NEW RESULTS FOR STOCHASTIC FRACTIONAL PSEUDO-PARABOLIC EQUATIONS WITH DELAYS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Dedicated to María J. Garrido-Atienza, in Memoriam.

Abstract. In this work, four problems for stochastic fractional pseudo-parabolic containing bounded and unbounded delays are investigated. The fractional derivative and the stochastic noise we consider here are the Caputo operator and the fractional Brownian motion. For the two problems involving bounded delays, we aim at establishing global existence, uniqueness, and regularity results under integral Lipschitz conditions for the non-linear source terms. Such behaviors of mild solutions are also analyzed in the unbounded delay cases but under globally and locally Lipschitz assumptions. We emphasize that our results are investigated in the novel spaces $\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))$, $\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))$, and the weighted space $\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))$, instead of usual ones $\mathcal{C}([-r,T];L^2(\Omega,\mathcal{H}))$, $\mathcal{C}_\mu((-\infty,T];L^2(\Omega,\mathcal{H}))$. The main technique allowing us to overcome the rising difficulties lies on some useful Sobolev embeddings between the Hilbert space $\mathcal{H}=L^2(\mathbb{D})$ and $W^{l,q}(\mathbb{D})$, and some well-known fractional tools. In addition, we also study the Hölder continuity for the mild solutions, which can be considered as one of the main novelties of this paper. Finally, we consider an additional result connecting delay stochastic fractional pseudo-parabolic equations and delay stochastic fractional parabolic equations. We show that the mild solution of the first model converges to the mild solution of the second one, in some sense, as the diffusion parameter $\beta\to 0^+$.

Keywords: fractional pseudo-parabolic equations, fractional Brownian motion, bounded delay, unbounded delay, stochastic fractional differential equations.

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1. Introduction

1.1. **Stochatics PDEs with fractional derivative and delays.** Over the recent decades, the researchers interest in partial differential equations with non-integer orders derivatives (e.g. Caputo, Riemann-Liouville, Weyl, etc), which are also known as fractional differential equations (FDEs), has experienced a big development (see, for instance, [15, 16, 18]). Such models play a crucial role to analyze diffusion phenomena, especially in processes involving the effects of power-law of memory such as rheology, transport theory, or viscoelasticity, which cannot be modeled exactly by equations with classical integer order derivatives, see [13, 26, 27, 29, 30, 37, 48, 50, 55, 66–68]. An illustration for the comparison between two types of derivatives is their applications in fluid flows. The fractional derivative is more suitable to describe the behavior of some non-Newtonian fluids, while the classical one is often used for Newtonian fluids. On the other hand, it cannot be ignored that the appearance of random noise coming from natural sources often make physical phenomena fluctuate. Hence, in order to have a more acurate model, it is required to add some stochastic terms in the main equations.

Stochastic partial differential equations (SPDE) have been well studied in mathematics and other sciences, see, for example, Khoshnevisan [64] for a long list of references. The field of SPDEs is very interesting and attractive to mathematicians because it contains many challenging open problems. In the framework of this paper, we only emphasize and mention SPDEs with fractional derivatives. According to our search and understanding, the number of studies on stochastic fractional differential equations (SFDEs) has increased significantly recently. Here, we would like to study SFDEs with

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respect to Caputo derivative due to the following reason. The Caputo time fractional derivative possesses some advantages, namely, the derivative of a constant is zero, it removes singularities at the origin, and especially, it is more appropriate for initial value problems compared with the Riemann-Liouville definition [2, 31, 48]. Some interesting works on SFDEs with Caputo derivative can be found in the work of S.V. Lotosky et al [28], Nane et al. [61–63] and references therein.

There are several and diverse fascinating topics for SFDEs and we aim to steer the reader towards *SFDEs models with delays.* It is well-known that, in reality, the future behavior of a dynamical system is often affected by the previous and current states, which should be taken into account in the formulation. Therefore, it is natural to add some essential controls including delays and memory terms depending on the history of solutions in mathematical models. As a result, the theory of delay partial differential equations (DPDEs) has become an active area and attracted much attention of researchers, see Marín-Rubio et al. [40–42], Caraballo and Han [10], Liu et al. [33], Liu and Caraballo [34], Marín-Rubio et al. [41], Xu et al. [59] and references therein.

Although deterministic PDEs with delay have been studied extensively, to the best of our knowledge, stochastic PDEs (and FDEs especially) with delay still offer many challenging problems to be investigated. We refer to some impressive studies for which they have developed many new models for delay stochastic fractional differential equations (DSFDEs), see Li and Wang [32], Xu and Caraballo [56], Xu et al. [57, 58], Wang et al. [65], Chen and Yang [7, 8] and references therein.

1.2. **Setting our problem.** Motivated by the aforementioned considerations, in this paper, we are interested in considering some partial differential equations containing both fractional derivative, stochastic noise, and delays. Our main purpose is to investigate the following four problems for stochastic fractional pseudo-parabolic equations involving delays, Caputo fractional derivative, and fractional Brownian motion (fBm for short).

- The first two problems contain finite delays:
$$\begin{cases} \mathcal{D}_t^\alpha\big(x-\beta\Delta x\big)(t)+(-\Delta)^\gamma x(t)=f(t,x_t)+\sigma(t)\dot{B}_Q^H(t), \quad t\in[0,T],\\ x(t)|_{\partial\mathbb{D}}=0, \quad t\in[0,T],\\ x(t)=v(t), \quad t\in[-r,0], \end{cases} \tag{P1}$$
and
$$\begin{cases} \mathcal{D}_t^\alpha\big(x-\beta\Delta x\big)(t)+(-\Delta)^\gamma x(t)=\mathcal{I}_t^{1-\alpha}f(t,x_t)+\big[\mathcal{I}_t^{1-\alpha}\sigma(t)\big]\dot{B}_Q^H(t), \quad t\in[0,T],\\ x(t)|_{\partial\mathbb{D}}=0, \quad t\in[0,T],\\ x(t)=v(t), \quad t\in[-r,0], \end{cases} \tag{P2}$$
where $r>0$.
- The second couple of problems involves infinite delays:
$$\begin{cases} \mathcal{D}_t^\alpha\big(x-\beta\Delta x\big)(t)+(-\Delta)^\gamma x(t)=f(t,x_t)+\sigma(t)\dot{B}_Q^H(t), \quad t\in[0,T],\\ x(t)|_{\partial\mathbb{D}}=0, \quad t\in[0,T],\\ x(t)=v(t), \quad t\in(-\infty,0], \end{cases} \tag{P3}$$
and
$$\begin{cases} \mathcal{D}_t^\alpha\big(x-\beta\Delta x\big)(t)+(-\Delta)^\gamma x(t)=\mathcal{I}_t^{1-\alpha}f(t,x_t)+\big[\mathcal{I}_t^{1-\alpha}\sigma(t)\big]\dot{B}_Q^H(t), \quad t\in[0,T],\\ x(t)|_{\partial\mathbb{D}}=0, \quad t\in[0,T],\\ x(t)=v(t), \quad t\in(-\infty,0]. \end{cases} \tag{P4}$$

Here, $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ is a filtered complete probability space and $\mathbb{D}$ is a bounded domain of $\mathbb{R}^d$ with sufficiently smooth boundary. Denote by $-\Delta$ the negative Laplacian operator, $\mathcal{H}:=L^2(\mathbb{D})$, and (λ_k,ϕ_k) an eigenpair of $-\Delta$ satisfying $\lambda_{k+1}\ge\lambda_k>0$ for all $k\in\mathbb{N}$, $\lambda_k\to\infty$ as $k\to\infty$, and (ϕ_k) forms an orthonormal basis in $\mathcal{H}$.

The constant $\beta > 0$ is called the diffusion parameter, $\mathcal{D}_t^\alpha$ stands for the Caputo fractional derivative of order $\alpha \in (0,1)$, $\mathcal{I}_t^{1-\alpha}$ is the Riesz fractional integral (see Definition 2.1-2.2), $(-\Delta)^\gamma$ is the fractional operator of order $\gamma \in (0,1)$ (see Subsection 2.2), which can be considered as the infinitesimal generators of Lévy stable diffusion processes [38, 46]. The terms f, σ, and the initial function v will be specified later, $B_Q^H(t)$ is an $\mathcal{H}$-valued Q-cylindrical fractional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, with Hurst parameter $H \in (\frac{1}{2}, 1)$, and the generalized derivative $\dot{B}_Q^H(t)$ of $B_Q^H(t)$ is called fractional noise. For any x defined on $(-\infty, T]$ and any $t \in [0,T]$ we define the function x_t on $(-\infty, 0]$ as follows

$$x_t(\tau) = x(t+\tau), \quad \tau \in (-\infty, 0].$$

The main reason why we consider the additional problems (P2), (P4) beside Problems (P1), (P3) is as follows. For the two models (P1), (P3), the existence and regularity results can only be guaranteed when $\alpha \in (\frac{1}{2}, 1)$ and p satisfies some strict conditions, since the integrals in the mild formulations contain the singular kernel $(t-s)^{\alpha-1}$. This leads us to investigate an additional couple (P2), (P4), where we can extend the results obtained in the two old problems to (α, p) belonging to the whole domain $(0,1) \times [2, \infty)$ (thanks to the higher regularity of the terms in the right hand side, with the presence of the operator $\mathcal{I}_t^{1-\alpha}$).

In the deterministic case, some studies have been reported to fractional pseudo-parabolic equation, for example Sousa and Oliveira [49], Tuan et al. [53], Tuan and Caraballo [54]. Fractional stochastic pseudo-parabolic equations driven by fractional Brownian motion were probably first studied by Thach and Tuan [52]. Indeed, they established the existence, uniqueness, regularity results for mild solutions to an initial value problem for considered equations in two cases of H, i.e, $H > 1/2$ and $H < 1/2$. However delay models were not considered in Thach and Tuan [52].

In spite of the necessity of discussing such model with both delays and fractional Brownian motion, as far as we know, there is no result on delay stochastic fractional pseudo-parabolic equations. This motivates us to establish some results on the behavior of solutions to such model with both bounded and unbounded delays. Our interest in considering the above models also comes from interesting articles on stochastic evolution equations with a fractional Brownian motion and delays, for example Caraballo et al. [12]

Our next goal is to introduce in more detail the main equations that appear in the four problems above. The fractional pseudo-parabolic equations have some connections with other traditional equations as follows. In the case $\alpha = 1$ and $\beta > 0$, the above equations become classical pseudo-parabolic equations with integer order derivative, which is a crucial issue in several fields thanks to its successful applications in describing some physical phenomena, namely, the leakage of liquid through cracks in rocks or materials [14, 60], long waves and dispersive [36], and the aggregation of populations [47]. In the case $\alpha \in (0,1)$ and $\beta = 0$, the equations turn to be fractional parabolic equations (also called fractional diffusion equations), which are well-known models describing several anomalous transport like finance, biology, hydrogeology, etc (see [24, 31, 51]). Due to successful applications in diffusion models containing the effects of power-law of memory, fractional pseudo-parabolic equations have witnessed a great development up to date.

The final purpose of this subsection is to highlight the fBm stochastic noise used in our models. The inspiration for us to choose fBm came from the strong growth and attraction of many excellent mathematicians to study this model. Fractional Brownian motions are important to model complex phenomena when the systems are subject to rough external forcing. Furthermore, it is worth mentioning that this model is a correlated subclass of Gaussian process that turns to be the standard Brownian motion if the scaling exponent H (also called Hurst parameter) equals to 1/2. We refer the readers to [6, 12, 44] for more complete presentations on fractional Brownian motion, where we emphasize that Caraballo and his co-authors (see Lemma 2 of [12]) constructed a crucial technique to deal with stochastic differential equations driven by fractional Brownian motion with

Hurst parameter $H \in (1/2, 1)$. Important and interesting contributions to the field of stochastic differential equations with fBm can be found in many interesting works by Garrido-Atienza and her colleagues [5, 17, 19–21] .

1.3. **Our challenge and contribution.** Let us now briefly describe the main results, difficulties, and some remarks on the current paper.

- For the two models containing bounded delay (P1) and (P2), we aim at investigating the global existence, uniqueness, and regularity results, under an integral Lipschitz condition on the non-linear term $f : [0, T] \times \mathcal{C}([-h, 0]; L^p(\Omega, W^{l,q}(\mathbb{D}))) \to L^p(\Omega, W^{l',q}(\mathbb{D}))$, where $W^{l,q}(\mathbb{D})$ is the fractional Sobolev space. We emphasize that, when dealing with bounded DPDEs with fractional Brownian motion, the usual space $\mathcal{C}([-r, T]; L^2(\Omega, \mathcal{H})$ is often adopted, see [12, 35, 57]. However, the existences here are established in a novel space that is $\mathcal{C}([-r, T]; L^p(\Omega, W^{l,q}(\mathbb{D})))$, which seems to be the first approach to this topic.
- Regarding the two models involving unbounded delay (P3) and (P4), we construct global existence, uniqueness, and regularity results, under both globally and locally Lipschitz conditions for the function $f : [0, T] \times \mathcal{C}_\mu((-\infty, 0]; L^p(\Omega, W^{l,q}(\mathbb{D}))) \to L^p(\Omega, W^{l',q}(\mathbb{D}))$. Such results are investigated in the new spaces
$$\mathcal{C}_\mu((-\infty, T]; L^p(\Omega, W^{l,q}(\mathbb{D}))) \quad \text{and} \quad \mathcal{F}^\varepsilon_\mu((-\infty, T]; L^p(\Omega, W^{l,q}(\mathbb{D}))),$$
(see Remark 5.3), instead of the usual one $\mathcal{C}_\mu((-\infty, T]; L^2(\Omega, \mathcal{H}))$ (see the papers [32, 34, 56–58] which inspire our results here). To the best of our knowledge, no results have been reported on the existence of mild solutions to unbounded DPDEs under the Lipschitz assumptions mentioned above.
- One of the novelties of our paper is the question of Hölder continuity. As far as we know, in the studies reported about bounded and unbounded delay stochastic partial (and fractional) differential equations, the Hölder continuity result seems not to be considered very often. Hence, such result can be considered as one of our new outcomes. Theorems 3.1-3.2 (res. Theorems 4.1-4.2) can be considered as the first work on the Hölder continuity result for bounded (res. unbounded) delay stochastic PDEs.
- It should be noted that if the diffusion parameter β equals to zero, then fractional pseudo-parabolic equations turn to be traditional fractional parabolic equations. Hence, we are interested in considering a question rising on the connection between the two models. In other words, we show that the mild solution of the first model tends to the mild solution of the second one, in some sense, as β tends to zero, which can be considered as another novelty of this paper. Let us illustrate here the importance of the positive parameter β appearing in pseudo-parabolic equations in the description of a physical process, namely, the seepage of inhomogeneous fluids through a fissured rock [4, 9]. In this process, β plays the role of characterizing the fissured rock. If this parameter decreases, then there would be a reduction in block dimension and an increase in the degree of fissuring.
- Compared with the usual space $L^2(\Omega, \mathcal{H})$, when working with the space $L^p(\Omega, W^{l,q}(\mathbb{D}))$, we have to face several integrals with singular kernels (which could not converge without flexible estimates) and some challenges coming from the Lipschitz assumptions. Fortunately, to overcome this trouble, we can design and apply some useful Sobolev embeddings techniques like $\mathcal{H}_{\nu_*} \hookrightarrow \mathbb{H}_{2\nu_*} = W^{2\nu_*,2}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D})$ and $W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*}$, provided suitable conditions for parameters hold.
- Unlike the very recent study on fractional stochastic pseudo-parabolic equations driven by fBm [52], the additional delays associated with the Lipschitz assumptions involving fractional Sobolev spaces in this paper generates several difficulties, which cannot be handled by a similar technique as in [52]. The most challenging part is that, the existence results on the continuous space $\mathcal{C}([-r, T]; L^p(\Omega, W^{l,q}(\mathbb{D})))$ (res. $\mathcal{C}_\mu((-\infty, T]; L^p(\Omega, W^{l,q}(\mathbb{D}))))$ could hardly

be obtained if we do not design extremely sharp estimates containing a variety of complex parameters, which is not easy to control to ensure the convergence of the integrals with singular kernels appearing when do calculations and estimations.

The rest of the present paper is organized as follows. In Section 2, some materials including fractional analysis, functional spaces, stochastic settings, mild solutions, and needed properties, are introduced. Section 3 and Section 4 are devoted to establish existence, uniqueness, and regularity results for the two couple of problems separately. In Section 5, we investigate additional global existence results for Problem (P3) and Problem (P4), under locally Lipschitz condition for f. The last section is concerned with the connection between fractional pseudo-parabolic equations and fractional parabolic equations; namely, we show that the mild solutions of fractional pseudo-parabolic equations converge to the mild solutions of fractional parabolic equations in particular senses.

2. Preliminaries

In this section, we introduce some preliminaries including fractional analysis, functional spaces, stochastic settings (fractional Brownian motion and the corresponding stochastic integral), and the definitions of mild solutions. From now on, we use C to denote a general positive constant which may different from line to line and $a \lesssim b$ to imply $a \le Cb$.

2.1. Fractional analysis.

We begin with the definitions of the Riesz fractional integral, the Caputo fractional derivative, the Mittag-Leffler function and the Wright type function, as well as some needed properties.

Definition 2.1. [26,48] The Riesz fractional integral $\mathcal{I}_t^\alpha$, with $\alpha > 0$, of a function g is defined by

$$\mathcal{I}_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau) d\tau,$$

where Γ is the standard Gamma function.

Definition 2.2. [26,48] The Caputo fractional derivative $\mathcal{D}_t^\alpha$, with $\alpha \in (0,1)$, of a function g is defined by

$$\mathcal{D}_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau g(\tau) d\tau,$$

Definition 2.3. [48] Let $(\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}$ and $z \in \mathbb{C}$. The Mittag-Leffler function and the Wright type function are defined by

$$E_{\alpha_1,\alpha_2}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha_1 k + \alpha_2)}, \quad M_{\alpha_1}(z) := \sum_{k=0}^\infty \frac{(-1)^k z^k}{k!\Gamma(1-\alpha_1 k - \alpha_1)}.$$

For the sake of convenience, we denote $E_\alpha(z) := E_{\alpha,1}(z)$ and $\overline{E}_\alpha(z) := E_{\alpha,\alpha}(z)$, for $\alpha \in \mathbb{R}^+$.

The relations between the two particular Mittag-Leffler functions $E_\alpha, \overline{E}_\alpha$ and the Wright type function [22] are as follows

$$E_\alpha(-z) = \int_0^\infty M_\alpha(\tau) \exp(-z\tau) d\tau, \quad \overline{E}_\alpha(-z) = \int_0^\infty \alpha\tau M_\alpha(\tau) \exp(-z\tau) d\tau. \tag{1}$$

The following lemmas describe some useful properties of the two mentioned functions, which will be used throughout this paper.

Lemma 2.1 (see [22]). *For $\alpha \in (0,1)$ and $\epsilon > -1$, there holds*

$$\int_0^\infty \tau^\epsilon M_\alpha(\tau) d\tau = \Gamma(\epsilon+1)\Gamma^{-1}(\epsilon\alpha+1).$$

Lemma 2.2 (see [25])**.** *For $\alpha \in (0,1)$ and $\lambda > 0$, there holds*

$$\partial_t\big(t^{\alpha-1}\overline{E}_\alpha(-\lambda t^\alpha)\big) = t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha).$$

For some further details about the fractional calculus, the readers can refer to [22, 25, 48].

2.2. **Functional spaces and Sobolev embeddings.** We now collect the definitions of functional spaces used in this work and introduce some Sobolev embeddings. We start by the Hilbert scale space and the fractional Sobolev space. For a given number $\nu \geq 0$, we denote $\mathcal{H}_\nu$ the Hilbert scale space

$$\mathcal{H}_\nu := \left\{ u \in \mathcal{H} : \|u\|_{\mathcal{H}_\nu} = \Big(\sum_{k=1}^{\infty} \lambda_k^{2\nu}(u,\phi_k)^2 \Big)^{\frac{1}{2}} < \infty \right\},$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathcal{H} = L^2(\mathbb{D})$. Denote by $\mathcal{H}_{-\nu}$ the dual space of $\mathcal{H}_\nu$, provided that the dual space of $\mathcal{H}$ is itself [43]. Then, $\mathcal{H}_{-\nu}$ is a Hilbert space endowed with the norm $\|u\|_{\mathcal{H}_{-\nu}} = \Big(\sum_{k=1}^{\infty} \lambda_k^{-2\nu}(u,\phi_k)^2_{-\nu,\nu} \Big)^{\frac{1}{2}}$, where $(\cdot,\cdot)_{-\nu,\nu}$ is the dual product between $\mathcal{H}_{-\nu}$ and $\mathcal{H}_\nu$, which possesses the property

$$(u_1,u_2)_{-\nu,\nu} = (u_1,u_2), \quad \text{for } (u_1,u_2) \in \mathcal{H} \times \mathcal{H}_\nu.$$

The fractional operator $(-\Delta)^\nu : \mathcal{H}_{-\nu} \to \mathcal{H}_\nu$ can be defined as $(-\Delta)^\nu u = \sum_{k=1}^{\infty} \lambda_k^\nu (u,\phi_k)\phi_k$.

For $l \in \mathbb{Z}^+$ and $q \geq 1$, we denote by $W^{l,q}(\mathbb{D})$ the standard Sobolev space [1]. In the case, $l \in [0,1]$, the fractional Sobolev-type space [39] can be defined as follows

$$W^{l,q}(\mathbb{D}) = \left\{ w \in L^q(\mathbb{D}) \quad \text{s.t.} \quad \frac{|w(\xi_1) - w(\xi_2)|}{|\xi_1 - \xi_2|^{\frac{d}{q}+l}} \in L^q(\mathbb{D}\times\mathbb{D}) \right\},$$

which is known as an intermediary Banach space between $L^q(\mathbb{D})$ and $W^{1,q}(\mathbb{D})$, endowed with the norms

$$\|w\|_{W^{l,q}(\mathbb{D})} := \Big(\int_{\mathbb{D}} |w|^q d\xi_1 + \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|w(\xi_1) - w(\xi_2)|}{|\xi_1 - \xi_2|^{d+lq}} d\xi_1 d\xi_2 \Big)^{\frac{1}{q}}.$$

If $q = 2$ then $W^{l,q}(\mathbb{D})$ turns to be a Hilbert space and we denote $\mathbb{H}_l = W^{l,2}(\mathbb{D})$ for short. For more details about this fractional Sobolev space, the readers can refer to [3, 39].

We collect some needed properties for this space [39] in the following lemmas:

Lemma 2.3. *Let ν be a non-negative number. The following Sobolev embeddings hold*

i. $\mathcal{H}_{\nu/2} \hookrightarrow \mathbb{H}_\nu = W^{\nu,2}(\mathbb{D})$,

ii. $W^{\nu,p}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D})$, *if* ν, p, l, q *satisfy* $\begin{cases} p,q \in [1,\infty), \\ 0 \leq l \leq \nu < \infty, \\ \nu - \frac{d}{p} \geq l - \frac{d}{q}. \end{cases}$

iii. *if* $0 \leq l < \frac{d}{2}$ *and* $1 \leq m \leq \frac{2d}{d-2l}$, *or* $l = \frac{d}{2}$ *and* $m \geq 1$, *then* $\mathbb{H}_l = W^{l,2}(\mathbb{D}) \hookrightarrow L^m(\mathbb{D})$.

iv. *if* $-\frac{d}{2} < l' \leq 0$ *and* $m' \geq \frac{2d}{d-2l'}$ *then* $L^{m'}(\mathbb{D}) \hookrightarrow W^{l',2}(\mathbb{D}) = \mathbb{H}_{l'}$.

Lemma 2.4. *Let $0 \leq \nu \leq \nu' \leq 2$. The following Sobolev embedding hold*

$$\mathcal{H}_{\nu'/2} \hookrightarrow \mathcal{H}_{\nu/2} \hookrightarrow \mathbb{H}_\nu \hookrightarrow \mathcal{H} \hookrightarrow \mathbb{H}_{-\nu} \hookrightarrow \mathcal{H}_{-\nu/2} \hookrightarrow \mathcal{H}_{-\nu'/2}.$$

Next, we continue to introduce some solution spaces, which will be used when investigating the existence and regularity of mild solutions to the four problems in the present paper.

Let $\mathbb{B}$ be an arbitrary Banach space. We denote by $\mathcal{C}([a,b];\mathbb{B})$ the space of all continuous functions from $[a,b]$ into $\mathbb{B}$ endowed with the sup norm. Additionally, for $\mu>0$, we define the two spaces

$$\mathcal{C}_\mu((-\infty,0];\mathbb{B}) := \Big\{u\in\mathcal{C}((-\infty,0];\mathbb{B}) : \|u\|_{\mathcal{C}_\mu((-\infty,0];\mathbb{B})} := \sup_{\tau\in(-\infty,0]} e^{\mu\tau}\|u(\tau)\|_{\mathbb{B}} < \infty\Big\},$$

$$\mathcal{C}_\mu((-\infty,T];\mathbb{B}) := \Big\{u\in\mathcal{C}((-\infty,T];\mathbb{B}) : \|u\|_{\mathcal{C}_\mu((-\infty,T];\mathbb{B})} := \sup_{\tau\in(-\infty,T]} e^{\mu\tau}\|u(\tau)\|_{\mathbb{B}} < \infty\Big\}.$$

For fixed $v\in\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$ (res. $v\in\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))))$, with $(p,l,q)\in\times[2,\infty)\times[0,1]\times[1,\infty)$, we define the following Banach spaces

$$\mathbb{C}^v_{p,l,q} := \Big\{x\in\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D}))) : x(t)=v(t) \text{ for } t\in[-r,0]\Big\},$$

$$\mathbb{V}^v_{\mu,p,l,q} := \Big\{x\in\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D}))) : x(t)=v(t) \text{ for } t\in(-\infty,0]\Big\}.$$

2.3. **Fractional Brownian motion and stochastic integral.** In this section, we recall the fractional Brownian motion (fBm), the Wiener integral with respect to fBm, and some related results.

Definition 2.4. The one-dimensional fractional Brownian motion $\varrho^H(t)$, with Hurst parameter $H\in(0,1)$, is a continuous centered Gaussian process with covariance function

$$\mathbb{E}\big[\varrho^H(t)\varrho^H(\tau)\big] = \frac{|t|^{2H}+|\tau|^{2H}-|t-\tau|^{2H}}{2}.$$

If $H=\frac{1}{2}$ then this motion becomes the one-dimensional standard Brownian motion $\varrho(t)$. Furthermore, it is known from [45] that $\varrho^H(t)=\int_0^t K(t,\tau)d\varrho(\tau)$, where

$$K(t,\tau) = c_H\tau^{\frac{1}{2}-H}\int_\tau^t (s-\tau)^{H-\frac{3}{2}}s^{H-\frac{1}{2}}ds, \quad H\in(\frac{1}{2},1),$$

which satisfies that $\partial_t K(t,\tau)=c_H\left(\frac{t}{\tau}\right)^{H-\frac{1}{2}}(t-\tau)^{H-\frac{3}{2}}$, where

$$c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-1/2)}}, \quad \text{with } B \text{ being the Beta function.}$$

For $\psi\in L^2([0,T])$, the Wiener integral of ψ with respect to ϱ^H (see [6, 11, 45]) can be defined as

$$\int_0^T \psi(s)d\varrho^H(s) = \int_0^T K_H^*\psi(s)d\varrho(s),$$

where $K_H^*\psi(s)=\int_s^T\psi(\tau)\partial_\tau K(\tau,s)d\tau$.

Now, we aim at introducing an fBm taking values in the Hilbert space $\mathcal{H}$ as well as the stochastic integral with respect to it. Let $L(\mathcal{H},\mathcal{H}_\nu)$ be the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_\nu$. Let $Q\in L(\mathcal{H},\mathcal{H})$ be a non-negative self-adjoint operator such that $Q\phi_k=\Lambda_k\phi_k$ and $Tr(Q)=\sum_{k=1}^\infty\Lambda_k$ is finite. The infinite dimensional fBm on $\mathcal{H}$ with covariance Q (see [6, 11]) is defined as follows

$$\dot{B}^H_Q(t) = \sum_{k=1}^\infty \varrho^H_k(t)Q^{\frac{1}{2}}\phi_k = \sum_{k=1}^\infty \sqrt{\Lambda_k}\phi_k\varrho^H_k(t),$$

where $\varrho^H_k(t)$ are one-dimensional fractional Brownian motions. To define the Wiener integral with respect to above fBm, we introduce the space $L^2_{Q,\nu}=L^2(Q^{\frac{1}{2}}(\mathcal{H}),\mathcal{H}^\nu)$ of all Q -Hilbert–Schmidt operators Ψ from $Q^{\frac{1}{2}}(\mathcal{H})$ to $\mathcal{H}_\nu$, endowed with the norm $\|\Psi\|_{L^2_{Q,\nu}} := \left(\sum_{k=1}^\infty\|\Psi Q^{\frac{1}{2}}\phi_k\|^2_{\mathcal{H}_\nu}\right)^{\frac{1}{2}}$. If $\nu=0$, we denote L^2_Q instead of $L^2_{Q,0}$ for short.

Definition 2.5. (see [12]) Let $\Psi : [0,T] \to L^2_Q$ such that $\sum_{k=1}^{\infty} \|K_H^*(\Psi Q^{\frac{1}{2}}\phi_k)\|_{L^2([0,T];\mathcal{H})} < \infty$. The Wiener integral with respect to the fBm B_Q^H is defined as

$$\int_0^t \Psi(\tau)dB_Q^H(\tau) := \sum_{k=1}^{\infty}\int_0^t \Psi(\tau)Q^{\frac{1}{2}}\phi_k d\varrho_k^H(\tau) = \sum_{k=1}^{\infty}\int_0^t \big(K_H^*(\Psi Q^{\frac{1}{2}}\phi_k)\big)(\tau)\varrho_k(\tau),$$

where $\varrho_k^H(\tau)$ and $\varrho_k(\tau)$ are one-dimensional fBms and standard Brownian motions respectively.

It should be noted that the Itô isometry can not be applied to the Wiener integral with respect to fBm. Fortunately, to overcome this difficulty, we can use the following property:

Lemma 2.5. *(see [12]) For $\Psi : [0,T] \to L^2_Q$ satisfying $\int_0^T \|\Psi(s)\|_{L^2_Q} ds < \infty$ and $0 \le a < b \le T$, there holds*

$$\mathbb{E}\Big\|\int_a^b \Psi(s)dB_Q^H(s)\Big\|_{\mathcal{H}}^2 \le c_H H(2H-1)(b-a)^{2H-1}\int_a^b \|\Psi(s)\|^2_{L^2_Q} ds.$$

The last lemma of this subsection is devoted to the Kahane-Khintchine inequality, which is a well-known tool used to pass from order p to order q when estimating stochastic terms, see [69–71] for example.

Lemma 2.6 (see Theorem 5.3 in [73], Theorem 3.12 and Corollary 4.13 in [74])**.** *Let X be a normed space, $(g_k)_{k\ge1}$ be a sequence of independent standard Gaussian random variables. Then, for any finite family $x_1,\dots,x_n \in X$ and $1 \le p,q < \infty$ we have*

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^n x_k g_k\Big\|_X^p\Big)^{\frac{1}{p}} \le c_{p,q}\Big(\mathbb{E}\Big\|\sum_{k=1}^n x_k g_k\Big\|_X^q\Big)^{\frac{1}{q}},$$

where $c_{p,q}$ is the Kahane-Khintchine constant. Consequently, for any X-valued Gaussian variable $\mathcal{X}$ there holds

$$\big(\mathbb{E}\|\mathcal{X}\|_X^p\big)^{\frac{1}{p}} \le c_{p,q}\big(\mathbb{E}\|\mathcal{X}\|_X^q\big)^{\frac{1}{q}}.$$

2.4. Mild solutions.

Definition 2.6. A stochastic process x is called a mild solution of Problem (P1) (res. Problem (P3)) if it satisfies

$$x(t) = \begin{cases} v(t),\ t\in[-r,0],\ \big(\text{res. } v(t),\ t\in(-\infty,0]\big),\\ E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)v_0+\\ +\int_0^t (t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,x_s)ds+\\ +\int_0^t (t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s),\ t\in[0,T],\mathbb{P}-\text{a.s.},\end{cases}$$

where $v_0 = v(0)$, $\mathcal{A} = (I-\beta\Delta)^{-1}$ and

$$E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big) = \int_0^\infty M_\alpha(\tau)\exp\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\tau\big)d\tau,$$
$$\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big) = \int_0^\infty \alpha\tau M_\alpha(\tau)\exp\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\tau\big)d\tau.$$

Definition 2.7. A stochastic process x is called a mild solution of Problem (P2) (res. Problem (P4)) if it satisfies

$$x(t) = \begin{cases} v(t),\ t\in[-r,0],\ \big(\text{res. } v(t),\ t\in(-\infty,0]\big),\\ E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)v_0+\\ +\int_0^t E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,x_s)ds+\\ +\int_0^t E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s),\ t\in[0,T],\mathbb{P}-\text{a.s.},\end{cases}$$

In what follows, we aim at explaining the way of establishing the mild formulations as in above. For short, we only consider Problem (P1). The representations of mild solutions to Problems (P2)-(P4) can be constructed similarly.

Out strategy here is to represent the solution to Problem (P1) in the form $x(t) = \sum_{k=1}^{\infty}(x(t),\phi_k)\phi_k$ for $t \in [0,T]$. By taking the inner product of the first equation of Problem (P1) with ϕ_k, we have

$$\mathcal{D}_t^{\alpha}\big(x(t),\phi_k\big) + \frac{\lambda_k^{\gamma}}{1+\beta\lambda_k}\big(x(t),\phi_k\big) = \frac{1}{1+\beta\lambda_k}\Big(\big(f(t,x_t),\phi_k\big) + \sum_{j=1}^{\infty}\sqrt{\Lambda_j}\big(\sigma(t)\phi_j,\phi_k\big)\dot{\varrho}_j^{H}(t)\Big).$$

The above differential equation can be solved by applying the method of Laplace transforms [26] as

$$\begin{aligned}
\big(x(t),\phi_k\big) &= E_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)\big(x(0),\phi_k\big)+\\
&+\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}(t-s)^{\alpha}\right)(1+\beta\lambda_k)^{-1}\big(f(s,x_s),\phi_k\big)ds+\\
&+\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}(t-s)^{\alpha}\right)(1+\beta\lambda_k)^{-1}\sum_{j=1}^{\infty}\sqrt{\Lambda_j}\big(\sigma(s)e_j,e_k\big)\dot{\varrho}_j^{H}(s)ds.
\end{aligned}$$

Replacing the above formula into $x(t) = \sum_{k=1}^{\infty}(x(t),\phi_k)\phi_k$, we directly obtain the following explicit formulation for $x(t)$ in the form of a Fourier series with $t \in [0,T]$

$$\begin{aligned}
x(t) &= \sum_{k=1}^{\infty} E_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)\big(x(0),\phi_k\big)\phi_k+\\
&+\sum_{k=1}^{\infty}\Big(\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)(1+\beta\lambda_k)^{-1}\big(f(s,x_s),\phi_k\big)ds\Big)\phi_k+\\
&+\sum_{k=1}^{\infty}\Big(\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)(1+\beta\lambda_k)^{-1}\sum_{j=1}^{\infty}\sqrt{\Lambda_j}\big(\sigma(s)e_j,e_k\big)\dot{\varrho}_j^{H}(s)ds\Big)\phi_k. \qquad (2)
\end{aligned}$$

By the definition $\mathcal{A} = (I-\beta\Delta)^{-1}$ and the following two representations

$$E_{\alpha}\big(-(-\Delta)^{\gamma}(\mathcal{A}t^{\alpha}\big)(\cdot) = \sum_{k=1}^{\infty} E_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)\big(\cdot,\phi_k\big)\phi_k,$$

$$\overline{E}_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}t^{\alpha}\big)(\cdot) = \sum_{k=1}^{\infty} \overline{E}_{\alpha}\left(-\lambda_k^{\gamma}(1+\beta\lambda_k)^{-1}t^{\alpha}\right)\big(\cdot,\phi_k\big)\phi_k,$$

and noting that $x(0) = v_0$, we can rewrite (2) in a short expression as follows

$$\begin{aligned}
x(t) &= E_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}t^{\alpha}\big)v_0+\\
&+\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}(t-s)^{\alpha}\big)\mathcal{A}f(s,x_s)ds+\\
&+\int_0^t (t-s)^{\alpha-1}\overline{E}_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}(t-s)^{\alpha}\big)\mathcal{A}\sigma(s)dB_Q^{H}(s). \qquad (3)
\end{aligned}$$

On the other hand, remember that $x(t) = v(t)$ for $t \in [-r,0]$. By the two latter observations, the mild solution to Problem (P1) can be defined as in Definition 2.6.

In the following lemma, we represent several useful properties of the operators appearing in the above mild formulations.

Lemma 2.7. *Let $\alpha \in (0,1)$, $\nu \in [0,1]$, $p \geq 2$, δ be a positive number small enough and ν' be a non-positive number satisfying $\nu' \leq 0 \leq \nu \leq \nu'+1$. Then, the following properties hold true:*

(E1) For $t \in [0,T]$, the operators $E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ and $\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ are linear and bounded

$$\big\|E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \|u\|_{L^p(\Omega,\mathcal{H}_\nu)},$$
$$\big\|\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.$$

(E2) For $t \in [0,T]$, the operators $E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ and $\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ are Hölder continuous of exponent α

$$\begin{aligned}\Big\|\Big(&E_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta)^\alpha\big)\\ &- E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\Big)u\Big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \delta^\alpha \|u\|_{L^p(\Omega,\mathcal{H}_\nu)},\\ \Big\|\Big(&\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta)^\alpha\big)\\ &- \overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\Big)u\Big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \delta^\alpha \|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.\end{aligned}$$

(E3) The operator $\mathcal{A}$ is linear, bounded and $\|\mathcal{A}u\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})}$, where the hidden constant is $\beta^{-(\nu-\nu')}$.

(E4) For $t \in (0,T]$, the following continuity hold

$$\begin{aligned}\Big\|\Big((t+\delta)^{\alpha-1}&\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta)^\alpha\big)-\\ &- t^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\Big)\mathcal{A}u\Big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim t^{\alpha-1-\theta}\delta^\theta \|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})},\end{aligned}$$

where $\theta > 0$ is sufficiently small such that $0 < \theta < 1-\alpha$.

The proof of Lemma 2.7 can be found in Appendix.

3. Existence, uniqueness, and regularity results for Problem (P1) and Problem (P2)

In this section, we shall investigate the existence, uniqueness, and regularity results for Problem (P1) and Problem (P2) (which contain finite delays) under the following assumption

(H1) $f(t,0) = 0$ for $t \in [0,T]$, and there exists $\mathcal{L}_f > 0$ such that, for any X, Y belonging to $C([-r,T]; L^p(\Omega, W^{l,q}(\mathbb{D})))$ and $t \in [0,T]$, there holds

$$\int_0^t \big\|f(s,X_s) - f(s,Y_s)\big\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))} ds \le \mathcal{L}_f \int_{-r}^t \big\|X(s) - Y(s)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} ds.$$

Our main results in this section are stated as follows:

Theorem 3.1. *Let $\alpha \in (\frac{1}{2},1)$. Let $d \ge 1$, $v \in \mathcal{C}([-r,0]; L^p(\Omega, W^{l,q}(\mathbb{D})))$, and (H1) hold for some (d,p,q,l,l') satisfying*

$$\begin{cases} 1 + d\Big(\frac{1}{q} - \frac{1}{2}\Big) \ge 0, \ q \ge 1, \\ l' \le 0 \le l \le 2 - d\Big|\frac{1}{q} - \frac{1}{2}\Big| + l', \end{cases} \tag{4}$$

and $2 \le p < \frac{1}{1-\alpha}$. Assume further that

$$\begin{cases} \sigma \in L^m(0,T; L^2(\Omega, L^2_{Q,\nu'})), \quad \text{for some } m > \frac{p}{p(\alpha-1)+1}, \ \nu' \in [\nu'_*, 0], \\ v_0 \in L^p(\Omega, \mathcal{H}_\nu), \quad \text{for some } \nu \in [\nu_*, 1]. \end{cases} \tag{5}$$

Here, ν_* and ν'_* are defined by

$$\begin{cases} \nu_* := \nu_*(d,q,l) = \frac{1}{2}\Big(l + d\Big|\frac{1}{q} - \frac{1}{2}\Big|\chi_{\{q\ge 2\}}(q)\Big), \\ \nu'_* := \nu'_*(d,q,l') = \frac{1}{2}\Big(l' - d\Big|\frac{1}{q} - \frac{1}{2}\Big|\chi_{\{q<2\}}(q)\Big), \end{cases} \tag{6}$$

where χ is the indicator function. Then, we have

(1) Problem (P1) has a unique solution $x \in \mathbb{C}^v_{p,l,q}$,

(2) the following regularity property holds

$$\begin{aligned}&\|x\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim \|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))},\end{aligned}\tag{7}$$

where the hidden constant depends on β.

(3) the following Hölder continuity result holds for $t\in[0,T]$ and $\delta>0$ small enough

$$\begin{aligned}&\big\|x(t+\delta)-x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\&\qquad\lesssim \delta^\eta\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big),\end{aligned}$$

for some $\eta>0$ small enough such that $\eta<(1-\alpha)\wedge\big(\alpha-1+\frac{m-p}{pm}\big)$.

Theorem 3.2. *Let $\alpha\in(0,1)$. Let $d\geq 1$, $v\in\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$, and (H1) hold for some (d,p,q,l,l') satisfying (4) and $p\geq 2$. Assume further that*

$$\begin{cases}\sigma\in L^p(0,T;L^2(\Omega,L^2_{Q,\nu'})), & \text{for some } \nu'\in[\nu'_*,0],\\ v_0\in L^p(\Omega,\mathcal{H}_\nu), & \text{for some } \nu\in[\nu_*,1].\end{cases}\tag{8}$$

Then, we have

(1) Problem (P2) has a unique solution $x\in\mathbb{C}^v_{p,l,q}$,

(2) the following regularity property holds

$$\begin{aligned}&\|x\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim \|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))},\end{aligned}$$

(3) the following Hölder continuity result holds for $t\in[0,T]$ and $\delta>0$ small enough

$$\begin{aligned}&\big\|x(t+\delta)-x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\&\qquad\lesssim \delta^{\overline{\eta}}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big),\end{aligned}$$

where $\overline{\eta}=\alpha\wedge(H-\frac{1}{p})$.

Remark 3.1. It is quite usual when dealing with stochastic differential equations containing bounded delay that the non-linear term is often assumed to satisfy the following integral Lipschitz condition [12,57]

$$\int_0^t\big\|f(s,X_s)-f(s,Y_s)\big\|^2_{L^2(\Omega,\mathcal{H})}ds\leq\mathcal{L}_f\int_{-r}^t\big\|X(s)-Y(s)\big\|^2_{L^2(\Omega,\mathcal{H})}ds.$$

Then, by applying a similar technique as in Caraballo et al. [12] and Xu et al. [57], the existence and uniqueness of the solution can be established in the usual space $\mathcal{C}([-r,T];L^2(\Omega,\mathcal{H}))$. However, inspired from [12,57], in this section, we consider the condition for f in a different point of view, in which $L^2(\Omega,\mathcal{H})$ is replaced by $L^p(\Omega,W^{l,q}(\mathbb{D}))$ (or $L^p(\Omega,W^{l',q}(\mathbb{D}))$), which generates some new difficulties. It is now worth clarifying the points mentioned above as follows.

- (i) Considering L^p type spaces is more difficult than considering L^2 ones, mainly because several integrals with singular kernels appearing when establishing the existence results are not easy to handle (they could not converge if we do not design flexible estimates) and Lemma 2.5 could not be applied directly;
- (ii) It is useful that we are fortunate to have some connections between the Hilbert space $\mathcal{H}$ and the fractional Sobolev space $W^{l,q}(\mathbb{D})$, $W^{l',q}(\mathbb{D})$ (designed in Lemma 3.1 and Lemma 2.4), which help us overcome the challenges.

Remark 3.2. To the best of our knowledge, in the studies reported on the topic of bounded delay stochastic partial (and fractional) differential equations, the Hölder continuity result has not been investigated. Hence, such result can be considered as one of our new outcomes.

Remark 3.3. • It can be seen the main advantage of Model (P2) is that we can extend the existence, uniqueness, and regularity results obtained in Theorem 3.1 for Model (P1) when $\alpha \in (\frac{1}{2}, 1)$ to Model (P2) for α belonging to the whole interval $(0,1)$. Additionally, the condition for the parameter p can be extended from $2 \le p < \frac{1}{1-\alpha}$ to $p \ge 2$. This leads to consider the additional problem (P2) to complete our study on the delay case.
• Another difference between the two models is that the results in Theorem 3.2 for Model (P2) are obtained when $\sigma \in L^p(0,T; L^2(\Omega, L^2_{Q,\nu'}))$ instead of the more strict condition $\sigma \in L^m(0,T; L^2(\Omega, L^2_{Q,\nu'}))$, for some $m > \frac{p}{p(\alpha-1)+1}$, as in Theorem 3.1. The main reason is that the mild formulation in Problem (P1) contains the singular kernel $(t-s)^{\alpha-1}$.
• Due to the two different expressions of the two mild solutions to Problems (P1) and (P2) and their two distinct assumptions, there will rise some different estimates. Hence, in the proof of the result for Problem (P2), we just focus on essential differences and omit similar parts as in the first problem (P1).

Remark 3.4. It is unfortunate that $E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ only satisfy the Hölder continuous property (E2), instead of the desired form

$$\Big\| \big(E_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta)^\alpha\big) \\ - E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\big) u \Big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \delta^\alpha \|u\|_{L^p(\Omega,\mathcal{H}_{\nu_-})},$$

where $\nu_- < \nu$ and the hidden constant does not depend on t. This is the main reason why we require the additional condition $v_0 \in L^p(\Omega, \mathcal{H}_\nu)$ to guarantee the existence result on the continuous space $\mathcal{C}([-r,T]; L^p(\Omega, W^{l,q}(\mathbb{D})))$.

Before proving the two previoius theorems, we first prepare some useful materials, including properties of the two parameters ν_*, ν'_*, and two Sobolev embeddings.

Lemma 3.1. *Let (d,q,l,l') satisfy (4) and ν_*, ν'_* be defined as in (6). Then, there holds*

$$\nu'_* \le 0 \le \nu_* \le \nu'_* + 1. \tag{9}$$

Furthermore, the following Sobolev embeddings hold true

$$\mathcal{H}_{\nu_*} \hookrightarrow \mathbb{H}_{2\nu_*} = W^{2\nu_*,2}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D}),$$
$$W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*}.$$

Proof of Lemma 3.1. Initially, we shall prove property (9). From the definitions of ν_* and ν'_* in (6), it is obvious that $\nu'_* \le 0 \le \nu_*$. Therefore, we only need to verify that $\nu_* \le \nu'_* + 1$. Indeed, due to (6) and the last condition in (4), we have

$$2(\nu_* - \nu'_*) = l - l' + d\Big|\frac{1}{q} - \frac{1}{2}\Big|\big(\chi_{\{q\ge 2\}}(q) + \chi_{\{q<2\}}(q)\big) = l - l' + d\Big|\frac{1}{q} - \frac{1}{2}\Big| \le 2,$$

which implies that $\nu_* \le \nu'_* + 1$.

Next, we continue to verify the two Sobolev embeddings. Thanks to Lemma 2.4, one can see $\mathcal{H}_{\nu_*} \hookrightarrow \mathbb{H}_{2\nu_*} = W^{2\nu_*,2}(\mathbb{D})$. On the other hand, due to the definition of ν_*, it is clear that $0 \le l \le 2\nu_*$ and

$$2\nu_* - \frac{d}{2} = l + d\Big|\frac{1}{q} - \frac{1}{2}\Big|\chi_{\{q\ge 2\}}(q) - \frac{d}{2} \ge l + d\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{d}{2} = l - \frac{d}{q}.$$

Applying Lemma 2.3, it is obvious that $W^{2\nu_*,2}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D})$ holds true. The embedding (16) can be proved by a similar way as above. Therefore, we omit the details here. ☐

Proof of part (1) of Theorem 3.1. For $y:(-\infty,T]\to L^p(\Omega,W^{l,q}(\mathbb{D}))$, we define the operator $\mathcal{N}$ as follows

$$(\mathcal{N}y)(t)=\begin{cases} v(t),\ t\in[-r,0],\\ E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)v_0+\\ +\int_0^t(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,y_s)ds+\\ +\int_0^t(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s),\ t\in[0,T].\end{cases}$$

It what follows, we shall show that $\mathcal{N}\big(\mathbb{C}^v_{p,l,q}\big)\subset\mathbb{C}^v_{p,l,q}$ and then prove this operator has a unique fixed point.
Step 1: Firstly, we verify that $\mathcal{N}y\in\mathbb{C}^v_{p,l,q}$ if $y\in\mathbb{C}^v_{p,l,q}$. For t and δ satisfying $0\le t<t+\delta\le T$, the Sobolev embedding

$$\mathcal{H}_\nu\hookrightarrow\mathcal{H}_{\nu_*}\hookrightarrow\mathbb{H}_{2\nu_*}=W^{2\nu_*,2}(\mathbb{D})\hookrightarrow W^{l,q}(\mathbb{D}),\tag{10}$$

and Property (E2) of Lemma 2.7 directly yield

$$\begin{aligned}&\Big\|\Big(E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta)^\alpha\big)-E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)\Big)v_0\Big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\ &\quad\lesssim\Big\|\Big(E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta)^\alpha\big)-E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)\Big)v_0\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})}\\ &\quad\lesssim\delta^\alpha\|v_0\|_{L^p(\Omega,\mathcal{H}_{\nu_*})}\lesssim\delta^\alpha\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}.\end{aligned}\tag{11}$$

For the sake of convenience, we define the operator $\mathcal{M}_f$ and function $\mathcal{M}_\sigma$ as follows

$$(\mathcal{M}_fy)(t):=\int_0^t(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,y_s)ds,\tag{12}$$

$$\mathcal{M}_\sigma(t):=\int_0^t(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s).\tag{13}$$

We shall verify the continuity of the two above terms. It should be noted that, due to conditions $p\in[2,\frac{1}{1-\alpha})$, $m>\frac{p}{p(\alpha-1)+1}$, there always exists $\eta>0$, small enough, such that

$$\eta<(1-\alpha)\wedge\Big(\alpha-1+\frac{m-p}{pm}\Big).$$

Since η satisfies the above condition, it can be seen that

$$\eta<\alpha-\frac{1}{p}+\frac{1}{p}-1+\frac{m-p}{pm}=\alpha-\frac{1}{p}-\frac{(p-2)m+p}{pm}<\alpha-\frac{1}{p}.$$

With the help of the Sobolev embedding $\mathcal{H}_{\nu_*}\hookrightarrow\mathbb{H}_{2\nu_*}=W^{2\nu_*,2}(\mathbb{D})\hookrightarrow W^{l,q}(\mathbb{D})$, and the triangle inequality, we obtain

$$\begin{aligned}&\big\|(\mathcal{M}_fy)(t+\delta)-(\mathcal{M}_fy)(t)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\lesssim\big\|(\mathcal{M}_fy)(t+\delta)-(\mathcal{M}_fy)(t)\big\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})}\\ &\lesssim\mathbb{E}\Big\|\int_0^t\Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\ &\qquad-(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}f(s,y_s)ds\Big\|^p_{\mathcal{H}_{\nu_*}}+\\ &+\mathbb{E}\Big\|\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)\mathcal{A}f(s,y_s)ds\Big\|^p_{\mathcal{H}_{\nu_*}}\\ &\lesssim\mathcal{M}_f^\dagger(t,\delta)+\mathcal{M}_f^\ddagger(t,\delta),\end{aligned}$$

where, as above, for the sake of brevity and readability, we are denoting

$$\mathcal{M}_f^{\dagger}(t,\delta) := \mathbb{E}\Bigg[\int_0^t \Big\|\Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta-s)^\alpha\big)- \\ -(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}f(s,y_s)\Big\|_{\mathcal{H}_{\nu_*}} ds\Bigg]^p \tag{14}$$

and

$$\mathcal{M}_f^{\ddagger}(t,\delta) := \mathbb{E}\Bigg[\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\Big\|\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta-s)^\alpha\big)\mathcal{A}f(s,y_s)\Big\|_{\mathcal{H}_{\nu_*}} ds\Bigg]^p \tag{15}$$

Let us first estimate the term $\mathcal{M}_f^{\dagger}(t,\delta)$ as in (14). Applying the property (E4) with $\theta=\eta$, the Hölder inequality, the Sobolev embedding

$$W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*}, \tag{16}$$

and Assumption (H1), we can estimate the first term as

$$\begin{aligned}
\mathcal{M}_f^{\dagger}(t,\delta) &\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1-\eta}\delta^\eta \big\|f(s,y_s)\big\|_{\mathcal{H}_{\nu'_*}} ds\Big]^p \\
&\lesssim \delta^{p\eta}\Big(\int_0^t (t-s)^{\frac{p(\alpha-1-\eta)}{p-1}} ds\Big)^{p-1}\int_0^t \mathbb{E}\|f(s,y_s)\|^p_{\mathcal{H}_{\nu'_*}} ds \\
&\lesssim \delta^{p\eta} t^{p(\alpha-\eta)-1}\int_0^t \|f(s,y_s)\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))} ds \lesssim \delta^{p\eta}\mathcal{L}_f \int_{-r}^t \|y(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} ds \\
&\lesssim \delta^{p\eta}(T+r)\sup_{t\in[-r,T]}\|y(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))},
\end{aligned} \tag{17}$$

where we have used the fact that $p(\alpha-\eta)>1$, which follows from $\eta<\alpha-\frac{1}{p}$.

The next purpose is to obtain an upper bound for the second component $\mathcal{M}_f^{\ddagger}(t,\delta)$ defined in the expression (15). This term can be estimated by the properties (E1),(E3), the Hölder inequality, the Sobolev embedding (16), and Assumption (H1) as

$$\begin{aligned}
\mathcal{M}_f^{\ddagger}(t,\delta) &\lesssim \mathbb{E}\Big[\int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\big\|f(s,y_s)\big\|_{\mathcal{H}_{\nu'_*}} ds\Big]^p \\
&\lesssim \Big(\int_t^{t+\delta} (t+\delta-s)^{\frac{p(\alpha-1)}{p-1}} ds\Big)^{p-1}\int_t^{t+\delta} \mathbb{E}\|f(s,y_s)\|^p_{\mathcal{H}_{\nu'_*}} ds \\
&\lesssim \delta^{p\alpha-1}\int_0^{t+\delta} \|f(s,y_s)\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))} ds \\
&\lesssim \delta^{p\alpha-1}\mathcal{L}_f\int_{-r}^{t+\delta} \|y(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} ds \lesssim \delta^{p\alpha-1}(T+r)\sup_{t\in[-r,T]}\|y(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))},
\end{aligned} \tag{18}$$

where we note that $p\alpha-1>0$, which follows from $p\geq 2>\frac{1}{\alpha}$.

From the three latter estimates and noting that $\eta<\alpha-\frac{1}{p}$, it is obvious that

$$\big\|(\mathcal{M}_f y)(t+\delta)-(\mathcal{M}_f y)(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \delta^\eta \|y\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D}))}. \tag{19}$$

With the help of the Sobolev embedding (10), the triangle inequality, and the Kahane–Khintchine inequality, we can see that

$$
\begin{aligned}
&\big\|\mathcal{M}_\sigma(t+\delta)-\mathcal{M}_\sigma(t)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \big\|\mathcal{M}_\sigma(t+\delta)-\mathcal{M}_\sigma(t)\big\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})}\\
&\quad\lesssim \mathbb{E}\Big\|\int_0^t \Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\
&\qquad\qquad -(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^p_{\mathcal{H}_{\nu_*}}+\\
&\quad+\mathbb{E}\Big\|\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^p_{\mathcal{H}_{\nu_*}}\\
&\quad\lesssim \Big[\mathbb{E}\Big\|\int_0^t \Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\
&\qquad\qquad -(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^2_{\mathcal{H}_{\nu_*}}\Big]^{\frac{p}{2}}+\\
&\quad+\Big[\mathbb{E}\Big\|\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^2_{\mathcal{H}_{\nu_*}}\Big]^{\frac{p}{2}}\\
&\quad=: \mathcal{M}^\dagger_\sigma(t,\delta)+\mathcal{M}^\ddagger_\sigma(t,\delta).
\end{aligned}
\tag{20}
$$

where the hidden constant only depend on the parameter q.

By applying Lemma 2.5 with $a=0$ and $b=t$, and the Hölder inequality, the first term $\mathcal{M}^\dagger_\sigma(t,\delta)$ can be estimated as

$$
\begin{aligned}
\mathcal{M}^\dagger_\sigma(t,\delta) &\lesssim t^{\frac{p}{2}(2H-1)}\Big[\int_0^t \mathbb{E}\Big\|(-\Delta)^{\nu_*}\Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\
&\qquad -(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}\sigma(s)\Big\|^2_{L^2_Q}ds\Big]^{\frac{p}{2}}\\
&= t^{\frac{p}{2}(2H-1)}t^{\frac{p-2}{2}}\int_0^t \Big\|(-\Delta)^{\nu_*}\Big((t+\delta-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\
&\qquad -(t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}\sigma(s)\Big\|^p_{L^2(\Omega,L^2_Q)}ds.
\end{aligned}
$$

On account of property (E4) and the Sobolev embedding $\mathcal{H}_{\nu'}\hookrightarrow\mathcal{H}_{\nu'_*}$, we deduce

$$
\begin{aligned}
\mathcal{M}^\dagger_\sigma(t,\delta) &\lesssim t^{pH-1}\int_0^t (t-s)^{p(\alpha-1-\eta)}\delta^{p\eta}\big\|(-\Delta)^{\nu'_*}\sigma(s)\big\|^p_{L^2(\Omega,L^2_Q)}ds\\
&\lesssim t^{pH-1}\delta^{p\eta}\int_0^t (t-s)^{p(\alpha-1-\eta)}\big\|\sigma(s)\big\|^p_{L^2(\Omega,L^2_{Q;\nu'})}ds\\
&\lesssim t^{pH-1}\delta^{p\eta}\Big(\int_0^t (t-s)^{\frac{mp(\alpha-1-\eta)}{m-p}}ds\Big)^{\frac{m-p}{m}}\Big(\int_0^t \big\|\sigma(s)\big\|^m_{L^2(\Omega,L^2_{Q,\nu'})}ds\Big)^{\frac{p}{m}},
\end{aligned}
$$

where we note that

$$
\frac{mp(\alpha-1-\eta)}{m-p}>-1,\ pH-1\ge 2H-1>0.
$$

It follows that

$$
\mathcal{M}^\dagger_\sigma(t,\delta)\lesssim t^{pH-1}\delta^{p\eta}t^{p(\alpha-1-\eta)+\frac{m-p}{m}}\|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\lesssim \delta^{p\eta}\|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}. \tag{21}
$$

By applying Lemma 2.5 with $a = t$ and $b = t + \delta$, the Hölder inequality, properties (E1),(E3), and the Sobolev embedding $\mathcal{H}_{\nu'} \hookrightarrow \mathcal{H}_{\nu'_*}$, the second term $\mathcal{M}^{\ddagger}_{\sigma}(t, \delta)$ can be estimated as

$$\begin{aligned}
\mathcal{M}^{\ddagger}_{\sigma}(t,\delta) &\lesssim \delta^{\frac{p}{2}(2H-1)}\mathbb{E}\Big[\int_t^{t+\delta} \Big\|(-\Delta)^{\nu_*}(t+\delta-s)^{\alpha-1}\times \\
&\qquad\qquad \times \overline{E}_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}(t+\delta-s)^{\alpha}\big)\mathcal{A}\sigma(s)\Big\|^2_{L^2_Q} ds\Big]^{\frac{p}{2}} \\
&\lesssim \delta^{\frac{p}{2}(2H-1)}\delta^{\frac{p-2}{2}}\int_t^{t+\delta}(t+\delta-s)^{p(\alpha-1)}\big\|(-\Delta)^{\nu'_*}\sigma(s)\big\|^p_{L^2(\Omega,L^2_Q)}ds \\
&\lesssim \delta^{pH-1}\Big(\int_t^{t+\delta}(t+\delta-s)^{\frac{mp(\alpha-1)}{m-p}}ds\Big)^{\frac{m-p}{m}}\Big(\int_t^{t+\delta}\|\sigma(s)\|^m_{L^2(\Omega,L^2_{Q,\nu'})}ds\Big)^{\frac{p}{m}},
\end{aligned}$$

where we note that $\frac{mp(\alpha-1)}{m-p} > -1$, $pH - 1 > 0$. This leads to

$$\mathcal{M}^{\ddagger}_{\sigma}(t,\delta) \lesssim \delta^{pH-1+p(\alpha-1)+\frac{m-p}{m}}\|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}. \tag{22}$$

Combining (20)-(22), and noting that $\eta < \alpha - 1 + \frac{m-p}{pm} < \frac{1}{p}\big(pH - 1 + p(\alpha-1) + \frac{m-p}{m}\big)$, we deduce that

$$\big\|\mathcal{M}_{\sigma}(t+\delta) - \mathcal{M}_{\sigma}(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \delta^{\eta}\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}. \tag{23}$$

Now, from (11), (19), (23), we conclude that $\mathcal{N}y$ is Hölder continuous of exponent η, namely

$$\begin{aligned}
&\big\|(\mathcal{N}y)(t+\delta) - (\mathcal{N}y)(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\
&\qquad \lesssim \delta^{\eta}\big(\|v_0\|_{L^p(\Omega,\mathcal{H}_{\nu})} + \big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|y\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\big).
\end{aligned}$$

Hence, we conclude that $\mathcal{N}(\mathbb{C}^v_{p,l,q}) \subset \mathbb{C}^v_{p,l,q}$.
Step 2: Next, we aim at verifying that $\mathcal{N} : \mathbb{C}^v_{p,l,q} \to \mathbb{C}^v_{p,l,q}$ has a unique fixed point. Indeed, for $y, y^* \in \mathbb{C}^v_{p,l,q}$ and $t \in [0, T]$, we have

$$(\mathcal{N}y)(t) - (\mathcal{N}y^*)(t) = (\mathcal{M}_f y)(t) - (\mathcal{M}_f y^*)(t). \tag{24}$$

The term on the right hand side can be estimated by the properties (E1),(E3), the Hölder inequality, and the two Sobolev embeddings (10),(16), as follows

$$\begin{aligned}
&\|(\mathcal{M}_f y)(t) - (\mathcal{M}_f y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \|(\mathcal{M}_f y)(t) - (\mathcal{M}_f y^*)(t)\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})} \\
&\quad\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1}\Big\|\overline{E}_{\alpha}\big(-(-\Delta)^{\gamma}\mathcal{A}(t-s)^{\alpha}\big)\mathcal{A}\big(f(s,y_s) - f(s,y^*_s)\big)\Big\|_{\mathcal{H}_{\nu_*}}ds\Big]^p \\
&\quad\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1}\big\|f(s,y_s) - f(s,y^*_s)\big\|_{\mathcal{H}_{\nu'_*}}ds\Big]^p \\
&\quad\lesssim \Big(\int_0^t (t-s)^{\frac{p(\alpha-1)}{p-1}}ds\Big)^{p-1}\int_0^t \mathbb{E}\|f(s,y_s) - f(s,y^*_s)\|^p_{\mathcal{H}_{\nu'_*}}ds \\
&\quad\lesssim t^{p\alpha-1}\int_0^t \|f(s,y_s) - f(s,y^*_s)\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))}ds.
\end{aligned}$$

By using Assumption (H1), $y(t) = y^*(t) = v(t)$ on $[-r, 0]$, and noting that $p\alpha - 1 > 0$, we obtain

$$\|(\mathcal{M}_f y)(t) - (\mathcal{M}_f y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \mathcal{L}_f\int_0^t \|y(s) - y^*(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds. \tag{25}$$

Combining (24)-(25), we deduce that there exists a constant $C > 0$ independent of t such that

$$\begin{aligned}\|(\mathcal{N}y)(t) - (\mathcal{N}y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\leq C\int_0^t \big\|y(s) - y^*(s)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds\\ &\leq C\int_0^t \sup_{\tau\in[0,s]}\|y(\tau) - y^*(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds,\end{aligned}$$

which shows that the following inequality holds for $k = 1$

$$\sup_{\tau\in[0,t]}\|(\mathcal{N}^k y)(\tau) - (\mathcal{N}^k y^*)(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq C^k(k!)^{-1}t^k \sup_{\tau\in[0,T]}\|y(\tau) - y^*(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}. \tag{26}$$

Assume that (26) holds true for $k \geq 1$, we shall show that it also holds for $k+1$. Indeed

$$\begin{aligned}\|(\mathcal{N}^{k+1}y)(t)-(\mathcal{N}^{k+1}y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\leq C\int_0^t \sup_{\tau\in[0,s]}\|(\mathcal{N}^k y)(\tau) - (\mathcal{N}^k y^*)(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds\\ &\leq C\int_0^t C^k(k!)^{-1}s^k \sup_{\tau\in[0,T]}\|y(\tau) - y^*(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds\\ &\leq C^{k+1}[(k+1)!]^{-1}t^{k+1}\sup_{\tau\in[0,T]}\|y(\tau) - y^*(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}.\end{aligned}$$

Inequality (26) leads to the following result, which is of importance in showing the contraction property of the operator $\mathcal{N}$

$$\sup_{\tau\in[0,T]}\|(\mathcal{N}^k y)(\tau) - (\mathcal{N}^k y^*)(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq C^k(k!)^{-1}T^k \sup_{\tau\in[0,T]}\|y(\tau) - y^*(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}.$$

With the help of the above inequality and noting that $C^k(k!)^{-1}T^k$ tends to zero as $k \to \infty$, we deduce that there exists $k \geq 1$ such that $\mathcal{N}^k$ is a contraction in $\mathbb{C}^v_{p,l,q}$, which implies $\mathcal{N}^k x = x$ has a unique solution $x \in \mathbb{C}^v_{p,l,q}$. Consequently, it holds $\mathcal{N}^k(\mathcal{N}x) = \mathcal{N}(\mathcal{N}^k x) = \mathcal{N}x$. Hence, we conclude that $\mathcal{N}x = x$ has a unique solution $x \in \mathbb{C}^v_{p,l,q}$. $\square$

Proof of part (2) of Theorem 3.1. By a similar way as in the Proof of part (1) of Theorem 3.1, one can easily verify that, for $t \in [0,T]$, there holds

$$\big\|x(t)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \int_{-r}^t \big\|x(s)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds + \big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}.$$

It follows that there exists a positive constant C independent of t and β such that

$$\begin{aligned}\sup_{\tau\in[0,t]}\big\|x(\tau)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\leq C\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + C\|v\|^p_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}+\\ &\quad + C\int_0^t \sup_{\tau\in[0,s]}\big\|x(\tau)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds + C\big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}.\end{aligned}$$

The Gronwall inequality allows us to obtain

$$\begin{aligned}&\sup_{\tau\in[0,t]}\big\|x(\tau)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\ &\qquad\leq Ce^{Ct}\Big(\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \|v\|^p_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D}))} + \big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\Big).\end{aligned}$$

Since $e^{Ct} \leq e^{CT}$ and $x = v \in \mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$ on $[-r,0]$, there holds

$$\begin{aligned}\big\|x\big\|^p_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D}))} &= \sup_{\tau\in[-r,T]}\big\|x(\tau)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\ &\lesssim \|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \|v\|^p_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D}))} + \big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))},\end{aligned}$$

which implies that the regularity (7) holds. □

Proof of part (3) of Theorem 3.1. This property can be proved easily by arguing similarly to the Proof of part (1) of Theorem 3.1. In this way, one arrives at

$$\begin{aligned}&\big\|x(t+\delta)-x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\&\qquad\lesssim \delta^{\eta}\big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|x\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}+\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\big).\end{aligned}$$

This together with property (7) leads to

$$\begin{aligned}&\big\|x(t+\delta)-x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\&\qquad\lesssim \delta^{\eta}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}+\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\Big).\end{aligned}$$

The proof is therefore complete. □

Proof of part (1) of Theorem 3.2. For $y:(-\infty,T]\to L^p(\Omega,W^{l,q}(\mathbb{D}))$, we define the operator $\overline{\mathcal{N}}$ as follows

$$(\overline{\mathcal{N}}y)(t)=\begin{cases}v(t),\ \ t\in[-r,0],\\ E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)v_0+\\ \ +\int_0^t E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,y_s)ds+\\ \ +\int_0^t E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s),\ \ t\in[0,T].\end{cases}$$

First, we verify that $\overline{\mathcal{N}}y\in\mathbb{C}^v_{p,l,q}$ if $y\in\mathbb{C}^v_{p,l,q}$. For the sake of convenience, for $t\in[0,T]$, we define the two operator $\overline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_\sigma$ as follows

$$\begin{aligned}(\overline{\mathcal{M}}_f y)(t)&:=\int_0^t\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s,y_s)ds,\\ \overline{\mathcal{M}}_\sigma(t)&:=\int_0^t\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB_Q^H(s).\end{aligned}$$

To verify the Höler continuity of the two above terms, one can use a similar technique as in Step 1 in the proof of part (1) of Theorem 3.1, but the properties (E2),(E3), and Assumption $\sigma\in L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))$ are used instead of the (E4) and Assumption $\sigma\in L^m(0,T;L^2(\Omega,L^2_Q\nu'))$. In this way, for t and δ satisfying $0\le t<t+\delta\le T$, one has the following estimate for the first term

$$\begin{aligned}\big\|(\overline{\mathcal{M}}_f y)(t+\delta)-(\overline{\mathcal{M}}_f y)(t)\big\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}&\lesssim\big\|(\overline{\mathcal{M}}_f y)(t+\delta)-(\overline{\mathcal{M}}_f y)(t)\big\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})}\\&\lesssim\mathbb{E}\Big[\int_0^t\Big\|\Big(E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)-\\&\qquad\qquad-E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}f(s,y_s)\Big\|_{\mathcal{H}_{\nu_*}}ds\Big]^p+\\&\quad+\mathbb{E}\Big[\int_t^{t+\delta}\Big\|E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta-s)^\alpha\big)\mathcal{A}f(s,y_s)\Big\|_{\mathcal{H}_{\nu_*}}ds\Big]^p\\&\lesssim\delta^{p\alpha}t^{p-1}\int_0^t\mathbb{E}\big\|f(s,y_s)\big\|^p_{\mathcal{H}_{\nu'_*}}ds+\delta^{p-1}\int_t^{t+\delta}\mathbb{E}\big\|f(s,y_s)\big\|^p_{\mathcal{H}_{\nu'_*}}ds\\&\lesssim\delta^{p\overline{\eta}_1}\int_0^{t+\delta}\big\|f(s,y_s)\big\|^p_{L^p(\Omega,W^{l',,q})}ds\lesssim\delta^{p\overline{\eta}_1}\mathcal{L}_f\int_{-r}^{t+\delta}\|y(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds,\end{aligned}\tag{27}$$

where we set $\overline{\eta}_1 := \alpha \wedge (1 - \frac{1}{p})$. On the other hand, the second term can be estimated as

$$\begin{aligned}
\big\|\overline{\mathcal{M}}_\sigma(t+\delta) - \overline{\mathcal{M}}_\sigma(t)\big\|^p_{L^p(\Omega, W^{l,q}(\mathbb{D}))} &\lesssim \big\|\overline{\mathcal{M}}_\sigma(t+\delta) - \overline{\mathcal{M}}_\sigma(t)\big\|^p_{L^p(\Omega, \mathcal{H}_{\nu_*})} \\
&\lesssim \Big[\mathbb{E}\Big\| \int_0^t \Big(E_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t+\delta-s)^\alpha\big) - \\
&\qquad - E_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t-s)^\alpha\big)\Big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^2_{\mathcal{H}_{\nu_*}}\Big]^{\frac{p}{2}} + \\
&\quad + \Big[\mathbb{E}\Big\| \int_t^{t+\delta} E_\alpha\big(-(-\Delta)^\gamma \mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s)dB^H_Q(s)\Big\|^2_{\mathcal{H}_{\nu_*}}\Big]^{\frac{p}{2}} \\
&\lesssim t^{pH-1}\int_0^t \delta^{p\alpha}\|\sigma(s)\|^p_{L^2(\Omega, L^2_{Q,\nu'})} ds + \delta^{pH-1}\int_t^{t+\delta} \|\sigma(s)\|^p_{L^2(\Omega, L^2_{Q,\nu'})} ds \\
&\lesssim \delta^{p\overline{\eta}}\|\sigma\|^p_{L^p(0,T;L^2(\Omega, L^2_{Q,\nu'}))}, \qquad (28)
\end{aligned}$$

where we set $\overline{\eta} := \alpha \wedge (H - \frac{1}{p})$, which is less than $\overline{\eta}_1$. Combining (11), (27), (28), and noting that

$$\int_{-r}^{t+\delta} \|y(s)\|^p_{L^p(\Omega, W^{l,q}(\mathbb{D}))} ds \le (T+r)\|y\|^p_{\mathcal{C}([-r,T];L^p(\Omega, W^{l,q}(\mathbb{D})))},$$

we conclude that $\overline{\mathcal{N}}y$ is Hölder continuous of exponent $\overline{\eta} = \alpha \wedge (H - \frac{1}{p})$

$$\begin{aligned}
&\big\|(\overline{\mathcal{N}}y)(t+\delta) - (\overline{\mathcal{N}}y)(t)\big\|_{L^p(\Omega, W^{l,q}(\mathbb{D}))} \\
&\quad \lesssim \delta^{\overline{\eta}}\big(\|v_0\|_{L^p(\Omega, \mathcal{H}_\nu)} + \|y\|_{\mathcal{C}([-r,T];L^p(\Omega, W^{l,q}(\mathbb{D})))} + \|\sigma\|_{L^p(0,T;L^2(\Omega, L^2_{Q,\nu'}))}\big),
\end{aligned}$$

which leads to $\overline{\mathcal{N}}(\mathbb{C}^v_{p,l,q}) \subset \mathbb{C}^v_{p,l,q}$.

Next, by a similar technique as in Step 2 in the proof of part (1) of Theorem 3.1, but Assumption $\sigma \in L^p(0,T;L^2(\Omega, L^2_{Q,\nu'}))$ is used instead of $\sigma \in L^m(0,T;L^2(\Omega, L^2_{Q,\nu'}))$, one can easily verify that $\overline{\mathcal{N}} : \mathbb{C}^v_{p,l,q} \to \mathbb{C}^v_{p,l,q}$ has a unique fixed point. Therefore, we omit the details here. □

Proof of parts (2) and (3) of Theorem 3.2. By applying the Gronwall inequality and the two following Sobolev embeddings

$$\mathcal{H}_{\nu_*} \hookrightarrow \mathbb{H}_{2\nu_*} = W^{2\nu_*,2}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D})$$

and

$$W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*},$$

and using a similar argument as in the proof of part (1) of Theorem 3.2, one can easily verify properties (2), (3). Therefore, we omit the details. □

4. Existence, uniqueness, and regularity results for Problem (P3) and Problem (P4) under globally Lipschitz condition

In this section, we shall investigate the existence, uniqueness, and regularity results for Problem (P3) and Problem (P4) (which contain infinite delays) under the following globally Lipschitz condition

(H2) $f(t,0) = 0$ for $t \in [0,T]$, and there exists $\mathcal{L}^*_f > 0$ such that, for any u, v belonging to $\mathcal{C}_\mu((-\infty,0];L^p(\Omega, W^{l,q}(\mathbb{D})))$ and $t \in [0,T]$, there holds

$$\big\|f(t,u) - f(t,v)\big\|_{L^p(\Omega, W^{l',q}(\mathbb{D}))} \le \mathcal{L}^*_f\|u - v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega, W^{l,q}(\mathbb{D})))}.$$

Our main results in this section are clearly stated in the following two theorems

Theorem 4.1. *Let $\alpha \in (\frac{1}{2}, 1)$. Let $d \ge 1$, $v \in \mathcal{C}_\mu((-\infty,0];L^p(\Omega, W^{l,q}(\mathbb{D})))$, and let (H2) hold for some (d,p,q,l,l') satisfying (4), $2 \le p < \frac{1}{1-\alpha}$, and $\mu > 0$. Assume further that σ, v_0 satisfy (5). Then,*

(1) Problem (P3) *has a unique solution* $x \in \mathbb{V}^v_{\mu,p,l,q}$,
(2) the following regularity property holds

$$\begin{aligned} &\|x\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\quad \lesssim \|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}, \end{aligned} \tag{29}$$

(3) the following Hölder continuity result holds for $t \in [0,T]$ *and* $\delta > 0$ *small enough*

$$\begin{aligned} &\big\|x(t+\delta) - x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\ &\quad \lesssim \delta^\eta \Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \Big), \end{aligned}$$

for some $\eta > 0$ *small enough such that* $\eta < (1-\alpha) \wedge (\alpha - 1 + \frac{m-p}{pm})$.

Theorem 4.2. *Let* $\alpha \in (0,1)$. *Let* $d \geq 1$, $v \in \mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$, *and let (H2) hold for some* (d,p,q,l,l') *satisfying* (4), $p \geq 2$, *and* $\mu > 0$. *Assume further that* σ, v_0 *satisfy* (8). *Then,*
(1) Problem (P4) *has a unique solution* $x \in \mathbb{V}^v_{\mu,p,l,q}$,
(2) the following regularity property holds

$$\begin{aligned} &\|x\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\quad \lesssim \|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}, \end{aligned}$$

(3) the following Hölder continuity result holds for $t \in [0,T]$ *and* $\delta > 0$ *small enough*

$$\begin{aligned} &\big\|x(t+\delta) - x(t)\big\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\ &\quad \lesssim \delta^{\overline{\eta}} \Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \Big), \end{aligned}$$

where $\overline{\eta} = \alpha \wedge \big(H - \frac{1}{p}\big)$.

Remark 4.1. Notice that Assumption (H2) used in the unbounded delay case is simple global Lipschitz condition, while Assumption (H1) imposed for bounded delay models is an integral Lipschitz condition. This reason justifying this difference comes from the fact that the following inequality holds in the infinite delay case

$$\sup_{\tau\leq 0} e^{\mu\tau}\|y_s(\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq e^{\mu(t-s)} \sup_{\tau\leq 0} e^{\mu\tau}\|y_t(\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}, \quad \text{for } s < t,$$

which is not guaranteed in the finite delay case.

Due to the previous difference and the two different Lipschitz assumptions adopted in both cases, the estimates and the solution space we work on in the infinite delay case are significantly different from the finite delay one, and it is therefore necessary to analyze the infinite delay case to complete our study.

Remark 4.2. To our knowledge, as in the bounded delay case, the Hölder continuity has not been investigated for stochastic partial (and fractional) differential equations with unbounded delay,. Hence, such result can be considered as one of our new outcomes.

Remark 4.3. Let us now briefly discuss some comments similar to those in Remark 3.1. It is quite usual when dealing with stochastic differential equations containing unbounded delay that the non-linear term is often assumed to satisfy the following Lipschitz condition [32, 34, 56, 57]

$$\big\|f(t,u) - f(t,v)\big\|_{L^2(\Omega,\mathcal{H})} \leq \mathcal{L}^*_f \|u - v\|_{\mathcal{C}_\mu((-\infty,0];L^2(\Omega,\mathcal{H}))}.$$

Then, by applying a similar technique as in [32,34,56,57], the existence and uniqueness of the solution can be established on the usual space $\mathcal{C}_\mu((-\infty,T];L^2(\Omega,\mathcal{H}))$. However, inspired from [32, 34, 56, 57], we make some differences, namely $L^2(\Omega,\mathcal{H})$ is replaced by $L^p(\Omega,W^{l,q}(\mathbb{D}))$ (or $L^p(\Omega,W^{l',q}(\mathbb{D}))$), see Remark 3.1 again for some discussions and relations between those spaces.

Remark 4.4. Like Model (P2), the main benefit of Model (P4) is that the results obtained in Theorem 4.1 for Model (P3) when $\alpha \in (\frac{1}{2}, 1)$ can be extended to Model (P4) for α belonging to the whole interval $(0,1)$. Furthermore, the condition for the parameter p can be extended from $2 \le p < \frac{1}{1-\alpha}$ to $p \ge 2$.

Proof of part (1) of Theorem 4.1. For $y : (-\infty, T] \to L^p(\Omega, W^{l,q}(\mathbb{D}))$, we define the operator $\mathcal{N}^*$ as follows

$$(\mathcal{N}^* y)(t) = \begin{cases} v(t), \ t \in (-\infty, 0], \\ E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big) v_0 + (\mathcal{M}_f y)(t) + \mathcal{M}_\sigma(t), \ t \in [0,T], \end{cases} \tag{30}$$

where $\mathcal{M}_f$ and $\mathcal{M}_\sigma$ are defined in (12)-(13).
Step 1: In this step, we aim at verifying that $\mathcal{N}^* y \in \mathbb{V}^v_{\mu,p,l,q}$ if $y \in \mathbb{V}^v_{\mu,p,l,q}$. As we have proved in Step 1 in the proof of part (1) of Theorem 3.1, for any $t \in [0,T]$ and $\eta < (1-\alpha) \wedge \big(\alpha - 1 + \frac{m-p}{pm}\big)$ there holds

$$\begin{aligned} \|(\mathcal{N}^* y)(t+\delta) - (\mathcal{N}^* y)(t)\|^p_{L^p(\Omega, W^{l,q}(\mathbb{D}))} &\lesssim \|(\mathcal{N}^* y)(t+\delta) - (\mathcal{N}^* y)(t)\|^p_{L^p(\Omega, \mathcal{H}_{\nu_*})} \\ \lesssim \delta^{p\alpha} \|v_0\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \|(\mathcal{M}_f y)(t+\delta) &- (\mathcal{M}_f y)(t)\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \\ &+ \|\mathcal{M}_\sigma(t+\delta) - \mathcal{M}_\sigma(t+\delta)\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*})} \\ \lesssim \delta^{p\eta}\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \mathcal{M}^\dagger_f(t,\delta) + \mathcal{M}^\ddagger_f(t,\delta) &+ \delta^{p\eta} \big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega, L^2_{Q,\nu'}))}, \end{aligned} \tag{31}$$

Next, we only need to bound the two terms $\mathcal{M}^\dagger_f(t,\delta)$ and $\mathcal{M}^\ddagger_f(t,\delta)$ under the new condition (H2). Recalling that in (17)-(18), we have shown

$$\begin{aligned} \mathcal{M}^\dagger_f(t,\delta) &\lesssim \delta^{p\eta} t^{p(\alpha-\eta)-1} \int_0^t \|f(s,y_s)\|^p_{L^p(\Omega, W^{l',q}(\mathbb{D}))} ds, \\ \mathcal{M}^\ddagger_f(t,\delta) &\lesssim \delta^{p\alpha-1} \int_0^{t+\delta} \|f(s,y_s)\|^p_{L^p(\Omega, W^{l',q}(\mathbb{D}))} ds. \end{aligned}$$

By Assumption (H2), the Sobolev embedding $W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*}$, and the property

$$\|y_s\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \le \|y\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))},$$

and noting that $p(\alpha-\eta) - 1 > 0$, $\eta < 1-\alpha < \alpha$, we obtain the following estimate

$$\begin{aligned} \mathcal{M}^\dagger_f(t,\delta) + \mathcal{M}^\ddagger_f(t,\delta) &\lesssim \delta^{p\eta} |\mathcal{L}^*_f|^p \int_0^t \|y_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} ds + \\ &\quad + \delta^{p\alpha-1} |\mathcal{L}^*_f|^p \int_t^{t+\delta} \|y_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,,W^{l,q}(\mathbb{D})))} ds \\ &\le \big(\delta^{p\eta} T + \delta^{p\alpha}\big) |\mathcal{L}^*_f|^p \|y\|^p_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\lesssim \delta^{p\eta} |\mathcal{L}^*_f|^p \|y\|^p_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}. \end{aligned} \tag{32}$$

Combining (31) and (32) and using the inequality $\sum_{j=1}^5 a_j^p \le \Big(\sum_{j=1}^5 a_j\Big)^p$ for $a_j \ge 0$, we deduce that for $t \in [0,T]$ there holds

$$\begin{aligned} &\|(\mathcal{N}^* y)(t+\delta) - (\mathcal{N}^* y)(t)\|^p_{L^p(\Omega, W^{l,q}(\mathbb{D}))} \\ &\quad \lesssim \delta^{p\eta} \big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|y\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\big). \end{aligned}$$

On the other hand, one can verify easily that

$$\sup_{\tau\in[0,T]} e^{\mu\tau}\|(\mathcal{N}^*y)(\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim e^{\mu T}\big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|y\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\big).$$

From the two latter estimates and $\mathcal{N}^*y = v$ on $(-\infty,0]$, we conclude $\mathcal{N}^*(\mathbb{V}^v_{\mu,p,l,q}) \subset \mathbb{V}^v_{\mu,p,l,q}$.
Step 2: Our purpose in this step is to show that $\mathcal{N}^*$ is a contraction. Let us take any $y, y^* \in \mathbb{V}^v_{\mu,p,l,q}$. As in Step 2 in the proof of part (1) of Theorem 3.1, one has the following inequality for $t \in [0,T]$

$$\|(\mathcal{N}^*y)(t) - (\mathcal{N}^*y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim t^{p\alpha-1}\int_0^t \|f(s,y_s) - f(s,y^*_s)\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))}ds.$$

Assumption (H2) and the fact that $p\alpha - 1 > 0$ allows us to obtain

$$\|(\mathcal{N}^*y)(t) - (\mathcal{N}^*y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \int_0^t \|y_s - y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}ds,$$

where the hidden constant depends on T, α, p and $\mathcal{L}^*_f$. Multiplying both sides by $e^{p\mu\theta}$, with $\theta \in (-t, 0]$, and replacing t by $t+\theta$, we obtain

$$\begin{aligned} e^{p\mu\theta}&\|(\mathcal{N}^*y)(t+\theta) - (\mathcal{N}^*y^*)(t+\theta)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\ &\lesssim e^{p\mu\theta}\int_0^{t+\theta} \|y_s - y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}ds \\ &\lesssim \int_0^t \|y_s - y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}ds. \end{aligned} \tag{33}$$

Since $(\mathcal{N}^*y)(t+\theta) - (\mathcal{N}^*y^*)(t+\theta) = 0$ for all $\theta \in (-\infty, -t]$, it is obvious that (33) holds true for all $\theta \in (-\infty, 0]$. it follows that for all $\theta \in (-\infty, 0]$ there holds

$$e^{p\mu\theta}\|(\mathcal{N}^*y)_t(\theta) - (\mathcal{N}^*y^*)_t(\theta)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \lesssim \int_0^t \|y_s - y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}ds.$$

Since the right-hand side does not depend on θ, it is clear that

$$\|(\mathcal{N}^*y)_t - (\mathcal{N}^*y^*)_t\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \lesssim \int_0^t \|y_s - y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))}ds,$$

which shows that the following result holds true for $k = 1$

$$\begin{aligned} \|((\mathcal{N}^*)^k y)_t - ((\mathcal{N}^*)^k y^*)_t\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ \leq \vartheta^k (k!)^{-1} t^k \|y - y^*\|^p_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}ds, \end{aligned}$$

where ϑ is a positive constant independent of t. By the principle of mathematical induction, one can verify that it also hold for all $k \geq 1$. This together with the property

$$e^{\mu t}\|y(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq e^{\mu T}\|y_t(0)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq e^{\mu T}\sup_{\tau\in(-\infty,0]} e^{\mu\tau}\|y_t(\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}, \text{ for } t \in [0,T],$$

allows us to obtain the following bound

$$\|(\mathcal{N}^*)^k y - (\mathcal{N}^*)^k y^*\|^p_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \leq \vartheta^k (k!)^{-1} T^k e^{p\mu T}\|y - y^*\|^p_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}.$$

By a similar argument as in Step 2 in the proof of part (1) of Theorem 3.1, one concludes that the operator $\mathcal{N}^*$ between $\mathbb{V}^v_{\mu,p,l,q}$ and $\mathbb{V}^v_{\mu,p,l,q}$ is a contraction. Hence, Problem (P3) has a unique mild solution in $\mathbb{V}^v_{\mu,p,l,q}$. $\square$

Proof of part (2) of Theorem 4.1. For $t \in [0,T]$, by a similar way as in the proof of part (1) of Theorem 4.1, one arrives at

$$\|x_t\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \lesssim \|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \\ + \int_0^t \|x_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} ds, \quad t \in [0,T].$$

Denote by $\overline{\vartheta}$ the hidden constant in the above inequality. The Gronwall inequality yields

$$\begin{aligned}\|x_t\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} &\leq \overline{\vartheta} e^{\overline{\vartheta} t}\big(\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\big) \\ &\leq \overline{\vartheta} e^{\overline{\vartheta} T}\big(\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\big).\end{aligned}$$

As a consequence, it can be observed that, for all $t \in [0,T]$, there holds

$$e^{\mu t}\|x(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq \overline{\vartheta} e^{(\mu+\overline{\vartheta})T}\big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\big).$$

On the other hand, $e^{\mu t}\|x(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \leq \|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}$ for all $t \in (-\infty,0]$. By the two latter inequalities, one concludes that the regularity (29) holds. ☐

Proof of part (3) of Theorem 4.1. Part (2) can be proved easily by arguing as in Step 1 in the proof of part (1) of Theorem 4.1 and using the regularity (29). Therefore, we omit the details here. ☐

Proof of three parts of Theorem 4.2. Theorem 4.2 can be proved by using a similar way as in the proof Theorem 3.2 associated with the new techniques in the proof of Theorem 4.1 (which are used to deal with the unbounded delay case under Assumption (H2)). Therefore, we omit the details here. ☐

5. Global existence results for Problem (P3) and Problem (P4) under locally Lipschitz conditions

In this section, we continue to investigate existence results for Problem (P3) and Problem (P4), but under the following locally Lipschitz condition

(H3) $f(t,0) = 0$ for $t \in [0,T]$, and there exists $\overline{\mathcal{L}}^*_f > 0$ such that, for any u, v belonging to $\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$ and $t \in [0,T]$, there holds

$$\begin{aligned}&\big\|f(t,u) - f(t,v)\big\|_{L^p(\Omega,W^{l',q}(\mathbb{D}))} \\ &\leq \overline{\mathcal{L}}^*_f\big(\|u\|^\vartheta_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} + \|v\|^\vartheta_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\big)\|u-v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))},\end{aligned}$$

where $\vartheta > 0$.

The main results are clearly stated in the following theorems:

Theorem 5.1. *Let $\alpha \in (\frac{1}{2},1)$. Assume that $d \geq 1$, v is reasonably small in the space*

$$\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))),$$

and (H3) hold for some (d,p,q,l,l') satisfying (4),$2 \leq p < \frac{1}{1-\alpha}$, and $\mu, \vartheta > 0$. Assume further that v_0 and σ are reasonably small in $L^p(\Omega,\mathcal{H}_\nu)$ and $L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))$, for some $\nu \in [\nu_*,1]$, $m > \frac{p}{p(\alpha-1)+1}$, $\nu' \in [\nu'_*,0]$, respectively. Then, Problem (P3) has a global mild solution in the space

$$\mathbb{W}^{v,\varepsilon}_{\mu,p,l,q} := \Big\{x \in \mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D}))) : x(t) = v(t) \ \textit{for}\ t \in (-\infty,0]\Big\}.$$

Theorem 5.2. *Let $\alpha \in (0,1)$. Assume that $d \geq 1$, v is reasonably small in the space*

$$\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D}))),$$

and (H3) hold for some (d,p,q,l,l') satisfying (4), $p \geq 2$, and $\mu,\vartheta > 0$. Assume further that v_0 and σ are reasonably small in $L^p(\Omega,\mathcal{H}_\nu)$ and $L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))$, for some $\nu \in [\nu_*,1]$, $\nu' \in [\nu'_*,0]$, respectively. Then, Problem (P4) has a global mild solution $x \in \mathbb{W}^{v,\varepsilon}_{\mu,p,l,q}$.

Remark 5.1. Some challenges appearing when establishing the above global results can be briefly described as follows. If we still apply the usual techniques used in the proofs of Theorem 4.1 and 4.2, the contraction property could not be obtained mainly because the hidden constant in the estimate $\|\mathcal{N}^*y - \mathcal{N}^*y^*\| \lesssim \|y-y^*\|$ is not necessarily less than 1. Hence, to achieve the global results, we have to handle some challenging points that are finding the right space to guarantee the contraction property and designing flexible estimates to ensure the aforementioned constant is less than 1. Due to this reason and the fact that no global result under the locally Lipschitz condition (H3) has been reported until now, we are strongly interested in dealing with our problems in this section.

Remark 5.2. Our goal now is to focus on the natural question: "whether the mild solutions of Problems (P3)-(P4) exist locally or globally under the two above assumptions?" In this section, we shall not mention the regularity and Hölder continuity results since they can be handled by a similar argument as in Section 4.

Remark 5.3. Unlike the existence results obtained under globally Lipschitz conditions in Section 4, the global existence results for mild solutions cannot be guaranteed in the space $\mathbb{V}^v_{\mu,p,l,q}$ under locally Lipschitz condition (H3). To overcome this challenge, we consider a new weighted space defined as follows

$$\mathcal{F}^\varepsilon_\mu((-\infty,T];\mathbb{B}) := \Big\{u \in \mathcal{C}((-\infty,T];\mathbb{B}) : \|u\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];\mathbb{B})} := \sup_{\tau\in(-\infty,T]} w_\varepsilon(\tau)e^{\mu\tau}\|u(\tau)\|_{\mathbb{B}} < \infty\Big\}.$$

where $w_\varepsilon(t) = \frac{t}{1-e^{-\varepsilon t}}$ if $t \in (0,T]$ and $w_\varepsilon(t) = \varepsilon^{-1}$ if $t \in (-\infty,0]$, and $\mathbb{B} := L^p(\Omega,W^{l,q}(\mathbb{D}))$. The reason we work with this space is that it possesses the following useful property. The weighted function w_ε is continuous and non-decreasing on $(-\infty,T]$. Furthermore, since the function $z \mapsto 2-2e^{-z}-z$ is increasing on $(0,\log 2)$, it can be seen that $w_\varepsilon(t) \leq 2\varepsilon^{-1}$ if $\varepsilon < \frac{\log 2}{T}$.

Remark 5.4. When dealing with delay stochastic partial (and fractional) differential equations, the non-linear term f is often assumed to satisfy a global Lipschitz assumption as we mentioned in Section 4. However, there are several terms that do not satisfy such global form. For instance, we will make the following remark to show that there exists f satisfying the local assumption (H3) but does not fulfill the global condition (H2).

For simplicity and the convenience of the reader, we consider here a simple case when $d \geq 2$ and $p = q = 2$. Let l, l', m, ν' satisfy the following condition

$$\begin{cases} 0 \leq l < \frac{d}{2}, \\ \frac{-d}{2} < l' \leq 0,\ \frac{l'}{2} \leq \nu' \leq 0 \\ 1 \leq m \leq \frac{d}{d-2l},\ m \geq \frac{d}{d-2(l'-1)} \end{cases} \tag{34}$$

Consider the non-linear term f of the polynomial form

$$f(t,u) = |\widetilde{u}|^{\rho-1}\widetilde{u}, \quad \text{with } \rho > 2,$$

where $\widetilde{u} = e^{\widetilde{\mu}\tau}u$, for $\tau \in (-\infty,0]$, $\widetilde{\mu} \geq \mu$. Then, f,σ satisfy the following locally form

$$\begin{aligned} &\big\|f(t,u) - f(t,v)\big\|_{L^2(\Omega,W^{l',2}(\mathbb{D}))} \\ &\quad \lesssim \big(\|u\|^{\rho-1}_{\mathcal{C}_\mu((-\infty,0];L^2(\Omega,W^{l,2}(\mathbb{D})))} + \|v\|^{\rho-1}_{\mathcal{C}_\mu((-\infty,0];L^2(\Omega,W^{l,2}(\mathbb{D})))}\big)\|u-v\|_{\mathcal{C}_\mu((-\infty,0];L^2(\Omega,W^{l,2}(\mathbb{D})))}. \end{aligned} \tag{35}$$

Indeed, thanks to the inequality (3.27) in page 136 of [72], namely,

$$\big|f(t,u)-f(t,v)\big| = \big||\widetilde{u}|^{\rho-1}\widetilde{u}-|\widetilde{v}|^{\rho-1}\widetilde{v}\big| \lesssim \big(|\widetilde{u}|^{\rho-1}+|\widetilde{v}|^{\rho-1}\big)|\widetilde{u}-\widetilde{v}|,$$

we find that

$$\begin{aligned}\big\|f(t,u)-f(t,v)\big\|_{L^2(\Omega,L^m(\mathbb{D}))} &\lesssim \big\||\widetilde{u}|^{\rho-1}|\widetilde{u}-\widetilde{v}|\big\|_{L^2(\Omega,L^m(\mathbb{D}))} + \big\||\widetilde{v}|^{\rho-1}|\widetilde{u}-\widetilde{v}|\big\|_{L^2(\Omega,L^m(\mathbb{D}))}\\ &= \Big[\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{u}|^{m(\rho-1)}|\widetilde{u}-\widetilde{v}|^m d\xi\Big)^{\frac{2}{m}}\Big]^{\frac{1}{2}} + \Big[\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{v}|^{m(\rho-1)}|\widetilde{u}-\widetilde{v}|^m d\xi\Big)^{\frac{2}{m}}\Big]^{\frac{1}{2}},\end{aligned}$$

The Hölder inequality and the Kahane-Khintchine inequality allow us to obtain

$$\begin{aligned}\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{u}|^{m(\rho-1)}|\widetilde{u}-\widetilde{v}|^m d\xi\Big)^{\frac{2}{m}} &\lesssim \mathbb{E}\Big[\Big(\int_{\mathbb{D}}|\widetilde{u}|^{2m(\rho-1)}d\xi\Big)^{\frac{1}{m}}\Big(\int_{\mathbb{D}}|\widetilde{u}-\widetilde{v}|^{2m}d\xi\Big)^{\frac{1}{m}}\Big]\\ &\lesssim \Big[\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{u}|^{2m(\rho-1)}d\xi\Big)^{\frac{2}{m}}\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{u}-\widetilde{v}|^{2m}d\xi\Big)^{\frac{2}{m}}\Big]^{\frac{1}{2}}\\ &\lesssim \Big[\mathbb{E}\|\tilde{u}\|^2_{L^{2m(\rho-1)}(\mathbb{D})}\Big]^{(\rho-1)}\mathbb{E}\big\||\tilde{u}-\tilde{v}|\big\|^2_{L^{2m}(\mathbb{D})}\\ &\lesssim \big\|\widetilde{u}\big\|^{2(\rho-1)}_{L^2(\Omega,L^{2m(\rho-1)}(\mathbb{D}))}\big\|\widetilde{u}-\widetilde{v}\big\|^2_{L^2(\Omega,L^{2m}(\mathbb{D}))},\end{aligned}$$

where the hidden constant only depend on m and ρ. By exactly the same way, one arrives at

$$\mathbb{E}\Big(\int_{\mathbb{D}}|\widetilde{v}|^{m(\rho-1)}|\widetilde{u}-\widetilde{v}|^m d\xi\Big)^{\frac{2}{m}} \lesssim \big\|\widetilde{v}\big\|^{2(\rho-1)}_{L^2(\Omega,L^{2m(\rho-1)}(\mathbb{D}))}\big\|\widetilde{u}-\widetilde{v}\big\|^2_{L^2(\Omega,L^{2m}(\mathbb{D}))}.$$

From the three above estimates, we deduce that

$$\big\|f(t,u)-f(t,v)\big\|_{L^2(\Omega,L^m(\mathbb{D}))} \lesssim \big(\big\|\widetilde{u}\big\|^{\rho-1}_{L^2(\Omega,L^{2m(\rho-1)}(\mathbb{D}))}+\big\|\widetilde{v}\big\|^{\rho-1}_{L^2(\Omega,L^{2m(\rho-1)}(\mathbb{D}))}\big)\big\|\widetilde{u}-\widetilde{v}\big\|_{L^2(\Omega,L^{2m}(\mathbb{D}))}.$$

Since the condition (34) is satisfied, by the properties iii) and iv) of Lemma 2.3, we can see that the three Sobolev embedding $W^{l,2}(\mathbb{D}) \hookrightarrow L^{2m}(\mathbb{D})$, $W^{l,2}(\mathbb{D}) \hookrightarrow L^{2m(\rho-1)}(\mathbb{D})$ and $L^m(\mathbb{D}) \hookrightarrow W^{l',2}(\mathbb{D})$ hold true. As a consequence, we obtain

$$\big\|f(t,u)-f(t,v)\big\|_{L^2(\Omega,W^{l',2}(\mathbb{D}))} \lesssim \Big(\big\|\widetilde{u}\big\|^{\rho-1}_{L^2(\Omega,W^{l,2}(\mathbb{D}))}+\big\|\widetilde{v}\big\|^{\rho-1}_{L^2(\Omega,W^{l,2}(\mathbb{D}))}\Big)\big\|\widetilde{u}-\widetilde{v}\big\|_{L^2(\Omega,W^{l,2}(\mathbb{D}))}.$$

With the help of the relationship

$$\|\widetilde{u}\|_{L^2(\Omega,W^{l,2}(\mathbb{D}))} = e^{\widetilde{\mu}\tau}\|u\|_{L^2(\Omega,W^{l,2}(\mathbb{D}))} \leq \|u\|_{\mathcal{C}_\mu(-\infty,0];L^2(\Omega,W^{l,2}(\mathbb{D}))},$$

for all $\tau \in (-\infty,0]$, it is obvious that f satisfies (35).

Proof of Theorem 5.1. Consider operator $\mathcal{N}^*$ defined as in (30). Let us set the ball in $\mathbb{W}^{v,\varepsilon}_{\mu,p,l,q}$ with radius $R>0$ as follows

$$\mathcal{B}(R) := \big\{y \in \mathbb{W}^{v,\varepsilon}_{\mu,p,l,q} : \|y\|_{\mathcal{F}^{\varepsilon}_{\mu}((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \leq R\big\}.$$

By a similar way as in Step 2 in the proof of part (1) of Theorem 4.1, but Assumption (H3) is used instead of Assumption (H2), one can easily verify that, for $t \in (0,T]$, there holds

$$\begin{aligned}&\|(\mathcal{N}^*y)(t)-(\mathcal{N}^*y^*)(t)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\ &\lesssim t^{p\alpha-1}|\overline{\mathcal{L}}^*_f|^p\int_0^t\Big(\|y_s\|^{\vartheta}_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}+\|y^*_s\|^{\vartheta}_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big)^p\times\\ &\qquad\qquad \times\|y_s-y^*_s\|^p_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}ds,\end{aligned}$$

where the hidden constant does not depend on t. Noting that $w_\varepsilon(\cdot)$ is a non-decreasing function on $(-\infty, T]$. Hence, the following inequality holds for all $\tau \in (-\infty, 0]$

$$\begin{aligned} e^{\mu\tau}\|y_t(\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &= e^{-\mu t}w_\varepsilon^{-1}(t+\tau)w_\varepsilon(t+\tau)e^{\mu(t+\tau)}\|y(t+\tau)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\ &\le e^{-\mu t}\varepsilon \sup_{\theta\in(-\infty,T]} w_\varepsilon(\theta)e^{\mu\theta}\|y(\theta)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}, \end{aligned}$$

which implies that $\|y_t\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))} \le \varepsilon\|y\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}$. By this inequality, we deduce that

$$\begin{aligned} \|(\mathcal{N}^*y)(t) - (\mathcal{N}^*y^*)(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\lesssim T^{\alpha-\frac{1}{p}}\overline{\mathcal{L}}^*_f\Big(\int_0^t 2^p e^{-p\mu s(\vartheta+1)}ds\Big)^{\frac{1}{p}}\varepsilon^{\vartheta+1}\times \\ \times\Big(\|y\|^\vartheta_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} &+ \|y^*\|^\vartheta_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big)\|y-y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}. \end{aligned}$$

By multiplying both sides by $e^{\mu t}w_\varepsilon(t)$ and noting that $e^{-p\mu s(\vartheta+)} \le 1$, for $s > 0$, and $y, y^* \in \mathcal{B}(R)$, we can see the following estimate holds if $\varepsilon < \frac{\log 2}{T}$

$$\begin{aligned} &e^{\mu t}w_\varepsilon(t)\|(\mathcal{N}^*y)(t) - (\mathcal{N}^*y^*)(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\ &\quad\lesssim e^{\mu t}w_\varepsilon(t)T^\alpha\overline{\mathcal{L}}^*_f\varepsilon^{\vartheta+1}R^\vartheta\|y-y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\quad\lesssim e^{\mu T}T^\alpha\overline{\mathcal{L}}^*_f\varepsilon^{\vartheta}R^\vartheta\|y-y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}, \end{aligned}$$

where we have used the fact that $w_\varepsilon(t) \le 2\varepsilon^{-1}$ if $\varepsilon < \frac{\log 2}{T}$. On the other hand, recalling $\mathcal{N}^*y = \mathcal{N}^*y^* = v$ on $(-\infty, 0]$. Hence, we conclude that there exists a positive constant C independent of T and ε such that

$$\|\mathcal{N}^*y - \mathcal{N}^*y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \le Ce^{\mu T}T^\alpha\overline{\mathcal{L}}^*_f\varepsilon^\vartheta R^\vartheta\|y-y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}.$$

Taking any $y \in \mathcal{B}(R)$, we also find that

$$\begin{aligned} \|\mathcal{N}^*y\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} &\le \Big[\varepsilon^{-1}\|v\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big]\wedge \\ &\wedge\Big[C\varepsilon^{-1}e^{\mu T}\big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}\big) + Ce^{\mu T}T^\alpha\overline{\mathcal{L}}^*_f\varepsilon^\vartheta R^{\vartheta+1}\Big]. \end{aligned}$$

To ensure that $\|\mathcal{N}^*y\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \le R$ and $\mathcal{N}^* : \mathcal{B}(R) \to \mathcal{B}(R)$ is a contraction, we need v_0, σ, v are sufficiently small in $L^p(\Omega,\mathcal{H}_\nu)$, $C_\mu((-\infty,0];L^p(\Omega,\mathcal{H}_\nu)))$, $L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))$, respectively, such that

$$\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} \le M_0,\ \|v\|_{C_\mu((-\infty,0];L^p(\Omega,\mathcal{H}_\nu)))} \le M_1$$

with M_0, M_1 are positive numbers chosen later. We shall choose R, M_0, M_1, and ε small enough such that the right hand side of the latter estimate is less than R and $\varepsilon < \frac{\log 2}{T}$. Our choice is as follows

$$M_0 < \Big[2^{\vartheta+1}C^{\vartheta+1}e^{(\vartheta+1)\mu T}T^\alpha\overline{\mathcal{L}}^*_f\Big]^{-\frac{1}{\vartheta}}, \quad M_1 < 2Ce^{\mu T}M_0,$$

and $R > \frac{2C}{\log 2}Te^{\mu T}M_0$, $\varepsilon = \frac{2Ce^{\mu T}M_0}{R}$. Then, is can be observed that $\varepsilon < \frac{\log 2}{T}$ and

$$\varepsilon^{-1}M_1 < R, \quad C\varepsilon^{-1}e^{\mu T}M_0 = \frac{R}{2}, \quad Ce^{\mu T}T^\alpha\overline{\mathcal{L}}^*_f\varepsilon^\vartheta R^{\vartheta+1} < \frac{R}{2},$$

which follows that $\|\mathcal{N}^*y\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} < R$ and $\|\mathcal{N}^*y - \mathcal{N}^*y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} < \frac{1}{2}\|y-y^*\|_{\mathcal{F}^\varepsilon_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}$. Thanks to the Banach fixed point theorem, Problem (P3) has a unique solution in $\mathcal{B}(R)$. □

Proof of Theorem 5.2. Theorem 5.2 can be proved by a similar way as in the proof of Theorem 5.1. Therefore, we omit the details here. □

6. The connection between fractional pseudo-parabolic equations and fractional parabolic equations

The main objective of this section is to investigate the relationship between the solutions of fractional pseudo-parabolic equations and fractional parabolic equations. Indeed, we will investigate the convergence of solutions of Problems (P1)-(P4) when $\beta \to 0^+$.

It should be noted that if $\beta = 0$ then fractional pseudo-parabolic equations (P1)-(P4) turn to be the following fractional traditional parabolic equations, respectively

$$\begin{cases} \mathcal{D}_t^\alpha(x)(t) + (-\Delta)^\gamma x(t) = f(t, x_t) + \sigma(t)\dot{B}_Q^H(t), \quad t \in [0,T], \\ x(t)|_{\partial\mathbb{D}} = 0, \quad t \in [0,T], \\ x(t) = v(t), \quad t \in [-r, 0], \end{cases} \tag{P1'}$$

$$\begin{cases} \mathcal{D}_t^\alpha(x)(t) + (-\Delta)^\gamma x(t) = \mathcal{I}_t^{1-\alpha} f(t, x_t) + \left[\mathcal{I}_t^{1-\alpha}\sigma(t)\right]\dot{B}_Q^H(t), \quad t \in [0,T], \\ x(t)|_{\partial\mathbb{D}} = 0, \quad t \in [0,T], \\ x(t) = v(t), \quad t \in [-r, 0], \end{cases} \tag{P2'}$$

and

$$\begin{cases} \mathcal{D}_t^\alpha(x)(t) + (-\Delta)^\gamma x(t) = f(t, x_t) + \sigma(t)\dot{B}_Q^H(t), \quad t \in [0,T], \\ x(t)|_{\partial\mathbb{D}} = 0, \quad t \in [0,T], \\ x(t) = v(t), \quad t \in (-\infty, 0], \end{cases} , \tag{P3'}$$

$$\begin{cases} \mathcal{D}_t^\alpha(x)(t) + (-\Delta)^\gamma x(t) = \mathcal{I}_t^{1-\alpha} f(t, x_t) + \left[\mathcal{I}_t^{1-\alpha}\sigma(t)\right]\dot{B}_Q^H(t), \quad t \in [0,T], \\ x(t)|_{\partial\mathbb{D}} = 0, \quad t \in [0,T], \\ x(t) = v(t), \quad t \in (-\infty, 0]. \end{cases} \tag{P4'}$$

Inspired from the mild formulation in [9], we introduce the definitions of mild solutions to the four traditional problems, which can be established by a similar way employed to obtain (3).

Definition 6.1. A stochastic process x is called a mild solution of Problem (P1') (res. Problem (P3')) if it satisfies

$$x(t) = \begin{cases} v(t),\ t \in [-r, 0],\ \big(\text{res. } v(t),\ t \in (-\infty, 0]\big), \\ E_\alpha\big(-(-\Delta)^\gamma t^\alpha\big)v_0 + \\ \quad + \int_0^t (t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma (t-s)^\alpha\big) f(s, x_s)ds + \\ \quad + \int_0^t (t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma (t-s)^\alpha\big)\sigma(s)dB_Q^H(s),\ t \in [0,T], \mathbb{P}-\text{a.s.}. \end{cases}$$

Definition 6.2. A stochastic process x is called a mild solution of Problem (P2') (res. Problem (P4')) if it satisfies

$$x(t) = \begin{cases} v(t),\ t \in [-r, 0],\ \big(\text{res. } v(t),\ t \in (-\infty, 0]\big), \\ E_\alpha\big(-(-\Delta)^\gamma t^\alpha\big)v_0 + \\ \quad + \int_0^t E_\alpha\big(-(-\Delta)^\gamma (t-s)^\alpha\big) f(s, x_s)ds + \\ \quad + \int_0^t E_\alpha\big(-(-\Delta)^\gamma (t-s)^\alpha\big)\sigma(s)dB_Q^H(s),\ t \in [0,T], \mathbb{P}-\text{a.s.}, \end{cases}$$

For Problems (P1')-(P4'), the existence results for the mild solutions can be verified as in Sections 3 and 4; therefore, we do not mention here. In this section, we aim at investigating the connection between the mild solutions of fractional pseudo-parabolic equations and fractional traditional parabolic equations. Denote by $x^{\beta,1}$, $x^{\beta,2}$, $x^{\beta,3}$, $x^{\beta,4}$ the solutions of (P1)-(P4), and $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, $X^{(4)}$ the solutions of (P1')-(P4'), respectively. Our goal is to find an answer for the natural question rising:

"Does $x^{\beta,j}$ converge to $X^{(j)}$, $j = \overline{1,4}$, in some appropriate sense as β tends to 0^+".

Remark 6.1. • It is necessary to verify the above convergence because if this result is true then the solution of the delay stochastic parabolic equations can be obtained as the limit of some sequence of solutions of the corresponding pseudo-parabolic models to any null sequence for the coefficient β. As a result, the solutions of the classical parabolic models can be approximated well by the solutions of (P1)-(P4).
• As we have mentioned in Introduction, when considering the seepage of inhomogeneous fluids through a fissured rock, the decreasing "$\beta \to 0^+$" implies the less in block dimension and the more in the degree of fissuring.

For the sake of estimating, we first define the following operators for $t \in [0, T]$

$$\mathcal{Z}_{\beta,\alpha,\gamma}(t) := E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big) - E_\alpha\big(-(-\Delta)^\gamma t^\alpha\big), \tag{36}$$

$$\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t) := \overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\mathcal{A} - \overline{E}_\alpha\big(-(-\Delta)^\gamma t^\alpha\big), \tag{37}$$

$$\widetilde{\mathcal{Z}}_{\beta,\alpha,\gamma}(t) := E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)\mathcal{A} - E_\alpha\big(-(-\Delta)^\gamma t^\alpha\big), \tag{38}$$

and prepare some useful properties for them.

Lemma 6.1. *Let $\alpha \in (0,1), \nu \geq 0, p \geq 2$, $\beta > 0$ and $\nu' \in (\nu, \nu+1]$. Then*

(Z1) For all $t \in [0,T]$, there holds

$$\big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \leq \beta^{\nu'-\nu}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})}.$$

(Z2) For all $t \in (0,T]$, there holds

$$\big\|\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} + \big\|\widetilde{\mathcal{Z}}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \beta^{\epsilon_4} t^{-\epsilon_3\alpha}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu-(\epsilon_3\gamma-\epsilon_4)})},$$

where ϵ_3, ϵ_4 are positive numbers small enough such that $0 < \epsilon_3 < 1$, $0 < \epsilon_4 < \epsilon_3\gamma$.

The proof of Lemma 6.1 can be found in Appendix.

The following theorem shall show the connection between the solutions of the pseudo-parabolic equations (P1) and the parabolic equations (P1') respectively.

Theorem 6.1. *Let $\alpha \in (\frac{1}{2}, 1)$, p, m be two positive numbers satisfying*

$$2 \leq p < \frac{1}{1-\alpha}, \qquad m > \frac{p}{p(\alpha-1)+1},$$

and κ_1, κ_1' be two positive numbers small enough such that

$$\kappa_1 < \frac{\alpha-1}{\alpha} + \frac{p-1}{p\alpha} \wedge \frac{m-p}{pm\alpha}, \qquad \kappa_1' < \kappa_1\gamma. \tag{39}$$

Let $d \geq 1$, $v \in \mathcal{C}([-r,0]; L^p(\Omega, W^{l,q}(\mathbb{D})))$, and (H1) hold for some (d,q,l,l') satisfying

$$\begin{cases} 1 + d\Big(\frac{1}{q} - \frac{1}{2}\Big) \geq 0, \ q \geq 1, \\ l' \leq 0 \leq l \leq 2(\kappa_1\gamma - \kappa_1') - d\Big|\frac{1}{q} - \frac{1}{2}\Big| + l'. \end{cases} \tag{40}$$

Assume further that σ, v_0 satisfy (5), where $\nu - \nu'$ is small enough such that $\nu - \nu' < \kappa_1'$. Then, the following convergence result holds

$$\begin{aligned} &\|x^{\beta,1} - X^{(1)}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\quad\lesssim \beta^{\widehat{\eta}_1}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big), \end{aligned} \tag{41}$$

where $\widehat{\eta}_1 = \big(\kappa_1' - (\nu - \nu')\big) \wedge (\nu - \nu_*)$.

Remark 6.2. • Since $p \geq 2 > \frac{1}{\alpha}$ and $\frac{pm(\alpha-1)}{m-p} > -1$, we know that $\frac{\alpha-1}{\alpha} > -\frac{p-1}{p\alpha}$ and $\frac{\alpha-1}{\alpha} > -\frac{m-p}{pm\alpha}$. Hence, there always exist a positive number $\kappa_1 > 0$ satisfying (39).
• The two parameters ν_*, ν'_* in Theorem 6.1 satisfy that

$$\nu'_* \leq 0 \leq \nu_* \leq \nu'_* + \kappa_1\gamma - \kappa'_1. \tag{42}$$

Indeed, due to their definitions and the last condition in (40), we have

$$2(\nu_* - \nu'_*) = l - l' + d\Big|\frac{1}{q} - \frac{1}{2}\Big|\big(\chi_{\{q\geq 2\}}(q) + \chi_{\{q<2\}}(q)\big) = l - l' + d\Big|\frac{1}{q} - \frac{1}{2}\Big| \leq 2(\kappa_1\gamma - \kappa'_1).$$

• Under condition (40), the following Sobolev embeddings also hold true

$$\mathcal{H}_{\nu_*} \hookrightarrow \mathbb{H}_{2\nu_*} = W^{2\nu_*,2}(\mathbb{D}) \hookrightarrow W^{l,q}(\mathbb{D}), \tag{43}$$

$$W^{l',q}(\mathbb{D}) \hookrightarrow W^{2\nu'_*,2}(\mathbb{D}) = \mathbb{H}_{2\nu'_*} \hookrightarrow \mathcal{H}_{\nu'_*}. \tag{44}$$

Remark 6.3. It should be noted that the convergence result in Theorem 6.1 is not guaranteed if l, l' only satisfy condition (4) as in Theorem 3.1. To ensure this result holds, a more restrictive condition for l, l', namely (40), needs to be imposed.

In the following theorem, we prove the convergence of solutions of the pseudo-parabolic equations (P2) and the parabolic equation (P2').

Theorem 6.2. *Let $\alpha \in (0,1)$, $p \geq 2$, and κ_2, κ'_2 be two positive numbers small enough such that*

$$\kappa_2 < \frac{p-1}{p\alpha}, \qquad \kappa'_2 < \kappa_2\gamma. \tag{45}$$

Let $d \geq 1$, $v \in \mathcal{C}([-r,0]; L^p(\Omega, W^{l,q}(\mathbb{D})))$, and (H1) hold for some (d, q, l, l') satisfying

$$\begin{cases} 1 + d\Big(\frac{1}{q} - \frac{1}{2}\Big) \geq 0, \ q \geq 1, \\ l' \leq 0 \leq l \leq 2(\kappa_2\gamma - \kappa'_2) - d\Big|\frac{1}{q} - \frac{1}{2}\Big| + l'. \end{cases} \tag{46}$$

Assume further that σ, v_0 satisfy (8), where $\nu - \nu'$ is small enough such that $\nu - \nu' < \kappa'_2$. Then, the following convergence result holds

$$\begin{aligned} &\|x^{\beta,2} - X^{(2)}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))} \\ &\qquad \lesssim \beta^{\widehat{\eta}_2}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)} + \|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))} + \|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big), \end{aligned}$$

where $\widehat{\eta}_2 = \big(\kappa'_2 - (\nu - \nu')\big) \wedge (\nu - \nu_*)$.

Proof of Theorem 6.1. From the Sobolev embedding (43), Definition 2.6, Definition 6.1, and (36), it is obvious that, for $t \in [0,T]$, there holds

$$\begin{aligned} \|x^{\beta,1}(t) - X^{(1)}(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\lesssim \|x^{\beta,1}(t) - X^{(1)}(t)\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} \lesssim \big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)v_0\big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \\ &+ \Big\|\int_0^t (t-s)^{\alpha-1}\Big(\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}f(s, x_s^{\beta,1}) - \\ &\qquad - \overline{E}_\alpha\big(-(-\Delta)^\gamma(t-s)^\alpha\big)f(s, X_s^{(1)})\Big)ds\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \\ &+ \Big\|\int_0^t (t-s)^{\alpha-1}\Big(\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t-s)^\alpha\big)\mathcal{A}\sigma(s) - \\ &\qquad - \overline{E}_\alpha\big(-(-\Delta)^\gamma(t-s)^\alpha\big)\sigma(s)\Big)dB_Q^H(s)\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})}. \end{aligned}$$

By the definition of $\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t)$ as (37) and the triangle inequality, it can be seen that

$$
\begin{aligned}
&\|x^{\beta,1}(t) - X^{(1)}(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))} \\
&\quad \lesssim \big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)v_0\big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \Big\| \int_0^t (t-s)^{\alpha-1}\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t-s) f(s,x_s^{\beta,1})ds\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \\
&\quad + \Big\| \int_0^t (t-s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma (t-s)^\alpha\big)\big(f(s,x_s^{\beta,1}) - f(s,X_s^{(1)})\big)ds\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \\
&\quad + \Big\| \int_0^t (t-s)^{\alpha-1}\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t-s)\sigma(s)dB_Q^H(s)\Big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} \\
&\quad =: \big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)v_0\big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.
\end{aligned}
\tag{47}
$$

The first term on the right hand side above can be bounded easily by applying Property (Z1) of Lemma 6.1 as

$$
\big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)v_0\big\|_{L^p(\Omega,\mathcal{H}_{\nu_*})} \lesssim \beta^{\nu-\nu_*}\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}. \tag{48}
$$

Now, we continue to estimate the last four terms in the right hand side of (47). We consider component $|\mathcal{E}_1|^p$ first. Noting that the assertion (42) allows that $\mathcal{H}_{\nu'_*} \hookrightarrow \mathcal{H}_{\nu_*+\kappa'_1-\kappa_1\gamma}$. By applying Property (Z2) of Lemma 6.1, the Hölder inequality, and the Sobolev embedding just mentioned in the previous line, we have that

$$
\begin{aligned}
|\mathcal{E}_1|^p &\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1}\big\|\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t-s)f(s,x_s^{\beta,1})\big\|_{\mathcal{H}_{\nu_*}}ds\Big]^p \\
&\lesssim \beta^{p\kappa'_1}\mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1-\kappa_1\alpha}\big\|f(s,x_s^{\beta,1})\big\|_{\mathcal{H}_{\nu_*+\kappa'_1-\kappa_1\gamma}}ds\Big]^p \\
&\lesssim \beta^{p\kappa'_1}\Big(\int_0^t (t-s)^{\frac{p(\alpha-1-\kappa_1\alpha)}{p-1}}ds\Big)^{p-1}\int_0^t \mathbb{E}\|f(s,x_s^{\beta,1})\|^p_{\mathcal{H}_{\nu_*+\kappa'_1-\kappa_1\gamma}}ds \\
&\lesssim \beta^{p\kappa'_1}t^{p(\alpha-1-\kappa_1\alpha)+p-1}\int_0^t \|f(s,x_s^{\beta,1})\|^p_{L^p(\Omega,\mathcal{H}_{\nu'_*})}ds,
\end{aligned}
$$

where we have use the fact that $\frac{p(\alpha-1-\kappa_1\alpha)}{p-1} > -1$ which follows from (39). The Sobolev embedding (44) associated with Assumption (H1) and $t^{p(\alpha-1-\kappa_1\alpha)+p-1} \le T^{p(\alpha-1-\kappa_1\alpha)+p-1}$ allows us to obtain

$$
\begin{aligned}
|\mathcal{E}_1|^p &\lesssim \beta^{p\kappa'_1}\int_0^t \|f(s,x_s^{\beta,1})\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))}ds \lesssim \beta^{p\kappa'_1}\mathcal{L}_f\int_{-r}^t \|x^{\beta,1}(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds \\
&\lesssim \beta^{p\kappa'_1}\mathcal{L}_f(T+r)\Big(\sup_{t\in[-r,T]}\|x^{\beta,1}(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\Big)^p.
\end{aligned}
\tag{49}
$$

Next, we look at the composition $|\mathcal{E}_2|^p$. Due to the fact that

$$
\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big) \lesssim (1+\lambda_k^\gamma t^\alpha)^{-1} \lesssim \lambda_k^{-\kappa_1\gamma}t^{-\kappa_1\alpha},
$$

for all $k \ge 1$ and $t \in (0,T]$, it can be seen $\|\overline{E}_\alpha\big(-(-\Delta)^\gamma t^\alpha\big)u\|_{\mathcal{H}_{\nu_*}} \lesssim t^{-\kappa_1\alpha}\|u\|_{\mathcal{H}_{\nu_*-\kappa_1\gamma}}$, for all $u \in \mathcal{H}_{\nu_*-\kappa_1\gamma}$. This together with the Sobolev embedding $\mathcal{H}_{\nu'_*} \hookrightarrow \mathcal{H}_{\nu_*+\kappa'_1-\kappa_1\gamma} \hookrightarrow \mathcal{H}_{\nu_*-\kappa_1\gamma}$ allows us to estimate the second term as

$$
\begin{aligned}
|\mathcal{E}_2|^p &\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1}\big\|\overline{E}_\alpha\big(-(-\Delta)^\gamma(t-s)^\alpha\big)\big(f(s,x_s^{\beta,1})-f(s,X_s^{(1)})\big)\big\|_{\mathcal{H}_{\nu_*}}ds\Big]^p \\
&\lesssim \mathbb{E}\Big[\int_0^t (t-s)^{\alpha-1-\kappa_1\alpha}\big\|f(s,x_s^{\beta,1})-f(s,X_s^{(1)})\big\|_{\mathcal{H}_{\nu_*+\kappa_1'-\kappa_1\gamma}}ds\Big]^p \\
&\lesssim \Big(\int_0^t (t-s)^{\frac{p(\alpha-1-\kappa_1\alpha)}{p-1}}ds\Big)^{p-1}\int_0^t \mathbb{E}\big\|f(s,x_s^{\beta,1})-f(s,X_s^{(1)})\big\|^p_{\mathcal{H}_{\nu_*+\kappa_1'-\kappa_1\gamma}}ds \\
&\lesssim t^{p(\alpha-1-\kappa_1\alpha)+p-1}\int_0^t \|f(s,x_s^{\beta,1})-f(s,X_s^{(1)})\|^p_{L^p(\Omega,\mathcal{H}_{\nu_*'})}ds,
\end{aligned}
\tag{50}
$$

where we have used Hölder inequality in the above evaluation. By the Sobolev embedding (44), Assumption (H1), $t^{p(\alpha-1-\kappa_1\alpha)+p-1}\le T^{p(\alpha-1-\kappa_1\alpha)+p-1}$, and using again the Hölder inequality, we deduce from (50) that

$$
\begin{aligned}
|\mathcal{E}_2|^p &\lesssim \int_0^t \|f(s,x_s^{\beta,1})-f(s,X_s^{(1)})\|^p_{L^p(\Omega,W^{l',q}(\mathbb{D}))}ds \\
&\lesssim \mathcal{L}_f\int_{-r}^t \|x^{\beta,1}(s)-X^{(1)}(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds \\
&= \mathcal{L}_f\int_0^t \|x^{\beta,1}(s)-X^{(1)}(s)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds.
\end{aligned}
\tag{51}
$$

The next aim is to estimate the component $|\mathcal{E}_3|^p$. Applying the Kahane–Khintchine inequality, Lemma 2.5, and the Hölder inequality, we can arrive at

$$
\begin{aligned}
|\mathcal{E}_3|^p &\lesssim t^{\frac{p}{2}(2H-1)}\Big[\int_0^t (t-s)^{\alpha-1}\mathbb{E}\big\|(-\Delta)^{\nu_*}\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t-s)\sigma(s)\big\|^2_{L_Q^2}ds\Big]^{\frac{p}{2}} \\
&\lesssim t^{\frac{p}{2}(2H-1)}t^{\frac{p-2}{2}}\int_0^t (t-s)^{p(\alpha-1)}\big\|(-\Delta)^{\nu_*}\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t-s)\sigma(s)\big\|^p_{L^2(\Omega,L_Q^2)}ds.
\end{aligned}
$$

Since κ_1 satisfies (39), it is obvious that $\frac{pm(\alpha-1-\kappa_1\alpha)}{m-1}>-1$. By using Property (Z2) of Lemma 6.1, the Hölder inequality, the Sobolev embedding $\mathcal{H}_{\nu_*'}\hookrightarrow\mathcal{H}_{\nu_*+\kappa_1'-\kappa_1\gamma}\hookrightarrow\mathcal{H}_{\nu_*-\kappa_1\gamma}$ and Assumption (H2), we deduce that

$$
\begin{aligned}
|\mathcal{E}_3|^p &\lesssim t^{pH-1}\beta^{p\kappa_1'}\int_0^t (t-s)^{p(\alpha-1-\kappa_1\alpha)}\big\|\sigma(s)\big\|^p_{L^2(\Omega,L^2_{Q,\nu_*'})}ds \\
&\lesssim t^{pH-1}\beta^{p\kappa_1'}\Big(\int_0^t (t-s)^{\frac{mp(\alpha-1-\kappa_1\alpha)}{m-p}}ds\Big)^{\frac{m-1}{m}}\Big(\int_0^t \|\sigma(s)\|^m_{L^2(\Omega,L^2_{Q;\nu'})}ds\Big)^{\frac{p}{m}} \\
&\lesssim t^{pH-1}\beta^{p\kappa_1'}t^{p(\alpha-1-\kappa_1\alpha)+\frac{m-p}{m}}\big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))} \\
&\lesssim \beta^{p\kappa_1'}T^{pH-1+p(\alpha-1-\kappa_1\alpha)+\frac{m-p}{m}}\big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}.
\end{aligned}
\tag{52}
$$

Now, combining (47)-(49) and (51)-(52), we deduce that

$$
\begin{aligned}
\sup_{\tau\in[0,t]}\|x^{\beta,1}(\tau)-X^{(1)}(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))} &\lesssim \beta^{p(\nu-\nu_*)}\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)}+\beta^{p\kappa_1'}\big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}+ \\
&+\beta^{p\kappa_1'}\Big(\sup_{t\in[-r,T]}\|x^{\beta,1}(t)\|_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\Big)^p+\int_0^t \sup_{\tau\in[0,s]}\|x^{\beta,1}(\tau-X^{(1)}(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}ds.
\end{aligned}
$$

Denote by C the hidden constant in the above inequality, which does not depend on t. The Gronwall inequality yields that

$$\begin{aligned}&\sup_{\tau\in[0,t]}\|x^{\beta,1}(\tau)-X^{(1)}(\tau)\|^p_{L^p(\Omega,W^{l,q}(\mathbb{D}))}\\&\le Ce^{Ct}\Big[\beta^{p(\nu-\nu_*)}\|v_0\|^p_{L^p(\Omega,\mathcal{H}_\nu)}+\beta^{p\kappa_1'}\big(\big\|\sigma\big\|^p_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}+\|x^{\beta,1}\|^p_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\big)\Big].\end{aligned}$$

Since $e^{Ct}\le e^{CT}$, which independent of t, we deduce that

$$\begin{aligned}&\|x^{\beta,1}-X^{(1)}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim\beta^{\nu-\nu_*}\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\beta^{\kappa_1'}\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}+\beta^{\kappa_1'}\|x^{\beta,1}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}.\end{aligned}$$

On the other hand, the property (7) of Theorem 3.1 implies

$$\begin{aligned}&\|x^{\beta,1}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\beta^{-(\nu-\nu')}\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}+\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))},\end{aligned}\tag{53}$$

where we note that the additional kernel $\beta^{-(\nu-\nu')}$ appears since it is the hidden constant in (E3). This addition is importance in describing that the right hand side of (53) depends on β with a negative exponent. From the last two observations, we conclude that

$$\begin{aligned}\|x^{\beta,1}-X^{(1)}\|_{\mathcal{C}([-r,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}&\lesssim(\beta^{\nu-\nu_*}+\beta^{\kappa_1'})\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\\&+\beta^{\kappa_1'}(1+\beta^{-(\nu-\nu')})\big\|\sigma\big\|_{L^m(0,T;L^2(\Omega,L^2_{Q;\nu'}))}+\beta^{\kappa_1'}\|v\|_{\mathcal{C}([-r,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}.\end{aligned}$$

By setting $\widehat{\eta}_1:=\big(\kappa_1'-(\nu-\nu')\big)\wedge(\nu-\nu_*)$ and noting that $\kappa_1'-(\nu-\nu')>0$, the above assertion leads to the convergence result (41). □

Proof of Theorem 6.2. The convergence result in Theorem 6.2 can be proved in a similar way as in the proof of Theorem 6.1, where we note that the parameter κ_2 only needs to satisfy (45) (instead of (39)) with $\alpha\in(0,1)$ and $p\ge 2$ mainly because the mild formulation of the solution to Problem (P2) does not contain the singular kernel $(t-s)^{\alpha-1}$. □

Next, we continue to state another couple of theorems, which describe the connection between the solutions of the pseudo-parabolic equations (P3)-(P4) and the parabolic ones (P3')-(P4'), respectively.

Theorem 6.3. *Let $\alpha\in(\frac{1}{2},1)$ and p,m,κ_1,κ_1' be positive numbers satisfying Theorem 6.1. Let $d\ge 1$, $v\in\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$, and (H2) hold for some (d,q,l,l') satisfying (40) and $\mu>0$. Assume further that σ,v_0 satisfy (5). Then, the following convergence result holds*

$$\begin{aligned}&\|x^{\beta,3}-X^{(3)}\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim\beta^{\widehat{\eta}_1}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^m(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big),\end{aligned}$$

where $\widehat{\eta}_1=\big(\kappa_1'-(\nu-\nu')\big)\wedge(\nu-\nu_*)$.

Theorem 6.4. *Let $\alpha\in(0,1)$, and p,κ_2,κ_2' be positive numbers satisfying Theorem 6.2. Let $d\ge 1$, $v\in\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))$, and (H2), hold for some (d,p,q,l,l') satisfying (46) and $\mu>0$. Assume further that σ,v_0 satisfy (8). Then, the following convergence result holds*

$$\begin{aligned}&\|x^{\beta,4}-X^{(4)}\|_{\mathcal{C}_\mu((-\infty,T];L^p(\Omega,W^{l,q}(\mathbb{D})))}\\&\qquad\lesssim\beta^{\widehat{\eta}_2}\Big(\|v_0\|_{L^p(\Omega,\mathcal{H}_\nu)}+\|\sigma\|_{L^p(0,T;L^2(\Omega,L^2_{Q,\nu'}))}+\|v\|_{\mathcal{C}_\mu((-\infty,0];L^p(\Omega,W^{l,q}(\mathbb{D})))}\Big),\end{aligned}$$

where $\widehat{\eta}_2=\big(\kappa_2'-(\nu-\nu')\big)\wedge(\nu-\nu_*)$. .

Proof of Theorems 6.3-6.4. The two theorems can be proved by using a similar way as in the proofs of Theorems 6.1-6.2, associated with the new techniques used to deal with the unbounded delay case (as in the proof of Theorem 4.1). Therefore, we omit the details here. $\square$

Remark 6.4. Overall, by comparing Theorems 6.1-6.4 (convergence results when $\beta \to 0^+$) with Theorems 3.1, 3.2, 4.1, 4.2 (existence results), one can see some differences as follows. The convergence results are harder to be established than the existence results. To this end, we need to design sharp estimates and adopt more strict conditions for the two parameters l, l'. Finding the right conditions for l, l' to guarantee not only the convergence results but also the existence of l, l' can be considered as a challenging point of the study here.

7. Conclusion

In this paper, we have investigated several problems for stochastic fractional pseudo-parabolic driven by fractional Brownian motion containing bounded or unbounded delays and the Caputo operator. The global existence, uniqueness, regularity, and Hölder continuity results have been established for such models under some new Lipschitz conditions involving the space $L^p(\Omega, W^{l,q}(\mathbb{D}))$. By designing some new techniques on Sobolev embeddings between the Hilbert space $\mathcal{H} = L^2(\mathbb{D})$ and $W^{l,q}(\mathbb{D})$ and applying some fractional tools, we overcome difficulties rising when proving our results. Additionally, we proved that the mild solution of the fractional pseudo-parabolic model converges to the mild solution of the fractional parabolic one, in some sense, as $\beta \to 0^+$.

Appendix

7.1. **Proof of Lemma 2.7.** • *Verify (E1):* For $u \in L^p(\Omega, \mathcal{H}_\nu)$, we have the following explicit formulations

$$E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u = \sum_{k=1}^{\infty} (u,\phi_k) E_\alpha\big(-\lambda_k^\gamma (1+\beta\lambda_k)^{-1} t^\alpha\big)\phi_k,$$
$$\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u = \sum_{k=1}^{\infty} (u,\phi_k) \overline{E}_\alpha\big(-\lambda_k^\gamma (1+\beta\lambda_k)^{-1} t^\alpha\big)\phi_k.$$

By the fact that $E_{a,b}(-z) \lesssim (1+z)^{-1}$ for $a \in (0,1), b \in \mathbb{R}, z > 0$ (see [23]), it is obvious to see that

$$E_\alpha\big(-\lambda_k^\gamma (1+\beta\lambda_k)^{-1} t^\alpha\big) \lesssim 1, \quad \overline{E}_\alpha\big(-\lambda_k^\gamma (1+\beta\lambda_k)^{-1} t^\alpha\big) \lesssim 1.$$

Hence, we have an estimate for the operator $E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)$ as follows

$$\begin{aligned}
\big\|E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} &= \Big[\mathbb{E}\big\|E_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u\big\|^p_{\mathcal{H}_\nu}\Big]^{\frac{1}{p}} \\
&= \Big[\mathbb{E}\Big(\sum_{k=1}^{\infty} |(u,\phi_k)|^2 \lambda_k^{2\nu} |E_\alpha\big(-\lambda_k^\gamma (1+\beta\lambda_k)^{-1} t^\alpha\big)|^2\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\lesssim \Big[\mathbb{E}\Big(\sum_{k=1}^{\infty} |(u,\phi_k)|^2 \lambda_k^{2\nu}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} = \|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.
\end{aligned}$$

By a similar technique as in above, it is easy to obtain

$$\big\|\overline{E}_\alpha\big(-(-\Delta)^\gamma \mathcal{A} t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.$$

• *Verify (E2):* By the first formulation in (1), the inequality $|e^{-z_1} - e^{-z_2}| \le |z_1 - z_2|$ for $z_1, z_2 > 0$, it can be observed that

$$\begin{aligned}
&\big|E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big) - E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)\big| \\
&\quad\le \int_0^\infty M_\alpha(\tau)\Big|\exp\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\tau\big) - \exp\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\tau\big)\Big|d\tau \\
&\quad\le \int_0^\infty M_\alpha(\tau)\lambda_k^\gamma(1+\beta\lambda_k)^{-1}\big|(t+\delta)^\alpha - t^\alpha\big|\tau d\tau \\
&\quad\lesssim \lambda_k^\gamma(1+\beta\lambda_k)^{-\gamma}(1+\beta\lambda_k)^{-(1-\gamma)}\big|(t+\delta)^\alpha - t^\alpha\big|\int_0^\infty M_\alpha(\tau)\tau d\tau.
\end{aligned}$$

Applying the Lemma 2.1 for $\epsilon = 1$ and noting that $\lambda_k^\gamma(1+\beta\lambda_k)^{-\gamma} \lesssim 1$, $(1+\beta\lambda_k)^{-(1-\gamma)} \le 1$, we directly obtain

$$\big|E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big) - E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)\big| \lesssim \delta^\alpha.$$

With the help of the above estimate, it is clear to see that

$$\begin{aligned}
&\big\|E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta)^\alpha\big) - E_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \\
&\quad= \Big[\mathbb{E}\Big(\sum_{k=1}^\infty |(u,\phi_k)|^2\lambda_k^{2\nu}|E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big) - E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)|^2\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\quad\lesssim \Big[\mathbb{E}\Big(\sum_{k=1}^\infty |(u,\phi_k)|^2\lambda_k^{2\nu}\delta^{2\alpha}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} = \delta^\alpha\|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.
\end{aligned}$$

Similarly, by the second property in (1), the inequality $|e^{-z_1} - e^{-z_2}| \le |z_1 - z_2|$ for $z_1, z_2 > 0$, and the property (2.1), one can verify that

$$\big\|\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta)^\alpha\big) - \overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \delta^\alpha\|u\|_{L^p(\Omega,\mathcal{H}_\nu)}.$$

• *Verify (E3):* By using the fact that

$$(1+\beta\lambda_k)^{-1} = (1+\beta\lambda_k)^{\nu-\nu'-1}(1+\beta\lambda_k)^{-(\nu-\nu')} \le (1+\beta\lambda_k)^{-(\nu-\nu')} \le \beta^{-(\nu-\nu')}\lambda_k^{-(\nu-\nu')},$$

one can verify that $\|\mathcal{A}u\|_{L^p(\Omega,\mathcal{H}_\nu)} \le \beta^{-(\nu-\nu')}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})}$ easily.

• *Verify (E4):* Applying Lemma 2.2 with $\lambda = \lambda_k^\gamma(1+\beta\lambda_k)^{-1} > 0$, we arrive at

$$\begin{aligned}
&\big|(t+\delta)^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big) - t^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)\big| \\
&\quad\le \int_t^{t+\delta}\Big|\partial_\tau\Big(\tau^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}\tau^\alpha\big)\Big)\Big|d\tau \\
&\quad\le \int_t^{t+\delta}\tau^{\alpha-2}\Big|E_{\alpha,\alpha-1}(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}\tau^\alpha)\Big|d\tau.
\end{aligned}$$

Using the property $E_{a,b}(-z) \lesssim (1+z)^{-1}$ for $a \in (0,1), b \in \mathbb{R}, z > 0$ again, we deduce that

$$\begin{aligned}
&\big|(t+\delta)^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big) - t^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)\big| \\
&\quad\le \int_t^{t+\delta}\tau^{\alpha-2}d\tau \lesssim \frac{1}{t^{1-\alpha}} - \frac{1}{(t+\delta)^{1-\alpha}} = \frac{(t+\delta)^{1-\alpha} - t^{1-\alpha}}{t^{1-\alpha}(t+\delta)^{1-\alpha}} \\
&\quad\le \frac{\delta^{1-\alpha}}{t^{1-\alpha}(t+\delta)^\theta(t+\delta)^{1-\alpha-\theta}} \le t^{\alpha-1-\theta}\delta^\theta.
\end{aligned}$$

By the above estimate and the Sobolev embedding $\mathcal{H}_{\nu'} \hookrightarrow \mathcal{H}_{\nu-1}$, we deduce

$$
\begin{aligned}
&\big\|\big((t+s)^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}(t+\delta)^\alpha\big)-t^{\alpha-1}\overline{E}_\alpha\big(-(-\Delta)^\gamma\mathcal{A}t^\alpha\big)\big)\mathcal{A}u\big\|_{L^p(\Omega,\mathcal{H}_\nu)}\\
&\quad=\Big[\mathbb{E}\Big(\sum_{k=1}^\infty|(u,\phi_k)|^2\lambda_k^{2\nu}\Big|(t+\delta)^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}(t+\delta)^\alpha\big)-\\
&\qquad\qquad-t^{\alpha-1}\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)\Big|^2|1-\beta\lambda_k|^{-2}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}\\
&\quad\lesssim\Big[\mathbb{E}\Big(\sum_{k=1}^\infty|(u,\phi_k)|^2\lambda_k^{2\nu-2}t^{2(\alpha-1-\theta)}\delta^{2\theta}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}\\
&\quad=t^{\alpha-1-\theta}\delta^\theta\|u\|_{L^p(\Omega,\mathcal{H}_{\nu-1})}\lesssim t^{\alpha-1-\theta}\delta^\theta\|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})}.
\end{aligned}
$$

The proof is completed.

7.2. **Proof of Lemma 6.1.** • *Verify (Z1):* By the property (1) and the inequalities $e^{-z}\le z^{-\epsilon_1}$, $1-e^{-z}\le z^{\epsilon_2}$, for $z>0$ and ϵ_1,ϵ_2 satisfying $\epsilon_1,\epsilon_2\ge 0$, $\epsilon_2-\epsilon_1>-1$, it is clear to see that

$$
\begin{aligned}
&\big|E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)-E_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|\\
&\quad\le\int_0^\infty M_\alpha(\tau)\Big|\exp\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\tau\big)-\exp\big(-\lambda_k^\gamma t^\alpha\tau\big)\Big|d\tau\\
&\quad=\int_0^\infty M_\alpha(\tau)\exp\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\tau\big)\Big[1-\exp\Big(-\frac{\beta\lambda_k^{\gamma+1}}{1+\beta\lambda_k}t^\alpha\tau\Big)\Big]d\tau\\
&\quad\le\int_0^\infty M_\alpha(\tau)\lambda_k^{-\epsilon_1\gamma}(1+\beta\lambda_k)^{\epsilon_1}t^{-\epsilon_1\alpha}\tau^{-\epsilon_1}\Big[\frac{\beta\lambda_k^{\gamma+1}}{1+\beta\lambda_k}t^\alpha\tau\Big]^{\epsilon_2}d\tau\\
&\quad=\beta^{\epsilon_2}\lambda_k^{(\epsilon_2-\epsilon_1)\gamma+\epsilon_2}(1+\beta\lambda_k)^{\epsilon_1-\epsilon_2}t^{(\epsilon_2-\epsilon_1)\alpha}\int_0^\infty M_\alpha(\tau)\tau^{\epsilon_2-\epsilon_1}d\tau.
\end{aligned}\tag{54}
$$

Applying the above inequality with $\epsilon_1=\epsilon_2=\nu'-\nu\in(0,1]$ and using Lemma 2.1, we obtain

$$
\begin{aligned}
&\big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)}=\Big[\mathbb{E}\big\|\mathcal{Z}_{\beta,\alpha,\gamma}(t)u\big\|^p_{\mathcal{H}_\nu}\Big]^{\frac{1}{p}}\\
&\quad=\Big[\mathbb{E}\Big(\sum_{k=1}^\infty|(u,\phi_k)|^2\lambda_k^{2\nu}\big|E_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)-E_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|^2\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}\\
&\quad\le\beta^{\nu'-\nu}\Big[\mathbb{E}\Big(\sum_{k=1}^\infty|(u,\phi_k)|^2\lambda_k^{2\nu'}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}=\beta^{\nu'-\nu}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu'})}.
\end{aligned}
$$

• *Verify (Z2):* By the triangle inequality and a similar technique as in (54), it can be seen that, for ϵ_1,ϵ_2 satisfying $\epsilon_1,\epsilon_2\ge 0$, $1+\epsilon_2-\epsilon_1>-1$, there holds

$$
\begin{aligned}
&\big|\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)(1+\beta\lambda_k)^{-1}-\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|\\
&\quad\le(1+\beta\lambda_k)^{-1}\big|\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)-\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|+\\
&\qquad\qquad+\big|(1+\beta\lambda_k)^{-1}\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)-\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|\\
&\quad\le\beta^{\epsilon_2}\lambda_k^{(\epsilon_2-\epsilon_1)\gamma+\epsilon_2}(1+\beta\lambda_k)^{\epsilon_1-\epsilon_2-1}t^{(\epsilon_2-\epsilon_1)\alpha}\times\\
&\qquad\qquad\times\int_0^\infty\alpha M_\alpha(\tau)\tau^{1+\epsilon_2-\epsilon_1}d\tau+\frac{\beta\lambda_k}{1+\beta\lambda_k}\big|\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|.
\end{aligned}
$$

On the other hand, since $E_{a,b}(-z) \lesssim (1+z)^{-1}$ for $a \in (0,1), b \in \mathbb{R}, z > 0$, the following property holds true for $\epsilon_3, \epsilon_4 \in [0,1]$ and $t \in (0,T]$

$$\frac{\beta\lambda_k}{1+\beta\lambda_k}\big|\overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big| \lesssim \frac{\beta\lambda_k}{(1+\beta\lambda_k)^{1-\epsilon_4}(1+\lambda_k^\gamma t^\alpha)^{\epsilon_3}} \lesssim \frac{\beta^{\epsilon_4}\lambda_k^{\epsilon_4}}{(1+\lambda_k^\gamma t^\alpha)^{\epsilon_3}} \lesssim \beta^{\epsilon_4}\lambda_k^{\epsilon_4-\epsilon_3\gamma}t^{-\epsilon_3\alpha}.$$

The two above observations and Lemma 2.1 yield

$$\begin{aligned} &\big|\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)(1+\beta\lambda_k)^{-1} - \overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big| \\ &\qquad \lesssim \beta^{\epsilon_2}\lambda_k^{(\epsilon_2-\epsilon_1)\gamma+\epsilon_2}(1+\beta\lambda_k)^{\epsilon_1-\epsilon_2-1}t^{(\epsilon_2-\epsilon_1)\alpha} + \beta^{\epsilon_4}\lambda_k^{\epsilon_4-\epsilon_3\gamma}t^{-\epsilon_3\alpha}. \end{aligned}$$

By choosing $\epsilon_2 = \epsilon_4, \epsilon_1 = \epsilon_2 + \epsilon_3$, with $0 < \epsilon_3 < 1$, $0 < \epsilon_4 < \epsilon_3\gamma$, we have

$$\begin{aligned} &\big|\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)(1+\beta\lambda_k)^{-1} - \overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big| \\ &\qquad \lesssim \beta^{\epsilon_4}\lambda_k^{-\epsilon_3\gamma+\epsilon_4}(1+\beta\lambda_k)^{\epsilon_3-1}t^{-\epsilon_3\alpha} + \beta^{\epsilon_4}\lambda_k^{\epsilon_4-\epsilon_3\gamma}t^{-\epsilon_3\alpha} \lesssim \beta^{\epsilon_4}\lambda_k^{-(\epsilon_3\gamma-\epsilon_4)}t^{-\epsilon_3\alpha}. \end{aligned}$$

Hence, for $\nu \geq 0$ and $t \in (0,T]$, the following estimate holds

$$\begin{aligned} &\big\|\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \\ &= \Big[\mathbb{E}\Big(\sum_{k=1}^\infty |(u,\phi_k)|^2\lambda_k^{2\nu}\big|\overline{E}_\alpha\big(-\lambda_k^\gamma(1+\beta\lambda_k)^{-1}t^\alpha\big)(1+\beta\lambda_k)^{-1} - \overline{E}_\alpha\big(-\lambda_k^\gamma t^\alpha\big)\big|^2\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\ &\lesssim \beta^{\epsilon_4}t^{-\epsilon_3\alpha}\Big[\mathbb{E}\Big(\sum_{k=1}^\infty |(u,\phi_k)|^2\lambda_k^{2\nu-2(\epsilon_3\gamma-\epsilon_4)}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} = \beta^{\epsilon_4}t^{-\epsilon_3\alpha}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu+\epsilon_4-\epsilon_3\gamma})}. \end{aligned}$$

Similarly, one can verify that $\big\|\overline{\mathcal{Z}}_{\beta,\alpha,\gamma}(t)u\big\|_{L^p(\Omega,\mathcal{H}_\nu)} \lesssim \beta^{\epsilon_4}t^{-\epsilon_3\alpha}\|u\|_{L^p(\Omega,\mathcal{H}_{\nu+\epsilon_4-\epsilon_3\gamma})}$. The two estimates imply that (Z2) holds true.

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