On a stochastic nonlocal system with discrete diffusion modeling life tables

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Dedicated to Prof. Bjorn Schmalfuss on the occasion of his 65-th birthday

Abstract

In this paper we study an stochastic system of differential equations with nonlocal discrete diffusion. For two type of noises we study the existence of either positive or probability solutions. Also, we analyse the asymptotic behaviour of solutions in the long term, showing that under suitable assumptions they tend to a neighborhood of the unique deterministic fixed point. Finally, we perform numerical simulations and discuss the application of the results to life tables for mortality in Spain.

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1 Introduction

Diffusion processes can be sometimes modeled by nonlocal operators, which take into account the influence of the value of the variable in the whole domain (or in part of them) and not only in the current point. For example, we can consider the following nonlocal parabolic problem

$$
\begin{aligned}
u_t(x,t) &= \int_{\mathbb{R}^N} J(x-y)\left(u(y,t)-u(x,t)\right)dy,\ x\in\Omega,\ t>0,\\
u(x,t) &= g(x,t),\ x\notin\Omega,\ t>0,\\
u(x,0) &= u_0(x),\ x\in\Omega,
\end{aligned}
$$

where $\Omega \subseteq \mathbb{R}$ is some interval (see [6], [13], [32]). Also, nonlinear nonlocal problems modeling phase transitions have been studied for example in [4], [5].

In [29] we considered the following nonlocal problem with discrete diffusion:

$$
\begin{aligned}
\frac{d}{dt}u_i(t) &= \sum_{r\in D} j_{i-r}u_r(t) - u_i(t) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r(t),\ i\in D,\ t>0, \qquad (1)\\
u_i(0) &= u_i^0,\ i\in D,
\end{aligned}
$$

where $D=\{m_1,\ldots,m_2\}$, $m_1<m_2$, $m_i\in\mathbb{Z}$, and $j:\mathbb{Z}\to\mathbb{R}^+$. The term $\sum_{r\in D} j_{i-r}u_r(t)-u_i(t)$ can be seen as a space discretization of the integral $\int_{\mathbb{R}^N} J(x-y)\left(u(y,t)-u(x,t)\right)dy$ (see e.g. [32]).

This problem was shown to be appropriate to model dynamical life tables by performing numerical simulations with data of mortality from Spanish population. In actuarial or demographic sciences life tables are used to study some biometrics functions, which are related, for example, to the probability of survival or death. They are important in order to calculate the insurance premium or to analyse the sustainability of the social welfare system

The predictions for mortality in Spain obtained from model (1) were shown to be adequate for a period of about three years but no longer. This is why in [30] we considered the following model with delay:

$$
\begin{aligned}
\frac{d}{dt}u_i(t) &= \sum_{r\in D}\int_{-h}^{0} j_{i-r}u^r(t+s)\alpha_i(s)d\mu(s) - u_i(t) \qquad (2)\\
&\quad + \sum_{r\in\mathbb{Z}\setminus D}\int_{-h}^{0} j_{i-r}g_r(t+s)\,\alpha_i(s)d\mu(s),\ i\in D,\ t>0,\\
u^i(\tau+s) &\equiv \phi^i(s),\ i\in D,\ s\in[-h,0],
\end{aligned}
$$

where $\alpha_i:[-s,0]\to\mathbb{R}^+$ and $d\mu(s)=\xi(s)\,ds$ being $\xi(\cdot)$ a probability density. In [30] it was shown that with this model the prediction horizon can be extended up to 8 years. In addition, it gives coherent values, in magnitude, when comparing it with other classical techniques such as the Lee-Carter model, up to 18 years.

However, in the real world there is always some kind of noise which can be intrinsic to the process under study or that can be related to errors in the observed data. Hence, we will consider now the following stochastic system of differential equations:

$$\frac{d}{dt}u_i(t) = \sum_{r\in D} j_{i-r}u_r(t) - u_i(t) + \sum_{r\in \mathbb{Z}\setminus D} j_{i-r}g_r(t) + b\sigma_i(u_i(t))\frac{dw_i}{dt},\ i\in D,\ t>0, \tag{3}$$

$$u_i(0) = u_i^0,\ i\in D,$$

where $w_i(t)$ are independent Brownian motions and $b>0$ is the intensity of the white noise. We will assume the following assumptions on the kernel $\{j_i\}$ and the functions $\{g_i(\cdot)\}$:

$(H1)$ $j_i \geq 0$ for all $i\in\mathbb{Z}$.

$(H2)$ $\sum_{i\in\mathbb{Z}} j_i = 1$.

$(H3)$ $g\in C([0,+\infty), l_2^\infty)$,
where $l_2^\infty = \{(u_i)_{i\in\mathbb{Z}\setminus D} : \sup_{i\in\mathbb{Z}\setminus D}|u_i|\}$.

We will consider two specific type of noises. Namely: 1) $\sigma_i(v)=v$ (linear case); 2) $\sigma_i(v)=v(1-v)$. The choice of the noise in the second case is motivated by the fact that we are interested in studying variables like the probability of death, which take values in the interval $[0,1]$. Noises of this type have been used for example for the logistic equation [22].

Our main aim in this paper is to present some theoretical results concerning the properties of solutions of the stochastic system. First, we prove in the linear case the existence of a unique positive solution whenever the initial datum is positive. Second, we establish in the second case that if the initial condition takes values in the interval $(0,1)$, then the solution remains in this interval for any future time. Third, we analyse the asymptotic behaviour of solutions, showing under certain assumptions that for large times they remain in a neighborhood of the unique fixed point of the deterministic system.

Finally, we perform numerical simulations of solutions and analyse the efficiency of model (3) for prediction of mortality using data from Spain.

2 Properties of solutions

In this section, under certain conditions on the functions of the problem, we will establish first that for any positive initial condition there exists a unique globally defined solution which is almost sure positive. Then we will prove that if the initial condition lies in the interval $(0,1)$, then the solution remains there for all positive times. This is important for the application to life tables.

Throughout this paper we will assume for the kernel $\{j_i\}$ and the functions $\{g_i(\cdot)\}$ the conditions $(H1)-(H3)$ given above.

System (3) can be rewritten in the matricial form

$$\begin{aligned} \frac{du}{dt} &= Bu\left(t\right) + h\left(t\right) + b\sigma(u(t))\frac{dw}{dt}, \\ u\left(0\right) &= u^{0} \in \mathbb{R}^{m}, \end{aligned} \tag{4}$$

where $u\left(t\right) = \left(u_{m_1}\left(t\right), ..., u_{m_2}(t)\right)^{T}$, $w\left(t\right) = \left(w_{m_1}\left(t\right), ..., w_{m_2}(t)\right)^{T}$, $h(t) = \left(h_{m_1}\left(t\right), ..., h_{m_2}(t)\right)^{T}$, $m = m_2 - m_1 + 1$, $\sigma : \mathbb{R}^{m} \to \mathbb{R}^{m \times m}$ and

$$\begin{aligned} \left(\sigma(v)\right)_{ij} &= 0 \text{ if } i \neq j, \\ \left(\sigma(v)\right)_{ii} &= \sigma_i(v_i) \text{ for } i = m_1, ..., m_2, \end{aligned}$$

$$h_i(t) = \sum_{r \in \mathbb{Z} \setminus D} j_{i-r} g_r\left(t\right),$$

$$B = \begin{pmatrix} j_0 & j_{-1} & j_{-2} & \cdots & \cdots & j_{m_1-m_2} \\ j_1 & j_0 & j_{-1} & j_{-2} & \cdots & j_{m_1-m_2+1} \\ j_2 & j_1 & j_0 & j_1 & \cdots & j_{m_1-m_2+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ j_{m_2-m_1-1} & \cdots & j_2 & j_1 & j_0 & j_1 \\ j_{m_2-m_1} & \cdots & \cdots & j_2 & j_1 & j_0 \end{pmatrix}.$$

2.1 A linear noise

First, we will consider a standard linear noise, that is, we study the system

$$\begin{aligned} \frac{d}{dt}u_i\left(t\right) &= \sum_{r \in D} j_{i-r} u_r\left(t\right) - u_i(t) + \sum_{r \in \mathbb{Z} \setminus D} j_{i-r} g_r\left(t\right) + bu_i(t)\frac{dw_i}{dt},\ i \in D,\ t > 0, \\ u_i\left(0\right) &= u_i^{0},\ i \in D. \end{aligned} \tag{5}$$

Denote $\mathbb{R}_{+}^{m} = \{v \in \mathbb{R}^{m} : v_j > 0 \text{ for all } j\}$. We will prove the existence of global positive solutions. For this aim we follow a standard procedure in stochastic differential equations (see e.g. [10], [11], [26]).

Lemma 1 *Assume that $g_r(t) \geq 0$ for all r and t. Then for any $u^0 \in \mathbb{R}_{+}^{m}$ there exists a unique globally defined solution $u\left(\cdot\right)$ such that $u\left(t\right) \in \mathbb{R}_{+}^{m}$ almost sure for $t \geq 0$.*

Proof. The existence of a unique locally defined solution for problem (4) follows from standard results as the functions involved in the systems are locally Lipschitz (see e.g. [15, Chapter 2, p.45]). If the solution is defined in the maximal interval $[0, \tau_e)$, then we need to establish that $\tau_e = +\infty$ and that $u\left(t\right) \in \mathbb{R}_{+}^{m}$ for all $t \geq 0$ a.s.

Let $k_0 > 0$ be large enough so that $u_i^0 \in [1/k_0, k_0]$ for all $i \in D$. As usual, we define the stopping time

$$\tau_k = \inf\{t \in [0,\infty) : u_i\left(t\right) \notin (\frac{1}{k}, k) \text{ for some } i \in D\}.$$

This sequence is increasing as $k \nearrow +\infty$. If $\tau_\infty = \lim_{k\to+\infty} \tau_k = +\infty$ a.s., then $\tau_e = +\infty$ and $u\left(t\right) \in \mathbb{R}_+^m$ for $t \geq 0$ almost sure, proving the assertion.

By contradiction, assume the existence of $T, \varepsilon > 0$ such that

$$P(\tau_\infty \leq T) > \varepsilon.$$

In such a case there exists $k_1 \geq k_0$ for which

$$P(\tau_k \leq T) \geq \varepsilon \text{ for } k \geq k_1.$$

Further, we consider the C^2-function $V : \mathbb{R}_+^m \to \mathbb{R}_+^1$ given by

$$V\left(u\right) = \sum_{i\in D}(u_i - 1 - \log(u_i)).$$

For $u\left(t\right) \in \mathbb{R}_+^m$ by Itô's formula we have

$$\begin{aligned} dV(u(t)) &= \sum_{i\in D}\left(1 - \frac{1}{u_i\left(t\right)}\right)\left(\sum_{r\in D} j_{i-r}u_r\left(t\right) - u_i(t) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\left(t\right)\right)dt \\ &\quad + \sum_{i\in D}\frac{1}{2}b^2dt + \sum_{i\in D}\left(1 - \frac{1}{u_i\left(t\right)}\right)bu_i(t)dw_i(t) \\ &= I(t)dt + \sum_{i\in D} I_i(t)dw_i(t), \end{aligned} \tag{6}$$

where $I_i(t) = \left(1 - \frac{1}{u_i(t)}\right)bu_i(t)$. By using $(H1) - (H3)$ and $g_r\left(t\right) \geq 0$ the first term is estimated as follows for $u\left(t\right) \in \mathbb{R}_+^m$:

$$\begin{aligned} I(t) &\leq \sum_{i\in D}\sum_{r\in D} j_{i-r}u_r\left(t\right) - \sum_{i\in D} u_i(t) + m + \sum_{i\in D}\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\left(t\right) + \frac{m}{2}b^2 \\ &\quad - \sum_{i\in D}\frac{1}{u_i\left(t\right)}\left(\sum_{r\in D} j_{i-r}u_r\left(t\right) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\left(t\right)\right) \\ &\leq K_T, \end{aligned}$$

where we have used that

$$\sum_{i\in D}\frac{1}{u_i\left(t\right)}\left(\sum_{r\in D} j_{i-r}u_r\left(t\right) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\left(t\right)\right) \geq 0,$$

$$\sum_{i\in D}\sum_{r\in D} j_{i-r}u_r(t)-\sum_{i\in D}u_i(t)=\sum_{r\in D}\sum_{i\in D} j_{i-r}u_r(t)-\sum_{i\in D}u_i(t)$$
$$\leq \sum_{r\in D}u_r(t)-\sum_{i\in D}u_i(t)=0,$$
$$\sum_{i\in D}\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r(t)\leq C_T\sum_{i\in D}\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}\leq mC_T.$$

Integrating in (6) over $(0,\tau_k\wedge T)$ and taking expectations we obtain that

$$\begin{aligned}
&0\leq \mathbb{E}V(u(\tau_k\wedge T))\\
&\leq V(u^0)+\mathbb{E}\int_0^{\tau_k\wedge T}K_T dt+\mathbb{E}\int_0^{\tau_k\wedge T}\sum_{i\in D}\left(1-\frac{1}{u_i(t)}\right)bu_i(t)dw_i(t)\\
&=V(u^0)+K_T\mathbb{E}(\tau_k\wedge T)\leq V(u^0)+K_T T.
\end{aligned}$$

Let $\Omega_k=\{\omega:\tau_k\leq T\}$, which satisfies $P(\Omega_k)\geq\varepsilon$ for $k\geq k_1$. For any $\omega\in\Omega_k$ there is $i\in D$ such that either $u_i(\tau_k,\omega)=k$ or $u_i(\tau_k,\omega)=1/k$, which implies that

$$V(u(\tau_k\wedge T,\omega))\geq (k-1-\log(k))\wedge(\frac{1}{k}-1+\log(k)).$$

Hence,

$$V(u^0)+K_T T\geq \mathbb{E}(1_{\Omega_k}V(u(\tau_k\wedge T)))\geq\varepsilon((k-1-\log(k))\wedge(\frac{1}{k}-1+\log(k)))=\varepsilon R(k),$$

where 1_A stands for the indicator function of the set A. Passing to the limit as $k\to+\infty$ we get a contradiction as $R(k)\to+\infty$. ■

Corollary 2 *Let $u^0,v^0\in\mathbb{R}^m$ be two initial conditions satisfying $u_i^0>v_i^0$ for any $i\in D$. Also, $g^u,g^v\in C([0,+\infty),l_2^\infty)$ are such that $g_i^u(t)\geq g_i^v(t)$ for all $i\in D$ and $t\geq 0$. Then, $u_i(t)>v_i(t)$, for all $i\in D$ and $t\geq0$, where $u(\cdot)$, $v(\cdot)$ are the unique solutions to problem (3) corresponding to $\{u^0,g^u\}$ and $\{v^0,g^v\}$, respectively.*

Proof. It follows immediately by defining the function $w(\cdot)=u(\cdot)-v(\cdot)$ and applying Lemma 1 to it. ■

2.2 A nonlinear noise

Second, we are going to consider the system

$$\begin{aligned}
&\frac{d}{dt}u_i(t)=\sum_{r\in D} j_{i-r}u_r(t)-u_i(t)+\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r(t)+bu_i(t)(1-u_i(t))\frac{dw_i}{dt},\\
&u_i(0)=u_i^0,
\end{aligned}\tag{7}$$

where $i \in D$, $t > 0$.

We are interesting in proving that if the components of the initial condition lie within the interval $(0,1)$ and the functions $g_r(t)$ take values in $[0,1]$, then the components of the solution $u(t)$ also lie in $(0,1)$ for any $t \geq 0$ a.s. This is important for applying the results to life tables, as in such situation the variable is a probability.

We start with the deterministic case.

Lemma 3 *Let $b = 0$. Assume that $g_r(t) \in [0,1]$ for all r and t. Then for any $u^0 \in \mathbb{R}^m$ such that $u_i^0 \in [0,1]$, for any $i \in D$, the unique solution $u(\cdot)$ to (7) satisfies that $u_i(t) \in [0,1]$ for all $i \in D$ and $t \geq 0$.*

Proof. The fact that $u_i(t) \geq 0$ was proved in [29], so it remains to check that $u_i(t) \leq 1$. Let $z^+ = \max\{z, 0\}$ for a real number z. Multiplying the equation by $(u-1)^+$ and using the equality [1, Lemma 2.2]

$$\left(\frac{dy}{dt}, y^+\right)_{\mathbb{R}^m} = \frac{1}{2}\frac{d}{dt}\left\|y^+\right\|^2_{\mathbb{R}^m}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(u(t)-1)^+\right\|^2_{\mathbb{R}^m} \\
&= \sum_{i\in D}\sum_{r\in D} j_{i-r}u_r(t)(u_i(t)-1)^+ \\
&+ \sum_{r\in \mathbb{Z}\setminus D} j_{i-r}g_r(t)(u_i(t)-1)^+ - \sum_{i\in D} u_i(t)(u_i(t)-1)^+ \\
&= \sum_{i\in D}\sum_{r\in D} j_{i-r}(u_r(t)-1)(u_i(t)-1)^+ + \sum_{i\in D}\sum_{r\in D} j_{i-r}(u^i(t)-1)^+ \\
&+ \sum_{i\in D}\sum_{r\in \mathbb{Z}\setminus D} j_{i-r}g_r(t)(u_i(t)-1)^+ - \sum_{i\in D}(u_i(t)-1)(u_i(t)-1)^+ - \sum_{i\in D}(u_i(t)-1)^+ .
\end{aligned}$$

Using $g_r(t) \leq 1$ and $(H1)-(H2)$ we obtain that

$$\begin{aligned}
&\sum_{i\in D}\sum_{r\in D} j_{i-r}(u_i(t)-1)^+ + \sum_{i\in D}\sum_{r\in \mathbb{Z}\setminus D} j_{i-r}g_r(t)(u_i(t)-1)^+ - \sum_{i\in D}(u_i(t)-1)^+ \\
&\leq \sum_{i\in D}\left(\sum_{r\in D} j_{i-r} + \sum_{r\in \mathbb{Z}\setminus D} j_{i-r}\right)(u_i(t)-1)^+ - \sum_{i\in D}(u_i(t)-1)^+ = 0.
\end{aligned}$$

Hence, by $(H2)$ we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(u(t)-1)^{+}\right\|_{\mathbb{R}^m}^2 \\
&\leq \sum_{i\in D}\sum_{r\in D} j_{i-r}(u_r(t)-1)(u_i(t)-1)^{+} \\
&\leq \sum_{i\in D}\sum_{r\in D} j_{i-r}(u_r(t)-1)^{+}(u_i(t)-1)^{+} \\
&\leq \frac{1}{2}\sum_{i\in D}\sum_{r\in D} j_{i-r}\left(\left((u_r(t)-1)^{+}\right)^2+\left((u_i(t)-1)^{+}\right)^2\right) \\
&\leq \left\|(u(t)-1)^{+}\right\|_{\mathbb{R}^m}^2 .
\end{aligned}$$

Thus,

$$\left\|(u(t)-1)^{+}\right\|_{\mathbb{R}^m}^2 \leq \left\|\left(u^0-1\right)^{+}\right\|_{\mathbb{R}^m}^2 e^{2t},$$

so $u(t)\leq 1$ for all $t\geq 0$. ■

Further, we study system (7).

Lemma 4 *Assume that $g_r(t)\in[0,1]$ for all r and t. Then for any $u^0\in\mathbb{R}^m$ such that $u_i^0\in(0,1)$, for any $i\in D$, the unique solution $u(\cdot)$ to (7) satisfies almost sure that $u_i(t)\in(0,1)$ for all $i\in D$ and $t\geq 0$.*

Proof. Let $k_0>0$ be large enough so that $u_i^0\in[1/k_0,1-1/k_0]$ for all $i\in D$. We define now the stopping time

$$\tau_k=\inf\{t\in[0,\infty): u_i(t)\notin(\frac{1}{k},1-\frac{1}{k}) \text{ for some } i\in D\}.$$

This sequence is increasing as $k\nearrow+\infty$. If $\tau_\infty=\lim_{k\to+\infty}\tau_k=+\infty$ a.s., then $0<u_i(t)<1$ almost sure for $t\geq 0$, proving the assertion.

By contradiction, assume the existence of $T,\varepsilon>0$ such that

$$P(\tau_\infty\leq T)>\varepsilon.$$

In such a case there exists $k_1\geq k_0$ for which

$$P(\tau_k\leq T)\geq\varepsilon \text{ for } k\geq k_1.$$

Let

$$K_0=\{u=(u_{m_1},...,u_{m_2})\in\mathbb{R}_+^m: 0<u_i<1\}.$$

We define the C^2-function $V:K_0\to\mathbb{R}_+^1$ given by

$$V(u)=-\sum_{i\in D}(\log(1-u_i)+\log(u_i)).$$

For $u(t) \in K_0$ by Itô's formula we have

$$
\begin{aligned}
dV(u(t)) &= \sum_{i\in D}\left(\frac{1}{1-u_i(t)}-\frac{1}{u_i}\right)\left(\sum_{r\in D} j_{i-r}u_r(t)-u_i(t)+\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r(t)\right)dt \\
&\quad+\sum_{i\in D}\frac{1}{2}\left(u_i^2(t)+(1-u_i(t))^2\right)b^2dt+\sum_{i\in D} b(2u_i(t)-1)dw_i(t) \\
&= I(t)dt+\sum_{i\in D} I_i(t)dw_i(t),
\end{aligned}
$$

where $I_i(t) = b(2u_i(t)-1)$. We note that

$$
\sum_{i\in D}\left(-\frac{1}{u_i}\right)\left(\sum_{r\in D} j_{i-r}u_r(t)-u_i(t)+\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r(t)\right)\le\sum_{i\in D}1.
$$

Then the term $I(t)$ is estimated as follows when $t \le \tau_k$:

$$
\begin{aligned}
I(t) &\le \sum_{i\in D}\left(\frac{1}{1-u_i(t)}\right)\left(\sum_{r\in D} j_{i-r}u_r(t)-u_i(t)+\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}\right)+\sum_{i\in D}\left(1+b^2\right) \\
&\le \sum_{i\in D}\left(\frac{1}{1-u_i(t)}\right)\left(\sum_{r\in D} j_{i-r}\left(u_r(t)-1\right)+\sum_{r\in D} j_{i-r}+\sum_{r\in\mathbb{Z}\setminus D} j_{i-r}-u_i(t)\right)+m\left(1+b^2\right) \\
&\le \sum_{i\in D}\left(\frac{1}{1-u_i(t)}\right)(1-u_i(t))+m\left(1+b^2\right)=m\left(2+b^2\right).
\end{aligned}
$$

Integrating over $(0,\tau_k\wedge T)$ and taking expectations we infer that

$$
\begin{aligned}
&0\le\mathbb{E}V(u(\tau_k\wedge T))\le V(u^0)+\mathbb{E}\int_0^{\tau_k\wedge T} m\left(2+b^2\right)dt+\mathbb{E}\int_0^{\tau_k\wedge T}\sum_{i\in D} b(2u_i(t)-1)dw_i(t) \\
&\quad=V(u^0)+m\left(2+b^2\right)\mathbb{E}(\tau_k\wedge T)\le V(u^0)+m\left(2+b^2\right)T.
\end{aligned}
$$

Let $\Omega_k=\{\omega:\tau_k\le T\}$, which satisfies $P(\Omega_k)\ge\varepsilon$ for $k\ge k_1$. For any $\omega\in\Omega_k$ there exists $i\in D$ such that either $u_i(\tau_k,\omega)=\frac{1}{k}$ or $u_i(\tau_k,\omega)=1-\frac{1}{k}$, so

$$
V(u(\tau_k\wedge T,\omega))\ge\log(k)-\log(1-\frac{1}{k}).
$$

Thus,

$$
V(u^0)+m\left(2+b^2\right)T\ge\mathbb{E}(1_{\Omega_k}V(u(\tau_k\wedge T)))\ge\varepsilon\left(\log(k)-\log(1-\frac{1}{k})\right).
$$

Passing to the limit as $k\to+\infty$ we arrive at a contradiction. ■

3 Asymptotic behaviour

We are now interested in studying the asymptotic behaviour of the solutions to problems (5)-(7) around an asymptotically stable equilibrium of the deterministic model in the autonomous case, that is, when $b=0$ and the function $g(\cdot)$ is constant, which means that $g_r(t) \equiv g_r \in \mathbb{R}$. In [29] the existence of such fixed point was proved. Namely, under suitable assumptions we show that the solutions of the stochastic system remain close for large times to the unique deterministic fixed point in some sense.

When $b=0$, a fixed point $u \in \mathbb{R}^m$ has to satisfy the following system:

$$-\sum_{r\in D} j_{i-r}u_r + u_i = \sum_{r\in \mathbb{Z}\setminus D} j_{i-r}g_r = h_i,\ i \in D,$$

which can be rewritten in the matricial form

$$Au = h, \tag{8}$$

where

$$A = \begin{pmatrix} 1-j_0 & -j_{-1} & -j_{-2} & \cdots & \cdots & -j_{m_1-m_2} \\ -j_1 & 1-j_0 & -j_{-1} & -j_{-2} & \cdots & -j_{m_1-m_2+1} \\ -j_2 & -j_1 & 1-j_0 & -j_{-1} & \cdots & -j_{m_1-m_2+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -j_{m_2-m_1-1} & \cdots & -j_2 & -j_1 & 1-j_0 & -j_{-1} \\ -j_{m_2-m_1} & \cdots & \cdots & -j_2 & -j_1 & 1-j_0 \end{pmatrix}.$$

If we assume that

$$\sum_{r\in D} j_{i-r} < 1,\ \forall i \in D, \tag{9}$$

which implies that A is diagonal dominant, then system (8) possesses a unique fixed point $\overline{u}$, which is exponentially stable and

$$\|u(t)-\overline{u}\|^2_{\mathbb{R}^m} \le \|u(0)-\overline{u}\|^2_{\mathbb{R}^m} e^{-2\delta t} \to 0 \text{ if } t \to +\infty, \tag{10}$$

for some $\delta > 0$ [29, Theorem 3].

We start with the linear case.

Theorem 5 *Assume that $g_r \ge 0$, for all r and t, $b<1$ and*

$$\sum_{r\in D} j_{i-r} < 1-b^2,\ \forall i \in D. \tag{11}$$

Then for any $u^0 \in \mathbb{R}^m_+$ the unique solution to problem (5) has the property

$$\lim\sup_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t \|u(s)-\overline{u}\|^2_{\mathbb{R}^m} ds \le \frac{b^2}{\delta}\|\overline{u}\|^2_{\mathbb{R}^m},$$

where $\overline{u}$ is the unique solution of system (8) and

$$\delta = \min_{i\in D}\left\{1-b^2-\sum_{r\in D} j_{i-r}\right\}.$$

Proof. We define the C^2-function $V : \mathbb{R}^m \to \mathbb{R}$ given by

$$V(u) = \sum_{i\in D} (u_i - \overline{u}_i)^2 .$$

By Itô's formula, we have

$$\begin{aligned} dV(u(t)) &= \sum_{i\in D} 2(u_i(t) - \overline{u}_i) \left(\sum_{r\in D} j_{i-r} u_r(t) - u_i(t) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r} g_r \right) dt \\ &\quad + \sum_{i\in D} b^2 u_i^2(t) dt + \sum_{i\in D} 2(u_i(t) - \overline{u}_i) b u_i(t) dw_i(t) \\ &= I(t)dt + \sum_{i\in D} I_i(t) dw_i(t). \end{aligned} \tag{12}$$

In view of $A\overline{u} = h$, we have that

$$-\sum_{r\in D} j_{i-r}\overline{u}_r + \overline{u}_i = \sum_{r\in\mathbb{Z}\setminus D} j_{i-r} g_r.$$

Then the term $I(t)$ is estimated as follows:

$$\begin{aligned} I(t) &= \sum_{i\in D} 2(u_i(t) - \overline{u}_i) \left(\sum_{r\in D} j_{i-r} (u_r(t) - \overline{u}_r) - (u_i(t) - \overline{u}_i) \right) \\ &\quad + \sum_{i\in D} b^2 (u_i(t) - \overline{u}_i + \overline{u}_i)^2 \\ &\le 2 \sum_{i\in D}\sum_{r\in D} j_{i-r} (u_r(t) - \overline{u}_r)(u_i(t) - \overline{u}_i) - 2(1-b^2) \|u(t) - \overline{u}\|^2_{\mathbb{R}^m} + 2b^2 \|\overline{u}\|^2_{\mathbb{R}^m} . \end{aligned}$$

We deal with the first term in the last inequality:

$$\begin{aligned} &\sum_{i\in D}\sum_{r\in D} j_{i-r} (u_r(t) - \overline{u}_r)(u_i(t) - \overline{u}_i) \\ &\le \frac{1}{2} \sum_{i\in D}\sum_{r\in D} j_{i-r} (u_r(t) - \overline{u}_r)^2 + \frac{1}{2} \sum_{i\in D}\sum_{r\in D} j_{i-r} (u_i(t) - \overline{u}_i)^2 \\ &= \frac{1}{2} \sum_{r\in D} (u_r(t) - \overline{u}_r)^2 \sum_{i\in D} j_{i-r} + \frac{1}{2} \sum_{i\in D} (u_i(t) - \overline{u}_i)^2 \sum_{r\in D} j_{i-r}. \end{aligned}$$

We observe (see [29, p.939]) that

$$\delta = \min_{i\in D} \left\{ 1 - b^2 - \sum_{r\in D} j_{i-r} \right\} = \min_{r\in D} \left\{ 1 - b^2 - \sum_{i\in D} j_{i-r} \right\}.$$

Hence,

$$I(t) \le -2\delta \|u(t) - \overline{u}\|^2_{\mathbb{R}^m} + 2b^2 \|\overline{u}\|^2_{\mathbb{R}^m} .$$

Integrating in (12) over $(0,t)$ we have

$$0 \leq V(u(t)) \leq V(u(0)) - 2\delta \int_0^t \|u(s) - \overline{u}\|_{\mathbb{R}^m}^2 ds + 2b^2 \|\overline{u}\|_{\mathbb{R}^m}^2 t$$
$$+ \sum_{i\in D} \int_0^t 2\left(u_i(t) - \overline{u}_i\right) b u_i(t) dw(t).$$

Taking expectations and dividing by t we obtain

$$\lim \sup_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t \|u(s) - \overline{u}\|_{\mathbb{R}^m}^2 ds \leq \frac{b^2}{\delta} \|\overline{u}\|_{\mathbb{R}^m}^2 .$$

■

Remark 6 *Since the unique solution to problem (5) with $b = 0$ satisfies $u(t) \geq 0$ [29], it is easy to see from (10) that $\overline{u}_r \geq 0$ for all $r \in D$.*

We consider now system (7).

Theorem 7 *Assume that $g_r(t) \in [0,1]$ for all r and t, $b < 1$ and*

$$\sum_{r\in D} j_{i-r} < 1 - b^2, \ \forall i \in D.$$

Then for any $u^0 \in \mathbb{R}^m$ such that $u_i^0 \in (0,1)$, for any $i \in D$, the unique solution to problem (7) has the property

$$\lim \sup_{t\to+\infty} \frac{1}{t}\mathbb{E}\int_0^t \|u(s) - \overline{u}\|_{\mathbb{R}^m}^2 ds \leq \frac{b^2}{\delta} \|\overline{u}\|_{\mathbb{R}^m}^2 ,$$

where $\overline{u}$ is the unique solution of system (8) and

$$\delta = \min_{i\in D}\left\{1 - b^2 - \sum_{r\in D} j_{i-r}\right\}.$$

Proof. We define the C^2-function $V : \mathbb{R}^m \to \mathbb{R}$ given by

$$V(u) = \sum_{i\in D} \left(u_i - \overline{u}_i\right)^2 .$$

By Itô's formula, we have

$$dV(u(t)) = \sum_{i\in D} 2\left(u_i(t) - \overline{u}_i\right)\left(\sum_{r\in D} j_{i-r}u_r(t) - u_i(t) + \sum_{r\in\mathbb{Z}\setminus D} j_{i-r}g_r\right) dt$$
$$+ \sum_{i\in D} b^2 u_i^2(t)\left(1 - u_i(t)\right)^2 dt + \sum_{i\in D} 2\left(u_i(t) - \overline{u}_i\right) b u_i(t)(1 - u_i(t)) dw_i(t)$$
$$= I(t)dt + \sum_{i\in D} I_i(t) dw_i(t).$$

Since $u_i(t) \in (0,1)$, we have

$$\sum_{i\in D} b^2 u_i^2(t)\left(1-u_i(t)\right)^2 \leq \sum_{i\in D} b^2 u_i^2(t).$$

Repeating the same arguments of Theorem 5 we obtain that

$$0 \leq V(u(t)) \leq V(u(0)) - 2\delta \int_0^t \left\| u\left(s\right) - \overline{u} \right\|_{\mathbb{R}^m}^2 ds + 2b^2 \left\| \overline{u} \right\|_{\mathbb{R}^m}^2 t$$
$$+ \sum_{i\in D} \int_0^t 2\left(u_i(t) - \overline{u}_i\right) b u_i(t)(1-u_i(t)) dw_i(t).$$

Taking expectations and dividing by t we finally have

$$\lim \sup_{t\to+\infty} \frac{1}{t} \mathbb{E} \int_0^t \left\| u\left(s\right) - \overline{u} \right\|_{\mathbb{R}^m}^2 ds \leq \frac{b^2}{\delta} \left\| \overline{u} \right\|_{\mathbb{R}^m}^2 .$$

■

Remark 8 *Since by Lemma 3 the unique solution to problem (7) with* $b = 0$ *satisfies* $u\left(t\right) \in [0,1]$, *it is easy to see from (10) that* $\overline{u}_r \in [0,1]$ *for all* $r \in D$.

4 Application to life tables

The mortality table is the most widely used tool for studying the survival of a population. This instrument is based on so-called biometric functions, among which we have life expectancy at birth at moment t, e_0^t; the number of living (deceased) individuals of an initial population, at a moment of time t, with completed age x, l_x^t (d_x^t); or the associated probabilities of survival or death, p_x^t and q_x^t, respectively. In this work we will focus on this last probability, q_x^t.

It is known that the probability of death is unknown a priori, and it is common to assume that deaths occur randomly and independently among individuals of the same age x. In this sense, we can only know particular values of mortality that have occurred at a specific moment in time and for the population being studied, usually from a region or country, although it is usually also used for specific groups, for example, the takers of a life insurance policy.

As the phenomenon of mortality is not reproducible in a natural way, it is usual to have a single estimate of q_x^t for each age x and each time interval $[t, t+1[$. This can cause a significant deviation of the estimation from the true probability of death, which can endanger all those processes in which it intervenes; for example, in calculating pension spending forecasts in the public welfare system; or in the calculation of the technical necessary provisions to deal with claims that may occur in, for example, the life insured portfolio that an insurance company owns.

The way to approach the problem of estimating mortality rates has been modified and adapted to the statistical and mathematical techniques that have emerged. Thus, at first the estimation of q_x was performed by using information from different periods, so that crude mortality rates ($\hat{q}_x$), which did not always respond to certain softness criteria, were obtained. Namely, "the probability of dying at an age x must be similar to that of dying at age $x-1$ and $x+1$". Failure to comply with this property implies that some techniques must be applied to dampen this effect, for example by adjusting classical survival laws such as Gompertz ([17]), Gompetz-Makeham ([24], [27]) or the Heligmann and Pollard laws [19]. However, these types of techniques deal with the problem from a static point of view, that is, q_x^t remains constant at close moments of time (see also [28], [31]). However, it is known that the probability of death does not remain constant over time, that is, $q_x^t \neq q_x^{t+h}$, so using information from different times either to estimate the true probabilities of death or to calculate the number of deaths within, say, 10 years is incorrect and can lead to serious errors. To solve this problem, dynamic models of mortality were introduced. The best known model of this type is the one introduced by Lee-Carter [23], which introduces mortality improvement factors for each calendar year. Subsequently, other similar models appeared, such as the CBD model [7], or the M3-M7 models ([8], [9]), all of which address the problem of the evolution of mortality from a stochastic point of view. In this line, in the previous work [29] we introduced a mathematical model of the evolution of q_x^t based on a system of differential equations with a non-local operator, which is based on the non-parametric estimation of mortality rates by kernel graduation ([2], [12]). This model reproduces qualitative aspects of mortality well, but has a short prediction horizon of three years. In order to solve this problem, in a later work [30] we proposed a second model, which improves the first one in the sense that it incorporates the previous information using a delay term. This model gives very good results and realizations, with the same valid prediction horizons as in the Lee-Carter models. Even though the model proposed in [30] provides adequate estimates, it does not provide alternative scenarios, that is, it provides a future point estimate of future mortality rates but does not provide uncertainty intervals for the estimates. In order to deal with this problem, in this work we have proposed a modification of the model described in [29] in which a stochastic term is introduced.

The model proposed in this work can be used to model (among others) the behavior of mortality rates, and can be deduced using a procedure analogous to the one in [29].

We solve numerically system (3) by using the Euler-Maruyama method (see e.g. [20]) and implementing it in the Matlab software, version R2020a. In the resolution of this system, the mortality data for the year 2010 from the National Institute of Statistics (of Spain) [21] are used as initial values. Also, the mortality values for 2019 are used as a reference for the suitability of the estimates.

In relation to the values of the parameters that are used for the resolution of the stochastic system, we observe first that ages are taken from 0 to 100 (thus, $m_1 = 0,\ m_2 = 100$). The function g is defined to be piecewise constant, with $g_i = 0.41$, for $i > 100$, and $g_i = 0.001802$, for $i < 0$; the choice of the value 0.41

is the one proposed in [14, 3] as the convergence value of the limit mortality. The election of the $g's$ values is motivated by the necessity of avoiding numerical perturbations in the method. The value 0.001802 is obtained applying static kernel graduation to the initial series of values, and taking the graduate value of the risk of death at birth, $\overset{\circ}{q}_0^{2010}$. Namely, we consider the observed values at the initial moment, q_x^{2010}, and then, in order to graduate this series we use static kernel estimation for $x = 0$ via the formula

$$\overset{\circ}{q}_i^{2010} = \frac{\sum_{r\in D} j_{i-r} q_r^{2010}}{\sum_{r\in D} j_{i-r}},$$

obtaining that $\overset{\circ}{q}_0^{2010} = 0.001802$. Other types of corrections at the boundary to handle this problem can be found for example in [16], [18].

The time step is one year, which is subdivided into 50 sub-periods for the intermediate calculations. As in previous papers, a Gaussian kernel with bandwidth equal to 1 is used.

Figure 1 shows (in logarithmic scale) the observed mortality values for the years 2010 and 2019, together with 50 trajectories obtained in the numerical resolution of the system. It can be seen that all of them contain the three parts in which the mortality curve is traditionally subdivided [19]: adaptation to the environment, social or accident hump, and natural longevity. Therefore we can say that the system preserves the qualitative properties of the mortality curve and, in this sense, is adequate.

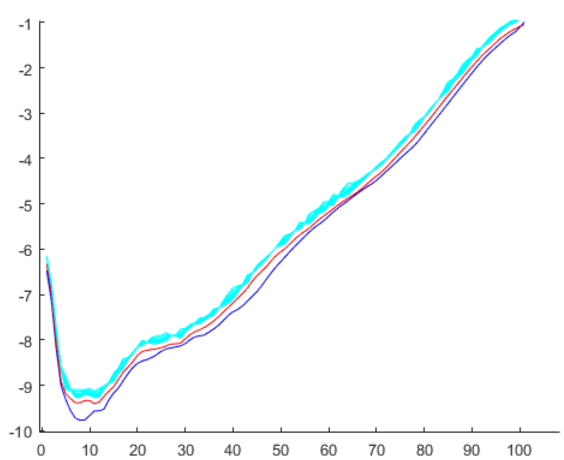

Figure 1: Ensemble of estimates of mortality

Although the proposed model is adequate from the qualitative point of view, it is evident that the estimated mortality rates are not between the initial values $\{q_x^{2010}\}$ and the observed limit values $\{q_x^{2019}\}$. This is due to the fact that the model does not take into account the improvements in mortality, which, as it was already mentioned, is the essential nature of dynamic mortality models. It is easy to obtain an initial correction of the estimates using the average annual improvement rate, $\overline{IR}$, which is calculated as the arithmetic mean of the improvement rates by age:

$$IR_x = \frac{q_x^t}{q_x^{t+1}}.$$

We have obtained that $\overline{IR} = 0.03$. Figure 2 shows the new graphic after this correction. Now, the set of corrected predictions lies between the observed values in 2010 and 2019.

Figure 2 shows how a simple correction provides reasonable estimates in magnitude for a short period of time. However, it can be seen that these estimates are not equally adequate for all ages. This is due to the fact that the improvement in mortality does not affect all ages in the same way and, therefore, it is necessary to introduce appropriate correction rates for each age, for example incorporating the history of mortality through delay terms as has benn carried out, in the deterministic system, in the previous work [30]. In this sense, the study of the stochastic model with delay would be an improvement of the current work. This is outside the scope of this study, which is a first approximation to the stochastic model, and will be considered in future works.

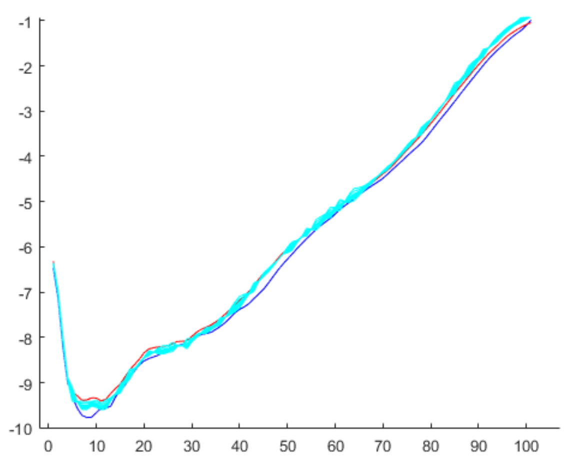

Figure 2: Ensemble of estimates of mortality

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