G-MEAN RANDOM ATTRACTORS FOR COMPLEX GINZBURG-LANDAU EQUATIONS WITH PROBABILITY-UNCERTAIN INIITAL DATA[*]

´A[†‡§]

Abstract. (??)$L^2_G(\Omega,L^2(\mathbb{R}))L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$

Keywords.

AMS subject classifications.

1. Introduction $\mathbb{R}$

$$\begin{cases} \frac{\partial u}{\partial t} = (\lambda+i\alpha)\Delta u - \rho u - (\kappa+i\beta)|u|^2u + f(x,t), x\in\mathbb{R}, t>\tau, \\ u(x,\tau) = u_\tau(x), x\in\mathbb{R}, \end{cases} \tag{1.1}$$

$\tau\in\mathbb{R}u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))u(x,t)i\lambda,\alpha,\rho,\kappa,\beta\lambda,\rho,\kappa>0fL^2_{loc}(\mathbb{R},L^2(\mathbb{R}))$

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?$L^2_G(\Omega,L^2(\mathbb{R}))(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F})(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathcal{F},\mathcal{F}_t)f$??

(??)$L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$????$L^2_G(\Omega,L^2(\mathbb{R}))L^2_G(\Omega,L^2_\sigma(\mathbb{R}))\sigma>\frac{1}{2}L^2_G(\Omega,L^2_\sigma(\mathbb{R}))f(x,t)L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$
$L^2_G(\Omega,L^2(\mathbb{R}))$

????(??)$L^2_G(\Omega,L^2(\mathbb{R}))$(??)$L^2_G(\Omega,L^2(\mathbb{R}))L^2_G(\Omega,L^2_\sigma(\mathbb{R}))$????

$\|\cdot\|_{L^p(\mathbb{R})}\langle\cdot,\cdot\rangle_{L^p(\mathbb{R})}L^p(\mathbb{R})p=2L^p(\mathbb{R})$

2. Preliminaries ?$(\Omega,\mathcal{F})\mathcal{H}(\Omega,\mathcal{F})\varphi(\xi_1,\xi_2,\cdots,\xi_n)\in\mathcal{H}\xi_1,\xi_2,\cdots,\xi_n\in\mathcal{H}\varphi\in C_{l,Lip}(\mathbb{R}^n)$
$C_{l,Lip}(\mathbb{R}^n)\varphi$

$$|\varphi(x)-\varphi(y)|\le C(1+|x|^m+|y|^m)|x-y| \text{for} x,y\in\mathbb{R}^n,$$

$C>0m\in\mathbb{N}\varphi$

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}}:\mathcal{H}\to\mathbb{R}$ satisfying the following properties: for all $\xi,\zeta\in\mathcal{H}$,*

(i) ***Monotonicity:*** *if $\xi\ge\zeta$, then $\hat{\mathbb{E}}[\xi]\ge\hat{\mathbb{E}}[\zeta]$;*

(ii) ***Constant preserving:*** *$\hat{\mathbb{E}}[c]=c$;*

(iii) ***Sub-additivity:*** *$\hat{\mathbb{E}}[\xi+\zeta]\le\hat{\mathbb{E}}[\xi]+\hat{\mathbb{E}}[\zeta]$;*

(iv) ***Positive homogeneity:*** *$\hat{\mathbb{E}}[\lambda\xi]=\lambda\hat{\mathbb{E}}[\xi]$ for $\lambda\ge 0$.*

The triple $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2. *A set function $\mathbb{C}$ on $\mathcal{F}$ is called a capacity if it satisfies*

(i) *$\mathbb{C}(\emptyset)=0$, $\mathbb{C}(\Omega)=1$;*

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(ii) $\mathbb{C}(A) \leq \mathbb{C}(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity $\mathbb{C}$ *is said to be sub-additive if it satisfies* $\mathbb{C}(A \cup B) \leq \mathbb{C}(A) + \mathbb{C}(B)$.

Definition 2.3. *Given a capacity* $\mathbb{C}$*, a set* $A \in \mathcal{F}$ *is said to be polar if* $\mathbb{C}(A) = 0$*. A property is said to hold quasi-surely (q.s.) if it holds outside a polar set.*

$(\Omega, \mathcal{H}, \hat{\mathbb{E}}) \mathbb{C}(A) := \hat{\mathbb{E}}(I_A) \forall A \in \mathcal{F} \mathbb{C}$**?**$\mathbb{E}_P : \mathcal{H} \to \mathbb{R} P \in \mathcal{P}$

$$\hat{\mathbb{E}}[u] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[u], u \in \mathcal{H}. \tag{2.1}$$

$L^0_G(\Omega, X)(\mathcal{F}, \mathcal{B}(X)) X \|\cdot\|_X \mathcal{L}^2 := \{u \in L^0_G(\Omega, X) : \hat{\mathbb{E}}[\|u\|^2_X] < \infty\} \mathcal{N}^2 := \{u \in L^0_G(\Omega, X) : \hat{\mathbb{E}}[\|u\|^2_X] = 0\} L^2_G(\Omega, X) := \mathcal{L}^2 / \mathcal{N}^2 L^2_G(\Omega, X) \|u\|_{L^2_G} := (\hat{\mathbb{E}}[\|u\|^2_X])^{\frac{1}{2}}$

(**??**)**???**$(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F})(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F}, \mathcal{F}_t) \mathfrak{X} := L^2_G((\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F}); X) = \{\phi | \phi \text{is} \mathcal{F}\text{-measurableand} \hat{\mathbb{E}}[\|\phi\|^2_X] < \infty\} \mathbb{R}^2_{\geq} := \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}$

Definition 2.4. *A family of mappings* $\Phi = \{\Phi(t, t_0, \xi_0) : (t, t_0) \in \mathbb{R}^2_{\geq}\}$ *on nonlinear expectation space is called a G-mean square random dynamical system over* $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathcal{F})$*, if the mapping* $\Phi(t, t_0, \cdot) : \mathfrak{X} \to \mathfrak{X}$*,* $(t, t_0) \in \mathbb{R}^2_{\geq}$ *satisfies:*

(i) ***Initial value property:*** $\Phi(t_0, t_0, \xi_0) = \xi_0$ *for any* $\xi_0 \in \mathfrak{X}$ *and* $t_0 \in \mathbb{R}$*;*

(ii) ***Two-parameter semigroup property:*** $\Phi(t_2, t_0, \xi_0) = \Phi(t_2, t_1, \Phi(t_1, t_0, \xi_0))$ *for every* $\xi_0 \in \mathfrak{X}$ *and* $(t_2, t_1), (t_1, t_0) \in \mathbb{R}^2_{\geq}$*;*

(iii) ***Continuity property:*** $(t, t_0, \xi_0) \mapsto \Phi(t, t_0, \xi_0)$ *is continuous in the space* $\mathbb{R}^2_{\geq} \times \mathfrak{X}$*.*

Definition 2.5. *A family* $K = \{K(t)\}_{t \in \mathbb{R}}$ *of nonempty subsets of* $\mathfrak{X}$ *for each* $t \in \mathbb{R}$ *is said to be* Φ*-invariant if*

$$\Phi(t, t_0, K(t_0)) = K(t), \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq},$$

and Φ*-positively invariant if*

$$\Phi(t, t_0, K(t_0)) \subseteq K(t), \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq}.$$

$D = \{D(t) \subseteq \mathfrak{X} : t \in \mathbb{R}\} \lambda > 0$

$$\lim_{t \to -\infty} e^{\lambda t} \|D(t)\|^2_{\mathfrak{X}} = 0, \tag{2.2}$$

$\|D(t)\|^2_{\mathfrak{X}} = \sup_{u \in D(t)} \hat{\mathbb{E}}[\|u\|^2_X]. D r \hat{\mathbb{E}}[\|u(t)\|^2_X] \leq r t \in \mathbb{R} u(t) \in D(t) \in D$

$$\mathcal{D} = \{D = \{D(t) \subseteq \mathfrak{X} : D(t) \neq \emptyset \text{bounded}, t \in \mathbb{R}\} : D \text{satisfies}(\mathbf{??})\}.$$

Definition 2.6. *A family* $K = \{K(t)\}_{t \in \mathbb{R}} \in \mathcal{D}$ *is called a* $\mathcal{D}$*-pullback absorbing family for* Φ *if for each* $t \in \mathbb{R}$ *and* $D \in \mathcal{D}$*, there exists* $T' = T'(t, D) > 0$ *such that*

$$\Phi(t, t - s, D(t - s)) \subseteq K(t), \quad s \geq T'.$$

Remark 2.1. *Compared with [**?**, **?**], in this paper, the absorbing family for G-mean random dynamical system* Φ *is not required to be uniformly bounded. Obviously,* D *is uniformly bounded, which implies that it satisfies* (**??**)*. This shows that a uniformly bounded family of nonempty closed subsets* $D = \{D(t)\}_{t \in \mathbb{R}}$ *belongs to* $\mathcal{D}$*. Therefore, the requirement conditions of absorbing set are weaker than those literature [**?**, **?**].*

Definition 2.7. *A family* $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ *is called a* $\mathcal{D}$*-pullback G-mean random attractor for* Φ *in* $\mathfrak{X}$ *if the following conditions are fulfilled:*

(i) $\mathcal{A}(t)$ *is a nonempty compact subset of* $\mathfrak{X}$ *for each* $t \in \mathbb{R}$*;*

(ii) $\mathcal{A}$ *is* Φ*-invariant, i.e.,* $\Phi(t, t_0, \mathcal{A}(t_0)) = \mathcal{A}(t)$*, for all* $(t, t_0) \in \mathbb{R}^2_{\geq}$*;*

(iii) $\mathcal{A}$ *pullback attracts every* $D\in\mathcal{D}$*, that is, for every* $t\in\mathbb{R}$*,*

$$\lim_{s\to+\infty} d(\Phi(t,t-s,D(t-s)),\mathcal{A}(t))=0,$$

where $d(A,B):=\sup_{x\in A}\inf_{y\in B}\|x-y\|_{\mathfrak{X}}$ *is the Hausdorff semi-distance, for any* $A,B\subseteq\mathfrak{X}$*.*

$\mathcal{D}$

Lemma 2.1 ([?] Gagliardo-Nirenberg's inequality). *Let* $u\in L^q(\mathbb{R})$ *and its derivatives of order* m*,* $D^m u\in L^r(\mathbb{R})$*,* $1\le q,r\le\infty$*. For the derivatives* $D^j u$*,* $1\le j<m$*, there exists* $c=c(m,j,q,r,\theta)$ *such that*

$$\|D^j u\|_{L^p(\mathbb{R})}\le c\|D^m u\|^{\theta}_{L^r(\mathbb{R})}\|u\|^{1-\theta}_{L^q(\mathbb{R})},$$

where $\frac{1}{p}=j+\theta(\frac{1}{r}-m)+(1-\theta)\frac{1}{q}$*, for all* θ *in the interval* $\frac{j}{m}\le\theta\le1$*.*

Lemma 2.2 ([?]). *For any* $-1<\mu<+\infty$ *and* $x,y\in\mathbb{C}$*, the following inequality holds*

$$|\mathrm{Im}(\bar{x}-\bar{y})(|x|^{\mu}x-|y|^{\mu}y)|\le\frac{\mu}{2\sqrt{\mu+1}}\mathrm{Re}(\bar{x}-\bar{y})(|x|^{\mu}x-|y|^{\mu}y).$$

3. Existence and uniqueness of solutions for system (??) (??)$(\Omega,\mathcal{H},\hat{\mathbb{E}})$(??) (??)

Definition 3.1. *Let* $\tau\in\mathbb{R}$ *and* $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$*. A continuous mapping* $u(\cdot)\doteq u(\cdot,\tau,u_\tau):[\tau,\infty)\to L^2_G(\Omega,L^2(\mathbb{R}))$ *is called a solution of system* (??) *if*

$$u(\cdot,\tau,u_\tau)\in C\big([\tau,\infty),L^2_G\big(\Omega,L^2(\mathbb{R})\big)\big)\bigcap L^2_G\big(\Omega,L^2_{loc}\big((\tau,\infty),H^1(\mathbb{R})\big)\big)$$
$$\bigcap L^4_G\big(\Omega,L^4_{loc}\big((\tau,\infty),L^4(\mathbb{R})\big)\big)$$

$ut>\tau\xi\in H^1(\mathbb{R})\cap L^4(\mathbb{R})$

$$\langle u(t),\xi\rangle=\langle u_\tau,\xi\rangle-(\lambda+i\alpha)\int_\tau^t\langle\nabla u,\nabla\xi\rangle ds-\rho\int_\tau^t\langle u,\xi\rangle ds$$
$$-(\kappa+i\beta)\int_\tau^t\langle|u|^2u,\xi\rangle ds+\int_\tau^t\langle f(s),\xi\rangle ds,\mathrm{q.s.},$$

$f(\cdot):\mathbb{R}\to L^2(\mathbb{R})\langle f(s),\xi\rangle=\int_{\mathbb{R}}f(x,s)\xi(x)dx$

(??)??

Theorem 3.1. *For every* $\tau\in\mathbb{R}$ *and* $u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$*, system* (??) *has a unique solution* $u(\cdot,\tau,u_\tau)$ *in the sense of Definition* ??*. Furthermore, this solution is* $\mathcal{F}$*-measurable with respect to* $\omega\in\Omega$*.*

Proof$O_k=\{x\in\mathbb{R},|x|<k\}k\in\mathbb{N}O_k$

$$\frac{\partial u_k}{\partial t}=(\lambda+i\alpha)\Delta u_k-\rho u_k-(\kappa+i\beta)|u_k|^2u+f(x,t),t>\tau,x\in O_k, \tag{3.1}$$

$$u_k(x,t)=0,t>\tau,|x|=k, \tag{3.2}$$

$$u_k(x,\tau)=u_\tau(x),x\in O_k. \tag{3.3}$$

$k>0$?(??)(??)u_k??$\mathbb{R}O_kt\ge\tau u_k(t,\omega)\mathcal{F}\omega\in\Omega$

u_k(??)(??)(??)

$$\frac{d}{dt}\|u_k\|^2 = -2\lambda\|\nabla u_k\|^2 - 2\rho\|u_k\|^2 - 2\kappa\|u_k\|^4_{L^4(O_k)} + 2\mathrm{Re}\int_{O_k} f(t,x)\overline{u_k}dx$$
$$\leq -2\lambda\|\nabla u_k\|^2 - \rho\|u_k\|^2 - 2\kappa\|u_k\|^4_{L^4(O_k)} + \frac{1}{\rho}\|f(t)\|^2. \quad (3.4)$$

$t \geq \tau \omega \in \Omega$

$$\|u_k(t,\omega)\|^2 + 2\lambda\int_\tau^t e^{-\rho(t-s)}\|\nabla u_k(s,\omega)\|^2 ds + 2\kappa\int_\tau^t e^{-\rho(t-s)}\|u_k(s,\omega)\|^4_{L^4(O_k)}ds$$
$$\leq e^{-\rho(t-\tau)}\|u_\tau(\omega)\|^2 + \frac{1}{\rho}\int_\tau^t e^{-\rho(t-s)}\|f(s)\|^2 ds.$$

$\tau \in \mathbb{R} \omega \in \Omega T > 0$

$$\|u_k(t,\omega)\|^2 \leq \|u_\tau(\omega)\|^2 + \frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2 ds, t \in [\tau,\tau+T], \quad (3.5)$$

$$\int_\tau^{\tau+T}\|\nabla u_k(s,\omega)\|^2 ds \leq \frac{e^{\rho T}}{2\lambda}\Big(\|u_\tau(\omega)\|^2 + \frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2 ds\Big), \quad (3.6)$$

$$\int_\tau^{\tau+T}\|u_k(s,\omega)\|^4_{L^4(O_k)} ds \leq \frac{e^{\rho T}}{2\kappa}\Big(\|u_\tau(\omega)\|^2 + \frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2 ds\Big). \quad (3.7)$$

(??)(??)$\tau \in \mathbb{R} \omega \in \Omega T > 0$

$$\{u_k(\cdot,\omega)\}_{k=1}^\infty \text{is bounded in} L^\infty\big((\tau,\tau+T),L^2(O_k)\big)\bigcap L^2\big((\tau,\tau+T),H_0^1(O_k)\big)$$
$$\bigcap L^4\big((\tau,\tau+T),L^4(O_k)\big) \quad (3.8)$$

$$\{|u_k|^2 u_k\}_{k=1}^\infty \text{is bounded in} L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(O_k)\big). \quad (3.9)$$

$$\left\{\frac{du_k}{dt}\right\}_{k=1}^\infty \text{is bounded in} L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(O_k)\big) + L^2\big((\tau,\tau+T),H^{-1}(O_k)\big). \quad (3.10)$$

$$\{u_k(\cdot,\omega)\}_{k=1}^\infty \text{is compact in} L^2\big((\tau,\tau+T),L^2(O_k)\big). \quad (3.11)$$

$u_k \mathbb{R} u_k(x,t) = 0 |x| > k t \in [\tau,\tau+T] t' \in (\tau,\tau+T]$(??)(??)$\{u_{k_l}\}_{l=1}^\infty \{u_k\}_{k=1}^\infty$

$$u_{k_l}(\cdot,\omega) \to u(\cdot,\omega) \text{weak star in} L^\infty\big((\tau,\tau+T),L^2(\mathbb{R})\big),$$
$$u_{k_l}(\cdot,\omega) \to u(\cdot,\omega) \text{weakly in} L^2\big((\tau,\tau+T),H^1(\mathbb{R})\big),$$
$$u_{k_l}(\cdot,\omega) \to u(\cdot,\omega) \text{weakly in} L^4\big((\tau,\tau+T),L^4(\mathbb{R})\big),$$
$$\frac{d}{dt}u_{k_l}(\cdot,\omega) \to \frac{d}{dt}u(\cdot,\omega) \text{weakly in} L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(\mathbb{R})\big) + L^2\big((\tau,\tau+T),H^{-1}(\mathbb{R})\big),$$

$$u_{k_l}(t',\omega)\to v\text{weakly in}L^2(\mathbb{R}),$$

$u\in L^{\infty}\big((\tau,\tau+T),L^2(\mathbb{R})\big)\cap L^2\big((\tau,\tau+T),H^1(\mathbb{R})\big)\cap L^4\big((\tau,\tau+T),L^4(\mathbb{R})\big)v\in L^2(\mathbb{R})$**?**

$$|u_{k_l}(\cdot,\omega)|^2u_{k_l}(\cdot,\omega)\to|u(\cdot,\omega)|^2u(\cdot,\omega)\text{weakly in}L^{\frac{4}{3}}\big((\tau,\tau+T),L^{\frac{4}{3}}(\mathbb{R})\big).$$

$l\to\infty\xi\in H^1(\mathbb{R})\cap L^4(\mathbb{R})$

$$\frac{d}{dt}\langle u,\xi\rangle=-(\lambda+i\alpha)\langle\nabla u,\nabla\xi\rangle-\rho\langle u,\xi\rangle-(\kappa+i\beta)\langle|u|^2u,\xi\rangle+\langle f(t),\xi\rangle, \tag{3.12}$$

$(\tau,\tau+T)u(\cdot,\omega)\in C([\tau,\tau+T],L^2(\mathbb{R}))u(\tau,\omega)=u_\tau(\omega)u(t',\omega)=v$

$$\frac{1}{2}\frac{d}{dt}\|u(t,\omega)\|^2=-\lambda\|\nabla u(t,\omega)\|^2-\rho\|u(t,\omega)\|^2-\kappa\|u(t,\omega)\|^2_{L^4(\mathbb{R})}+\langle f(t),u(t,\omega)\rangle. \tag{3.13}$$

$t'\in(\tau,\tau+T]$

$$u_{k_l}(t',\omega)\to u(t',\omega)\text{weakly in}L^2(\mathbb{R}), \tag{3.14}$$

$u(\cdot,\omega)$(**??**)$u_\tau(\omega)\omega\in\Omega$
$\tau\in\mathbb{R}\omega\in\Omega T>0$

$$\|u(t,\omega)\|^2\le\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds,t\in[\tau,\tau+T], \tag{3.15}$$

$$\int_\tau^{\tau+T}\|\nabla u(s,\omega)\|^2ds\le\frac{e^{\rho T}}{2\lambda}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big), \tag{3.16}$$

$$\int_\tau^{\tau+T}\|u(s,\omega)\|^4_{L^4(\mathbb{R})}ds\le\frac{e^{\rho T}}{2\kappa}\Big(\|u_\tau(\omega)\|^2+\frac{1}{\rho}\int_\tau^{\tau+T}\|f(s)\|^2ds\Big). \tag{3.17}$$

$u_\tau\in L^2_G(\Omega,L^2(\mathbb{R}))$(**??**)$u\in L^{\infty}_{loc}\big((\tau,\infty),L^2_G(\Omega,L^2(\mathbb{R}))\big)$(**??**)(**??**)

$$L^2_G\big(\Omega,L^2_{loc}\big((\tau,\infty),H^1(\mathbb{R})\big)\big)\bigcap L^4_G\big(\Omega,L^4_{loc}\big((\tau,\infty),L^4(\mathbb{R})\big)\big).$$

$C\big([\tau,\infty),L^2_G\big(\Omega,L^2(\mathbb{R})\big)\big)$(**??**)

$$\hat{\mathbb{E}}[\|u(t)\|^2-\|u(s)\|^2]\le\frac{1}{\rho}\int_s^t\|f(r)\|^2dr. \tag{3.18}$$

$t,s\ge\tau$

$$\hat{\mathbb{E}}[\|u(t)-u(s)\|^2]=\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u(s)\rangle]. \tag{3.19}$$

$L^2_G(\Omega,H^1(\mathbb{R}))\cap L^2_G(\Omega,L^4(\mathbb{R}))L^2_G(\Omega,L^2(\mathbb{R}))\{u_n(s)\}_{n=1}^{\infty}\in L^2_G(\Omega,H^1(\mathbb{R}))\cap L^2_G(\Omega,L^4(\mathbb{R}))$

$$\hat{\mathbb{E}}[\|u(s)-u_n(s)\|^2]\to0,\text{as}n\to\infty. \tag{3.20}$$

(??) $s \geq \tau$

$$
\begin{aligned}
&\hat{\mathbb{E}}[\|u(t)-u(s)\|^2] \\
=&\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u_n(s)+u(s)-u_n(s)\rangle] \\
\leq&\hat{\mathbb{E}}[\|u(t)\|^2+\|u(s)\|^2-2\langle u(t),u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle] \\
\leq&\hat{\mathbb{E}}\left[2\|u(s)\|^2-2\langle u(t),u_n(s)\rangle+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr\right]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle].
\end{aligned}
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\begin{aligned}
&\langle u(t,\omega),u_n(s,\omega)\rangle \\
=&\langle u(s,\omega),u_n(s,\omega)\rangle-\int_s^t(\lambda+i\alpha)\langle\nabla u(r,\omega),\nabla u_n(r,\omega)\rangle dr-\int_s^t\rho\langle u(r,\omega),u_n(s,\omega)\rangle dr \\
&-\int_s^t(\kappa+i\beta)\langle|u(r,\omega)|^2u(r,\omega),u_n(s,\omega)\rangle dr+\int_s^t\langle f(r),u_n(s,\omega)\rangle dr.
\end{aligned}
$$

$t \geq s$

$$
\begin{aligned}
&\hat{\mathbb{E}}[\|u(t)-u(s)\|^2] \\
\leq&\hat{\mathbb{E}}\Big[2\|u(s)\|^2-2\langle u(s),u_n(s)\rangle-\int_s^t(\lambda+i\alpha)\langle\nabla u(r),\nabla u_n(s)\rangle dr \\
&-\int_s^t\rho\langle u(r),u_n(s)\rangle dr-\int_s^t(\kappa+i\beta)\langle|u(r)|^2u(r),u_n(s)\rangle dr \\
&+\int_s^t\langle f(r),u_n(s)\rangle dr+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr\Big]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle] \\
\leq&\hat{\mathbb{E}}[2\langle u(s),u(s)-u_n(s)\rangle]+C\hat{\mathbb{E}}\left[\int_s^t\|\nabla u(r)\|\|\nabla u_n(s)\|dr\right] \\
&+C\hat{\mathbb{E}}\left[\int_s^t\|u(r)\|\|u_n(s)\|ds\right]+C\hat{\mathbb{E}}\left[\int_s^t\|u(r)\|_{L^4(\mathbb{R})}^3\|u_n(s)\|_{L^4(\mathbb{R})}dr\right] \\
&+\hat{\mathbb{E}}\left[\int_s^t\|f(r)\|\|u_n(s)\|dr\right]+\frac{1}{\rho}\int_s^t\|f(r)\|^2dr+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle].
\end{aligned}
\tag{3.21}
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\begin{aligned}
&2\hat{\mathbb{E}}[\langle u(s),u(s)-u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle] \\
\leq&2(\hat{\mathbb{E}}[\|u(s)\|^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u(s)-u_n(s)\|^2])^{\frac{1}{2}}+2(\hat{\mathbb{E}}[\|u(t)\|^2])^{\frac{1}{2}}(\hat{\mathbb{E}}[\|u_n(s)-u(s)\|^2])^{\frac{1}{2}}.
\end{aligned}
\tag{3.22}
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$$
\hat{\mathbb{E}}[\langle u(s),u(s)-u_n(s)\rangle]+2\hat{\mathbb{E}}[\langle u(t),u_n(s)-u(s)\rangle]\to 0,\text{as} n\to\infty.
\tag{3.23}
$$

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$$
\begin{aligned}
&C\hat{\mathbb{E}}\left[\int_s^t\|\nabla u(r)\|\|\nabla u_n(s)\|dr\right] \\
\leq&C\hat{\mathbb{E}}\left[\left(\int_s^t\|\nabla u(r)\|^2dr\right)^{\frac{1}{2}}\left(\int_s^t\|\nabla u_n(s)\|^2dr\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

$$\leq C\left(\hat{\mathbb{E}}\Big[\int_s^t \|\nabla u(r)\|^2 dr\Big]\right)^{\frac{1}{2}} (\hat{\mathbb{E}}[\|\nabla u_n(s)\|^2])^{\frac{1}{2}} |t-s|^{\frac{1}{2}}. \tag{3.24}$$

$$C\hat{\mathbb{E}}\left[\int_s^t \|u(r)\|\|u_n(s)\|ds\right] \leq C\left(\hat{\mathbb{E}}\Big[\int_s^t \|u(r)\|^2 dr\Big]\right)^{\frac{1}{2}} (\hat{\mathbb{E}}[\|u_n(s)\|^2])^{\frac{1}{2}} |t-s|^{\frac{1}{2}}, \tag{3.25}$$

$$\begin{aligned} &C\hat{\mathbb{E}}\left[\int_s^t \|u(r)\|^3_{L^4(\mathbb{R})}\|u_n(s)\|_{L^4(\mathbb{R})} dr\right] \\ \leq &C\left(\hat{\mathbb{E}}\Big[\int_s^t \|u(r)\|^4_{L^4(\mathbb{R})} dr\Big]\right)^{\frac{4}{3}} (\hat{\mathbb{E}}[\|u_n(s)\|^4_{L^4(\mathbb{R})}])^{\frac{1}{4}} |t-s|^{\frac{1}{4}}, \end{aligned} \tag{3.26}$$

$$\hat{\mathbb{E}}\left[\int_s^t \|f(r)\|\|u_n(s)\|dr\right] \leq C\left(\hat{\mathbb{E}}\Big[\int_s^t \|f(r)\|^2 dr\Big]\right)^{\frac{1}{2}} (\hat{\mathbb{E}}[\|u_n(s)\|^2])^{\frac{1}{2}} |t-s|^{\frac{1}{2}}. \tag{3.27}$$

(**??**)(**??**)

$$\begin{aligned} &\hat{\mathbb{E}}[\|u(t)-u(s)\|^2] \to 0, \text{as} t \to s, \\ &u \in C([\tau,\infty), L^2_G(\Omega, L^2(\mathbb{R}))). \end{aligned} \tag{3.28}$$

u(**??**)**??**

$u_1 u_2$(**??**)$v = u_1 - u_2$

$$\begin{aligned} \frac{d}{dt}\|v\|^2 = &-2\lambda\|\nabla v\|^2 - 2\rho\|v\|^2 - 2\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1 - |u_2|^2u_2)\bar{v}dx \\ \leq &-2\lambda\|\nabla v\|^2 + 2\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1 - |u_2|^2u_2)\bar{v}dx\right|. \end{aligned}$$

??

$$\begin{aligned} &\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2u_1 - |u_2|^2u_2)\bar{v}dx\right| \\ =&\left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}\left[|u_1|^2(u_1-u_2) + (|u_1|^2 - |u_2|^2)u_2\right]\bar{v}dx\right| \\ \leq&\kappa\int_{\mathbb{R}}|u_1|^2|v|^2dx + \left|\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}(|u_1|^2 - |u_2|^2)u_2\bar{v}dx\right| \\ \leq&3\sqrt{\kappa^2+\beta^2}\int_{\mathbb{R}}(|u_1|^2+|u_2|^2)|v|^2dx \\ \leq&3\sqrt{2(\kappa^2+\beta^2)}\Big(\int_{\mathbb{R}}(|u_1|^4+|u_2|^4)dx\Big)^{\frac{1}{2}}\|v\|^2_{L^4(\mathbb{R})} \\ \leq&3c\sqrt{2(\kappa^2+\beta^2)}\big(\|u_1\|^2_{L^4(\mathbb{R})} + \|u_2\|^2_{L^4(\mathbb{R})}\big)\|\nabla v\|^{\frac{1}{2}}\|v\|^{\frac{3}{2}} \\ \leq&3c\sqrt{2(\kappa^2+\beta^2)}\left[\varepsilon\|\nabla v\|^2 + \frac{3}{4}(2\varepsilon)^{-\frac{1}{3}}\big(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})} + \|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})}\big)\|v\|^2\right], \end{aligned}$$

ε ?? $\varepsilon = \frac{}{3c\sqrt{2(\kappa^2+\beta^2)}}$

$$\frac{d}{dt}\|v\|^2 \leq \frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(2\varepsilon)^{-\frac{1}{3}}\left(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})} + \|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})}\right)\|v\|^2.$$

$$\begin{aligned}\|u_1(t,\omega) - u_2(t,\omega)\|^2 &\leq \|u_{\tau,1}(\omega) - u_{\tau,2}(\omega)\|^2 e^{\frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(2\varepsilon)^{-\frac{1}{3}}\int_\tau^t(\|u_1\|^{\frac{8}{3}}_{L^4(\mathbb{R})}+\|u_2\|^{\frac{8}{3}}_{L^4(\mathbb{R})})ds} \\ &\leq \|u_{\tau,1}(\omega) - u_{\tau,2}(\omega)\|^2 e^{C[(\int_\tau^t\|u_1\|^4_{L^4(\mathbb{R})}ds)^{\frac{2}{3}}+(\int_\tau^t\|u_2\|^4_{L^4(\mathbb{R})}ds)^{\frac{2}{3}}]}, \quad (3.29)\end{aligned}$$

$C = \frac{9\sqrt{2}c}{4}\sqrt{\kappa^2+\beta^2}(\frac{t-\tau}{2\varepsilon})^{\frac{1}{3}}$

(**??**)$u_k(t,\omega) \to u(t,\omega)L^2(\mathbb{R})u_k(t,\omega)\omega u(t,\omega)$.

REMARK 3.1. *In classical probability space, the continuity of the solution with respect to time in the mean sense can be proved by the dominated convergence theorem, see, e.g., [?, ?, ?]. Nevertheless, different from the classical probability space, the dominated convergence theorem usually does not hold in the framework of nonlinear expectation. This gives rise to some difficulties in proving $u \in C([\tau,\infty), L^2_G(\Omega, L^2(\mathbb{R})))$, which is necessary to establish the G-mean square dynamical system associated with* (**??**) *in the next section.*

REMARK 3.2. *Inequality* (**??**) *shows the uniqueness of the solution. However, it does not indicate the continuity of solutions with respect to initial conditions, which will be proved in Lemma* **??**.

4. G-mean random attractors for (??) in $L^2_G(\Omega, L^2(\mathbb{R}))$ (??)$L^2_G(\Omega, L^2(\mathbb{R}))$

$$|\beta| \leq \sqrt{3}\kappa, \quad (4.1)$$

$$\int_{-\infty}^{\tau} e^{\rho s}\|f(s)\|^2 ds < \infty, \forall \tau \in \mathbb{R}. \quad (4.2)$$

4.1. G-mean random dynamical systems. (??)

$\Phi\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))$

$$\Phi(t,\tau,u_\tau) = u(t,\tau,u_\tau), t \geq \tau,$$

$u_\tau \in L^2_G(\Omega, L^2(\mathbb{R}))u$(**??**)$u_\tau\Phi$

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_\tau)\|^2] = \hat{\mathbb{E}}[\|u(t,\tau,u_\tau)\|^2] < \infty,$$

$\Phi(t,\tau,u_\tau)\mathcal{F}u(t)\mathcal{F}\Phi\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))L^2_G(\Omega, L^2(\mathbb{R}))$

$$\Phi(t,\tau,u_\tau) = \Phi(t,s,\Phi(s,\tau,u_\tau)), \quad (4.3)$$

$(t,s)(s,\tau) \in \mathbb{R}^2_{\geq} u_\tau \in L^2_G(\Omega, L^2(\mathbb{R}))$

$$\Phi(\tau,\tau,u_\tau) = u_\tau.$$

??(**??**)$\Phi\mathbb{R}^2_{\geq} \times L^2_G(\Omega, L^2(\mathbb{R}))$

LEMMA 4.1. *Assume* (**??**) *holds. Then, the mapping Φ is uniformly strictly contracting, i.e., for the different initial values $u_{\tau,1}$, $u_{\tau,2} \in L^2_G(\Omega, L^2(\mathbb{R}))$, we have*

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1}) - \Phi(t,\tau,u_{\tau,2})\|^2] \leq \hat{\mathbb{E}}[\|u_{\tau,1} - u_{\tau,2}\|^2]e^{-2\rho(t-\tau)}, \quad (4.4)$$

for all $t \geq \tau$.

Proof $u_1(t,\tau,u_{\tau,1})u_2(t,\tau,u_{\tau,2})$(**??**)$u_{\tau,1}u_{\tau,2}\in L^2_G(\Omega,L^2(\mathbb{R}))\Phi$

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1})-\Phi(t,\tau,u_{\tau,2})\|^2]=\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|^2].$$

$$\begin{aligned}&\frac{d}{dt}\|u_1-u_2\|^2\\&=-2\lambda\|\nabla(u_1-u_2)\|^2-2\rho\|u_1-u_2\|^2-2\mathrm{Re}(\kappa+i\beta)\langle|u_1|^2u_1-|u_2|^2u_2,u_1-u_2\rangle\\&\le-2\rho\|u_1-u_2\|^2-2\mathrm{Re}(\kappa+i\beta)\langle|u_1|^2u_1-|u_2|^2u_2,u_1-u_2\rangle.\end{aligned}\tag{4.5}$$

??(**??**)(**??**)

$$\begin{aligned}&-2\mathrm{Re}(\kappa+i\beta)\langle|u_1|^2u_1-|u_2|^2u_2,u_1-u_2\rangle\\&=-2\kappa\int_{\mathbb{R}}\mathrm{Re}\big(\bar{u}_1(x)-\bar{u}_2(x)\big)\big(|u_1(x)|^2u_1(x)-|u_2(x)|^2u_2(x)\big)dx\\&\quad+2\beta\int_{\mathbb{R}}\mathrm{Im}\big(\bar{u}_1(x)-\bar{u}_2(x)\big)\big(|u_1(x)|^2u_1(x)-|u_2(x)|^2u_2(x)\big)dx\\&\le2\kappa\Big(-1+\frac{|\beta|}{\kappa\sqrt{3}}\Big)\int_{\mathbb{R}}\mathrm{Re}\big(\bar{u}_1(x)-\bar{u}_2(x)\big)\big(|u_1(x)|^2u_1(x)-|u_2(x)|^2u_2(x)\big)dx\\&\le0.\end{aligned}\tag{4.6}$$

$$\frac{d}{dt}\|u_1-u_2\|^2\le-2\rho\|u_1-u_2\|^2.$$

$$\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|^2]\le\hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|^2]e^{-2\rho(t-\tau)}.$$

.

REMARK 4.1. *It follows from Lemma* **??** *that $\Phi(t,\tau,\cdot)$ maps $L^2_G(\Omega,L^2(\mathbb{R}))$ to $L^2_G(\Omega,L^2(\mathbb{R}))$ continuously. Then combining $u\in C([\tau,\infty),L^2_G(\Omega,L^2(\mathbb{R})))$ with* (**??**)*, we can deduce that the mapping Φ is continuous in the space $\mathbb{R}^2_{\ge}\times L^2_G(\Omega,L^2(\mathbb{R}))$. Therefore, Φ is a G-mean random dynamical system associated with problem* (**??**).

4.2. Existence of G-mean random attractors. $\mathcal{D}$(**??**)$L^2_G(\Omega,L^2(\mathbb{R}))$

LEMMA 4.2. *Let* (**??**) *and* (**??**) *hold. Then for every $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, there exist $T=T(\tau,D)>0$ and $R(\tau)>0$ such that for all $t\ge T$,*

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2]\le R(\tau),$$

where $u_{\tau-t}\in D(\tau-t)$.

Proof

$$\begin{aligned}\frac{d}{dt}\|u\|^2&=-2\lambda\|\nabla u\|^2-2\rho\|u\|^2-2\kappa\|u\|^4_{L^4(\mathbb{R})}+2\mathrm{Re}\langle f(t),u\rangle\\&\le-2\lambda\|\nabla u\|^2-2\rho\|u\|^2-2\kappa\|u\|^4_{L^4(\mathbb{R})}+\rho\|u\|^2+\frac{1}{\rho}\|f(t)\|^2\\&\le-\rho\|u\|^2+\frac{1}{\rho}\|f(t)\|^2.\end{aligned}$$

$$\frac{d}{dt}e^{\rho t}\|u\|^2=\rho e^{\rho t}\|u\|^2+e^{\rho t}\frac{d}{dt}\|u\|^2\le\frac{1}{\rho}e^{\rho t}\|f(t)\|^2.$$

$(\tau-t,\tau)\, t\geq 0$

$$\|u(\tau,\tau-t,u_{\tau-t})\|^2 \leq e^{-\rho t}\|u_{\tau-t}\|^2+\frac{1}{\rho}e^{-\rho\tau}\int_{\tau-t}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

$$\hat{\mathbb{E}}[\|u(\tau,\tau-t,u_{\tau-t})\|^2] \leq e^{-\rho t}\hat{\mathbb{E}}[\|u_{\tau-t}\|^2]+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds, \tag{4.7}$$

$u_{\tau-t}\in D(\tau-t)\in\mathcal{D} T=T(\tau,D)>0 t\geq T$

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2] \leq R(\tau),$$

$$R(\tau)=1+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

.

REMARK 4.2. *Define the family of sets $B(\tau)=B_{r(\tau)}$, where $B_{r(\tau)}$ is the ball in $L_G^2(\Omega,L^2(\mathbb{R}))$ centered on the origin with radius $r(\tau)$ specified by*

$$r(\tau):=\sqrt{1+\frac{1}{\rho}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds.}$$

*It is not difficult to prove that the family $B=\{B(\tau):\tau\in\mathbb{R}\}$ belongs to $\mathcal{D}$. Indeed, choosing $\lambda=\rho$ in (**??**), we have*

$$\lim_{\tau\to-\infty}e^{\rho\tau}\|B(\tau)\|_{\mathfrak{X}}^2=\lim_{\tau\to-\infty}e^{\rho\tau}+\lim_{\tau\to-\infty}\frac{e^{\rho\tau}e^{-\rho\tau}}{\rho}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds,$$

*which together with (**??**) implies*

$$\lim_{\tau\to-\infty}e^{\rho\tau}\|B(\tau)\|_{\mathfrak{X}}^2=0.$$

*Therefore, $B\in\mathcal{D}$. Further, by Lemma **??**, we find that for every $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$, there exists $T=T(\tau,D)>0$ such that for all $t\geq T$, $\Phi(\tau,\tau-t,u_{\tau-t})\subseteq B(\tau)$. Therefore, B is a $\mathcal{D}$-pullback absorbing family for Φ.*

THEOREM 4.1. *Assume (**??**) and (**??**) hold. Then, problem (**??**) has a unique $\mathcal{D}$-pullback G-mean random attractor $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}$ in $L_G^2(\Omega,L^2(\mathbb{R}))$. Furthermore, if there exists a positive number ϖ such that $f:\mathbb{R}\to L^2(\mathbb{R})$ is ϖ-periodic, then such attractor $\mathcal{A}$ is also ϖ-periodic; that is, $\mathcal{A}(\tau+\varpi)=\mathcal{A}(\tau)$ for all $\tau\in\mathbb{R}$.*

Proof. **??** $\{B(\tau):\tau\in\mathbb{R}\}\mathcal{D}\Phi$

$\{\Phi(0,t_n,x_n)\}_{n\in\mathbb{N}}L_G^2(\Omega,L^2(\mathbb{R}))B(0)x_n\in D(t_n)\in\mathcal{D}n\in\mathbb{N}\{t_n\}_{n\in\mathbb{N}}-\infty t_1=0 t_n-t_{n+1}\geq$ $T^*(t_n)T^*(t_n)m>n$

$$\begin{aligned}\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_m,x_m)\|^2\big]&=\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_n,\Phi(t_n,t_m,x_m))\|^2\big]\\&\leq e^{\rho t_n}\hat{\mathbb{E}}\big[\|x_n-\Phi(t_n,t_m,x_m)\|^2\big].\end{aligned} \tag{4.8}$$

$\Phi(t_n,t_m,x_m)\in B(t_n)$

$$\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-\Phi(0,t_m,x_m)\|^2\big] \leq 4e^{\rho t_n}\left(1+\frac{1}{\rho}e^{-\rho t_n}\int_{-\infty}^{t_n}e^{\rho s}\|f(s)\|^2ds\right)$$

$$=4e^{\rho t_n}+\frac{4}{\rho}\int_{-\infty}^{t_n} e^{\rho s}\|f(s)\|^2 ds, \qquad (4.9)$$

$\{\Phi(0,t_n,x_n)\}_{n\in\mathbb{N}} t_n \to -\infty n \to \infty$
$L_G^2(\Omega, L^2(\mathbb{R})) x^*(0) \in B(0)$

$$\hat{\mathbb{E}}\big[\|\Phi(0,t_n,x_n)-x^*(0)\|^2\big] \to 0 \text{as} n \to \infty.$$

$x^*(t) \doteq \Phi(t,0,x^*(0)) t \geq 0 0 - 1 x^*(-1) \in B(-1)$

$$\hat{\mathbb{E}}\big[\|\Phi(-1,t_n,x_n)-x^*(-1)\|^2\big] \to 0 \text{as} n \to \infty.$$

$x^*(0)=\Phi(0,-1,x^*(-1))$

$$\begin{aligned}
&\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,-1,x^*(-1))\|^2\big]\\
&=\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)+\Phi(0,t_n,x_n)-\Phi(0,-1,x^*(-1))\\
&+\Phi(0,-1,\Phi(-1,t_n,x_n))-\Phi(0,-1,\Phi(-1,t_n,x_n))\|^2\big]\\
&=\hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)-\Phi(0,-1,x^*(-1))+\Phi(0,-1,\Phi(-1,t_n,x_n))\|^2\big]\\
&\leq \hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)\|^2\big]+\hat{\mathbb{E}}\big[\|\Phi(0,-1,\Phi(-1,t_n,x_n))-\Phi(0,-1,x^*(-1))\|^2\big]\\
&\leq \hat{\mathbb{E}}\big[\|x^*(0)-\Phi(0,t_n,x_n)\|^2\big]+e^{-\rho}\hat{\mathbb{E}}\big[\|\Phi(-1,t_n,x_n)-x^*(-1)\|^2\big].
\end{aligned}$$

$-n - n-1 x^*(-n-1) \in B(-n-1) x^*(-n)=\Phi(-n,-n-1,x^*(-n-1)) x^*(t)=\Phi(t,-n-1,x^*(-n-1)) -n-1<t<-n x^*(t) \Phi x^*(t)=\Phi(t,s,x^*(s)) (t,s)\in\mathbb{R}^2_{\geq}$
$\Phi x^*(t) x^*(t) \mathcal{D} \Phi \mathcal{A}=\{x^*(t)\} \bar{x}^*(t) \bar{x}^*(t)\in\mathcal{A}(t) t\in\mathbb{R} \hat{\mathbb{E}}\big[\|x^*(0)-\bar{x}^*(0)\|^2\big]\geq\varepsilon>0$
????

$$\begin{aligned}
&\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\\
&\leq e^{\rho t}\hat{\mathbb{E}}\big[\|x^*(-t)-\bar{x}^*(-t)\|^2\big]\\
&\leq 4\left(1+\frac{1}{\rho}e^{\rho t}\int_{-\infty}^{-t} e^{\rho s}\|f(s)\|^2 ds\right)e^{-\rho t}\\
&\leq 4e^{-\rho t}+\frac{4}{\rho}\int_{-\infty}^{-t} e^{\rho s}\|f(s)\|^2 ds
\end{aligned}$$

$t\geq 0 T>0 t\geq T$

$$\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\leq\frac{1}{2}\varepsilon.$$

$x^*(0)=\Phi(0,-t,x^*(-t)) \bar{x}^*(0)=\Phi(0,-t,\bar{x}^*(-t))$

$$\varepsilon\leq\hat{\mathbb{E}}\big[\|x^*(0)-\bar{x}^*(0)\|^2\big]=\hat{\mathbb{E}}\big[\|\Phi(0,-t,x^*(-t))-\Phi(0,-t,\bar{x}^*(-t))\|^2\big]\leq\frac{1}{2}\varepsilon,$$

$t\geq T$
$\Phi \mathcal{D} \mathcal{A}=\{x^*(t)\}_{t\in\mathbb{R}}$
$x^*(t) t\in\mathbb{R} x^*(t+\varpi)=\Phi(t+\varpi,t,x^*(t)) f(x,t) \varpi \Phi \varpi \Phi x^*(t)=x^*(t+\varpi) \mathcal{A} \varpi$.

Remark 4.3. *If $f(x,t)\equiv 0$ for $x\in\mathbb{R}$, we can obtain that 0 is the solution of system* (**??**). *Then combining with Theorem* **??**, *we have $x^*(t)\equiv 0$.*

Remark 4.4. *In [**?**, **?**, **?**, **?**, **?**], the existence of mean random attractors has been investigated for mean random dynamical system in classical probability space, which cannot be applied to the system with probability-uncertain initial data. However, Theorem* **??** *may be applied to prove the existence of pullback G-mean random attractors for G-mean random dynamical system, so it can be applied to more cases.*

5. G-mean random attractors for (??) in $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$ (??) $L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$

$\phi(x) = (1+|x|^2)^{-\sigma} x \in \mathbb{R} \sigma > \frac{1}{2} L^2_\sigma(\mathbb{R})$

$$L^2_\sigma(\mathbb{R}) = \left\{ u : \mathbb{R} \to \mathbb{R} \text{is measurable}, \int_{\mathbb{R}} (1+|x|^2)^{-\sigma} |u(x)|^2 dx < \infty \right\}$$

$$\|u\|_\sigma = \Big(\int_{\mathbb{R}} (1+|x|^2)^{-\sigma} |u(x)|^2 dx \Big)^{1/2}.$$

(??) $L^2_G(\Omega, L^2_\sigma(\mathbb{R})) \Phi L^2_G(\Omega, L^2(\mathbb{R})) L^2_G(\Omega, L^2_\sigma(\mathbb{R})) L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$
$L^2_G(\Omega, L^2_\sigma(\mathbb{R}))$

Lemma 5.1. *Suppose $f_1, f_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$, $u_{\tau,1}, u_{\tau,2} \in L^2_G(\Omega, L^2(\mathbb{R}))$ and (??) hold. Let u_1, u_2 be solutions of problem (??) with f replaced by f_1 and f_2, respectively. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists a positive constant $C = C(\tau, T)$ such that for all $t \in [\tau, \tau+T]$:*

$$\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1}) - u_2(t,\tau,u_{\tau,2})\|^2_\sigma] \le C\hat{\mathbb{E}}[\|u_{\tau,1} - u_{\tau,2}\|^2_\sigma] + C\int_\tau^t \|f_1(s) - f_2(s)\|^2_\sigma ds. \quad (5.1)$$

Proof(??)

$$\begin{aligned}\frac{\partial(u_1-u_2)}{\partial t} =& (\lambda + i\alpha)\triangle(u_1-u_2) - \rho(u_1-u_2) \\ & - (\kappa+i\beta)(|u_1|^2 u_1 - |u_2|^2 u_2) + f_1(t,x) - f_2(t,x).\end{aligned}$$

$u_1 - u_2 L^2_\sigma(\mathbb{R})$

$$\begin{aligned}\frac{d}{dt}\|u_1-u_2\|^2_\sigma =& 2\mathrm{Re}(\lambda+i\alpha)\big\langle \triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma - 2\rho\|u_1-u_2\|^2_\sigma \\ & - 2\mathrm{Re}(\kappa+i\beta)\big\langle |u_1|^2u_1 - |u_2|^2u_2, u_1-u_2\big\rangle_\sigma + 2\mathrm{Re}\langle f_1(t)-f_2(t), u_1-u_2\rangle_\sigma.\end{aligned} \quad (5.2)$$

(??)

$$\begin{aligned}& 2\mathrm{Re}(\lambda+i\alpha)\big\langle \triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma \\ =& -2\lambda\big\langle \nabla(u_1-u_2), \nabla(u_1-u_2)\cdot\phi(x)\big\rangle - 2\mathrm{Re}(\lambda+i\alpha)\big\langle \nabla(u_1-u_2), \nabla\phi(x)\cdot(u_1-u_2)\big\rangle.\end{aligned}$$

$$|\nabla\phi(x)| \le \sigma\phi(x), \forall x \in \mathbb{R},$$

$$\begin{aligned}& 2\mathrm{Re}(\lambda+i\alpha)\big\langle \triangle(u_1-u_2),(u_1-u_2)\big\rangle_\sigma \\ \le& -2\lambda\|\nabla(u_1-u_2)\|^2_\sigma + 2\sigma\sqrt{\lambda^2+\alpha^2}\big\langle \nabla(u_1-u_2), \phi(x)(u_1-u_2)\big\rangle \\ \le& -2\lambda\|\nabla(u_1-u_2)\|^2_\sigma + \lambda\|\nabla(u_1-u_2)\|^2_\sigma + \frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}\|u_1-u_2\|^2_\sigma \\ \le& -\lambda\|\nabla(u_1-u_2)\|^2_\sigma + \frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}\|u_1-u_2\|^2_\sigma.\end{aligned} \quad (5.3)$$

(??)??

$$-2\mathrm{Re}(\kappa+i\beta)\big\langle |u_1|^2u_1 - |u_2|^2u_2, u_1-u_2\big\rangle_\sigma$$

$$
\begin{aligned}
&=-2\mathrm{Re}(\kappa+i\beta)\int_{\mathbb{R}}\phi(x)(|u_1|^2u_1-|u_2|^2u_2)(\bar{u}_1-\bar{u}_2)dx\\
&=-2\kappa\int_{\mathbb{R}}\phi(x)\mathrm{Re}Ldx+2\beta\int_{\mathbb{R}}\phi(x)\mathrm{Im}Ldx\\
&\le 2\kappa\Big(-1+\frac{|\beta|}{\kappa\sqrt{3}}\Big)\int_{\mathbb{R}}\phi(x)\mathrm{Re}Ldx, \qquad (5.4)
\end{aligned}
$$

$L=\big(|u_1(x)|^2u_1(x)-|u_2(x)|^2u_2(x)\big)\big(\bar{u}_1(x)-\bar{u}_2(x)\big)$(**??**)(**??**)(**??**)

$$
\frac{d}{dt}\|u_1-u_2\|_\sigma^2\le c\|u_1-u_2\|_\sigma^2+\frac{1}{\rho}\|f_1(t)-f_2(t)\|_\sigma^2, \qquad (5.5)
$$

$c=\frac{\sigma^2(\lambda^2+\alpha^2)}{\lambda}-\rho\tau\in\mathbb{R}T>0t\in[\tau,\tau+T]$

$$
\begin{aligned}
&\hat{\mathbb{E}}[\|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})\|_\sigma^2]\\
\le&\hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|_\sigma^2]e^{c(t-\tau)}+\frac{1}{\rho}\int_\tau^t\|f_1(s)-f_2(s)\|_\sigma^2e^{c(t-s)}ds.
\end{aligned}
$$

.

$\Phi L_G^2(\Omega,L^2(\mathbb{R}))L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$(**??**)$L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$

THEOREM 5.1. *Suppose $f\in L_{loc}^2(\mathbb{R},L_\sigma^2(\mathbb{R}))$ and (**??**) hold. Then one can associate problem (**??**) with a continuous system $\Phi:\mathbb{R}_\ge^2\times L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\to L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$ such that for every $(\tau,t)\in\mathbb{R}_\ge^2$ and $u_\tau\in L_G^2(\Omega,L^2(\mathbb{R}))$, $\Phi(t,\tau,u_\tau)=u(t,\tau,u_\tau)$, where $\Phi(t,\tau,u_\tau)$ is the solution of problem (**??**) with initial time τ and initial condition u_τ.*

Proof$L_G^2(\Omega,L^2(\mathbb{R}))L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\forall u\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))u_n(x)=u(x)I_{\{|x|<n\}}n\in\mathbb{N}^+n$

$$
\begin{aligned}
\hat{\mathbb{E}}\Big[\int_{\mathbb{R}}|u(x)I_{\{|x|<n\}}|^2dx\Big]&=\hat{\mathbb{E}}\Big[\int_{|x|<n}|u(x)|^2dx\Big]\\
&=(1+n^2)^\sigma\hat{\mathbb{E}}\Big[\int_{|x|<n}(1+n^2)^{-\sigma}|u(x)|^2dx\Big]\\
&\le(1+n^2)^\sigma\hat{\mathbb{E}}\Big[\int_{|x|<n}(1+x^2)^{-\sigma}|u(x)|^2dx\Big]\\
&\le(1+n^2)^\sigma\hat{\mathbb{E}}\Big[\int_{\mathbb{R}}(1+x^2)^{-\sigma}|u(x)|^2dx\Big]\\
&<\infty,
\end{aligned}
$$

$u_n\in L_G^2(\Omega,L^2(\mathbb{R}))$

$$
\hat{\mathbb{E}}\Big[\|u(x)-u_n(x)\|_\sigma^2\Big]=\hat{\mathbb{E}}\Big[\int_{|x|\ge n}(1+x^2)^{-\sigma}|u(x)|^2dx\Big],
$$

$u\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$

$$
\hat{\mathbb{E}}\Big[\|u(x)-u_n(x)\|_\sigma^2\Big]\to 0,\mathrm{as}n\to\infty.
$$

$L_G^2(\Omega,L^2(\mathbb{R}))L_G^2(\Omega,L_\sigma^2(\mathbb{R}))L_G^2(\Omega,L^2(\mathbb{R}))\times L^2((\tau,\tau+T),L^2(\mathbb{R}))L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))$

$(u_\tau,f)\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))(u_n,f_n)\in L_G^2(\Omega,L^2(\mathbb{R}))\times L^2((\tau,\tau+T),L^2(\mathbb{R}))$ $(u_n,f_n)\to(u_\tau,f)L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))\tau\in\mathbb{R}T>0$**??**$\{u(\cdot,\tau,(u_n,f_n))\}_{n=1}^\infty C([\tau,\tau+$

$T], L_G^2(\Omega, L_\sigma^2(\mathbb{R}))) \lim_{n\to\infty} u(\cdot,\tau,(u_n,f_n)) C([\tau,\tau+T], L_G^2(\Omega, L_\sigma^2(\mathbb{R})))(u_n,f_n)\Phi L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times$
$L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))C([\tau,\tau+T],L_G^2(\Omega,L_\sigma^2(\mathbb{R})))(u_\tau,f)\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))$
$(u_n,f_n)\in L_G^2(\Omega,L^2(\mathbb{R}))\times L^2((\tau,\tau+T),L^2(\mathbb{R}))$

$$\tilde{\Phi}(\cdot,\tau,(u_\tau,f)) = \lim_{n\to\infty} u(\cdot,\tau,(u_n,f_n)), \tag{5.6}$$

$(u_n,f_n)\to(u_\tau,f)L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))$**??**$\tilde{\Phi}(\cdot,\tau,(u_\tau,f))(u_\tau,f)L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\times$
$L^2((\tau,\tau+T),L_\sigma^2(\mathbb{R}))$
$t\geq\tau u(t,\tau,\omega,(u_n,f_n))(\mathcal{F},\mathcal{B}(L^2(\mathbb{R})))L^2(\mathbb{R})\hookrightarrow L_\sigma^2(\mathbb{R})u(t,\tau,\omega,(u_n,f_n))(\mathcal{F},\mathcal{B}(L_\sigma^2(\mathbb{R})))$(**??**)
$\tilde{\Phi}(\cdot,\tau,(u_\tau,f))(\mathcal{F},\mathcal{B}(L_\sigma^2(\mathbb{R})))t\geq\tau\tilde{\Phi}\mathbb{R}_\geq^2\times L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\tilde{\Phi}\Phi L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\Phi\tilde{\Phi}$.
$\Phi L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\Phi L_G^2(\Omega,L_\sigma^2(\mathbb{R}))f$

$$\int_{-\infty}^{\tau} e^{\rho s}\|f(s)\|_\sigma^2 ds < \infty, \forall \tau\in\mathbb{R}, \tag{5.7}$$

(**??**)
REMARK 5.1. *As we will see later, when studying the existence of pullback mean attractors of Φ in $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$, it is convenient to use an equivalent norm for $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$ which is defined by the weight function $\phi_\delta(x)=(1+|\delta x|^2)^{-\sigma}$, $x\in\mathbb{R}$, where*

$$\delta=\min\Big\{1,\frac{\sqrt{\lambda\rho}}{\sqrt{(\lambda^2+\alpha^2)\sigma}}\Big\}. \tag{5.8}$$

By simple calculations, we can obtain that

$$|\nabla\phi_\delta(x)|\leq\sigma\delta\phi_\delta(x),\ \forall\ x\in\mathbb{R}, \tag{5.9}$$

and

$$\hat{\mathbb{E}}\left[\|u\|_\sigma^2\right]\leq\hat{\mathbb{E}}\Big[\int_{\mathbb{R}}\phi_\delta(x)|u(x)|^2dx\Big]\leq\delta^{-2\sigma}\hat{\mathbb{E}}\left[\|u\|_\sigma^2\right],\ \forall\ u\in L_G^2(\Omega,L_\sigma^2(\mathbb{R})), \tag{5.10}$$

which shows that the weighted space $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$ has an equivalent norm which is given by $\left(\hat{\mathbb{E}}\Big[\int_{\mathbb{R}}\phi_\delta(x)|u(x)|^2dx\Big]\right)^{\frac{1}{2}}$ for $u\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$.
LEMMA 5.2. *Assume* (**??**) *holds. Then the G-mean random dynamical system Φ is uniformly strictly contracting in $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$, i.e., for the different initial values $u_{\tau,1}$, $u_{\tau,2}\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$, there exists a constant M (which is independent of τ) such that*

$$\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1})-\Phi(t,\tau,u_{\tau,2})\|_\sigma^2]\leq M\hat{\mathbb{E}}[\|u_{\tau,1}-u_{\tau,2}\|_\sigma^2]e^{-\rho(t-\tau)}, \tag{5.11}$$

for all $t\geq\tau$.
Proof$u_1(t,\tau,u_{\tau,1})u_2(t,\tau,u_{\tau,2})$(**??**)$u_{\tau,1}u_{\tau,2}\in L_G^2(\Omega,L_\sigma^2(\mathbb{R}))\tau u_1-u_2\phi_\delta(u_1-u_2)L^2(\mathbb{R})$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u_1-u_2|^2dx=&-2\lambda\int_{\mathbb{R}}\phi_\delta(x)|\nabla(u_1-u_2)|^2dx-2\rho\int_{\mathbb{R}}|u_1-u_2|^2\phi_\delta(x)dx\\&-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}(\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\cdot\nabla\phi_\delta(x)dx\\&-2\mathrm{Re}(\kappa+i\beta)\big\langle|u_1|^2u_1-|u_2|^2u_2,\phi_\delta(x)(u_1-u_2)\big\rangle.\end{aligned} \tag{5.12}$$

$$
\begin{aligned}
&-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}(\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\cdot\nabla\phi_\delta(x)dx\\
\le&2\sigma\delta\sqrt{\lambda^2+\alpha^2}\int_{\mathbb{R}}\left|(\bar{u}_1-\bar{u}_2)\nabla(u_1-u_2)\phi_\delta(x)\right|dx\\
\le&\frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}}|u_1-u_2|^2\phi_\delta(x)dx+\lambda\int_{\mathbb{R}}|\nabla(u_1-u_2)|^2\phi_\delta(x)dx, \qquad (5.13)
\end{aligned}
$$

(??)(??)??

$$
\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u_1-u_2|^2dx\le-\rho\int_{\mathbb{R}}|u_1-u_2|^2\phi_\delta(x)dx \qquad (5.14)
$$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_1(t,\tau,u_{\tau,1})-u_2(t,\tau,u_{\tau,2})|^2dx\right]\\
\le&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_{\tau,1}-u_{\tau,2}|^2dx\right]e^{-\rho(t-\tau)}, \qquad (5.15)
\end{aligned}
$$

(??)Φ

$$
\hat{\mathbb{E}}[\|\Phi(t,\tau,u_{\tau,1})-\Phi(t,\tau,u_{\tau,2})\|_\sigma^2]\le\delta^{-2\sigma}\hat{\mathbb{E}}\left[\|u_{\tau,1}-u_{\tau,2}\|_\sigma^2\right]e^{-\rho(t-\tau)}. \qquad (5.16)
$$

.

Lemma 5.3. *Let (??) and (??) hold. Then for every $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, there exist $T=T(\tau,D)>0$ and $\tilde{R}(\tau)>0$ such that for all $t\ge T$,*

$$
\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|_\sigma^2]\le\tilde{R}(\tau), \qquad (5.17)
$$

where $u_{\tau-t}\in D(\tau-t)$.

Proof$u\phi_\delta uL^2(\mathbb{R})$

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx=&-2\lambda\int_{\mathbb{R}}\phi_\delta(x)|\nabla u|^2dx-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}\bar{u}\nabla u\cdot\nabla\phi_\delta(x)dx\\
&-2\rho\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx-2\kappa\int_{\mathbb{R}}|u|^4\phi_\delta(x)dx+2\mathrm{Re}\int_{\mathbb{R}}f(x,t)\phi_\delta(x)\bar{u}dx.
\end{aligned}
$$

(??)

$$
\begin{aligned}
-2\mathrm{Re}(\lambda+i\alpha)\int_{\mathbb{R}}\bar{u}\nabla u\cdot\nabla\phi_\delta(x)dx\le&2\sigma\delta\sqrt{\lambda^2+\alpha^2}\int_{\mathbb{R}}\left|\bar{u}\nabla u\cdot\nabla\phi_\delta(x)\right|dx\\
\le&\frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx+\lambda\int_{\mathbb{R}}|\nabla u|^2\phi_\delta(x)dx.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx\le&\frac{\sigma^2\delta^2(\lambda^2+\alpha^2)}{\lambda}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx-2\rho\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx\\
&+\frac{2}{\rho}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx+\frac{\rho}{2}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx
\end{aligned}
$$

$$\leq -\frac{\rho}{2}\int_{\mathbb{R}}|u|^2\phi_\delta(x)dx + \frac{2}{\rho}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx. \quad (5.18)$$

$$\begin{aligned}\frac{d}{dt}e^{\frac{\rho t}{2}}\int_{\mathbb{R}}\phi_\delta(x)|u|^2dx = &\frac{\rho}{2}e^{\frac{\rho t}{2}}\|u\|^2 + e^{\frac{\rho t}{2}}\frac{d}{dt}\|u\|^2\\ \leq &\frac{2}{\rho}e^{\frac{\rho t}{2}}\int_{\mathbb{R}}|f(x,t)|^2\phi_\delta(x)dx.\end{aligned}$$

$(\tau-t,\tau) t\geq 0$

$$\begin{aligned}&\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u(\tau,\tau-t,u_{\tau-t})|^2dx\right]\\ \leq &e^{-\frac{\rho t}{2}}\hat{\mathbb{E}}\left[\int_{\mathbb{R}}\phi_\delta(x)|u_{\tau-t}|^2dx\right] + \frac{2}{\rho}e^{-\frac{\rho\tau}{2}}\int_{\tau-t}^{\tau}\int_{\mathbb{R}}e^{\rho s}|f(x,s)|^2\phi_\delta(x)dxds. \quad (5.19)\end{aligned}$$

(**??**)Φ

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|_\sigma^2] \leq e^{-\frac{\rho t}{2}}\delta^{-2\sigma}\hat{\mathbb{E}}[\|u_{\tau-t}\|_\sigma^2] + \frac{1}{\rho\delta^{2\sigma}}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds,$$

$u_{\tau-t}\in D(\tau-t)\in\mathcal{D} T=T(\tau,D)>0 t\geq T$

$$\hat{\mathbb{E}}[\|\Phi(\tau,\tau-t,u_{\tau-t})\|^2] \leq \tilde{R}(\tau),$$

$$\tilde{R}(\tau) = 1 + \frac{1}{\rho\delta^{2\sigma}}e^{-\rho\tau}\int_{-\infty}^{\tau}e^{\rho s}\|f(s)\|^2ds.$$

.

THEOREM 5.2. *Under assumptions* (**??**) *and* (**??**)*, then the G-mean random dynamical system* Φ *associated with problem* (**??**) *has a unique* $\mathcal{D}$*-pullback G-mean random attractor* $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}$ *in* $L_G^2(\Omega,L_\sigma^2(\mathbb{R}))$*. Furthermore, if there exists a positive number* ϖ *such that* $f:\mathbb{R}\to L^2(\mathbb{R})$ *is* ϖ*-periodic, then such attractor* $\mathcal{A}$ *is also* T*-periodic; that is,* $\mathcal{A}(\tau+\varpi)=\mathcal{A}(\tau)$ *for all* $\tau\in\mathbb{R}$.

Proof **??????**.

Acknowledgements.óíií

Regularity analysis for stochastic complex Landau-Ginzburg equation with Dirichlet white-noise boundary conditions
Random attractors for stochastic reaction-diffusion equations on unbounded domains
Random data Cauchy theory for supercritical wave equations I: local theory
ß*Exponentially stable stationary solutions for stochastic evolution equations and their perturbation*
í*Autonomous and non-autonomous attractors for differential equations with delays*
Pullback attractors for stochastic heat equations in materials with memory
Nonlocal stochastic functional differential equations driven by G-Brownian motion and mean random dynamical systems
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Global existence theory for a generalized Ginzburg-Landau equation

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ö*Random attractors for a class of stochastic partial differential equations driven by general additive noise*

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Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains

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Pullback attractors of non-autonomous stochastic degenerate parabolic equations on unbounded domains

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Dynamics of the 3D fractional Ginzburg-Landau equation with multiplicative noise on an unbounded domain

Well-posedness and dynamics for the fractional Ginzburg-Landau equation

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One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation

Random attractors for non-autonomous fractional stochastic Ginzburg-Landau equations on unbounded domains

A study of Landau damping with random initial inputs

Global solution for a stochastic Ginzburg-Landau equation with multiplicative noise

Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity

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Compactness in Lebesgue-Bochner spaces of random variables and the existence of mean-square random attractors

Existence and uniqueness of weak solutions to Ginzburg-Landau equation with external noise and stochastic perturbation

Mean-square random attractors of stochastic delay differential equations with random delay

Nonlocal stochastic differential equations with time-varying delay driven by G-Brownian motion

Weak pullback attractors for stochastic Ginzburg-Landau equations in Bochner space