Practical stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion

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Abstract

In this paper, practical stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion (G-SDEs) is studied. The analysis of the global practical uniform pth moment exponential stability, as well as the global practical uniform exponential stability with respect to a part of the variables of G-SDEs are investigated by means of the G-Lyapunov functions. An illustrative example to show the usefulness of the practical stability with respect to a part of the variables notion is also provided.

Keywords: G-Stochastic differential equations, G-Lyapunov techniques, G-Itô formula, G-Brownian motion, Nontrivial solution, Practical stability with respect to a part of the variables.
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1 Introduction

Stochastic differential equations are widely used for modeling physical, technical, ecological, economical dynamical systems, wherein significant uncertainty is present. With the rapid development of stochastic analysis theory, the stability of SDEs has become a very prevalent theme of recent research in Mathematics and its applications.

Some stochastic systems do not fulfill stability properties with respect to all the unknown variables of the system. Nevertheless, it is very crucial to investigate if it is still possible to prove some stability criteria with respect to some of the variables in the problem, which is called "Stability with respect to a part of the variables."

In 1892, Lyapunov introduced the theory of stability for deterministic dynamical systems and firstly established the so-called Lyapunov exponent method. Since then, the method of Lyapunov functions has become one of the most powerful tools to study the stability of stochastic dynamical systems, we would like to mention here the references [5, 6, 19], among others. With the development of the second method of Lyapunov, Ignatyev [2], Peiffer and Rouche [23], Rouche et al. [27], Rumyantsev [28], Rumyantsev and Oziraner [29], Savchenko and Ignatyev [30], Vorotnikov [31], Vorotnikov and Rumyantsev [32] investigated the concept of stability with respect to a part of the variables especially is supposed to be an equilibrium point. The stability with respect to a part of the variables has been used in analyzing the qualitative properties of equilibria and boundedness properties of motions of dynamical systems determined by ordinary differential equations, difference equations, functional differential equations, stochastic differential equations, etc. It involves a notion of stability with respect to only a prespecified subset of the state variables describing the motions of the system under investigation.

Several intrinsic variants to Lyapunov's original concepts of practical stability were proposed in [2, 4]. In the case that the origin is not an equilibrium point, we can investigate the stability of the SDEs in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball. This property is defined as "Practical stability". The practical stability, in the sense introduced in [3, 9, 20, 21]. In fact, it is very important and very useful for analyzing the stability or for designing practical controllers of dynamical systems since controlling system to an idealized point are either expensive or impossible in the presence of disruptions and the best which we can hope in such situations is to use practical stability. In practice, we may only need to stabilize a system into the region of phase space in which the implementation is still acceptable. It is well known that asymptotic stability is more important than stability. Also, the desired system may oscillate near the origin. Thus the notion of practical stability is more suitable in several situations than Lyapunov stability. In this case, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner especially in presence of perturbations. In general, we know some information on the upper bound of the term of perturbation whose size influences the size of the ball.

Recently, in [7, 8], the practical stability of stochastic stability with respect to a part of the variables is investigated.

For the dormant applications in uncertainty problems, risk measures, and superhedging in finance, considerable attention has been paid to the theory of nonlinear expectation. Notably, Peng [24] built the fundamental basics of time-consistent G-expectation as well as G-conditional expectation where G is the infinitesimal generator of a nonlinear heat equation. Under the G-framework, Peng [22, 24] introduced the notion of G-normal distribution, G-Brownian motion, and he also established the corresponding stochastic calculus of Itô's type. Since then, much researches have been carried out on the stochastic analysis concerning the G-Brownian motion. On that basis, Gao [11] and Peng [22] studied the existence and uniqueness of the solution of stochastic differential equations driven by G-Brownian motion (G-SDEs, in short) under a standard Lipschitz condition. Moreover, Lin [17] obtained the existence and uniqueness of the solution of G-SDEs with reflecting boundary. The G-Brownian motion has a very rich and interesting new structure which non-trivially, for a recent account and development of this theory, we refer the reader to see [1, 15, 16].

To the best of our knowledge, no work has been published about the practical stability of stochastic differential equations driven by G-Brownian motion. Our main objective in this paper is to extend the results in [7], which investigated the practical stability with respect to a part of the variables of stochastic differential equations driven by the standard Brownian motion, to the case of G-Brownian motion. Based on the method of Lyapunov and recently developed Itô calculus for SDE driven by G-Brownian motion, we establish some criteria for the global practical uniform pth moment exponential stability, and the global practical uniform exponential stability with respect to a part of the variables of a class of nonlinear stochastic differential equations driven by G-Brownian motion.

This paper is structured in the following way: In Section 2, we recall some necessary preliminaries and results. In Sections 3 and 4, we establish sufficient conditions to ensure the global practical uniform pth moment exponential stability, and the global practical uniform exponential stability with respect to a part of the variables of G-SDEs by using G-Lyapunov techniques and Itô's calculus for G-SDEs. Moreover, we exhibit an illustrative example to show the applicability of our abstract theory. Eventually, in Section 6, some conclusions are included.

2 Preliminaries

In this section, we briefly recall some notations and preliminaries about sublinear expectations and G-Brownian motions. For more details, one can see [14, 22, 24, 25].

Notations on G-stochastic calculus

- $\mathbb{R}^n$: the space of n-dimensional real column vectors.
- $\langle x, y\rangle$: the scalar product of two vectors $x, y \in \mathbb{R}^n$.
- If $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.

- $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$.
- $\mathcal{F}_t = \mathcal{B}(\Omega_t)$.
- $\mathcal{B}(\Omega)$: the Borel σ-algebra of Ω.
- $\mathcal{C}_{b,Lip(\mathbb{R}^n)}$: the space of all bounded real-valued Lipschitz continuous functions.
- $\mathbb{L}^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions.
- $\mathbb{L}^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real functions.
- $B_b(\Omega)$: all bounded elements in $\mathbb{L}^0(\Omega)$, $B_b(\Omega_t) := B_b(\Omega) \cap \mathbb{L}^0(\Omega_t)$.
- $\mathbb{L}^p_G(\Omega)$: Banach space under the natural norm $||X||^p = \widehat{\mathbb{E}}\left(|X|^p\right)^{\frac{1}{p}}$.
- $M^{p,0}_G([0,T]) = \{\zeta := \zeta_t(\omega) = \sum_{j=0}^{N-1} \zeta_j \mathbb{1}_{[t_j,t_{j+1})}(t),\ \forall N > 0,$
$$0 = t_0 < ... < t_N = T,\ \zeta_j \in \mathbb{L}^p_G(\omega_{t_j}),\ j = 0, 1, 2, .., N-1\}.$$
- $M^p_G([0,T])$: the completion of $M^{p,0}_G$ under $||\eta||_{M^p_G} = |\displaystyle\int_0^T \widehat{\mathbb{E}}\left(|\eta(t)|^p\right) dt|^{\frac{1}{p}}$.

Let Ω be a given set and $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We further suppose that $\mathcal{H}$ satisfies $a \in \mathcal{H}$ for each constant a and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$.

Definition 2.1. A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,
i) Monotonicity: if $X \geq Y$, then $\widehat{\mathbb{E}}(X) \geq \widehat{\mathbb{E}}(Y)$.
ii) Constant preserving: $\widehat{\mathbb{E}}(a) = a,\ \forall\, a \in \mathbb{R}$.
iii) Sub-additivity: $\widehat{\mathbb{E}}(X+Y) \leq \widehat{\mathbb{E}}(X) + \widehat{\mathbb{E}}(Y)$.
iv) Positive homogeneity: $\widehat{\mathbb{E}}(\lambda X) = \lambda \widehat{\mathbb{E}}(X), \quad \lambda \geq 0$.
The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. $Y = (Y_1, ..., Y_n)$, where $Y_i \in \mathcal{H}$ is called an n-dimensional random vector in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.

Definition 2.2. Weakly compact sets are defined by sets which are compact with respect to the weak topology of a Banach space.

The representation of a sublinear expectation can be expressed as a supermum of linear expectations.

Theorem 2.1. *[24] There exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$, such that*
$$\widehat{\mathbb{E}}(X) = \sup_{p \in \mathcal{P}} \mathbb{E}_p(X), \quad X \in \mathbb{L}^1_G(\Omega).$$

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, an n-dimensional random vector $Y = (Y_1, .., Y_n) \in \mathcal{H}$ is said to be independent from an m-dimensional random vector $X = (X_1, .., X_m) \in \mathcal{H}$ under the sublinear expectation $\widehat{\mathbb{E}}$, if for any test function $\varphi \in \mathcal{C}_{b,Lip}(\mathbb{R}^{m+n})$

$$\widehat{\mathbb{E}}(\varphi(X, Y)) = \widehat{\mathbb{E}}\left(\widehat{\mathbb{E}}(\varphi(x, Y))|_{x=X}\right).$$

Definition 2.4. Let X_1 and X_2 be two n-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\widehat{\mathbb{E}}_1(\varphi(X_1)) = \widehat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in \mathcal{C}_{b,Lip(\mathbb{R}^n)}.$$

$\bar{X}$ is said to be an independent copy of X if $\bar{X} \overset{d}{=} X$ and $\bar{X}$ is independent from X.

Definition 2.5. (G-Normal Distribution) A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called G-normal distributed, denoted by $X \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, for a given pair $0 \leq \underline{\sigma} \leq \bar{\sigma}$, if for any $a, b \geq 0$,

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of X.

Definition 2.6. A stopping time τ relative to the filtration $\mathcal{F}_t$ in a map on Ω with values in $[0, T]$, such that for every $t \geq T$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Let Ω be the space of all $\mathbb{R}^d-$valued continuous paths $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$. We assume moreover that Ω is a metric space equipped with the following distance:

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}\left(\max_{0\leq t\leq N}(|\omega_t^1 - \omega_t^2|) \wedge 1\right),$$

and consider the canonical process $B_t(\omega) = \omega_t,\ t \in [0, +\infty),$ for $\omega \in \Omega$; then for each fixed $T \in [0, +\infty)$, we have

$$L_{ip}^0(\Omega_T) := \left\{\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}) :\ n \geq 1,\ 0 \leq t_1 \leq ... \leq t_n \leq T,\ \varphi \in C_{b,lip}(\mathbb{R}^{d\times n})\right\}.$$

Definition 2.7. On the sublinear expectation space $(\Omega, L_{ip}^0(\Omega_T), \widehat{\mathbb{E}})$, the canonical process $(B_t)_{t\geq 0}$ is called a G-Brownian motion, if the following properties are satisfied:
(i) $B_0 = 0$.
(ii) for $t, s \geq 0$, the increment $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$, where X is G-normal distributed.
(iii) for $t, s \geq 0$, the increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ for each $n \in \mathbb{N}$, and $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$.
Moreover, the sublinear expectation $\widehat{\mathbb{E}}(\cdot)$ is called G-expectation.

Let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion. The letter G denotes the function

$$G(a) := \frac{1}{2}\widehat{\mathbb{E}}(aB_1^2) = \frac{1}{2}\left(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\right), \quad a \in \mathbb{R},$$

with $\underline{\sigma}^2 := -\widehat{\mathbb{E}}(-B_1^2) \leq \widehat{\mathbb{E}}(B_1^2) := \overline{\sigma}^2, \quad 0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$. Recall that $a^+ = \max\{0, a\}$ and $a^- = -\min\{0, a\}$.

Definition 2.8. For $p \geq 1$ and $T \in \mathbb{R}_+$ fixed, we first consider the following type of simple processes:

$$M_{b,0}([0,T]) = \{\eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \zeta_j \mathbb{1}_{[t_j, t_{j+1})}(t), \quad \forall N > 0,$$
$$0 = t_0 < ... < t_N = T,\ \zeta_j \in B_b(\Omega_{t_j}),\ j = 0, 1, 2, .., N-1\}.$$

For each $p \geq 1$, we denote by $M_\star^p([0,T])$ the completion of $M_{b,0}([0,T])$ under the following norm:

$$||\eta||_{M^p([0,T])} = \left(\widehat{\mathbb{E}}\left(\int_0^T |\eta_t|^p dt\right)\right)^{\frac{1}{p}}.$$

We have $M_\star^p([0,T]) \supset M_\star^q([0,T])$ for $p \leq q$.

Definition 2.9. We define $C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ by the family of all real-valued functions $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ which are once continuously differentiable in t and twice in x.

Now, we introduce the following G-Itô formula that can be found in [15].

Lemma 2.2. *Let $V \in C^{1.2}(\mathbb{R}_+, \mathbb{R}^n)$, and*

$$x(t) = x(0) + \int_0^t \widetilde{\varphi}(s)ds + \int_0^t \widetilde{\chi}(s)d\langle B\rangle_s + \int_0^t \widetilde{\psi}(s)dB_s,$$

where $\widetilde{\varphi},\ \widetilde{\chi} \in M_\star^1([0,T]),\ \widetilde{\psi} \in M_\star^2([0,T])$. Then, for $t > 0$, we have

$$\begin{aligned}
&V(t, x(t)) - V(0, x(0)) \\
&\int_0^t V_x(s, x(s))\widetilde{\psi}(s)dB_s + \int_0^t \left(V_t(s, x(s)) + V_x(s, x(s))\widetilde{\varphi}(s)\right) ds \\
&\int_0^t \left(V_x(s, x(s))\widetilde{\chi}(s) + \frac{1}{2}\widetilde{\psi}^T(s)V_{xx}(s, x(s))\widetilde{\psi}(s)\right) d\langle B\rangle_s,
\end{aligned}$$

where $V_t(t,x) = \dfrac{\partial V}{\partial t}(t,x)$; $V_x(t,x) = (\dfrac{\partial V}{\partial x_1}(t,x), .., \dfrac{\partial V}{\partial x_n}(t,x))$; $V_{xx}(t,x) = \left(\dfrac{\partial^2 V}{\partial x_i \partial x_j}(t,x)\right)_{n\times n}$.

Now, we introduce the natural Choquet capacity.

Definition 2.10. Let $\mathcal{B}(\Omega)$ the Borel $\sigma-$algebra and $\mathcal{P}$ be a weakly compact collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$, then the capacity $\hat{C}(\cdot)$ associated to $\mathcal{P}$ is defined by

$$\hat{C}(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 2.11. A set $A \subset \mathcal{B}(\Omega)$ is polar if $\hat{C}(A) = 0$. A property holds "quasi-surely" (q.s.), if it holds outside a polar set.

Lemma 2.3. *[10] Let $\{A_k\} \subset \mathcal{B}(\Omega)$, such that*

$$\sum_{k=1}^{\infty} \hat{C}(A_k) < \infty.$$

Then, $\lim_{k\to\infty} \sup A_k$ is polar.

3 Global practical uniform pth moment exponential stability of G-SDEs with respect to a part of the variables

Several authors handled the problem of stability with respect to a part of variables of stochastic differential equations driven by standard Brownian motion when the origin is an equilibrium point in the sense of Lyapunov, see [7, 12, 13, 31]. The innovation in our work is to tackle the problem of stability with respect to a part of variables of stochastic differential equations driven by G-Brownian motion when the origin is not a trivial solution by using the method of G-Lyapunov functions and recently developed Itô's calculus for SDEs driven by G-Brownian motion.

Consider the following SDE driven by m-dimensional G-Brownian motion:

$$dx(t) = f(t, x(t))dt + h(t, x(t))d\langle B\rangle_t + g(t, x(t))dB_t, \quad \forall x \in \mathbb{R}^n, \quad \forall t \geq 0, \tag{3.1}$$

where, $B_t = (B_1(t), ..., B_m(t))^T$ is an m-dimensional G-Brownian motion, and $(\langle B\rangle)_{t\geq 0}$ is the quadratic variation process of B.

Assume that $f, h,$ and $g \in M_\star^p([0, T], \mathbb{R}^n)$, $p \geq 2$, satisfy the following condition:

$$|f(t, x) - f(t, \widetilde{x})|^2 + |h(t, x) - h(t, \widetilde{x})|^2 + |g(t, x) - g(t, \widetilde{x})|^2 \leq K_2|x - \widetilde{x}|^2, \text{ for all } t \geq 0,\ x,\ \widetilde{x} \in \mathbb{R}^n,$$

where K_2 is a given positive real constant.

Denote $x=(y,z)^T$ where $y=(y_1,...,y_{n_1})^T\in\mathbb{R}^{n_1},\ z=(z_1,...,z_{n_2})^T\in\mathbb{R}^{n_2},\ n_1>0,\ n_2\geq 0;\ n_1+n_2=n;$

$$|y|=\sqrt{y_1^2+...+y_{n_1}^2},\,|z|=\sqrt{z_1^2+...+z_{n_2}^2},\ |x|=\left(|y|^2+|z|^2\right)^{\frac{1}{2}}.$$

Under the precedent assumptions, there exists a unique global solution $x(t,t_0,x_0)\in M_\star^p([0,T],\mathbb{R}^n)$, with $x(t,t_0,x_0)=(y(t,t_0,x_0),z(t,t_0,x_0))$ corresponding to the initial condition $x_0=(y_0,z_0)\in\mathbb{R}^n$ (see [18] for more details). In what follows, we use $x(t,t_0,x_0)=(y(t,t_0,x_0),z(t,t_0,x_0))$, or simply $x(t)=(y(t),z(t))$ to denote a solution of our system.

Suppose that there exists t such that $f(t,0)\neq 0$ or $h(t,0)\neq 0$ or $g(t,0)\neq 0$, i.e., the G-stochastic differential equation (3.1) does not have the trivial solution $x\equiv 0$. Therefore, we will study the pth moment exponential stability with respect to y of the G-SDE in a small neighborhood of the origin. The study of the practical stability with respect to a part of the variables of G-SDE lead back to the study of the stability with respect to a part of the variables of a small ball $B_r:=\{x\in\mathbb{R}^n:|x|\leq r\}$, $r>0$.

Definition 3.1. The solutions of (3.1) are said to be quasi surely globally uniformly bounded, if for each $\alpha>0$, there exists $c=c(\alpha)>0$ (independent of t_0), such that

$$\text{for every } t_0\geq 0, \text{ and all } x_0\in\mathbb{R}^n \text{ with } |x_0|\leq\alpha,\quad \sup_{t\geq t_0}|x(t,t_0,x_0)|\leq c(\alpha),\quad \text{q.s.} \tag{3.2}$$

Definition 3.2. *i*) Let $p>0$, the ball B_r is said to be globally uniformly pth moment exponentially stable with respect to y, if there exist $\lambda_1>0$ and $\lambda_2>0$, such that

$$\widehat{\mathbb{E}}(|y(t,t_0,x_0)|^p)\leq\lambda_1|x_0|^pe^{-\lambda_2(t-t_0)}+r,\quad\forall t\geq t_0,\quad\forall x_0\in\mathbb{R}^n. \tag{3.3}$$

ii) The system (3.1) is said to be globally practically uniformly pth moment exponentially stable with respect to y if there exists $r>0$, such that B_r is globally uniformly pth moment exponentially stable with respect to y.

Definition 3.3. For each $V\in C^{1.2}(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R})$, we define an operator L (called as G-Lyapunov function) from $\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ as follows:

$$LV(t,x):=V_t(t,x)+V_xf(t,x)+G\left(\langle V_x(t,x),2h(t,x)\rangle+\langle V_{xx}(t,x)g(t,x),g(t,x)\rangle\right).$$

Now, we aim to prove the global practical pth moment exponential stability of stochastic differential equations driven by G-Brownian motion with respect to a part of the variables, via the G-Lyapunov functions.

Theorem 3.1. *Assume that there exist $V\in C^{1.2}(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}_+)$ and nonnegative constants c_i $(i=1,2,3), p$ and ρ, such that for all $t\geq 0$, and all $x=(y,z)\in\mathbb{R}^n$,*

$$c_1|y|^p\leq V(t,x)\leq c_2|y|^p, \tag{3.4}$$

$$LV(t,x) \leq -c_3|y|^p + \rho. \tag{3.5}$$

Moreover, we suppose that $z(t,t_0,x_0)$ is quasi surely globally uniformly bounded.
Then, system (3.1) is globally practically uniformly pth moment exponentially stable with respect to y, with $r = \dfrac{\rho c_2}{c_1 c_3}$.

In order to prove this theorem, we need to introduce the Gronwall lemma proved in [26].

Lemma 3.2. *Let $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ be a continuous function, ε is a positive real number and λ is a strictly positive real number. Assume that for all $t \in [0,+\infty)$ and $0 \leq v \leq t$, we have*

$$\varphi(t) - \varphi(v) \leq \int_v^t (-\lambda\varphi(s) + \varepsilon)ds.$$

Then,

$$\varphi(t) \leq \frac{\varepsilon}{\lambda} + \varphi(0)\exp(-\lambda t).$$

Proof of Theorem 3.3. Applying the G-Itô's formula 2.2 on V for any $t \geq v \geq 0$, leads to

$$V(t,x(t)) = V(v,x(v)) + \int_v^t LV(s,x(s))ds + \int_v^t V_x(s,x(s))g(s,x(s))dB_s.$$

Taking the G-expectation on both sides and taking into consideration conditions (3.4) and (3.5), we obtain

$$\begin{aligned} 0 \leq \widehat{\mathbb{E}}(V(t,x(t))) - \widehat{\mathbb{E}}(V(v,x(v))) &\leq \int_v^t \widehat{\mathbb{E}}(LV(s,x(s)))ds \\ &\leq \int_v^t -\frac{c_3}{c_2}\widehat{\mathbb{E}}(V(s,x(s))) + \rho ds. \end{aligned}$$

In order to use Lemma 3.2, we need to prove that

$$t \mapsto \hat{\mathbb{E}}(V(t,x(t))) \text{ is continuous on } \mathbb{R}_+.$$

In fact, we denote by $F(t,\omega) = V(t,x(t))$. Then, from the representation of the sublinear expectation (2.1), it follows that,

$$\widehat{\mathbb{E}}\left(F(t,\omega)\right) = \sup_{p\in\mathcal{P}} \mathbb{E}_p\left(F(t,\omega)\right),$$

where $\mathbb{E}_p(F(t,\omega)) = \displaystyle\int_\Omega F(t,\omega)dP(\omega)$, for each $P \in \mathcal{P}$. Thus, our target now is to prove that

$$t \mapsto \int_\Omega F(t,\omega)dP(\omega) \text{ is continuous on } \mathbb{R}_+.$$

To this end, it is enough to prove that is continuous in $[0,T]$ for all $T > 0$.

Observe that:

For $\omega \in \Omega$ fixed, we have $t \in [0,T] \mapsto F(t,\omega)$ is continuous.

For $t \in \mathbb{R}_+$ fixed, we have $t \mapsto F(t,\omega)$ measurable.

Moreover, based upon condition (3.4), we have for $t \in [0,T]$

$$\begin{aligned} V(t, x(t,0,x_0)) &\leq c_2 |y(t,0,x_0)|^p \\ &\leq \sup_{t\in[0,T]} |y(t,0,x_0)|^p = G_T(\omega). \end{aligned}$$

It is obvious that,

$$\int_\Omega G_T(\omega) dP(\omega) < \infty.$$

Therefore, thanks to the standard results on continuity of integrals depending on parameters,

$$t \in [0,T] \mapsto \int_\Omega f(t,\omega) dP(\omega) \text{ is continuous.}$$

Eventually, we conclude that

$$t \mapsto \widehat{\mathbb{E}}(V(t,x(t))) \text{ is continuous on } \mathbb{R}_+.$$

Then, all assumptions of the Gronwall-lemma 3.2 are satisfied, and as a consequence

$$\begin{aligned} \widehat{\mathbb{E}}(V(t,x(t))) &\leq \frac{\rho c_2}{c_3} + \widehat{\mathbb{E}}(V(0,x_0)) e^{-\frac{c_3}{c_2}t} \\ &\leq \frac{\rho c_2}{c_3} + c_2 |y_0|^p e^{-\frac{c_3}{c_2}t}. \end{aligned}$$

Due to the fact that $|y_0| \leq |x_0|$ and condition (3.4), we deduce that

$$\widehat{\mathbb{E}}(|y(t)|^p) \leq \frac{c_2}{c_1} |x_0|^p e^{-\frac{c_3}{c_2}t} + \frac{\rho c_2}{c_1 c_3}.$$

Setting $\lambda_1 = \dfrac{c_2}{c_1}$ and $\lambda_2 = \dfrac{c_3}{c_2}$, we conclude that system (3.1) is globally practically uniformly pth moment exponentially stable with respect to y, with $r = \dfrac{\rho c_2}{c_3 c_1}$. $\square$

Theorem 3.3. *Let c_1, c_2, c_3 and p be nonnegative numbers and assume that there exist $V \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and a function ϕ fulfilling $e^{\frac{c_3}{c_2}t}\phi(t) \in \mathbb{L}^1([0,+\infty[)$, such that for all $t \geq t_0 \geq 0$, and all $x = (y,z) \in \mathbb{R}^n$,*

(i) $c_1\,|y|^p \le V(t,x) \le c_2\,|y|^p$,

(ii) $LV(t,x) \le -c_3|y|^p + \phi(t)$.

Moreover, we suppose that $z(t,t_0,x_0)$ is quasi surely globally uniformly bounded. Then, system (2.1) is globally practically uniformly pth moment exponentially stable with respect to y, with $r = \dfrac{1}{c_1}\displaystyle\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\phi(s)ds$.

In order to prove this theorem we need to recall the following lemma:

Lemma 3.4. *[18] Let $\eta \in M_\star^1([0,T])$ and $M_t = \displaystyle\int_0^t \eta(s)d\langle B\rangle_s - \int_0^t 2G(\eta(s))ds$. Then, for each $t\in[0,T]$, $\quad \widehat{\mathbb{E}}(M_t) \le 0$.*

Proof of Theorem 3.3. Applying G-Itô's formula 2.2 to $e^{\frac{c_3}{c_2}t}V(t,x(t))$, we obtain

$$
\begin{aligned}
&d\left(e^{\frac{c_3}{c_2}t}V(t,x(t))\right)\\
&= e^{\frac{c_3}{c_2}t}\left(\frac{c_3}{c_2}V(t,x(t)) + V_t(t,x(t)) + \langle V_x(t,x(t)), f(t,x(t))\rangle\right)dt\\
&+ e^{\frac{c_3}{c_2}t}\langle V_x(t,x(t)), h(t,x(t))\rangle d\langle B\rangle_t + e^{\frac{c_3}{c_2}t}\langle V_x(t,x(t)), g(t,x(t))\rangle dB_t\\
&+ \frac{1}{2}e^{\frac{c_3}{c_2}t}\langle V_{xx}(t,x(t))g(t,x(t)), g(t,x(t))\rangle d\langle B\rangle_t.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&e^{\frac{c_3}{c_2}t}V(t,x(t))\\
=&e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \int_{t_0}^t e^{\frac{c_3}{c_2}s}\Big[\frac{c_3}{c_2}V(s,x(s)) + V_s(s,x(s)) + \langle V_x(s,x(s)), f(s,x(s))\rangle\\
&+ G\left(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle\right)\Big]ds\\
&- \int_{t_0}^t e^{\frac{c_3}{c_2}s}G\left(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle\right)ds\\
&+ \int_{t_0}^t e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)), h(s,x(s))\rangle d\langle B\rangle_s + \int_{t_0}^t e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s\\
&+ \frac{1}{2}\int_{t_0}^t e^{\frac{c_3}{c_2}s}\langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle d\langle B\rangle_s\\
&= e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \int_{t_0}^t e^{\frac{c_3}{c_2}s}\Big(\frac{c_3}{c_2}V(s,x(s)) + LV(s,x(s))\Big)ds + M_t^{t_0}\\
&+ \int_{t_0}^t e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s,
\end{aligned}
$$

where

$$M_t^{t_0} = \int_{t_0}^{t} e^{\dfrac{c_3}{c_2}s} \langle V_x(s,x(s)), h(s,x(s))\rangle d\langle B\rangle_s$$
$$+ \frac{1}{2}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s} \langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle d\langle B\rangle_s$$
$$- \int_{t_0}^{t} e^{\frac{c_3}{c_2}s} G\left(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle\right) ds.$$

Taking, G-expectation on both sides, it follows that

$$\begin{aligned}
&\widehat{\mathbb{E}}\left(e^{\frac{c_3}{c_2}t}V(t,x(t))\right)\\
&\le e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) + LV(s,x(s))\right)\right) ds\\
&+ \widehat{\mathbb{E}}(M_t^{t_0}) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s\right).
\end{aligned}$$

On the other hand, since $\widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s\right) = 0$, and by Lemma 3.4, we obtain $\widehat{\mathbb{E}}(M_t^{t_0}) \le 0$. Consequently,

$$\widehat{\mathbb{E}}\left(e^{\frac{c_3}{c_2}t}V(t,x(t))\right) \le e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) + LV(s,x(s))\right) ds\right).$$

This, together with (3.4) and (3.5), implies

$$\begin{aligned}
\widehat{\mathbb{E}}\left(e^{\frac{c_3}{c_2}t}V(t,x(t))\right) &\le e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) - c_3|y(s)|^p + \phi(s)\right) ds\right).\\
&\le e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) - \frac{c_3}{c_2}V(s,x(s)) + \phi(s)\right) ds\right)\\
&\le e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\phi(s)ds\right).
\end{aligned}$$

Therefore, by inequality (i), we obtain

$$c_1\widehat{\mathbb{E}}\left(e^{\frac{c_3}{c_2}t}|y(t)|^p\right) \le c_2|y_0|^p e^{\frac{c_3}{c_2}t_0} + \widehat{\mathbb{E}}\left(\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\phi(s)ds\right).$$

Thus,

$$\begin{aligned}
\widehat{\mathbb{E}}(|y(t)|^p) &\le \frac{c_2}{c_1}|y_0|^p e^{-\frac{c_3}{c_2}(t-t_0)} + \frac{e^{-\frac{c_3}{c_2}t}}{c_1}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\phi(s)ds\\
&\le \frac{c_2}{c_1}|x_0|^p e^{-\frac{c_3}{c_2}(t-t_0)} + \frac{1}{c_1}\int_{0}^{+\infty} e^{\frac{c_3}{c_2}s}\phi(s)ds.
\end{aligned}$$

Setting $\lambda_1 = \frac{c_2}{c_1}$, $\lambda_2 = \frac{c_3}{c_2}$, and $r = \frac{1}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\phi(s)ds$, we conclude that system (3.1) is globally practically uniformly pth moment exponentially stable with respect to y. □

4 Global practical uniform exponential stability of G-stochastic differential equation with respect to part of the variables

In this section, we are interested in studying the global practical uniform exponential stability with respect to a part of the variables of stochastic differential systems driven by G-Brownian motion, based on G-Lyapunov's techniques.

In the sequel, we will suppose that B_t is a one-dimensional G-Brownian motion, with $B_t \sim N(0, [\underline{\sigma}^2 t, \bar{\sigma}^2 t])$.

Definition 4.1. *i*) The ball B_r is said to be quasi surely globally uniformly exponentially stable with respect to y, if for any x_0 such that $0 < |y(t, t_0, x_0)| - r$ for all $t \geq 0$, it holds that

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln\left(|y(t, t_0, x_0)| - r\right) < 0, \text{ q.s.} \tag{4.1}$$

ii) The system (3.1) is said to be quasi surely globally practically uniformly exponentially stable with respect to y, if there exists $r > 0$ such that B_r is quasi surely globally uniformly exponentially stable with respect to y.

Before we establish the main result in this section, we need to prove the following lemma, which has its own importance.

Lemma 4.1. *For all $x_0 \in \mathbb{R}^n$ such that $x_0 \neq 0$ it holds*

$$\hat{C}\left(x(t, t_0, x_0) \neq 0, \quad \forall 0 \geq t \geq T\right) = 1. \tag{4.2}$$

That is, almost all the sample paths of any solution starting from a non-zero state will never reach the origin.

Proof. Arguing by contradiction, if the lemma were false, there would exist some $x_0 \neq 0$ such that $\hat{C}\{\tau < \infty\} > 0$, where

$$\tau = \inf\{t : x(t) = 0\} \wedge T,$$

which satisfies $\{\tau \leq t\} \in \mathcal{F}_t$. We can find a pair of constants $\widetilde{T} > 0$ and $\theta > 1$ sufficiently large, such that $\hat{C}(B) > 0$, where

$$B = \{\tau \leq \widetilde{T} \quad \text{and} \quad |x(t)| \leq \theta - 1, \quad \text{for all} \quad 0 \leq t \leq \tau\}.$$

However, by the standing hypotheses, there exists a positive constant K_θ, such that

$$|f(t,x)| + |h(t,x)| + |g(t,x)| \le K_\theta |x|, \quad \text{for all } |x| \le \theta, \quad 0 \le t \le \widetilde{T}.$$

Let $V(t,x) = |x|^{-1}$. Then by the G-Itô's formula 2.2, for $0 \le |x| \le \theta$ and $0 \le t \le \widetilde{T}$, it follows that

$$\begin{aligned}
LV(t,x) &= V_x f(t,x) + G\left(\langle V_x, 2h\rangle + \langle V_{xx} g(t,x), g(t,x)\rangle\right) \\
&= -|x|^{-3} x^T f(t,x) + G\left(-2|x|^{-3} x^T h(t,x) + (-|x|^{-3}|g(t,x)|^2 + 3|x|^{-5}|x^T g(t,x)|^2)\right) \\
&\le |x|^{-2}|f(t,x)| + G\left(2|x|^{-2}|h(t,x)| + 2|x|^{-3}|g(t,x)|^2\right) \\
&\le |x|^{-2}|f(t,x)| + \bar{\sigma}^2|x|^{-2}|h(t,x)| + \bar{\sigma}^2|x|^{-3}|g(t,x)|^2 \\
&\le K_\theta|x|^{-1} + K_\theta\bar{\sigma}^2|x|^{-1} + K_\theta^2\bar{\sigma}^2|x|^{-1} \\
&= K_\theta(1 + \bar{\sigma}^2 + K_\theta\bar{\sigma}^2)|x|^{-1} = \alpha V(t,x),
\end{aligned}$$

where $\alpha = K_\theta(1 + \bar{\sigma}^2 + K_\theta\bar{\sigma}^2\big)$.

Now, for any $\varepsilon \in]0, |x_0|[$, define the stopping time

$$\tau_\varepsilon = \inf\{t : |x(t)| \notin]\varepsilon, \theta[\} \wedge T.$$

Applying G-Itô's formula 2.2, we obtain

$$\begin{aligned}
& e^{-\alpha(\tau_\varepsilon \wedge \widetilde{T})} V(\tau_\varepsilon \wedge \widetilde{T}, x(\tau_\varepsilon \wedge \widetilde{T})) \\
=& V(0,x_0) + \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\Big[-\alpha V(s,x(s)) + V_s(s,x(s)) + \langle V_x(s,x(s)), f(s,x(s))\rangle \\
& + G\left(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s)) g(s,x(s)), g(s,x(s))\rangle\right)\Big] ds \\
& - \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s} G\left(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s)) g(s,x(s)), g(s,x(s))\rangle\right) ds \\
& + \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_x(s,x(s)), h(s,x(s))\rangle d\langle B\rangle_s + \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s \\
& + \frac{1}{2}\int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_{xx}(s,x(s)) g(s,x(s)), g(s,x(s))\rangle d\langle B\rangle_s \\
& = V(0,x(0)) + \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\left(-\alpha V(s,x(s)) + LV(s,x(s))\right) ds + M_t^0 \\
& + \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_x(s,x(s)), g(s,x(s))\rangle dB_s,
\end{aligned}$$

where

$$
\begin{aligned}
M_t^0 = &\int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_x(s,x(s)), h(s,x(s))\rangle d\langle B\rangle_s \\
&+ \frac{1}{2}\int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s}\langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle d\langle B\rangle_s \\
&- \int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha s} G\Big(\langle V_x(s,x(s)), 2h(s,x(s))\rangle + \langle V_{xx}(s,x(s))g(s,x(s)), g(s,x(s))\rangle\Big) ds.
\end{aligned}
$$

Taking, G-expectation on both sides,

$$
\begin{aligned}
&\widehat{\mathbb{E}}\left(e^{-\alpha(\tau_\varepsilon \wedge \widetilde{T})} V(\tau_\varepsilon \wedge \widetilde{T}, x(\tau_\varepsilon \wedge \widetilde{T}))\right) \\
&\leq V(0,x_0) + \widehat{\mathbb{E}}\left(\int_0^{\tau_\varepsilon \wedge \widetilde{T}} e^{-\alpha(\tau_\varepsilon \wedge \widetilde{T})}\left(-\alpha V(s,x(s)) + LV(s,x(s))\right) ds\right) + \widehat{\mathbb{E}}(M_t^0).
\end{aligned}
$$

By Lemma 3.4, we have $\widehat{\mathbb{E}}(M_t^0) \leq 0$. Then, we obtain

$$
\widehat{\mathbb{E}}(e^{-\alpha(\tau_\varepsilon \wedge \widetilde{T})} V(\tau_\varepsilon \wedge T, x(\tau_\varepsilon \wedge \widetilde{T}))) \leq V(0,x_0) = |x_0|^{-1}.
$$

Note that for $\omega \in B$, this implies $\tau_\varepsilon \leq \widetilde{T}$ and $|x(\tau_\varepsilon)| = \varepsilon$. Therefore, the above inequality involves that

$$
\widehat{\mathbb{E}}\left(\exp\left\{-\alpha \widetilde{T}\right\} \varepsilon^{-1} \mathbb{1}_B\right) \leq |x_0|^{-1}.
$$

Hence,

$$
\hat{C}(B) \leq \varepsilon e^{\alpha \widetilde{T}} |x_0|^{-1}.
$$

Letting $\varepsilon \longrightarrow 0$ gives $\hat{C}(B) = 0$, but this contradicts the definition of B.
The proof is complete. □

We need to recall the following lemma, which will be useful in our analysis.

Lemma 4.2. *[33] Let B_t be a one-dimensional G-Brownian motion, suppose that there exist constants $\varepsilon > 0$ and $\alpha > 0$, such that*

$$
\widehat{\mathbb{E}}\left(\exp\left[\frac{\alpha^2}{2}(1+\varepsilon)\int_0^T g^2(s) d\langle B\rangle_s\right]\right) < \infty.
$$

Then, for any $T > 0$ and $\beta > 0$,

$$
\hat{C}\left(\sup_{0 \leq t \leq T}\left[\int_0^t g(s) dB_s - \frac{\alpha}{2}\int_0^t g^2(s) d\langle B\rangle_s\right] > \beta\right) \leq \exp(-\alpha\beta).
$$

Now, we can establish and prove our main result in this section.

Theorem 4.3. *Assume that there exist a function $V \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and constants $p \in \mathbb{N}^*$, $\beta_1 \geq 1$, $\gamma \geq \beta_1$, $\zeta \geq 0$ and $\beta_2 \in \mathbb{R}$, $\beta_3 \geq 0$, such that for all $t \geq t_0 \geq 0$, and $x = (y, z) \in \mathbb{R}^n$,*

1. $\beta_1 |y|^p \leq V(t, x)$,
2. $V_t(t, x) + V_x(t, x) f(t, x) \leq \beta_2 V(t, x)$,
3. $V_x(t, x) h(t, x) \leq \gamma$,
4. $|V_x(t, x) g(t, x)|^2 \geq \beta_3 V^2(t, x) + \zeta$.

Moreover, we suppose that $z(t, t_0, x_0)$ is quasi surely globally uniformly bounded.

Then,

$$\lim_{t \to +\infty} \sup \frac{1}{t} \ln \left(|y(t, t_0, x_0)| - \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right) \leq - \frac{[\underline{\sigma}^2 \beta_3 - 2 (\beta_2 + \bar{\sigma}^2)]}{2} \quad q.s.$$

In particular, if $\underline{\sigma}^2 \beta_3 > 2(\beta_2 + \bar{\sigma}^2)$, then system (3.1) is said to be quasi surely globally practically uniformly exponentially stable with respect to y, with $r = \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}}$.

Proof. Fix $x_0 \neq 0$ in $\mathbb{R}^n$. By Lemma 4.1, $x(t) \neq 0$, for all $t \geq 0$ quasi surely.

Suppose that $\beta_1 |y(t)|^p > \gamma$, then we have

$$\begin{aligned} \beta_1 |y(t)|^p - \gamma &= \beta_1 \left(|y(t)|^p - \frac{\gamma}{\beta_1} \right) \\ &= \beta_1 \left(|y(t)|^p - \left(\left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right)^p \right). \end{aligned}$$

Based on the following inequality,

$$a^p - b^p = (a - b)(a^{p-1} + a^{p-2} b + a^{p-3} b^2 + ... + a^0 b^{p-1}),$$

we deduce

$$\begin{aligned} \beta_1 |y(t)|^p - \gamma &= \beta_1 \left(|y(t)| - (\frac{\gamma}{\beta_1})^{\frac{1}{p}} \right) \left(|y(t)|^{p-1} + |y(t)|^{p-2} \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} + ... + \left(\frac{\gamma}{\beta_1} \right)^{\frac{p-1}{p}} \right) \\ &\geq \beta_1 \left(|y(t)| - \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right) \left(\frac{\gamma}{\beta_1} \right)^{\frac{p-1}{p}}. \end{aligned}$$

Since $\beta_1 \geq 1$ and $\gamma \geq \beta_1$, we obtain

$$V(t, x(t)) \geq \beta_1 |y(t)|^p \geq \beta_1 |y(t)|^p - \gamma \geq \beta_1 \left(|y(t)| - \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right) \left(\frac{\gamma}{\beta_1} \right)^{\frac{p-1}{p}} \geq \left(|y(t)| - \left(\frac{\gamma}{\beta_1} \right)^{\frac{1}{p}} \right).$$

Thus,

$$\left(|y(t)| - \left(\frac{\gamma}{\beta_1}\right)^{\frac{1}{p}}\right) \leq V(t, x(t)),$$

and

$$\ln\left(|y(t)| - \left(\frac{\gamma}{\beta_1}\right)^{\frac{1}{p}}\right) \leq \ln\big(V(t, x(t))\big), \quad \forall t \geq 0.$$

The application of G-Itô's formula 2.2 leads to, for all $t \geq 0$,

$$\begin{aligned} d\big(\ln(V(t, x(t)))\big) &= \frac{V_t(t, x(t)) + V_x(t, x(t))f(t, x(t))}{V(t, x(t))}dt + \frac{V_x(t, x(t))g(t, x(t))}{V(t, x(t))}dB_t \\ &+ \frac{V_x(t, x(t))h(t, x(t))}{V(t, x(t))}d\langle B\rangle_t - \frac{1}{2}\frac{|V_x(t, x(t))g(t, x(t))|^2}{V^2(t, x(t))}d\langle B\rangle_t. \end{aligned}$$

That is,

$$\begin{aligned} &\ln(V(t, x(t))) = \ln(V(0, x(0))) + \int_0^t \frac{V_s(s, x(s)) + V_x(s, x(s))f(s, x(s))}{V(s, x(s))}ds + \\ &\int_0^t \frac{V_x(s, x(s))g(s, x(s))}{V(s, x(s))}dB_s + \int_0^t \frac{V_x(s, x(s))h(s, x(s))}{V(s, x(s))}d\langle B\rangle_s - \frac{1}{2}\int_0^t \frac{|V_x(s, x(s))g(s, x(s))|^2}{V^2(s, x(s))}d\langle B\rangle_s. \end{aligned} \tag{4.3}$$

Assign $\varepsilon \in]0, 1[$ arbitrarily and let $n = 1, 2, ...$ It follows from Lemma 4.2 that,

$$\widehat{C}\left\{\sup_{0\leq t\leq n}\left[\int_0^t \frac{V_x(s, x(s))g(s, x(s))}{V(s, x(s))}dB_s - \frac{\varepsilon}{2}\int_0^t \frac{(V_x(x(s), s)g(x(s), s))^2}{V^2(x(s), s)}d\langle B\rangle_s\right] > \frac{2}{\varepsilon}\ln(n)\right\} \leq \frac{1}{n^2}.$$

Using the Borel-Cantelli lemma (Lemma 2.3) for capacity, we deduce that there exists $n_0 \in \mathbb{N}$, such that

$$\int_0^t \frac{V_x(s, x(s))g(s, x(s))}{V(s, x(s))}dB_s \leq \frac{2}{\varepsilon}\ln(n) + \frac{\varepsilon}{2}\int_0^t \frac{\left(V_x(x(s), s)g(x(s), s)\right)^2}{V^2(x(s), s)}d\langle B\rangle_s, \text{ for all } n \geq n_0,\ 0 \leq t \leq n.$$

Combining this result together with (4.3), we obtain

$$\begin{aligned} \ln(V(t, x(t))) \leq \ln(V(0, x(0))) &+ \int_0^t \frac{V_s(s, x(s)) + V_x(s, x(s))f(s, x(s))}{V(s, x(s))}ds \\ &+ \int_0^t \frac{V_x(s, x(s))h(s, x(s))}{V(s, x(s))}d\langle B\rangle_s + \frac{2}{\varepsilon}\ln(n) \\ &+ \frac{\varepsilon - 1}{2}\int_0^t \frac{\left(V_x(s, x(s))g(s, x(s))\right)^2}{V^2(s, x(s))}d\langle B\rangle_s, \quad 0 \leq t \leq n,\ n \geq n_0, \quad \text{q.s.} \end{aligned}$$

By Peng [22, Chapter III], we have for each $0 \le s \le t \le T$,

$$\underline{\sigma}^2(t-s) \le \langle B\rangle_t - \langle B\rangle_s \le \bar{\sigma}^2(t-s).$$

Based on this fact, we obtain

$$\begin{aligned}\ln(V(t,x(t))) &\le \ln(V(0,x(0))) + \int_0^t \frac{V_s(s,x(s)) + V_x(s,x(s))f(s,x(s))}{V(s,x(s))}ds \\ &+ \bar{\sigma}^2 \int_0^t \frac{V_x(s,x(s))h(s,x(s))}{V(s,x(s))}ds + \frac{2}{\varepsilon}\ln(n) \\ &+ \frac{\varepsilon - 1}{2}\underline{\sigma}^2 \int_0^t \frac{\left(V_x(s,x(s))g(s,x(s))\right)^2}{V^2(s,x(s))}ds, \quad 0 \le t \le n,\ n \ge n_0, \quad \text{q.s.}\end{aligned}$$

Taking into account the assumptions, it follows that

$$\begin{aligned}\ln(V(t,x(t))) &\le \ln(V(0,x(0))) + \int_0^t \frac{\beta_2 V(s,x(s)) + \bar{\sigma}^2\gamma}{V(s,x(s))}ds + \frac{2}{\varepsilon}\ln(n) \\ &+ \frac{\varepsilon - 1}{2}\underline{\sigma}^2 \int_0^t \frac{\beta_3 V^2(s,x(s)) + \zeta}{V^2(s,x(s))}ds \\ &\le \ln(V(0,x(0))) + \beta_2 t + \int_0^t \frac{\bar{\sigma}^2\gamma}{\beta_1|y(s)|^p}ds + \frac{2}{\varepsilon}\ln(n) \\ &- \frac{1-\varepsilon}{2}\underline{\sigma}^2\beta_3 t - \frac{1-\varepsilon}{2}\underline{\sigma}^2 \int_0^t \frac{\zeta}{V^2(s,x(s))}ds \\ &\le \ln(V(0,x(0))) + (\beta_2 + \bar{\sigma}^2)t + \frac{2}{\varepsilon}\ln(n) - \frac{1-\varepsilon}{2}\underline{\sigma}^2\beta_3 t - \frac{1-\varepsilon}{2}\underline{\sigma}^2 \int_0^t \frac{\zeta}{V^2(s,x(s))}ds,\end{aligned}$$

for all $0 \le t \le n,\ n \ge n_0,$ q.s.

Since $0 < \varepsilon < 1$, the following inequality holds

$$-\frac{1-\varepsilon}{2}\underline{\sigma}^2 \int_0^t \frac{\zeta}{V^2(s,x(s))}ds \le 0.$$

As a consequence, we obtain

$$\ln(V(t,x(t))) \le \ln(V(0,x(0))) + (\beta_2 + \bar{\sigma}^2)t + \frac{2}{\varepsilon}\ln(n) - \frac{1-\varepsilon}{2}\underline{\sigma}^2\beta_3 t, \quad 0 \le t \le n,\ n \ge n_0, \quad \text{q.s.}$$

Then, it follows that

$$\frac{1}{t}\ln(V(t,x(t))) \le -\frac{1}{2}\left[(1-\varepsilon)\underline{\sigma}^2\beta_3 - 2(\beta_2 + \bar{\sigma}^2)\right] + \frac{\ln(V(0,x(0))) + \frac{2}{\varepsilon}\ln(n)}{n-1}.$$

The inequality above implies that,

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln(V(t,x(t))\leq-\frac{1}{2}\left[(1-\varepsilon)\underline{\sigma}^2\beta_3-2(\beta_2+\bar{\sigma}^2)\right],\quad \text{q.s.}$$

Putting $\varepsilon\longrightarrow 0$ gives,

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln\left(|y(t)|-\left(\frac{\gamma}{\beta_1}\right)^{\frac{1}{p}}\right)\leq-\frac{[\underline{\sigma}^2\beta_3-2(\beta_2+\bar{\sigma}^2)]}{2},\quad \text{q.s.}$$

If the inequality $\underline{\sigma}^2\beta_3>2(\beta_2+\bar{\sigma}^2)$ is satisfied, then system (3.1) is quasi surely globally practically uniformly exponentially stable with respect to y. □

5 Example

Consider the following G-stochastic system:

$$\begin{cases} dx_1(t)=-2x_1(t)dt+\dfrac{1}{2}\sin^2(x_2)e^{-2t}x_1d\langle B\rangle_t+(1+e^{-t})^{\frac{1}{2}}dB_t \\ dx_2(t)=-2x_2(t)dt+\dfrac{1}{2}\cos^2(x_1)e^{-t}x_2d\langle B\rangle_t+\dfrac{1}{\sqrt{\operatorname{ch}(t)}}dB_t \\ dx_3(t)=Ax_3(t)dt, \end{cases} \tag{5.1}$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^4$, $x_3=(z_1,z_2)\in\mathbb{R}^2$, B_t is a one-dimensional G-Brownian motion and $B_t\sim N(0,[\frac{1}{2},1])$,

$$A_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

with initial value $x_0=(x_{1_0},x_{2_0},x_{3_0})$, and $x_{3_0}=(z_{1_0},z_{2_0})$.

By simple computations, we obtain

$$x_3(t)=\begin{pmatrix} z_{1_0}(t,w)\cos(t)-z_{2_0}(t,w)\sin(t)\\ z_{1_0}(t,w)\sin(t)+z_{2_0}(t,w)\cos(t)\end{pmatrix}.$$

Using the Euclidean norm leads to

$$|x_3(t)|=|x_{3_0}|.$$

It is clear that $x_3(t)$ is quasi surely globally uniformly bounded. In fact, for all $\alpha'>0$ (independent of t_0), all $t\geq t_0\geq 0$, and all $x_{3_0}\in\mathbb{R}^2$ with $|x_{3_0}|\leq\alpha'$, we have $|x_3(t)|\leq\alpha'$, q.s.

Let consider the following G-Lyapunov-like function:

$$V(t,x)=x_1^2+x_2^2.$$

It is easy to verify that,

$$LV(t,x) = -4x_1^2 - 4x_2^2 + G\left(2\sin^2(x_2)e^{-2t}x_1^2 + 2\cos^2(x_1)e^{-t}x_2^2 + 2(1+e^{-t}) + 2\frac{1}{\mathrm{ch}(t)}\right).$$

By the sub-additivity of the function G, we obtain

$$\begin{aligned} LV(t,x) &\leq -4x_1^2 - 4x_2^2 + G\left(2\sin^2(x_2)e^{-2t}x_1^2\right) + G\left(2\cos^2(x_1)e^{-t}x_2^2\right) + G\left(2(1+e^{-t})\right) + G\left(2\frac{1}{\mathrm{ch}(t)}\right) \\ &\leq -4x_1^2 - 4x_2^2 + x_1^2\sin^2(x_2)e^{-2t} + \cos^2(x_1)e^{-t}x_2^2 + (1+e^{-t}) + \frac{1}{\mathrm{ch}(t)} \\ &\leq -3x_1^2 - 3x_2^2 + 3. \end{aligned}$$

That is,

$$LV(t,x) \leq -4x_1^2 - 4x_2^2 + x_1^2 + x_2^2 + 3.$$

Thus, the constants in Theorem 3.1 become $c_1 = c_2 = 1$, $c_3 = \rho = 3$, $p = 2$, and all conditions of Theorem 3.1 are satisfied, hence system (5.1) is globally practically uniformly pth moment exponentially stable with respect to (x_1, x_2), with $r = 1$.

6 Conclusion

In this paper, practical stability with respect to a part of the variables procedure for certain classes of nonlinear stochastic differential equations driven by G-Brownian motion was developed. Some stability criteria are established. Then, using G-Lyapunov techniques, the notion of practical stability with respect to a part of the variables is studied for G-SDEs. In this context, some sufficient conditions are presented to ensure the desired stability property. An example has been introduced to validate the developed methods.

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