theta_{-\tau}\omega, e^{-\alpha z(\theta_{t-\tau}\omega)}v'(t)) \quad \text{in} \quad H, \tag{6.22}$$

where u is the solution of (2.11). Note that

$$u_{\delta_n}(\tau, t, \theta_{-\tau}\omega, u_{\delta_n}(t, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n)) = u_{\delta_n}(\tau, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n).$$

Therefore, by (6.22), we have

$$u_{\delta_n}(\tau, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_n) \to u(\tau, t, \theta_{-\tau}\omega, e^{-\alpha z(\theta_{t-\tau}\omega)}v') \quad \text{in} \quad H,$$

which along with (6.16) completes the proof. □

Theorem 6.4 *Assume the conditions of Lemma 6.3 are true. Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have*

$$\lim_{\delta \to 0} dist_H(\mathcal{A}_\alpha^\delta(\tau,\omega), \mathcal{A}_\alpha(\tau,\omega)) = 0. \tag{6.23}$$

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let

$$K_0(\tau,\omega) = \{u \in H : \|u\|^2 \le e^{2\alpha z(\omega)} R_0(\tau,\omega)\},$$

where $R_0(\tau,\omega)$ is defined by

$$R_0(\tau,\omega) = 1 + \frac{2\kappa_1|\mathcal{O}|}{m\mu} + \frac{4}{m\mu}\int_{-\infty}^0 e^{\frac{5}{4}m\mu s - 2\alpha\int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} \|h(s+\tau)\|^2 ds.$$

By (4.3), we see the family $K_0 = \{K_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $\mathcal{D}$. Moreover, by (5.16) and (5.6), we have

$$\lim_{\delta \to 0} \|K_\alpha^\delta(\tau,\omega)\| = \|K_0(\tau,\omega)\| \quad \text{for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega,$$

which, together with Corollary 6.2 and Lemma 6.3, concludes this proof by [36, Theorem 3.1] immediately. □

7 Final comments and remarks

In this paper, we have analyzed a kind of fractional stochastic nonlocal partial differential equations driven by multiplicative noise, showing first the existence and uniqueness of solution to our model (2.11). Next, we also proved the existence and uniqueness of tempered pullback random attractor. When the fractional nonlocal partial differential equations are driven by colored noise, we investigated the convergence of solutions of Wong-Zakai approximations and the upper semicontinuity of random attractors of the approximated random system when $\delta \to 0$.

Actually, instead of studying the linear multiplicative noise $\alpha u \circ \frac{dW}{dt}$, we could consider a more general nonlinear one $g(u)\frac{dW}{dt}$ (with an appropriate nonlinear function g). It is worth recalling that Imkeller and Schmalfuß [22] proved that a stochastic ordinary differential equation, with locally Lipschitz g, always generates a random dynamical system. However, it is not known yet whether the same happens for general stochastic partial differential equations driven by nonlinear white noise. Therefore, we can consider the Wong-Zakai approximate system via replacing $g(u)\frac{dW}{dt}$ by $g(u)\zeta_\delta(\theta_t\omega)$. Thus, the similar analysis carried out in our paper is useful to prove the existence and uniqueness of solution and the existence of random attractors to this approximate problem. So far, it is unknown how to prove the convergence of solutions and attractors of approximated problems driven by nonlinear noise towards the original stochastic ones. However, as in some cases (multiplicative or additive noise) we are able to prove these convergences (see [43]), it seems sensible to think that the attractors of the Wong-Zakai approximations provide us interesting information about the dynamics of the stochastic problem. We plan to drive deeper into this analysis in the near future.

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