Stochastic fractional diffusion equations containing finite and infinite delays with multiplicative noise

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October 4, 2022

Abstract

In this work, we investigate stochastic fractional diffusion equations with Caputo-Fabrizio fractional derivatives and multiplicative noise, involving finite and infinite delays. Initially, the existence and uniqueness of the mild solution in the spaces $\mathcal{C}^p([-a,b];L^q(\Omega,\dot{H}^r))$ and $\mathcal{C}^\delta((-\infty,b];L^q(\Omega,\dot{H}^r))$ are established. Next, besides investigating the regularity properties, we show the continuity of mild solutions with respect to the initial functions and the order of the fractional derivative for both cases of delay separately.

Keywords: fractional diffusion equations, standard Brownian motion, finite delay, infinite delay, stochastic equations.

2010 MSC: 60G15, 60G22, 60G52, 60G57.

1 Introduction

Let $\mathbb{X}$ be a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}^+$. Assume that the boundary of $\mathbb{X}$, namely $\partial\mathbb{X}$, is smooth enough. In this paper, we aim at investigating the existence, uniqueness, regularity and continuity results for two problems for stochastic fractional diffusion equations containing delays with Caputo-Fabrizio derivatives and multiplicative noise.

- The first problem we are interested in involves a finite or bounded delay:

$$\begin{cases} \left({}^{CF}\mathcal{D}_t^\beta + (-\triangle)^\alpha\right)x(t) = A(t,x_t) + B(t,x_t)\dot{\omega}_t, \quad t \in J := [0,b], \\ x(t)|_{\partial\mathbb{X}} = 0, \quad t \in J = [0,b], \\ x(s) = \chi(s), \quad s \in J_0 := [-a,0], \quad a > 0. \end{cases} \tag{1}$$

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- The second model is concerned with an infinite (or unbounded) delay:

$$\begin{cases} \left({}^{CF}\mathcal{D}_t^\beta + (-\triangle)^\alpha\right)x(t) = A(t,x_t) + B(t,x_t)\dot{\omega}_t, & t \in J = [0,b], \\ x(t)|_{\partial\mathbb{X}} = 0, & t \in J := [0,b], \\ x(s) = \chi(s), & s \in J_\infty := (-\infty, 0], \end{cases} \tag{2}$$

where we notice that the condition $x = \chi$ on $J_0 = [-a, 0]$ is now replaced by $x = \chi$ on $J_\infty = (-\infty, 0]$.

In the above models, α is a positive number, $b > 0$ is the final time of observations, ${}^{CF}\mathcal{D}_t^\beta$ is the Caputo-Fabrizio fractional derivative [14] of order $0 < \beta < 1$

$${}^{CF}\mathcal{D}_t^\beta f(t) = \mathcal{E}_\beta(t) * \dot{f}(t), \qquad t \geq 0,$$

where $\mathcal{E}_\beta(t) := \frac{\mathcal{M}(\beta)}{1-\beta}\exp\left(-\frac{\beta}{1-\beta}t\right)$, with $\mathcal{M}(\beta)$ is a normalization function satisfying $\mathcal{M}(0) = \mathcal{M}(1) = 1$ ([14, 15]), $\dot{f}(t) := \frac{\partial}{\partial t}f(t)$ is the classical derivative of f, and $*$ denotes the convolution, $\dot{\omega}_t = \frac{\partial}{\partial t}\omega_t$ (called white noise) stands for the generalized derivative of ω_t, which is the standard Brownian motion (also called Wiener process) defined on a completed probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The initial function $\chi \in \mathcal{C}(J_0; L^2(\Omega, L^2(\mathbb{X})))$. The fractional operator $(-\triangle)^\alpha$, the function x_t, the non-linear source A, and the non-linear space-time-noise B will be specified later.

Initially, let us mention about the classical diffusion equations, the paramount importance of the Caputo-Fabrizio operator, and some related studies on time fractional diffusion equations in the deterministic case. It should be noted that if the fractional derivative ${}^{CF}\mathcal{D}_t^\beta$ is replaced by the integer order derivative ∂_t then the equations we consider turn to be the primitive diffusion models (also called typical heat equations and classical parabolic equations), which are traditional and have been much studied previously due to their theoretical interest and essential applications in various fields of science such as heat transfer and image processing [3, 28, 35, 37]. Regarding the fractional derivative ${}^{CF}\mathcal{D}_t^\beta$, the presence of this derivative plays the role of modeling several practical phenomena in physics, control systems, biology, fluid dynamics and material science [5, 6, 7, 24]. The readers can refer to impressive studies [1, 36] for more details about its physical interpretation and an application in mass–spring–damper motion. It is worth mentioning that such derivative possesses the advantage of not having singular kernel [14, 38]. We can list here some recent studies on useful properties of the Caputo-Fabrizio derivative [2, 7, 14] and fractional differential equations containing such operator [1, 22, 38, 40]. For some other recent results on deterministic fractional diffusion equations, the readers can refer to [10, 11, 23, 34], where fundamental solutions are constructed, and the existence and behaviors of solutions are investigated.

Next, let us explain the description of our stochastic model with delays in details. Since uncontrollable sources in natural generate distinct random noises, it is essential to consider our problems containing stochastic perturbations. In our models, we would like to deal with a usual stochastic term that is a standard Brownian motion ω_t, which is the classical and well-known noise studied in various papers recently [8, 9, 16, 26, 27, 43, 47, 48]. Additionally, due to the fact that, in some practical situations, the current behavior is affected by the previous states, it is required to include some delays and external forces depending on history state in our models. In the last decades, the number of articles dealing with delay partial differential equations (DPDEs) has increased significantly. Some DPDEs with finite delay can be found in [13, 17, 20, 21, 30, 31, 41]. As regards the infinite delay case, we can list here several considerable results [12, 25, 32, 33, 39, 42, 44, 45, 46].

Despite of the importance of the appearances of stochastic perturbations and delays mentioned above, to the best of our knowledge, fractional diffusion equations with Caputo-Fabrizio derivative containing delays and multiplicative noise have not been studied in the literature until now. Therefore, the present paper is concerned with problems (1) and (2), being our main goals as follows.

- Firstly, the existence and uniqueness of mild solutions will be proved. The results are inspired in some previous papers [17, 45, 46] but constructed in the subspaces $\mathcal{C}^p([-a,b];L^q(\Omega,\dot{H}^r)))$ and $\mathcal{C}_\delta((-\infty,b];L^q(\Omega,\dot{H}^r)))$, with $r\geq 0, q\geq 2, p\in(0,\frac{1}{2}], \delta>0$, instead of the two usual spaces $\mathcal{C}([-a,b];L^2(\Omega,L^2(\mathbb{X}))))$ and $\mathcal{C}_\delta((-\infty,b];L^2(\Omega,L^2(\mathbb{X}))))$ respectively (notice that if $q=2$ and $r=0$ then $L^q(\Omega,\dot{H}^r)$ becomes $L^2(\Omega,L^2(\mathbb{X})))$, under more generalized assumptions for the nonlinear source A and space-time-noise B. From our perception, this is one of positive points of the present paper.

- Secondly, we aim at describing some regularity properties for mild solutions.

- Lastly, for each problem, besides verifying the continuity of the mild solution with respect to initial functions, we are strongly interested in investigating the continuity with respect to the order of the fractional derivative. As far as we know, until now, no one dealt with this type of continuity for mild solutions to stochastic fractional differential equations involving delays.

The organization of the present paper is as follows. In Section 2, we recall some notations including the fractional operator, the expression of the standard Brownian motion, and some necessary functional spaces. Furthermore, the definition of mild solutions to our problems and some properties of resolvent operators are presented in this section. In Section 3, the first results on the existence, uniqueness, regularity properties, and the continuity of mild solutions with respect to the initial function and the order of the fractional derivative in the case of finite delay are stated. In Section 4, we continue to investigate the behaviors of mild solutions in the case of infinite delay including the existence, uniqueness, regularity and continuity properties, but in different spaces and under different conditions for the initial function, source term, and space-time-noise.

2 Preliminaries

2.1 Notations

To make easier the reading of the paper, in this section, we introduce some notations and functional spaces.

Let us first consider the negative Laplacian operator $\mathbb{L}:=-\triangle$ defined on $H_0^1(\mathbb{X})\cap H^2(\mathbb{X})$ as well as recall the definition of fractional operators. Denote by (λ_k,ξ_k) an eigenpair of $\mathbb{L}$ satisfying that (λ_k) is a positive non-decreasing sequence, which tends to infinity, and (ξ_k) form an orthonormal basis in $L^2(\mathbb{X})$. We also recall $\dot{H}^r$, $r\geq 0$, the subspace of $L^2(\mathbb{X})$ satisfying

$$\|f\|_{\dot{H}^r}:=\Big(\sum_{k\in\mathbb{N}^+}\lambda_k^{2r}|\langle f,\xi_k\rangle|\Big)^{\frac{1}{2}}<\infty,$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $L^2(\mathbb{X})$. Identifying the dual space $\big(L^2(\mathbb{X})\big)^*=L^2(\mathbb{X})$, it can be set $\dot{H}^{-r}=\big(\dot{H}^r\big)^*$ and the fractional operator $\mathbb{L}_r:=(-\triangle)^r:\dot{H}^{-r}\to\dot{H}^r$ can be defined by $\mathbb{L}_r:=\sum_{k\in\mathbb{N}^+}\lambda_k^r\langle\cdot,\xi_k\rangle\xi_k$ (see [19], [29]).

Next, we describe some functional spaces necessary for our main results. Let $\mathcal{Q}\in\mathcal{L}\left(L^2(\mathbb{X})\right)$ be the linear operator defined by $\mathcal{Q}\xi_k=\Lambda_k\xi_k$, with $Tr(\mathcal{Q}):=\sum_{k\in\mathbb{N}^+}\Lambda_k$ finite, and let the $L^2(\mathbb{X})$-valued Brownian motion defined by

$$\omega_t:=\sum_{k\in\mathbb{N}^+}\mathcal{Q}^{\frac{1}{2}}\xi_k w_t^{(k)},\quad t\geq 0,$$

where $w_t^{(k)}$ are one-dimensional standard Brownian motions. Let $L^2_{\mathcal{Q},r} = L^2(\mathcal{Q}^{\frac{1}{2}}(L^2(\mathbb{X})), \dot{H}^r)$ be the space of all Hilbert-Schmidt operators $\mathbf{T} : Q^{\frac{1}{2}}(L^2(\mathbb{X})) \to \dot{H}^r$ satisfying

$$\|\mathbf{T}\|_{L^2_{\mathcal{Q},r}} := \Big(\sum_{k\in\mathbb{N}^+} \|\mathbf{T}\mathcal{Q}^{\frac{1}{2}}\xi_k\|^2_{\dot{H}^r} \Big)^{\frac{1}{2}} < \infty.$$

For short, we denote $L^2_{\mathcal{Q}} = L^2_{\mathcal{Q},r}$ if $r = 0$.

Let U be an arbitrary Banach space. We denote by $L^q(\Omega, U)$ the space of U-valued random variables ϱ such that

$$\|\varrho\|_{L^q(\Omega,U)} := \big(\mathbb{E}\|\varrho\|^q_U\big)^{\frac{1}{q}} < \infty.$$

Additionally, we denote by $\mathcal{C}(\mathcal{I}; U)$ the space of continuous functions y from $\mathcal{I} \subset \mathbb{R}$ into U with the sup norm, and let $\mathcal{C}^p(\mathcal{I}; U)$ be the subspace of $\mathcal{C}(\mathcal{I}; U)$ equipped with the norm

$$\|y\|_{\mathcal{C}^p(\mathcal{I};U)} := \sup_{t_1,t_2\in\mathcal{I}} \frac{\|y(t_1) - y(t_2)\|_U}{|t_1 - t_2|^p} < \infty.$$

If $x \in \mathcal{C}([-a, b]; L^2(\Omega, L^2(\mathbb{X})))$, then for $t \in J$ we denote by x_t the function on $[-a, 0]$ as

$$x_t(s) = x(t + s), \quad s \in [-a, 0],$$

where a stands for the finite delay if $a < \infty$ or the infinite delay if $a = \infty$.

Let us now introduce the Burkholder-Davis-Gundy-type inequality [18], which is of paramount importance in estimating the stochastic term appearing in the expression of solutions.

Proposition 2.1. *Let $q \geq 2$, $t_0, t \in J$ and $\varphi : J \to L^2_{\mathcal{Q}}$ satisfy $\mathbb{E}\Big[\int_{t_0}^t \|\varphi(\zeta)\|^2_{L^2_{\mathcal{Q}}} d\zeta\Big]^{\frac{q}{2}} < \infty$. Then, the following inequality holds*

$$\mathbb{E}\Big\| \int_{t_0}^t \varphi(\zeta) d\omega_\zeta \Big\|^q \leq c(q)\mathbb{E}\Big[\int_{t_0}^t \|\varphi(\zeta)\|^2_{L^2_{\mathcal{Q}}} d\zeta \Big]^{\frac{q}{2}},$$

where $c(q) = \sqrt{\frac{q(q-1)}{2} \left(\frac{q}{q-1}\right)^{q-2}}$.

2.2 Mild solutions and properties of resolvent operator

Our goal in this subsection is to construct a mild formulation for solutions to Problem (1), which is of the form $x(t) = \sum_{k\in\mathbb{N}^+} \langle x(t), \xi_k\rangle \xi_k$ for $t \in J$. For the sake of convenience, let us first consider the following problem

$$\begin{cases} \big({}^{CF}\mathcal{D}_t^\beta + (-\triangle)^\alpha\big)x(t) = F(t), \quad t \in J := [0, b], \\ x(t)|_{\partial\mathbb{X}} = 0, \quad t \in J = [0, b], \\ x(0) = x_0. \end{cases} \tag{3}$$

By taking the inner product of the first equation in Problem (3) and then taking the Laplace transform, one arrives at

$$\frac{\zeta\mathfrak{L}\{\langle x(t), \xi_k\rangle\}(\zeta) - \langle x(0), \xi_k\rangle}{\zeta + \beta(1 - \zeta)} = -\lambda_k^\alpha \mathfrak{L}\{\langle x(t), \xi_k\rangle\}(\zeta) + \mathfrak{L}\{\langle F(t), \xi_k\rangle\}(\zeta),$$

which implies

$$\begin{aligned}&\big(1+\lambda_k^\alpha(1-\beta)\big)\mathfrak{L}\{\langle x(t),\xi_k\rangle\}(\zeta)\\&\quad=\frac{1}{\zeta+\frac{\lambda_k^\alpha\beta}{1+\lambda_k^\alpha(1-\beta)}}\langle x(0),\xi_k\rangle+\frac{\zeta+\beta(1-\zeta)}{\zeta+\frac{\lambda_k^\alpha\beta}{1+\lambda_k^\alpha(1-\beta)}}\mathfrak{L}\{\langle F(t),\xi_k\rangle\}(\zeta).\end{aligned}$$

Now, with the help of the inverse Laplace transform and the condition $x(0)=x_0$, one obtains the following expression for the Fourier coefficients

$$\begin{aligned}\big(1+\lambda_k^\alpha(1-\beta)\big)\langle x(t),\xi_k\rangle&=\exp\Big(-\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}\Big)\langle x_0,\xi_k\rangle+\\&\quad+\frac{\beta}{1+\lambda_k^\alpha(1-\beta)}\int_0^t\exp\Big(-\frac{\beta\lambda_k^\alpha(\zeta-t)}{1+\lambda_k^\alpha(1-\beta)}\Big)\langle F(\zeta),\xi_k\rangle d\zeta.\end{aligned}$$

For the sake of convenience, we set $m_{\alpha,\beta}(\lambda_k):=\frac{\beta}{1+\lambda_k^\alpha(1-\beta)}$ and $g_{\alpha,\beta}(t,\lambda_k)$ as follows

$$g_{\alpha,\beta}(t,\lambda_k):=(1+\lambda_k^\alpha(1-\beta))^{-1}\exp\Big(-\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}\Big),\quad t\in J,k\in\mathbb{N}^+.\tag{4}$$

Then, the following expression of solutions to Problem (3) is obtained

$$x(t)=\sum_{k\in\mathbb{N}^+}g_{\alpha,\beta}(t,\lambda_k)\langle x_0,\xi_k\rangle\xi_k+\sum_{k\in\mathbb{N}^+}\Big(m_{\alpha,\beta}(\lambda_k)\int_0^t g_{\alpha,\beta}(t-\zeta,\lambda_k)\langle F(\zeta),\xi_k\rangle d\zeta\Big)\xi_k,\quad t\in J.$$

Inspired by the above formulation of solutions to Problem (3), we give the following definition of the mild solution to Problem (1).

Definition 2.1. An $\mathbb{X}$-valued process $\{x(t)\}$ is said to be a mild solution of Problem (1) (resp. Problem (2)) if

(i) $x\in\mathcal{C}([-a,b];L^2(\Omega,L^2(\mathbb{X})))$ (resp. $x\in\mathcal{C}((-\infty,b];L^2(\Omega,L^2(\mathbb{X})))$),

(ii) $x(s)=\chi(s)$, for $s\in J_0$ (resp. $s\in J_\infty$),

(iii) For $t\in J$, $x(t)$ satisfies

$$x(t)=\mathcal{G}_{\alpha,\beta}(t)\chi(0)+\int_0^t\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)A(\zeta,x_\zeta)d\zeta+\mathcal{Z}_{\alpha,\beta}(t),\quad\mathbb{P}-a.e,\tag{5}$$

where the two operators $\mathcal{G}_{\alpha,\beta}(t):L^2(\mathbb{X})\to L^2(\mathbb{X})$, $\tilde{\mathcal{G}}_{\alpha,\beta}(t):L^2(\mathbb{X})\to L^2(\mathbb{X})$ and the stochastic term $\mathcal{Z}_{\alpha,\beta}(t)$ are defined by

$$\mathcal{G}_{\alpha,\beta}(t)(\cdot):=\sum_{k\in\mathbb{N}^+}g_{\alpha,\beta}(t,\lambda_k)\langle\cdot,\xi_k\rangle\xi_k,\quad\tilde{\mathcal{G}}_{\alpha,\beta}(t)(\cdot):=\sum_{k\in\mathbb{N}^+}m_{\alpha,\beta}(\lambda_k)g_{\alpha,\beta}(t,\lambda_k)\langle\cdot,\xi_k\rangle\xi_k$$

$$\mathcal{Z}_{\alpha,\beta}(t):=\int_0^t\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)B(\zeta,x_\zeta)d\omega_\zeta,\quad t\in J,\tag{6}$$

where the coefficients $m_{\alpha,\beta}(\lambda_k)$ and $g_{\alpha,\beta}(t,\lambda_k)$ are defined in (4).

In the following propositions, some properties of the aforementioned operator are presented.

Proposition 2.2. *Let $\alpha>0$, $0<\beta<1$, and $q\geq 2$. Then*

(G1) The two operators $\mathcal{G}_{\alpha,\beta}(t)$, $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ are linear, bounded and satisfies the following property for any $0 \le \eta \le \alpha$

$$\|\mathbb{L}_\eta \mathcal{G}_{\alpha,\beta}(t)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le C_1(\alpha,\beta,\eta), \quad \text{with } C_1(\alpha,\beta,\eta) = (1-\beta)^{-\eta/\alpha}$$
$$\|\mathbb{L}_\eta \tilde{\mathcal{G}}_{\alpha,\beta}(t)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le \overline{C}_1(\alpha,\beta,\eta), \quad \text{with } \tilde{C}_1(\alpha,\beta,\eta) = \beta(1-\beta)^{-\eta/\alpha},$$

(G2) The two operators $\mathcal{G}_{\alpha,\beta}(t)$, $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ satisfies the following Hölder continuity of exponent $\gamma \in (0,1]$ for any $0 \le t_1 < t_2 \le b$ and $0 \le \eta \le \alpha$

$$\|\mathbb{L}_\eta\big(\mathcal{G}_{\alpha,\beta}(t_2) - \mathcal{G}_{\alpha,\beta}(t_1)\big)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le C_2(\alpha,\beta,\eta,\gamma)(t_2-t_1)^\gamma,$$
$$\|\mathbb{L}_\eta\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t_2) - \tilde{\mathcal{G}}_{\alpha,\beta}(t_1)\big)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le \tilde{C}_2(\alpha,\beta,\eta,\gamma)(t_2-t_1)^\gamma$$

with $C_2(\alpha,\beta,\eta,\gamma) = \beta^\gamma(1-\beta)^{-(\eta/\alpha)-\gamma}$ and $\tilde{C}_2(\alpha,\beta,\eta,\gamma) = \beta^{1+\gamma}(1-\beta)^{-(\eta/\alpha)-\gamma}$.

Remark 2.1. *Assume that $\eta = 0$ in Proposition 2.2, then the following properties hold*

$$\|\mathcal{G}_{\alpha,\beta}(t)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le M_1(\alpha,\beta), \quad \text{for } t \in J,$$
$$\|\mathcal{G}_{\alpha,\beta}(t_1) - \mathcal{G}_{\alpha,\beta}(t_2)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le M_2(\alpha,\beta,\gamma)(t_2-t_1)^\gamma, \quad \text{for } 0 \le t_1 < t_2 \le b,$$
$$\|\tilde{\mathcal{G}}_{\alpha,\beta}(t)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le \tilde{M}_1(\alpha,\beta), \quad \text{for } t \in J,$$
$$\|\tilde{\mathcal{G}}_{\alpha,\beta}(t_1) - \tilde{\mathcal{G}}_{\alpha,\beta}(t_2)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))} \le \tilde{M}_2(\alpha,\beta,\gamma)(t_2-t_1)^\gamma, \quad \text{for } 0 \le t_1 < t_2 \le b,$$

where the exponent $\gamma \in (0,1]$, $M_1(\alpha,\beta) = C_1(\alpha,\beta,0)$, $M_2(\alpha,\beta,\gamma) = C_2(\alpha,\beta,\gamma,0)$, $\tilde{M}_1(\alpha,\beta) = \tilde{C}_1(\alpha,\beta,0)$, $\tilde{M}_2(\alpha,\beta,\gamma) = \tilde{C}_2(\alpha,\beta,\gamma,0)$.

Proof. To prove two above properties, we first estimate the coefficients $g_{\alpha,\beta}(t,\lambda_k)$ defined in (4). It is obvious that for any $k \in \mathbb{N}^+$

$$|g_{\alpha,\beta}(t,\lambda_k)| \le (1+\lambda_k^\alpha(1-\beta))^{-1} = (1+\lambda_k^\alpha(1-\beta))^{-\eta/\alpha}(1+\lambda_k^\alpha(1-\beta))^{-(\alpha-\eta)/\alpha}.$$

Since $0 \le \eta \le \alpha$, it is clear that

$$(1+\lambda_k^\alpha(1-\beta))^{-\eta/\alpha} \le \lambda_k^{-\eta}(1-\beta)^{-\eta/\alpha} \quad \text{and} \quad (1+\lambda_k^\alpha(1-\beta))^{-(\alpha-\eta)/\alpha} \le 1, \tag{7}$$

which allow us to estimate the coefficients $g_{\alpha,\beta}(t,\lambda_k)$ as

$$|g_{\alpha,\beta}(t,\lambda_k)| \le \lambda_k^{-\eta}(1-\beta)^{-\eta/\alpha}, \qquad \text{for any } k \in \mathbb{N}^+. \tag{8}$$

Let $x \in L^q(\Omega, L^2(\mathbb{X}))$, the above estimate directly yields that

$$\begin{aligned}\mathbb{E}\|\mathbb{L}_\eta \mathcal{G}_{\alpha,\beta}(t)x\|^q_{L^2(\mathbb{X})} &= \mathbb{E}\Big(\sum_{k\in\mathbb{N}^+} \lambda_k^{2\eta}|g_{\alpha,\beta}(t,\lambda_k)|^2|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}} \\ &\le (1-\beta)^{-q\eta/\alpha}\mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}}. \end{aligned} \tag{9}$$

Since $\mathbb{E}\|x\|^q_{L^2(\mathbb{X})} = \mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}|\langle x,\xi_k\rangle|^2\Big)^{q/2}$, (9) shows that the property (G1) holds, which implies that the resolvent operator $\mathcal{G}_{\alpha,\beta}(t)$ is linear and bounded. Similarly, the operator $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ is linear and bounded as in the property (G1).

Next, we continue to verify property (G2) by taking into account

$$\begin{aligned}
&g_{\alpha,\beta}(t_2,\lambda_k)-g_{\alpha,\beta}(t_1,\lambda_k)\\
&\quad=(1+\lambda_k^\alpha(1-\beta))^{-1}\exp\Big(-\frac{\beta\lambda_k^\alpha t_1}{1+\lambda_k^\alpha(1-\beta)}\Big)\Big[\exp\Big(-\frac{\beta\lambda_k^\alpha(t_2-t_1)}{1+\lambda_k^\alpha(1-\beta)}\Big)-1\Big]\\
&\quad=g_{\alpha,\beta}(t_1,\lambda_k)\Big[\exp\Big(-\frac{\beta\lambda_k^\alpha(t_2-t_1)}{1+\lambda_k^\alpha(1-\beta)}\Big)-1\Big].
\end{aligned}$$

Using property (8) again and the inequality $1-e^{-y}\le y^\gamma$, $\gamma\in(0,1]$, for $y>0$, it follows, for $0\le t_1\le t_2\le T$, that

$$\begin{aligned}
\big|g_{\alpha,\beta}(t_2,\lambda_k)-g_{\alpha,\beta}(t_1,\lambda_k)\big|&\le\lambda_k^{-\eta}(1-\beta)^{-\eta/\alpha}\Big|\frac{\beta\lambda_k^\alpha(t_2-t_1)}{1+\lambda_k^\alpha(1-\beta)}\Big|^\gamma\\
&\le\beta^\gamma(1-\beta)^{-(\eta/\alpha)-\gamma}\lambda_k^{-\eta}(t_2-t_1)^\gamma.
\end{aligned}$$

For $x\in L^q(\Omega,L^2(\mathbb{X}))$, the above estimate directly implies

$$\begin{aligned}
\mathbb{E}\|\mathbb{L}_\eta\big(\mathcal{G}_{\alpha,\beta}(t_2)-\mathcal{G}_{\alpha,\beta}(t_1)\big)x\|^q_{L^2(\mathbb{X})}&=\mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}\lambda_k^{2\eta}\big|g_{\alpha,\beta}(t_2,\lambda_k)-g_{\alpha,\beta}(t_1,\lambda_k)\big|^2|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}}\\
&\le\beta^{q\gamma}(1-\beta)^{-q(\eta/\alpha)-q\gamma}(t_2-t_1)^{q\gamma}\mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}},
\end{aligned}$$

whence property (G2) holds and the resolvent operator $\mathcal{G}_{\alpha,\beta}(t)$ is Hölder continuous of exponent $\gamma\in(0,1]$. Similarly, the resolvent operator $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ is Hölder continuous of exponent $\gamma\in(0,1]$. □

Proposition 2.3. *Let $\alpha>0$ and $q\ge2$. Then, the two operators $\mathcal{G}_{\alpha,\beta}(t)$ and $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ are continuous with respect to the order β. Namely, for $\beta,\beta'\in[\beta_0,\beta_1]\in(0,1)$, $0\le\eta\le\alpha$, and $t\in J$, there exists two positive constants $C_3(\alpha,\beta_1,b,\lambda_1,\eta),\tilde{C}_3(\alpha,\beta_1,b,\lambda_1,\eta)>0$ such that*

$$\begin{aligned}
&\|\mathbb{L}_\eta\big(\mathcal{G}_{\alpha,\beta}(t)-\mathcal{G}_{\alpha,\beta'}(t)\big)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))}\le C_3(\alpha,\beta_1,b,\lambda_1,\eta)|\beta-\beta'|,\\
&\|\mathbb{L}_\eta\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t)-\tilde{\mathcal{G}}_{\alpha,\beta'}(t)\big)\|_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))}\le\tilde{C}_3(\alpha,\beta_1,b,\lambda_1,\eta)|\beta-\beta'|.
\end{aligned}\tag{10}$$

Proof. For $\beta,\beta'\in(0,1)$, it can be seen from (4) that

$$\begin{aligned}
&|g_{\alpha,\beta}(t,\lambda_k)-g_{\alpha,\beta'}(t,\lambda_k)|\\
&=\Big|(1+\lambda_k^\alpha(1-\beta))^{-1}\exp\Big(-\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}\Big)-(1+\lambda_k^\alpha(1-\beta'))^{-1}\exp\Big(-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big)\Big|\\
&\le(1+\lambda_k^\alpha(1-\beta))^{-1}\Big|\exp\Big(-\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}\Big)-\exp\Big(-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big)\Big|+\\
&+\big|(1+\lambda_k^\alpha(1-\beta))^{-1}-(1+\lambda_k^\alpha(1-\beta'))^{-1}\big|\exp\Big(-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big)\\
&=:(I)+(II).
\end{aligned}$$

By using the inequality $|e^{-c}-e^{-d}|\le|c-d|$ for $c,d\in\mathbb{R}^+$, one directly obtain

$$\begin{aligned}
\Big|\exp\Big(-\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}\Big)-\exp\Big(-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big)\Big|&\le\Big|\frac{\beta\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta)}-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big|\\
&=\Big|\frac{\beta\lambda_k^\alpha t+\beta(1-\beta')\lambda_k^{2\alpha}t-\beta'\lambda_k^\alpha t-\beta'(1-\beta)\lambda_k^{2\alpha}t}{\big(1+\lambda_k^\alpha(1-\beta)\big)\big(1+\lambda_k^\alpha(1-\beta')\big)}\Big|\\
&\le\Big|\frac{(\beta-\beta')(\lambda_k^\alpha+\lambda_k^{2\alpha})t}{(1-\beta)(1-\beta')\lambda_k^{2\alpha}}\Big|,
\end{aligned}$$

which together with $(1+\lambda_k^\alpha(1-\beta))^{-1} \leq \lambda_k^{-\eta}(1-\beta_1)^{-\eta/\alpha}$, for $\eta \in [0,\alpha]$, and $\lambda_k^{-\alpha} \leq \lambda_1^{-\alpha}$, implies that

$$(I) \leq \frac{(\lambda_1^{-\alpha}+1)b}{(1-\beta_1)^{2+\eta/\alpha}}|\beta-\beta'|\lambda_k^{-\eta}.$$

On the other hand, one can verify that

$$\begin{aligned}\big|(1+\lambda_k^\alpha(1-\beta))^{-1}-(1+\lambda_k^\alpha(1-\beta'))^{-1}\big| &= \Big|\frac{\lambda_k^\alpha(\beta-\beta')}{\big(1+\lambda_k^\alpha(1-\beta)\big)\big(1+\lambda_k^\alpha(1-\beta')\big)}\Big| \\ &\leq \frac{|\beta-\beta'|}{(1-\beta)(1-\beta')^{\eta/\alpha}\lambda_k^\eta},\end{aligned}$$

which together with $\exp\Big(-\frac{\beta'\lambda_k^\alpha t}{1+\lambda_k^\alpha(1-\beta')}\Big) \leq 1$ yields

$$(II) \leq \frac{|\beta-\beta'|}{(1-\beta_1)^{1+\eta/\alpha}}\lambda_k^{-\eta}.$$

From all the above observations, one deduces that there exist $C_3(\alpha,\beta_1,b,\lambda_1,\eta)>0$ such that

$$|g_{\alpha,\beta}(t,\lambda_k)-g_{\alpha,\beta'}(t,\lambda_k)| \leq C_3(\alpha,\beta_1,b,\lambda_1,\eta)|\beta-\beta'|\lambda_k^{-\eta}.$$

Let $x \in L^q(\Omega, L^2(\mathbb{X}))$, with the help of the above estimate, one obtains

$$\begin{aligned}\mathbb{E}\|\mathbb{L}_\eta\big(\mathcal{G}_{\alpha,\beta}(t)-\mathcal{G}_{\alpha,\beta'}(t)\big)x\|^q_{L^2(\mathbb{X})} &= \mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}\lambda_k^{2\eta}|g_{\alpha,\beta}(t,\lambda_k)-g_{\alpha,\beta'}(t,\lambda_k)|^2|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}} \\ &\leq |C_3(\alpha,\beta_1,b,\lambda_1,\eta)|^q|\beta-\beta'|^q\mathbb{E}\Big(\sum_{k\in\mathbb{N}^+}|\langle x,\xi_k\rangle|^2\Big)^{\frac{q}{2}}, \qquad (11)\end{aligned}$$

which implies property (10) holds. The operator $\tilde{\mathcal{G}}_{\alpha,\beta}(t)$ can be estimated similarly; therefore, we omit the detail here. □

3 Existence, uniqueness, and regularity results in the case of finite delay

In this section, we aim at studying Problem (1), which is included in the finite delay case. For $r \geq 0, q \geq 2, p \in (0,\frac{1}{2}]$, let us define the following Banach space

$$\mathcal{B}_\chi^{p,q,r} := \Big\{x \in \mathcal{C}^p([-a,b];L^q(\Omega,\dot{H}^r)) : x(s)=\chi(s), \text{for } s \in J_0\Big\},$$

endowed with the norm

$$\|x\|_{\mathcal{B}_\chi^{p,q,r}} = \sup_{-a\leq t<t+\theta\leq b}\frac{\|x(t+\theta)-x(t)\|_{L^q(\Omega,\dot{H}^r)}}{\theta^p}.$$

Our goal now is to establish existence, uniqueness and regularity results for Problem (1), provided that $\chi \in \mathcal{C}^p(J_0;L^q(\Omega,\dot{H}^r))$, for $r\geq 0, q\geq 2, p\in(0,\frac{1}{2}]$. Suppose that the non-linear source A and the non-linear space-time-noise B satisfy the following assumptions for $\mu,\nu \in [r-\alpha, r]$:

(A1) There exists $K_A>0$ such that for any $x, x^\dagger \in \mathcal{C}^p([-a;b];L^q(\Omega,\dot{H}^r))$ and $t \in J$

$$\int_0^t \mathbb{E}\|A(\zeta,x_\zeta)-A(\zeta,x_\zeta^\dagger)\|^q_{\dot{H}^\mu}d\zeta \leq K_A\int_{-a}^t \mathbb{E}\|x(\zeta)-x^\dagger(\zeta)\|^q_{\dot{H}^r}d\zeta,$$

(A2) There exists $\overline{K}_A > 0$ such that $\|A(\cdot, 0)\|_{L^q(J;L^q(\Omega,\dot{H}^\mu))} \leq \overline{K}_A$,

(B1) There exists $K_A > 0$ such that for any $x, x^\dagger \in \mathcal{C}^p([-a;b];L^q(\Omega,\dot{H}^r))$ and $t \in J$

$$\int_0^t \mathbb{E}\|B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\|^q_{L^2_{\mathcal{Q},\nu}} d\zeta \leq K_B \int_{-a}^t \mathbb{E}\|x(\zeta) - x^\dagger(\zeta)\|^q_{\dot{H}^r} d\zeta,$$

(B2) There exists $\overline{K}_B > 0$ such that $\|B(\cdot, 0)\|_{L^q(J;L^q(\Omega,L^2_{\mathcal{Q},\nu}))} \leq \overline{K}_B$.

Remark 3.1. *The above conditions for A and B are inspired on assumptions $(H1)$, $(H2)$ in [45]. The novel point here is that subspaces $\dot{H}^\mu, \dot{H}^\nu, \dot{H}^r$ are considered instead of the usual Hilbert space $L^2(\mathbb{X})$. Furthermore, a generalized version of the expectation $\mathbb{E}\|\cdot\|^2$, that is, $\mathbb{E}\|\cdot\|^q$, $q \geq 2$ is considered.*

The following theorem states the existence and uniqueness result in the space $\mathcal{B}^{p,q,r}_\chi$.

Theorem 3.1. *Let $\chi \in \mathcal{C}^p(J_0;L^q(\Omega,\dot{H}^r))$, for $r \geq 0, q \geq 2, p \in (0, \frac{1}{2}]$. Assume that (A1), (A2), (B1), (B2) hold. Then, Problem (1) has a unique mild solution in the space $\mathcal{B}^{p,q,r}_\chi$. Furthermore, the following regularity properties hold for $-a \leq t < t+\sigma \leq b$*

i) $\mathbb{E}\big\|x(t)\big\|^q_{\dot{H}^r} \leq M_3(\alpha,\beta)\big(1 + \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r}\big)$,

ii) $\mathbb{E}\big\|x(t+\sigma) - x(t)\big\|^q_{\dot{H}^r} \leq M_4(\alpha,\beta)\sigma^{pq}\big(1 + \|\chi\|^q_{\mathcal{C}^p(J_0;L^q(\Omega,\dot{H}^r))}\big)$,

where $M_3(\alpha,\beta)$, $M_4(\alpha,\beta)$ depend on $\alpha, \beta, r, \mu, \nu, q, a, b, K_A, K_B, \overline{K}_A, \overline{K}_B$.

Corollary 3.1 (The existence and uniqueness results on the usual space $\mathcal{B}_\chi$)**.** *Notice that if Assumptions (A1), (A2), (B1), (B2) hold for $r = \mu = \nu = 0$, $q = 2$, and the initial condition χ belongs to $\mathcal{C}(J_0;L^2(\Omega,L^2(\mathbb{X})))$, then Problem (1) has a unique mild solution in the usual space*

$$\mathcal{B}_\chi = \left\{x \in \mathcal{C}([-a,b];L^2(\Omega,L^2(\mathbb{X}))) : x(s) = \chi(s), \text{ for } s \in J_0\right\}.$$

Furthermore, the following regularity properties hold for $-a \leq t < t+\sigma \leq b$

$$\mathbb{E}\big\|x(t)\big\|^2_{L^2(\mathbb{X})} \leq \overline{M_3}(\alpha,\beta)\big(1 + \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^2_{L^2(\mathbb{X})}\big)$$
$$\mathbb{E}\big\|x(t+\sigma) - x(t)\big\|^2_{L^2(\mathbb{X})} \leq \overline{M_4}(\alpha,\beta)\sigma^{2p}\big(1 + \|\chi\|^2_{\mathcal{C}(J_0;L^2(\Omega,L^2(\mathbb{X})))}\big)$$

where $\overline{M}_3(\alpha,\beta)$, $\overline{M}_4(\alpha,\beta)$ depend on $\alpha, \beta, a, b, K_A, K_B, \overline{K}_A, \overline{K}_B$.

Proof. Let us define the operator Ψ on $\mathcal{B}^{p,q,r}_\chi$ by

$$\Psi(x)(s) = \chi(s), \quad \text{for all } s \in J_0, \tag{12}$$

and for all $t \in J$

$$\Psi(x)(t) = \mathcal{G}_{\alpha,\beta}(t)\chi(0) + \int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)A(\zeta, x_\zeta)d\zeta + \mathcal{Z}_{\alpha,\beta}(t). \tag{13}$$

It is obvious that to show the existence of a mild solution to Problem (1) is equivalent to find a fixed point of the operator Ψ. To do this end, we aim to use a well-known method that is Banach fixed point theorem. Our strategy here is to prove that Ψ is well-defined, i.e. $\Psi(\mathcal{B}^{p,q,r}_\chi) \subset \mathcal{B}^{p,q,r}_\chi$, and then verify that Ψ is a contraction.

Claim 1. $\Psi(x) \in \mathcal{B}^{p,q,r}_\chi$, for all $x \in \mathcal{B}^{p,q,r}_\chi$.

For $x \in \mathcal{B}_\chi^{p,q,r}$, we shall show that $t \mapsto \Psi(x)(t)$ is Hölder continuous on J. Indeed, for $t \in J$ and $\sigma > 0$ small enough, it can be seen that

$$\begin{aligned}\mathbb{E}\Big\|\Psi(x)(t+\sigma) - \Psi(x)(t)\Big\|^q_{\dot H^r} \le\ & 3^{q-1}\mathbb{E}\|\big(\mathcal{G}_{\alpha,\beta}(t+\sigma) - \mathcal{G}_{\alpha,\beta}(t)\big)\chi(0)\|^q_{\dot H^r} + \\ & + 3^{q-1}\mathbb{E}\Big\|\int_0^{t+\sigma}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)A(\zeta,x_\zeta)d\zeta - \int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)A(\zeta,x_\zeta)d\zeta\Big\|^q_{\dot H^r} + \\ & + 3^{q-1}\mathbb{E}\|\mathcal{Z}_{\alpha,\beta}(t+\sigma) - \mathcal{Z}_{\alpha,\beta}(t)\|^q_{\dot H^r} \\ & =: J_1(t,\sigma) + J_2(t,\sigma) + J_3(t,\sigma). \end{aligned} \tag{14}$$

Thanks to (G2), one can estimate the first term in the right hand side as follows

$$J_1(t,\sigma) = 3^{q-1}\mathbb{E}\|\mathbb{L}_r\big(\mathcal{G}_{\alpha,\beta}(t+\sigma) - \mathcal{G}_{\alpha,\beta}(t)\big)\chi(0)\|^q_{L^2(\mathbb{X})} \le 3^{q-1}|M_2(\alpha,\beta,\gamma)|^q\sigma^{\gamma q}\mathbb{E}\|\mathbb{L}_r\chi(0)\|^q_{L^2(\mathbb{X})},$$

where $\gamma \in (0,1]$. Since $\mathbb{E}\|\mathbb{L}_r\chi(0)\|^q_{L^2(\mathbb{X})} = \mathbb{E}\|\chi(0)\|^q_{\dot H^r} \le \sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r}$, it is easy to see that

$$J_1(t,\sigma) \le 3^{q-1}|M_2(\alpha,\beta,\gamma)|^q\sigma^{\gamma q}\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} \longrightarrow 0, \quad \text{as } \sigma \to 0. \tag{15}$$

For the second term, let us split it into $J_{2,1}(t,\sigma)$ and $J_{2,2}(t,\sigma)$ defined as follows

$$\begin{aligned} J_2(t,\sigma) \le\ & 6^{q-1}\mathbb{E}\Big\|\int_0^t \mathbb{L}_r\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)A(\zeta,x_\zeta)d\zeta\Big\|^q_{L^2(\mathbb{X})} + \\ & + 6^{q-1}\mathbb{E}\Big\|\int_t^{t+\sigma}\mathbb{L}_r\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)A(\zeta,x_\zeta)d\zeta\Big\|^q_{L^2(\mathbb{X})} \\ & =: J_{2,1}(t,\sigma) + J_{2,2}(t,\sigma). \end{aligned}$$

In order to show $J_{2,1}(t,\sigma)$ tends to zero as $\sigma \to 0$, we first use the Hölder inequality and property (G2) in turns

$$\begin{aligned} J_{2,1}(t,\sigma) &\le 6^{q-1}\mathbb{E}\Big|\int_0^t \big\|\mathbb{L}_\mu\mathbb{L}_{r-\mu}\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)A(\zeta,x_\zeta)\big\|_{L^2(\mathbb{X})}d\zeta\Big|^q \\ &\le 6^{q-1}t^{q-1}\mathbb{E}\Big|\int_0^t \big\|\mathbb{L}_\mu\mathbb{L}_{r-\mu}\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)A(\zeta,x_\zeta)\big\|^q_{L^2(\mathbb{X})}d\zeta\Big| \\ &\le 6^{q-1}b^{q-1}|\tilde C_2(\alpha,\beta,r-\mu,\gamma)|^q\sigma^{\gamma q}\int_0^t \mathbb{E}\big\|\mathbb{L}_\mu A(\zeta,x_\zeta)\big\|^q_{L^2(\mathbb{X})}d\zeta, \end{aligned}$$

where we have used $r - \mu \le \alpha$. On the other hand, it follows from Assumption (A2) that

$$\begin{aligned} &\int_0^t \mathbb{E}\big\|\mathbb{L}_\mu A(\zeta,x_\zeta)\big\|^q_{L^2(\mathbb{X})}d\zeta \\ &\quad\le 2^{q-1}\int_0^t \mathbb{E}\big\|\mathbb{L}_\mu\big(A(\zeta,x_\zeta) - A(\zeta,0)\big)\big\|^q_{L^2(\mathbb{X})}d\zeta + 2^{q-1}\int_0^t \mathbb{E}\big\|\mathbb{L}_\mu A(\zeta,0)\big\|^q_{L^2(\mathbb{X})}d\zeta \\ &\quad\le 2^{q-1}K_A\int_{-a}^t \mathbb{E}\big\|x(\zeta)\big\|^q_{\dot H^r}d\zeta + 2^{q-1}\int_0^t \mathbb{E}\big\|A(\zeta,0)\big\|^q_{\dot H^\mu}d\zeta \\ &\quad\le 2^{q-1}K_A\big(a\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + t\sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r}\big) + 2^{q-1}\overline{K}^q_A. \end{aligned} \tag{16}$$

From two latter observations, one deduces that there exists $\kappa^{(1)}_{\alpha,\beta}$ depending on $\alpha, \beta, r, \mu, \gamma, q, a, b, K_A$ such that

$$J_{2,1}(t,\sigma) \le \kappa^{(1)}_{\alpha,\beta}\sigma^{\gamma q}\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + \sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r} + \overline{K}^q_A\big), \tag{17}$$

which implies that $J_{2,1}(t,\sigma)$ tends to zero as $\sigma \to 0$. We continue to estimate $J_{2,2}(t,\sigma)$ by using property (G1) and a similar technique as above. In this way, one arrives at

$$\begin{aligned}
J_{2,2}(t,\sigma) &\leq 6^{q-1}\mathbb{E}\Big|\int_t^{t+\sigma} \|\mathbb{L}_\mu \mathbb{L}_{r-\mu}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)A(\zeta,x_\zeta)\|_{L^2(\mathbb{X})}d\zeta\Big|^q_{L^2(\mathbb{X})} \\
&\leq 6^{q-1}\sigma^{q-1}\mathbb{E}\Big|\int_t^{t+\sigma} \|\mathbb{L}_\mu \mathbb{L}_{r-\mu}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)A(\zeta,x_\zeta)\|^q_{L^2(\mathbb{X})}d\zeta\Big| \\
&\leq 6^{q-1}\sigma^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q\int_t^{t+\sigma} \mathbb{E}\|\mathbb{L}_\mu A(\zeta,x_\zeta)\|^q_{L^2(\mathbb{X})}d\zeta.
\end{aligned}$$

Reasoning similarly to estimate (16), one can verify that

$$\begin{aligned}
&\int_t^{t+\sigma} \mathbb{E}\big\|\mathbb{L}_\mu A(\zeta,x_\zeta)\big\|^q_{L^2(\mathbb{X})}d\zeta \\
&\qquad \leq 2^{q-1}K_A\big(a\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + b\sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot{H}^r}\big) + 2^{q-1}\|A(\cdot,0)\|^q_{L^q(J;L^q(\Omega,\dot{H}^\mu))}.
\end{aligned}$$

Hence, there exists $\kappa^{(2)}_{\alpha,\beta}$ depending on $\alpha,\beta,r,\mu,q,a,b,K_A$ such that

$$J_{2,2}(t,\sigma) \leq \kappa^{(2)}_{\alpha,\beta}\sigma^{q-1}\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot{H}^r} + \overline{K}^q_A\big), \tag{18}$$

which implies that $J_{2,2}(t,\sigma)$ tends to zero as $\sigma \to 0$.

We now estimate the last term on the right hand side of (14) as

$$\begin{aligned}
J_3(t,\sigma) &= 3^{q-1}\mathbb{E}\|\mathcal{Z}_{\alpha,\beta}(t+\sigma) - \mathcal{Z}_{\alpha,\beta}(t)\|^q_{\dot{H}^r} \\
&= 3^{q-1}\mathbb{E}\Big\|\int_0^{t+\sigma}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)B(\zeta,x_\zeta)d\omega_\zeta - \int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)B(\zeta,x_\zeta)d\omega_\zeta\Big\|^q_{\dot{H}^r}.
\end{aligned}$$

It can be seen that $J_3(t,\sigma) \leq J_{3,1}(t,\sigma) + J_{3,2}(t,\sigma)$, where two new non-linear terms $J_{3,1}(t,\sigma)$ and $J_{3,2}(t,\sigma)$ are defined by

$$\begin{aligned}
J_{3,1}(t,\sigma) &:= 6^{q-1}\mathbb{E}\Big\|\int_0^t \mathbb{L}_r\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)B(\zeta,x_\zeta)d\omega_\zeta\Big\|^q_{L^2(\mathbb{X})}, \\
J_{3,2}(t,\sigma) &:= 6^{q-1}\mathbb{E}\Big\|\int_t^{t+\sigma} \mathbb{L}_r\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)B(\zeta,x_\zeta)d\omega_\zeta\Big\|^q_{L^2(\mathbb{X})}.
\end{aligned}$$

Using the Burkholder-Davis-Gundy-type inequality, the Hölder inequality, and property (G2), one can arrive at

$$\begin{aligned}
J_{3,1}(t,\sigma) &\leq 6^{q-1}c(q)\mathbb{E}\Big(\int_0^t \|\mathbb{L}_\nu\mathbb{L}_{r-\nu}\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)B(\zeta,x_\zeta)\|^2_{L^2_\mathcal{Q}}d\zeta\Big)^{\frac{q}{2}} \\
&\leq 6^{q-1}c(q)t^{1-\frac{2}{q}}\int_0^t \mathbb{E}\|\mathbb{L}_\nu\mathbb{L}_{r-\nu}\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)B(\zeta,x_\zeta)\|^q_{L^2_\mathcal{Q}}d\zeta \\
&\leq 6^{q-1}c(q)t^{1-\frac{2}{q}}\int_0^t \|\mathbb{L}_{r-\nu}\big(\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta) - \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big)\|^q_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))}\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_\mathcal{Q}}d\zeta \\
&\leq 6^{q-1}c(q)|\tilde{C}_2(\alpha,\beta,r-\nu,\gamma)|^q\sigma^{q\gamma}t^{1-\frac{2}{q}}\int_0^t \mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_\mathcal{Q}}d\zeta,
\end{aligned}$$

where we note that $r-\nu \leq \alpha$ and $\gamma \in (0,1]$. In addition, Assumption (B1) allows us to obtain

$$\begin{aligned}
&\int_0^t \mathbb{E}\big\|\mathbb{L}_\nu B(\zeta,x_\zeta)\big\|^q_{L^2_{\mathbb{Q}}} d\zeta \\
&\qquad \leq 2^{q-1}\int_0^t \mathbb{E}\big\|\mathbb{L}_\nu\big(B(\zeta,x_\zeta)-B(\zeta,0)\big)\big\|^q_{L^2_{\mathbb{Q}}} d\zeta + 2^{q-1}\int_0^t \mathbb{E}\big\|\mathbb{L}_\nu B(\zeta,0)\big\|^q_{L^2_{\mathbb{Q}}} d\zeta \\
&\qquad \leq 2^{q-1}K_B\int_{-a}^t \mathbb{E}\big\|x(\zeta)\big\|^q_{\dot H^r} d\zeta + 2^{q-1}\int_0^t \mathbb{E}\big\|B(\zeta,0)\big\|^q_{\dot H^\nu} d\zeta \\
&\qquad \leq 2^{q-1}K_B\big(a\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + t\sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r}\big) + 2^{q-1}\|B(\cdot,0)\|^q_{L^q(J;L^q(\Omega,L^2_{\mathbb{Q},\nu}))}.
\end{aligned} \tag{19}$$

From two latter observations, one deduces that there exists $\kappa^{(3)}_{\alpha,\beta}$ depending on $\alpha,\beta,r,\nu,\gamma,q,a,b,K_B$ such that

$$J_{3,1}(t,\sigma) \leq \kappa^{(3)}_{\alpha,\beta}\sigma^{\gamma q}\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + \sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r} + \overline{K}^q_B\big), \tag{20}$$

which implies that $J_{3,1}(t,\sigma)$ tends to zero as $\sigma\to 0$. We continue to estimate $J_{3,2}(t,\sigma)$ by using the property (G1) and a similar technique as in above. In this way, we have

$$\begin{aligned}
J_{3,2}(t,\sigma) &\leq 6^{q-1}c(q)\mathbb{E}\Big(\int_t^{t+\sigma}\|\mathbb{L}_\nu\mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)B(\zeta,x_\zeta)\|^2_{L^2_{\mathbb{Q}}} d\zeta\Big)^{\frac{q}{2}} \\
&\leq 6^{q-1}c(q)\sigma^{\frac{q}{2}-1}\int_t^{t+\sigma}\mathbb{E}\|\mathbb{L}_\nu\mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}} d\zeta \\
&\leq 6^{q-1}c(q)\sigma^{\frac{q}{2}-1}\int_t^{t+\sigma}\|\mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t+\sigma-\zeta)\|^q_{\mathcal{L}(L^q(\Omega,L^2(\mathbb{X})))}\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}} d\zeta \\
&\leq 6^{q-1}c(q)|\tilde C_1(\alpha,\beta,r-\nu)|^q\sigma^{\frac{q}{2}-1}\int_t^{t+\sigma}\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}} d\zeta.
\end{aligned}$$

By using a similar way as in the estimate (19), one can verify that

$$\begin{aligned}
&\int_t^{t+\sigma}\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}} d\zeta \\
&\qquad \leq 2^{q-1}K_B\big(a\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + b\sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r}\big) + 2^{q-1}\|B(\cdot,0)\|^q_{L^q(J;L^q(\Omega,L^2_{\mathbb{Q},\nu}))}.
\end{aligned}$$

Hence, there exists $\kappa^{(4)}_{\alpha,\beta}$ depending on $\alpha,\beta,r,\nu,q,a,b,K_B$ such that

$$J_{3,2}(t,\sigma) \leq \kappa^{(4)}_{\alpha,\beta}\sigma^{\frac{q}{2}}\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r} + \sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r} + \overline{K}^q_B\big), \tag{21}$$

which implies that $J_{3,2}(t,\sigma)$ tends to zero as $\sigma\to 0$. Therefore, we conclude that $J_3(t,\sigma)\to 0$ as $\sigma\to 0$.

Since $J_1(t,\sigma)$, $J_2(t,\sigma)$, and $J_3(t,\sigma)$ tend to zero as $\sigma\to 0$, the map $t\mapsto \Psi(x)(t)$ is continuous on J in $L^q(\Omega,\dot H^r)$ sense. Furthermore, results (15)-(21) imply that the terms $J_1(t,\sigma)$, $J_2(t,\sigma)$, and $J_3(t,\sigma)$ are of order σ^p, with $p\in(0,\frac{1}{2}]$. From this together with (12) it follows that $\Psi(\mathcal{B}^{p,q,r}_\chi)\subset\mathcal{B}^{p,q,r}_\chi$.
Claim 2. Operator Ψ is a contraction in $\mathcal{B}^{p,q,r}_\chi$.

The present claim can be verified by showing that for $x,x^\dagger\in\mathcal{B}^{p,q,r}_\chi$ there exists a positive constant $\Pi_{\alpha,\beta}$ depending on $\alpha,\beta,r,\mu,\nu,q,a,b,K_A,K_B$ such that, for $t\in J$,

$$\sup_{0\leq\theta\leq t}\mathbb{E}\big\|\Psi^m(x)(\theta)-\Psi^m(x^\dagger)(\theta)\big\|^q_{\dot H^r} \leq \frac{|\Pi_{\alpha,\beta}t|^m}{m!}\sup_{0\leq\theta\leq b}\mathbb{E}\|x(\theta)-x^\dagger(\theta)\|^q_{\dot H^r}, \tag{22}$$

for any $m \in \mathbb{N}^+$, which leads to

$$\sup_{0\le t\le b} \mathbb{E}\big\|\Psi^m(x)(t) - \Psi^m(x^\dagger)(t)\big\|^q_{\dot{H}^r} \le \frac{|\Pi_{\alpha,\beta} b|^m}{m!} \sup_{0\le t\le b} \mathbb{E}\|x(t) - x^\dagger(t)\|^q_{\dot{H}^r}. \tag{23}$$

Indeed, it can be seen from equation (13) that, for $t \in J$,

$$\begin{aligned}\mathbb{E}\big\|\Psi(x)(t) - \Psi(x^\dagger)(t)\big\|^q_{\dot{H}^r} &\le 2^{q-1}\mathbb{E}\Big\| \int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(A(\zeta, x_\zeta) - A(\zeta, x^\dagger_\zeta)\big)d\zeta\Big\|^q_{\dot{H}^r} + \\ &\quad + 2^{q-1}\mathbb{E}\big\|\mathcal{Z}_{\alpha,\beta}(t) - \mathcal{Z}_{\alpha,\beta}(t)\big\|^q_{\dot{H}^r} \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{24}$$

The first term can be estimated by using the Hölder inequality and property (G1):

$$\begin{aligned}\mathcal{I}_1 &\le 2^{q-1}\mathbb{E}\Big|\int_0^t \|\mathbb{L}_\mu \mathbb{L}_{r-\mu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(A(\zeta, x_\zeta) - A(\zeta, x^\dagger_\zeta)\big)\|_{L^2(\mathbb{X})} d\zeta\Big|^q \\ &\le 2^{q-1}t^{q-1}\mathbb{E}\Big|\int_0^t \|\mathbb{L}_\mu \mathbb{L}_{r-\mu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(A(\zeta, x_\zeta) - A(\zeta, x^\dagger_\zeta)\big)\|^q_{L^2(\mathbb{X})} d\zeta\Big| \\ &\le 2^{q-1}b^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q \int_0^t \mathbb{E}\|\mathbb{L}_\mu\big(A(\zeta, x_\zeta) - A(\zeta, x^\dagger_\zeta)\big)\|^q_{L^2(\mathbb{X})} d\zeta.\end{aligned}$$

Now, Assumption (A2) allows us to obtain

$$\mathcal{I}_1 \le 2^{q-1}b^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q K_A \int_{-a}^t \mathbb{E}\|x(\zeta) - x^\dagger(\zeta)\|^q_{\dot{H}^r} d\zeta. \tag{25}$$

The second term can be estimated by using the Burkholder-Davis-Gundy-type inequality, the Hölder inequality, and property (G1) as follows:

$$\begin{aligned}\mathcal{I}_2 &\le 2^{q-1}\mathbb{E}\Big\|\int_0^t \mathbb{L}_\nu \mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\big)d\omega_\zeta\Big\|^q_{L^2(\mathbb{X})} \\ &\le 2^{q-1}c(q)\mathbb{E}\Big(\int_0^t \|\mathbb{L}_\nu \mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\big)\|^2_{L^2_Q} d\zeta\Big)^{\frac{q}{2}} \\ &\le 2^{q-1}c(q)t^{1-\frac{2}{q}} \int_0^t \mathbb{E}\|\mathbb{L}_\nu \mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\big(B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\big)\|^q_{L^2_Q} d\zeta \\ &\le 2^{q-1}c(q)t^{1-\frac{2}{q}} \int_0^t \|\mathbb{L}_{r-\nu}\tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)\|^q_{\mathcal{L}(L^q(\Omega, L^2(\mathbb{X})))}\mathbb{E}\|\mathbb{L}_\nu\big(B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\big)\|^q_{L^2_Q} d\zeta \\ &\le 2^{q-1}c(q)t^{1-\frac{2}{q}}|\tilde{C}_1(\alpha,\beta,\nu-r)|^q \int_0^t \mathbb{E}\|\mathbb{L}_\nu\big(B(\zeta, x_\zeta) - B(\zeta, x^\dagger_\zeta)\big)\|^q_{L^2_Q} d\zeta.\end{aligned}$$

This together with Assumption (B2) yields

$$\mathcal{I}_2 \le 2^{q-1}c(q)b^{1-\frac{2}{q}}|\tilde{C}_1(\alpha,\beta,\nu-r)|^q K_B \int_{-a}^t \mathbb{E}\|x(\zeta) - x^\dagger(\zeta)\|^q_{\dot{H}^r} d\zeta. \tag{26}$$

Combining (24), (25), (26), and noting $x(s) = x^\dagger(s) = \chi(s)$ for $s \in J_0$, one concludes that

$$\begin{aligned}&\mathbb{E}\big\|\Psi(x)(t) - \Psi(x^\dagger)(t)\big\|^q_{\dot{H}^r} \\ &\le 2^{q-1}\big(K_A b^{q-1}|\tilde{C}_1(\alpha,\beta,\mu-r)|^q + K_B c(q) b^{1-\frac{2}{q}}|\tilde{C}_1(\alpha,\beta,\nu-r)|^q\big)\int_0^t \mathbb{E}\|x(\zeta) - x^\dagger(\zeta)\|^q_{\dot{H}^r} d\zeta,\end{aligned}$$

which implies that there exists $\Pi_{\alpha,\beta}$ depending on $\alpha, \beta, r, \mu, \nu, \gamma, q, a, b, K_A, K_B$ such that, for $t \in J$,

$$\sup_{0\le\theta\le t} \mathbb{E}\big\|\Psi(x)(\theta) - \Psi(x^\dagger)(\theta)\big\|^q_{\dot{H}^r} \le \Pi_{\alpha,\beta} \int_0^t \sup_{0\le\theta\le\zeta} \mathbb{E}\|x(\theta) - x^\dagger(\theta)\|^q_{\dot{H}^r} d\zeta$$
$$\le \Pi_{\alpha,\beta} t \sup_{\theta\in J} \mathbb{E}\|x(\theta) - x^\dagger(\theta)\|^q_{\dot{H}^r} d\zeta,$$

which implies that (22) holds for $m = 1$. We now aim at showing that if it holds in the case $m = m_0 \in \mathbb{N}^+$ then it also holds when $m = m_0 + 1$. Indeed, by using a similar technique as above, one arrives at

$$\begin{aligned}
&\sup_{0\le\theta\le t} \mathbb{E}\big\|\Psi^{m_0+1}(x)(\theta) - \Psi^{m_0+1}(x^\dagger)(\theta)\big\|^q_{\dot{H}^r} \\
&\qquad \le \Pi_{\alpha,\beta} \int_0^t \sup_{0\le\theta\le\zeta} \mathbb{E}\|\Psi^{m_0}(x)(\theta) - \Psi^{m_0}(x^\dagger)(\theta)\|^q_{\dot{H}^r} d\zeta \\
&\qquad \le \Pi_{\alpha,\beta} \int_0^t \frac{|\Pi_{\alpha,\beta}\zeta|^{m_0}}{m_0!} \sup_{\theta\in J} \mathbb{E}\|x(\theta) - x^\dagger(\theta)\|^q_{\dot{H}^r} d\zeta \\
&\qquad \le \frac{|\Pi_{\alpha,\beta} b|^{m_0+1}}{(m_0+1)!} \sup_{\theta\in J} \mathbb{E}\|x(\theta) - x^\dagger(\theta)\|^q_{\dot{H}^r} d\zeta.
\end{aligned}$$

Hence, (22) holds for any $m \in \mathbb{N}^+$ and the inequality (23) is true as desired. Since $\frac{|\Pi_{\alpha,\beta} b|^m}{m!}$ tends to zero as $m \to \infty$, there exists $m \in \mathbb{N}^+$ such that Ψ^m is a contraction. This procedure can be repeated in order to obtain that Ψ is a contraction too. In other words, $\Psi(x) = x$ has a unique solution in the space $\mathcal{B}^{p,q,r}_\chi$.

Claim 3. The solution x satisfies regularity properties i) and ii).

Initially, we aim at proving that the solution x satisfies the regularity property i). Indeed, for $t \in J$, by using a similar argument as in the proof of Claim 1, one can easily arrive at

$$\begin{aligned}
\mathbb{E}\big\|x(t)\big\|^q_{\dot{H}^r} &\le 3^{q-1}|M_1(\alpha,\beta)|^q \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \\
&+ 6^{q-1} b^{q-1} |\tilde{C}_1(\alpha,\beta,r-\mu)|^q \Big(K_A \int_{-a}^t \mathbb{E}\big\|x(\zeta)\big\|^q_{\dot{H}^r} d\zeta + \int_0^t \mathbb{E}\big\|A(\zeta,0)\big\|^q_{\dot{H}^\mu} d\zeta\Big) + \\
&+ 6^{q-1} c(q) |\tilde{C}_1(\alpha,\beta,r-\nu)|^q b^{1-\frac{2}{q}} \Big(K_B \int_{-a}^t \mathbb{E}\big\|x(\zeta)\big\|^q_{\dot{H}^r} d\zeta + \int_0^t \mathbb{E}\big\|B(\zeta,0)\big\|^q_{L^2_{\mathcal{Q},\nu}} d\zeta\Big),
\end{aligned}$$

which implies that there exists positive constants $\kappa^{(5)}_{\alpha,\beta}$, $\kappa^{(6)}_{\alpha,\beta}$, $\kappa^{(7)}_{\alpha,\beta}$ depending on $\alpha, \beta, r, \mu, \nu, q, a, b$, and $K_A, K_B, \overline{K}_B, \overline{K}_B$ such that

$$\sup_{0\le\theta\le t} \mathbb{E}\big\|x(\theta)\big\|^q_{\dot{H}^r} \le \kappa^{(5)}_{\alpha,\beta} + \kappa^{(6)}_{\alpha,\beta} \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \kappa^{(7)}_{\alpha,\beta} \int_0^t \sup_{0\le\theta\le\zeta} \mathbb{E}\|x(\theta)\|^q_{\dot{H}^r} d\zeta.$$

By using the Grönwall inequality, one obtains

$$\sup_{0\le\theta\le t} \mathbb{E}\big\|x(\theta)\big\|^q_{\dot{H}^r} \le \big(\kappa^{(5)}_{\alpha,\beta} + \kappa^{(6)}_{\alpha,\beta} \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r}\big) \exp\big(\kappa^{(7)}_{\alpha,\beta} t\big),$$

which yields that

$$\sup_{\theta\in J} \mathbb{E}\big\|x(\theta)\big\|^q_{\dot{H}^r} \le \big(\kappa^{(5)}_{\alpha,\beta} + \kappa^{(6)}_{\alpha,\beta} \sup_{\theta\in J_0} \mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r}\big) \exp\big(\kappa^{(7)}_{\alpha,\beta} b\big),$$

which implies that property i) holds true.

Next, we shall employ the result we have proved above to show that the solution x satisfies the following regularity property ii). By the same way as in Claim 1, one can check that there exists a positive constant $\kappa^{(8)}_{\alpha,\beta}$ depending on $\alpha,\beta,r,\mu,\nu,q,a,b,K_A,K_B,\overline{K}_A,\overline{K}_B$ such that

$$\mathbb{E}\big\|x(t+\sigma)-x(t)\big\|^q_{\dot H^r}\le \kappa^{(8)}_{\alpha,\beta}\sigma^{pq}\Big(1+\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r}+\sup_{\theta\in J}\mathbb{E}\|x(\theta)\|^q_{\dot H^r}\Big),\quad p\in(0,1/2]. \tag{27}$$

From (27) and property i), it is clear that there exists a positive constant $\kappa^{(9)}_{\alpha,\beta}$ depending only on $\alpha,\beta,r,\mu,\nu,q,a,b,K_A,K_B,\overline{K}_A,\overline{K}_B$ such that

$$\mathbb{E}\big\|x(t+\sigma)-x(t)\big\|^q_{\dot H^r}\le \kappa^{(9)}_{\alpha,\beta}\sigma^{pq}\Big(1+\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r}\Big),\quad \text{for } 0\le t<t+\sigma\le b.$$

Furthermore, in the case $-a\le t<t+\sigma\le 0$, we note that

$$\mathbb{E}\big\|x(t+\sigma)-x(t)\big\|^q_{\dot H^r}=\mathbb{E}\big\|\chi(t+\sigma)-\chi(t)\big\|^q_{\dot H^r}\le \sigma^{pq}\|\chi\|^p_{\mathcal{C}^p(J_0;L^q(\Omega,\dot H^r))}.$$

Hence, it can be seen that property ii) holds. This completes the proof. □

In what follows, we are interested in considering the continuity of mild solutions in the case of finite delay with respect to the initial function and the order of the fractional derivative separately. Initially, the following theorem describes the continuity result in the first sense.

Theorem 3.2. *Assume that (A1), (A2), (B1), (B2) hold. Then, the mild solution to Problem* (1) *is continuous with respect to the initial function. Namely, if $\chi_1,\chi_2\in\mathcal{C}(J_0;L^q(\Omega,\dot H^r))$, for $r\ge 0$ and $q\ge 2$, and x_1,x_2 are mild solutions to Problem* (1) *with respect to the initial functions χ_1,χ_2 respectively. Then, there exists $M_5(\alpha,\beta)>0$ depending on $\alpha,\beta,r,\mu,\nu,q,a,b,K_A,K_B$ such that*

$$\big\|x_1(t)-x_2(t)\big\|_{L^q(\Omega,\dot H^r)}\le M_5(\alpha,\beta)\|\chi_1-\chi_2\|_{\mathcal{C}(J_0;L^q(\Omega,\dot H^r))}, \tag{28}$$

for all $t\in J$.

Proof. By a similar argument to that one used to obtain the uniqueness result, we have

$$\sup_{0\le\theta\le t}\mathbb{E}\big\|x_1(\theta)-x_2(\theta)\big\|^q_{\dot H^r}\le 3^{q-1}|M_1(\alpha,\beta)|^q\sup_{\theta\in J_0}\mathbb{E}\|\chi_1(\theta)-\chi_2(\theta)\|^q_{\dot H^r}+$$
$$+\Pi_{\alpha,\beta}\int_0^t\sup_{0\le\theta\le\zeta}\mathbb{E}\|x_1(\theta)-x_2(\theta)\|^q_{\dot H^r}d\zeta.$$

By using the Grönwall inequality, we obtain

$$\sup_{0\le\theta\le t}\mathbb{E}\big\|x_1(\theta)-x_2(\theta)\big\|^q_{\dot H^r}\le 3^{q-1}|M_1(\alpha,\beta)|^q\sup_{\theta\in J_0}\mathbb{E}\|\chi_1(\theta)-\chi_2(\theta)\|^q_{\dot H^r}\exp\big(\Pi_{\alpha,\beta}t\big),$$

which implies that property (28) holds. □

Next, the following theorem gives the continuity of mild solutions with respect to the order of fractional derivative.

Theorem 3.3. *Let $\chi\in\mathcal{C}(J_0;L^q(\Omega,\dot H^r))$, for $r\ge 0,q\ge 2$. Assume that (A1), (A2), (B1), (B2) hold. Then, the mild solution to Problem* (1) *is continuous with respect to the order of the fractional derivative. Namely, if $\beta,\beta'\in[\beta_0,\beta_1]\subset(0,1)$ and $x^{(\beta)},x^{(\beta')}$ are mild solutions to Problem* (1) *with respect to the orders β,β' respectively. Then, there exists $M_6(\alpha)>0$ depending on $\alpha,\beta_1,r,\lambda_1,\mu,\nu,q,a,b,K_A,K_B$, such that*

$$\begin{aligned}&\big\|x^{(\beta)}(t)-x^{(\beta')}(t)\big\|_{L^q(\Omega,\dot H^r)}\\&\qquad\le M_6(\alpha)|\beta-\beta'|\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot H^r}+\sup_{\theta\in J}\mathbb{E}\|x^{(\beta)}(\theta)\|^q_{\dot H^r}+\overline{K}^q_A+\overline{K}^q_B\big)^{\frac1q},\end{aligned} \tag{29}$$

for all $t\in J$.

Proof. For the sake of convenience, let us set $\overline{\mathcal{G}}_{\alpha,\beta,\beta'}(t) := \mathcal{G}_{\alpha,\beta}(t) - \mathcal{G}_{\alpha,\beta'}(t)$ and $\hat{\mathcal{G}}_{\alpha,\beta,\beta'}(t) := \tilde{\mathcal{G}}_{\alpha,\beta}(t) - \tilde{\mathcal{G}}_{\alpha,\beta'}(t)$. It can be seen from the integral equations (5)-(6) that, for $t \in J$,

$$\begin{aligned}
\mathbb{E}\|x^{(\beta)}(t) - x^{(\beta')}(t)\|^q_{\dot{H}^r} &\le 3^{q-1}\mathbb{E}\|\overline{\mathcal{G}}_{\alpha,\beta,\beta'}(t)\chi(0)\|^q_{\dot{H}^r} + 6^{q-1}\mathbb{E}\Big\|\int_0^t \hat{\mathcal{G}}_{\alpha,\beta,\beta'}(t-\zeta)A(\zeta, x^{(\beta)}_\zeta)d\zeta\Big\|^q_{\dot{H}^r} + \\
&+ 6^{q-1}\mathbb{E}\Big\|\int_0^t \tilde{\mathcal{G}}_{\alpha,\beta'}(t-\zeta)\big(A(\zeta, x^{(\beta)}_\zeta) - A(\zeta, x^{(\beta')}_\zeta)\big)d\zeta\Big\|^q_{\dot{H}^r} + \\
&+ 6^{q-1}\mathbb{E}\Big\|\int_0^t \hat{\mathcal{G}}_{\alpha,\beta,\beta'}(t-\zeta)B(\zeta, x^{(\beta)}_\zeta)d\omega_\zeta\Big\|^q_{\dot{H}^r} + \\
&+ 6^{q-1}\mathbb{E}\Big\|\int_0^t \tilde{\mathcal{G}}_{\alpha,\beta'}(t-\zeta)\big(B(\zeta, x^{(\beta)}_\zeta) - B(\zeta, x^{(\beta')}_\zeta)\big)d\omega_\zeta\Big\|^q_{\dot{H}^r}.
\end{aligned}$$

It is obvious that the five previous terms can be estimated similarly as the estimates for $J_1(t,\sigma)$, $J_2(t,\sigma)$, $J_3(t,\sigma)$ defined in (14), but one needs to use property (G1) and Proposition 2.3 instead of property (G2). Therefore, in what follows, we omit the details and show directly the results, which can be verified easily

$$\begin{aligned}
\mathbb{E}\|x^{(\beta)}(t) - x^{(\beta')}(t)\|^q_{\dot{H}^r} &\le \overline{\kappa}^{(1)}_\alpha|\beta-\beta'|^q\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \sup_{\theta\in J}\mathbb{E}\|x^{(\beta)}(\theta)\|^q_{\dot{H}^r} + \overline{K}^q_A + \overline{K}^q_B\big) \\
&+ \overline{\kappa}^{(2)}_\alpha\int_{-a}^t \mathbb{E}\|x^{(\beta)}(\zeta) - x^{(\beta')}(\zeta)\|^q_{\dot{H}^r}d\zeta,
\end{aligned}$$

where $\overline{\kappa}^{(1)}_\alpha$ and $\overline{\kappa}^{(2)}_\alpha$ depend on $\alpha, \beta_1, \lambda_1, r, \mu, \nu, q, a, b, K_A, K_B$. Since $x^{(\beta)}(\zeta) = x^{(\beta')}(\zeta)$ for $\zeta \in J_0$, the above inequality yields that

$$\begin{aligned}
\sup_{0\le\theta\le t}\mathbb{E}\|x^{(\beta)}(\theta) - x^{(\beta')}(\theta)\|^q_{\dot{H}^r} &\le \overline{\kappa}^{(1)}_\alpha|\beta-\beta'|^q\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \sup_{\theta\in J}\mathbb{E}\|x^{(\beta)}(\theta)\|^q_{\dot{H}^r} + \overline{K}^q_A + \overline{K}^q_B\big) \\
&+ \overline{\kappa}^{(2)}_\alpha\int_0^t \sup_{0\le\theta\le\zeta}\mathbb{E}\|x^{(\beta)}(\theta) - x^{(\beta')}(\theta)\|^q_{\dot{H}^r}d\zeta.
\end{aligned}$$

With the help of the Grönwall inequality, one obtains

$$\begin{aligned}
&\sup_{0\le\theta\le t}\mathbb{E}\|x^{(\beta)}(\theta) - x^{(\beta')}(\theta)\|^q_{\dot{H}^r} \\
&\qquad \le \overline{\kappa}^{(1)}_\alpha|\beta-\beta'|^q\big(\sup_{\theta\in J_0}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \sup_{\theta\in J}\mathbb{E}\|x^{(\beta)}(\theta)\|^q_{\dot{H}^r} + \overline{K}^q_A + \overline{K}^q_B\big)\exp(\overline{\kappa}^{(2)}_\alpha t),
\end{aligned}$$

which implies that property (29) holds. □

4 Existence, uniqueness, and regularity results in the case of infinite delay

In this section, we continue to investigate stochastic fractional diffusion equations, but in the case of infinite delay. Existence, uniqueness, and regularity results for Problem (2) are verified from now on, provided that $\chi \in \mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))$, for $r \ge 0, q \ge 2, p \in (0, \frac{1}{2}]$, and $\delta > 0$.

To this end, let us prepare some materials including some notations of necessary functional spaces. We first introduce the spaces $\mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))$ and $\mathcal{C}_\delta((-\infty, b]; L^q(\Omega, \dot{H}^r))$, which are subspaces of $\mathcal{C}(J_\infty; L^q(\Omega, \dot{H}^r))$ and $\mathcal{C}((-\infty, b]; L^q(\Omega, \dot{H}^r))$ endowed with the norms

$$\|x\|_{\mathcal{C}_\delta(J_\infty;L^q(\Omega,\dot{H}^r))} := \Big(\sup_{\theta\in J_\infty} e^{\delta\theta}\mathbb{E}\|x(\theta)\|^q_{\dot{H}^r}\Big)^{\frac{1}{q}} < \infty,$$

$$\|x\|_{\mathcal{C}_\delta((-\infty,b];L^q(\Omega,\dot{H}^r))} := \Big(\sup_{\theta\in(-\infty,b]} e^{\delta\theta}\mathbb{E}\|x(\theta)\|^q_{\dot{H}^r}\Big)^{\frac{1}{q}} < \infty.$$

Additionally, let us introduce the following Banach space

$$\mathcal{S}^{q,r}_{\chi,\delta} := \left\{ x \in \mathcal{C}_\delta((-\infty, b]; L^q(\Omega, \dot{H}^r)) : x(s) = \chi(s), \text{for } s \in J_\infty \right\},$$

endowed with the norm of the space $\mathcal{C}_\delta((-\infty, b]; L^q(\Omega, \dot{H}^r))$.

Suppose that the non-linear source A and the non-linear space-time-noise B satisfy the following assumptions for $\mu, \nu \in [r - \alpha, r]$

(A3) $A(\cdot, 0) = 0$ and there exists $k_A : J \to \mathbb{R}^+$ such that for any $x, x^\dagger \in \mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))$ and $t \in J$

$$\mathbb{E}\|A(t, x) - A(t, x^\dagger)\|^q_{\dot{H}^\mu} \le k_A(t)\|x - x^\dagger\|^q_{\mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))},$$

(B3) $B(\cdot, 0) = 0$ and there exists $k_B : J \to \mathbb{R}^+$ such that for any $x, x^\dagger \in \mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))$ and $t \in J$

$$\mathbb{E}\|B(t, x) - B(t, x^\dagger)\|^q_{L^2_{\mathcal{Q},\nu}} \le k_B(t)\|x - x^\dagger\|^q_{\mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))},$$

The following theorem state the existence, uniqueness, and regularity results for Problem (2).

Theorem 4.1. *Let $\chi \in \mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))$, for $r \ge 0, q \ge 2$, and $\delta > 0$. Assume that (A3), (B3) hold and*

$$\mathbb{K}_A := \int_0^b k_A(\zeta)d\zeta < \infty, \quad \mathbb{K}_B := \int_0^b k_B(\zeta)d\zeta < \infty.$$

Then, Problem (2) *has a unique mild solution in the space $\mathcal{S}^{q,r}_{\chi,\delta}$. Furthermore, the following regularity property holds for $t \in J$*

$$\|x_t\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} \le M_7(\alpha, \beta)\|\chi\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))}, \tag{30}$$

where $M_7(\alpha, \beta)$ depends on $\alpha, \beta, r, \mu, \nu, q, a, b, \mathbb{K}_A, \mathbb{K}_B$.

Corollary 4.1. *As a consequence, property* (30) *implies*

$$\mathbb{E}\|x(t)\|^q_{L^q(\Omega, \dot{H}^r)} \le M_7(\alpha, \beta)\|\chi\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))}, \quad \textit{for all } t \in J,$$

which together with $x(s) = \chi(s)$, for $s \in J_\infty$, leads to the existence of $M_8(\alpha, \beta)$, depending on $\alpha, \beta, r, \mu, \nu$, $q, a, b, \mathbb{K}_A, \mathbb{K}_B$, such that

$$\mathbb{E}\|x(t)\|^q_{L^q(\Omega, \dot{H}^r)} \le M_8(\alpha, \beta)\|\chi\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))}, \quad \textit{for all } t \in (-\infty, b],$$

Corollary 4.2 (Existence and uniqueness results on the usual space $\mathcal{PC}^b$)**.** *If Assumptions (A3), (B3) hold for $r = \mu = \nu = 0$, $q = 2$, and the initial condition χ belongs to $C_\delta(J_\infty; L^2(\Omega, L^2(\mathbb{X})))$, then Problem* (2) *has a unique mild solution in the usual space $\mathcal{PC}^b$ (see [42, 46]) defined by*

$$\mathcal{PC}^b := \left\{ x \in \mathcal{C}_\delta((-\infty, b]; L^2(\Omega, L^2(\mathbb{X}))) : x(s) = \chi(s), \textit{for } s \in J_\infty \right\},$$

Furthermore, the following regularity properties hold for $t \in J$

$$\|x_t\|^2_{C_\delta(J_\infty; L^2(\Omega, L^2(\mathbb{X}))} \le \overline{M}_7(\alpha, \beta)\|\chi\|^2_{C_\delta(J_\infty; L^2(\Omega, L^2(\mathbb{X}))},$$

where $\overline{M}_7(\alpha, \beta)$ depends on $\alpha, \beta, a, b, \mathbb{K}_A, \mathbb{K}_B$.

Proof. Let us define the operator Φ on $\mathcal{S}^{q,r}_{\chi,\delta}$ by

$$\Phi(x)(s)=\chi(s),\quad \text{for all } s\in J_\infty, \tag{31}$$

and for all $t\in J$

$$\Phi(x)(t)=\mathcal{G}_{\alpha,\beta}(t)\chi(0)+\int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)A(\zeta,x_\zeta)d\zeta+\mathcal{Z}_{\alpha,\beta}(t). \tag{32}$$

where $\mathcal{Z}_{\alpha,\beta}$ is defined by

$$\mathcal{Z}_{\alpha,\beta}(t):=\int_0^t \tilde{\mathcal{G}}_{\alpha,\beta}(t-\zeta)B(\zeta,x_\zeta)d\omega_\zeta.$$

In what follows, we aim at proving that Ψ is well-defined, i.e. $\Phi(\mathcal{S}^{q,r}_{\chi,\delta})\subset\mathcal{S}^{q,r}_{\chi,\delta}$, and then verify that Φ is a contraction.

Claim 1. $\Phi(x)\in\mathcal{S}^{q,r}_{\chi,\delta}$, for all $x\in\mathcal{S}^{q,r}_{\chi,\delta}$.

Let us verify the continuity of the map $t\mapsto\Phi(x)(t)$ firstly. For $t\in J$ and $\sigma>0$ small enough, by a similar argument as in the proof of Theorem 3.1, one arrives at

$$\begin{aligned}\mathbb{E}\Big\|\Phi(x)(t+\sigma)-\Phi(x)(t)\Big\|^q_{\dot{H}^r}&\le 3^{q-1}|M_2(\alpha,\beta,\gamma)|^q\sigma^{\gamma q}\sup_{\theta\in J_\infty}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r}+\\&+6^{q-1}b^{q-1}|\tilde{C}_2(\alpha,\beta,r-\mu,\gamma)|^q\sigma^{\gamma q}\int_0^t\mathbb{E}\|\mathbb{L}_\mu A(\zeta,x_\zeta)\|^q_{L^2(\mathbb{X})}d\zeta+\\&+6^{q-1}\sigma^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q\int_t^{t+\sigma}\mathbb{E}\|\mathbb{L}_\mu A(\zeta,x_\zeta)\|^q_{L^2(\mathbb{X})}d\zeta+\\&+6^{q-1}c(q)|\tilde{C}_2(\alpha,\beta,r-\nu,\gamma)|^q\sigma^{q\gamma}b^{1-\frac{2}{q}}\int_0^t\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}}d\zeta+\\&+6^{q-1}c(q)|\tilde{C}_1(\alpha,\beta,r-\nu)|^q\sigma^{\frac{q}{2}-1}\int_t^{t+\sigma}\mathbb{E}\|\mathbb{L}_\nu B(\zeta,x_\zeta)\|^q_{L^2_{\mathbb{Q}}}d\zeta\\&=:\mathcal{N}_1(t,\sigma)+\mathcal{N}_2(t,\sigma)+\mathcal{N}_3(t,\sigma)+\mathcal{N}_4(t,\sigma)+\mathcal{N}_5(t,\sigma).\end{aligned}$$

Assumption (A3) allows us to obtain

$$\begin{aligned}\mathcal{N}_2(t,\sigma)&\le 6^{q-1}b^{q-1}|\tilde{C}_2(\alpha,\beta,r-\mu,\gamma)|^q\sigma^{\gamma q}\int_0^t k_A(\zeta)\|x_\zeta\|^q_{\mathcal{C}_\delta(J_\infty,L^q(\Omega,\dot{H}^r))}d\zeta,\\ \mathcal{N}_3(t,\sigma)&\le 6^{q-1}\sigma^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q\int_t^{t+\sigma}k_A(\zeta)\|x_\zeta\|^q_{\mathcal{C}_\delta(J_\infty,L^q(\Omega,\dot{H}^r))}d\zeta.\end{aligned}$$

On the other hand, with the help of Assumption (B3), we have

$$\begin{aligned}\mathcal{N}_4(t,\sigma)&\le 6^{q-1}c(q)|\tilde{C}_2(\alpha,\beta,r-\nu,\gamma)|^q\sigma^{q\gamma}b^{1-\frac{2}{q}}\int_0^t k_B(\zeta)\|x_\zeta\|^q_{\mathcal{C}_\delta(J_\infty,L^q(\Omega,\dot{H}^r))}d\zeta,\\ \mathcal{N}_5(t,\sigma)&\le 6^{q-1}c(q)|\tilde{C}_1(\alpha,\beta,r-\nu)|^q\sigma^{\frac{q}{2}-1}\int_t^{t+\sigma}k_B(\zeta)\|x_\zeta\|^q_{\mathcal{C}_\delta(J_\infty,L^q(\Omega,\dot{H}^r))}d\zeta.\end{aligned}$$

Noting that for $0<\zeta<b$, there holds $\|x_\zeta\|^q_{\mathcal{C}_\delta(J_\infty,L^q(\Omega,\dot{H}^r))}\le\sup_{\theta\in(-\infty,\zeta]}\mathbb{E}\|x(\theta)\|^q_{L^q(\Omega,\dot{H}^r)}$. By all the above observations, one can see that there exists $\kappa^{(10)}_{\alpha,\beta}>0$ depending on $\alpha,\beta,r,\mu,\nu,q,a,b$ and

$p \in (0, \frac{1}{2}]$ such that

$$\mathbb{E}\Big\|\Phi(x)(t+\sigma) - \Phi(x)(t)\Big\|^q_{\dot{H}^r} \le \kappa^{(10)}_{\alpha,\beta}\sigma^{pq}\Big(\sup_{\theta\in J_\infty}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + \\ + \int_0^t \big(k_A(\zeta)+k_B(\zeta)\big)\sup_{\theta\in(-\infty,\zeta]}\mathbb{E}\|x(\theta)\|^q_{L^q(\Omega,\dot{H}^r)}d\zeta + \\ + \int_t^{t+\sigma}\big(k_A(\zeta)+k_B(\zeta)\big)\sup_{\theta\in(-\infty,\zeta]}\mathbb{E}\|x(\theta)\|^q_{L^q(\Omega,\dot{H}^r)}d\zeta\Big).$$

Since $\mathbb{K}_A = \int_0^b k_A(\zeta)d\zeta < \infty$ and $\mathbb{K}_B = \int_0^b k_B(\zeta)d\zeta < \infty$, one concludes that

$$\mathbb{E}\Big\|\Phi(x)(t+\sigma) - \Phi(x)(t)\Big\|^q_{\dot{H}^r} \le \kappa^{(10)}_{\alpha,\beta}\sigma^{pq}\Big(\sup_{s\in J_\infty}\mathbb{E}\|\chi(s)\|^q_{\dot{H}^r} + (\mathbb{K}_A+\mathbb{K}_B)\sup_{\theta\in(-\infty,b]}\mathbb{E}\|x(\theta)\|^q_{L^q(\Omega,\dot{H}^r)}\Big),$$

which implies that the map $t \mapsto \Phi(x)(t)$ is continuous on J in $L^q(\Omega, \dot{H}^r)$ sense.

Similarly, one can verify there exists $\kappa^{(11)}_{\alpha,\beta} > 0$ depending on $\alpha, \beta, r, \mu, \nu, q, a, b$ such that

$$\sup_{t\in J} e^{\delta t}\mathbb{E}\|\Phi(x)(t)\|^q_{\dot{H}^r} \le \kappa^{(11)}_{\alpha,\beta}e^{\delta b}\Big(\sup_{\theta\in J_\infty}\mathbb{E}\|\chi(\theta)\|^q_{\dot{H}^r} + (\mathbb{K}_A+\mathbb{K}_B)\sup_{\theta\in(-\infty,b]}\mathbb{E}\|x(\theta)\|^q_{L^q(\Omega,\dot{H}^r)}\Big).$$

From two observations and noting that $\Phi(x) = \chi$ on J_∞, one concludes that the map $t \mapsto \Phi(x)(t)$ belongs to the space $\mathcal{S}^{q,r}_{\chi,\delta}$ as desired.

Claim 2. The operator Φ is a contraction in $\mathcal{S}^{q,r}_{\chi,\delta}$.

For $t \in J$, by arguing as in Theorem 3.1, one arrives at

$$\mathbb{E}\big\|\Phi(x)(t) - \Phi(x^\dagger)(t)\big\|^q_{\dot{H}^r} \\ \le 2^{q-1}b^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q\int_0^t \mathbb{E}\|\mathbb{L}_\mu\big(A(\zeta,x_\zeta)-A(\zeta,x^\dagger_\zeta)\big)\|^q_{L^2(\mathbb{X})}d\zeta + \\ + 2^{q-1}c(q)t^{1-\frac{2}{q}}|\tilde{C}_1(\alpha,\beta,\nu-r)|^q\int_0^t \mathbb{E}\|\mathbb{L}_\nu\big(B(\zeta,x_\zeta)-B(\zeta,x^\dagger_\zeta)\big)\|^q_{L^2_{\mathbb{Q}}}d\zeta. \tag{33}$$

Setting $\kappa^{(12)}_{\alpha,\beta} = 2^{q-1}\big(b^{q-1}|\tilde{C}_1(\alpha,\beta,r-\mu)|^q + c(q)b^{1-\frac{2}{q}}|\tilde{C}_1(\alpha,\beta,\nu-r)|^q\big)$, Assumption (A3) and Assumption (B3) yield that

$$\mathbb{E}\big\|\Phi(x)(t) - \Phi(x^\dagger)(t)\big\|^q_{\dot{H}^r} \le \kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))\|x_\zeta - x^\dagger_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot{H}^r))}d\zeta$$

Multiplying both sides of the above inequality by $e^{\delta h}$ and replacing t by $t+h$, with $h \in (-t, 0]$, one obtains

$$e^{\delta h}\mathbb{E}\big\|\Phi(x)(t+h) - \Phi(x^\dagger)(t+h)\big\|^q_{\dot{H}^r} \le \kappa^{(12)}_{\alpha,\beta}e^{\delta h}\int_0^{t+h}(k_A(\zeta)+k_B(\zeta))\|x_\zeta - x^\dagger_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot{H}^r))}d\zeta. \tag{34}$$

On the other hand, it should be noted that

$$\Phi(x)(t+h) = \Phi(x^\dagger)(t+h) = \chi(t+h), \quad \text{for all } h \in (-\infty, -t].$$

Hence, one deduces that for all $h \in J_\infty$ there holds

$$e^{\delta h}\mathbb{E}\big\|\big(\Phi(x)\big)_t(h) - \big(\Phi(x^\dagger)\big)_t(h)\big\|^q_{\dot{H}^r} \le \kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))\|x_\zeta - x^\dagger_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot{H}^r))}d\zeta,$$

which implies that

$$\|\big(\Phi(x)\big)_t - \big(\Phi(x^\dagger)\big)_t\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))} \le \kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))\|x_\zeta - x^\dagger_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}d\zeta$$

Now, by using a similar method as in the proof of Claim 2 of Theorem 3.1, one can obtain that $\Phi(x)=x$ has a unique solution in the space $\mathcal{S}^{q,r}_{\chi,\delta}$.

Claim 3. The solution x satisfies the regularity property (30).

By using a similar way employed to obtain (34), one arrives at for $t\in J$

$$\begin{aligned}\mathbb{E}\big\|x(t)\big\|^q_{\dot H^r} &\le 3^{q-1}|M_1(\alpha,\beta)|^q\mathbb{E}\|\mathbb{L}_r\chi(0)\|^q_{L^2(\mathbb{X})}+\\ &+\kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))\|x_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}d\zeta.\end{aligned}$$

Multiplying both sides of the above inequality by $e^{\delta h}$ and replacing t by $t+h$, with $h\in(-t,0]$,

$$\begin{aligned}e^{\delta h}\mathbb{E}\big\|x(t+h)\big\|^q_{\dot H^r} &\le 3^{q-1}e^{\delta h}|M_1(\alpha,\beta)|^q\mathbb{E}\|\mathbb{L}_r\chi(0)\|^q_{L^2(\mathbb{X})}+\\ &+\kappa^{(12)}_{\alpha,\beta}e^{\delta h}\int_0^{t+h} (k_A(\zeta)+k_B(\zeta))\|x_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}d\zeta\\ &\le 3^{q-1}|M_1(\alpha,\beta)|^q\|\chi\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}+\\ &+\kappa^{(12)}_{\alpha,\beta}\int_0^{t} (k_A(\zeta)+k_B(\zeta))\|x_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}d\zeta.\end{aligned}$$

In addition, for $h\in(-\infty,-t]$, it is obvious that

$$e^{\delta h}\mathbb{E}\big\|x(t+h)\big\|^q_{\dot H^r} = e^{-\delta t}e^{\delta(t+h)}\mathbb{E}\big\|\chi(t+h)\big\|^q_{\dot H^r}\le \|\chi\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}.$$

Hence, for all $h\in J_\infty$, there exists $\kappa^{(13)}_{\alpha,\beta}>0$ depending on $\alpha,\beta,r,\mu,\nu,q,a,b$ such that

$$\|x_t\|_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}\le \kappa^{(13)}_{\alpha,\beta}\|\chi\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}+\kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))\|x_\zeta\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}d\zeta$$

With the help of the Grönwall inequality, we obtain

$$\begin{aligned}\|x_t\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))} &\le \kappa^{(13)}_{\alpha,\beta}\|\chi\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))}\exp\left(\kappa^{(12)}_{\alpha,\beta}\int_0^t (k_A(\zeta)+k_B(\zeta))d\zeta\right)\\ &\le \kappa^{(13)}_{\alpha,\beta}\exp\left(\kappa^{(12)}_{\alpha,\beta}(\mathbb{K}_A+\mathbb{K}_B)\right)\|\chi\|^q_{C_\delta(J_\infty;L^q(\Omega,\dot H^r))},\end{aligned}$$

which implies that the regularity property (30) holds. □

In the following couple of theorems, we shall investigate the continuity of mild solutions in the case of infinite delay with respect to the initial function and the order of the fractional derivative separately.

Theorem 4.2. *Assume that (A3), (B3) hold. Then, the mild solution to Problem* (2) *is continuous with respect to the initial function. Namely, let* $\chi_1,\chi_2\in\mathcal{C}_\delta(J_\infty;L^q(\Omega,\dot H^r))$, *for* $r\ge 0$, $q\ge 2$, $\delta>0$, *and let* x_1,x_2 *be the mild solutions to Problem* (2) *with respect to the initial functions* χ_1,χ_2 *respectively. Then, there exists* $M_8(\alpha,\beta)$ *depending on* $\alpha,\beta,r,\mu,\nu,q,b,\mathbb{K}_A,\mathbb{K}_B$ *such that*

$$\big\|x_1(t)-x_2(t)\big\|_{L^q(\Omega,\dot H^r)}\le M_8(\alpha,\beta)\|\chi_1-\chi_2\|_{\mathcal{C}_\delta(J_\infty;L^q(\Omega,\dot H^r))},\tag{35}$$

for all $t\in J$.

Proof. The proof follows by the similar technique as for Claim 2 of Theorem 4.1. Therefore, we skip here. □

Theorem 4.3. *Let $\chi \in \mathcal{C}_\delta(J_0; L^q(\Omega, \dot{H}^r))$, for $r \geq 0, q \geq 2$ and $\delta > 0$. Assume that (A3), (B3) hold. Then, the mild solution to Problem* (2) *is continuous with respect to the order of the fractional derivative. Namely, if $\beta, \beta' \in [\beta_0, \beta_1] \subset (0,1)$ and $x^{(\beta)}, x^{(\beta')}$ are the mild solutions to Problem* (2) *with respect to the orders β, β' respectively. Then, there exists $M_9(\alpha, \beta)$ depending on $\alpha, \beta_1, \lambda_1, \mu, \nu, q, b, \mathbb{K}_A, \mathbb{K}_B$ such that*

$$\left\| x^{(\beta)}(t) - x^{(\beta')}(t) \right\|_{L^q(\Omega, \dot{H}^r)} \leq M_9(\alpha, \beta)|\beta - \beta'| \|\chi\|_{\mathcal{C}_\delta(J_\infty; L^q(\Omega, \dot{H}^r))},$$

for all $t \in J$.

Proof. By a similar technique to the one in Theorem 4.1, and using Proposition 2.3 instead of property (G2), one can easily arrive at

$$\|x_t^{(\beta)} - x_t^{(\beta')}\|_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} \leq \overline{\kappa}_\alpha^{(3)} |\beta - \beta'|^q \big(\|\chi\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} + \sup_{\theta \in (-\infty, b]} \mathbb{E}\|x(\theta)\|^q_{L^q(\Omega, \dot{H}^r)} \big) +$$
$$+ \overline{\kappa}_\alpha^{(4)} \int_0^t (k_A(\zeta) + k_B(\zeta)) \|x_\zeta^{(\beta)} - x_\zeta^{(\beta')}\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} d\zeta$$

where $\overline{\kappa}_\alpha^{(3)}$ and $\overline{\kappa}_\alpha^{(4)}$ depend on $\alpha, \beta_1, \lambda_1, r, \mu, \nu, q, a, b, \mathbb{K}_A, \mathbb{K}_B$. With the help of the Grönwall inequality, one obtains

$$\begin{aligned} &\|x_t^{(\beta)} - x_t^{(\beta')}\|_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} \\ &\quad \leq \overline{\kappa}_\alpha^{(3)} |\beta - \beta'|^q \big(\|\chi\|^q_{C_\delta(J_\infty; L^q(\Omega, \dot{H}^r))} + \sup_{\theta \in (-\infty, b]} \mathbb{E}\|x(\theta)\|^q_{L^q(\Omega, \dot{H}^r)} \big) \exp(\overline{\kappa}_\alpha^{(4)} (\mathbb{K}_A + \mathbb{K}_B)), \end{aligned}$$

which implies that property (29) holds. □

5 Conclusion

In the present paper, two problems for stochastic fractional differential equations are considered, one involving finite delay and another with infinite delay. The main contributions here are to establish the existence, uniqueness, regularity properties of the mild solution to such problems. In the case of finite delay, the existence result is obtained on the space $\mathcal{C}^p([-a, b]; L^q(\Omega, \dot{H}^r))$, which is a subspace of the usual one $\mathcal{C}([-a, b]; L^2(\Omega, L^2(\mathbb{X}))$. In the case of infinite delay, we show the existence of the mild solution on the space $\mathcal{C}_\delta((-\infty, b]; L^q(\Omega, \dot{H}^r))$, which is a subspace of the usual one $\mathcal{C}_\delta((-\infty, b]; L^2(\Omega, L^2(\mathbb{X}))$. Besides of constructing the continuity with respect to the initial function, we investigated a novel result for stochastic fractional differential equations involving delays that is the continuity with respect to the order of the fractional derivative.

Acknowledgements

The first author (Nguyen Huy Tuan) is supported by Van Lang University. T.C. has been partially supported by the Spanish Ministerio de Ciencia e Innovación, Agencia Estatal de Investigación (AEI), Fondo Europeo de Desarrollo Regional (FEDER) under the project PID2021-122991NB-C21 and the Junta de Andalucía and FEDER under the project P18-FR-4509.

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