Stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion

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Abstract

In this paper, we investigate the pth moment exponential stability of stochastic differential equations driven by G-Brownian motion (G-SDEs) with respect to a part of the variables by means of the G-Lyapunov functions and recently developed Itô's calculus for SDEs driven by G-Brownian motion, as well as Gronwall's inequalities. We establish sufficient conditions to ensure the quasi sure exponential stability of stochastic differential equations perturbed by G-Brownian motion with respect to a part of the variables. Some illustrative examples to show the usefulness of the stability with respect to a part of the variables notion are also provided.

Keywords: G-Stochastic differential equations, G-Itô formula, G-Brownian motion, Pth moment exponential stability with respect to a part of the variables, Quasi sure exponential stability with respect to a part of the variables .

Mathematics Subject Classification Primary 93E03 · Secondary 60H10.

*The research of Tomás Caraballo has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00, and by Junta de Andalucía (Consejería de Economía y Conocimiento) and FEDER under projects US-1254251 and P18-FR-4509.

1 Introduction

Stability of stochastic differential equations (SDEs) has become a very prevalent theme of recent research in Mathematics and its applications. Stochastic systems are used to model problems from the real world in which some kind or randomness or noise must be taken into account.
Some stochastic models cannot be proved to fulfill stability properties with respect to all the unknown variables of the system. However, it is very interesting in some situations to analyze if it is still possible to prove some stability properties with respect to some of the variables in the problem. It is worth mentioning that, recently considerable attention has been paid to the concept of stability with respect to a part of the system states. Such concept arises from the study of combustion systems [1], vibrations in rotating machinery [14], biocenology [24], inertial navigation systems [29], electro-magnetics [35], and spacecraft stabilization via gimballed gyroscopes and/or flywheels [31].

The method of Lyapunov functions is one of the most powerful tool to study the stability of stochastic dynamical systems. Lyapunov stability of stochastic dynamical systems has attracted the attention of several authors, we would like to mention here the references [4, 5, 10, 19], among others.

With the emergence of the second method of Lyapunov as an essential means in science, engineering, and applied mathematics, numerous exciting and important variants to Lyapunov's original concept stability were proposed. One of these involves the notion of stability with respect to a part of the variables, Peiffer and Rouche [21], Rouche et al. [24], Rumyantsev [25], Rumyantsev and Oziraner [26], Savchenko and Ignatyev [30], Vorotnikov [31], Vorotnikov and Rumyantsev [32]. This type of stability has been used in investigating the qualitative properties of equilibria and boundedness properties of motions of dynamical systems determined by ordinary differential equations, difference equations, functional differential equations, stochastic differential equations driven by standard Brownian motion, etc. It involves a notion of stability with respect to only a prespecified subset of the state variables characterizing the motions of the system under investigation.

For the dormant applications in risk measures, and superhedging in finance, considerable attention has been paid to the theory of nonlinear expectation. Notably, Peng [22] built the fundamental theory of time-consistent G-expectation and G-conditional expectation, where G is the infinitesimal generator of a nonlinear heat equation. Under the G-framework, Peng [20, 22] introduced the notion of G-normal disribution, G-Brownian motion and he also established the corresponding stochastic calculus of Itô's type. Since then, many researches have been carried out on the stochastic analysis with respect to the G-Brownian motion. On that basis, Gao [9] and Peng [20] studied the existence and uniqueness of the solution of G-SDE under a standard Lipschitz condition. Moreover, Lin [17] obtained the existence and uniqueness of the solution of G-SDE with reflecting boundary. The G-Brownian motion has a very rich and interesting new structure which non-trivially, for a recent account and development of this theory we refer the reader to see [2],[16], and [15].

Stability analysis of G-SDEs became increasingly significant since it has been extensively used in many branches of science. The corresponding study of the stability properties of solutions has received much attention during the last decades, see [28, 33, 34].

Several works concentrated on the study of stability with respect to a part of variables of stochastic differential equations driven by standard Brownian motion. However, to the best of our knowledge, no work has been brought about stability with respect to a part of variables of stochastic differential equations driven by G-Brownian motion (GSDEs, in short). Consequently, this paper is devoted to establishing some criteria for the pth moment exponential stability and the quasi sure exponential stability with respect to a part of the variables of GSDEs by using the method of Lyapunov's functions and recently developed Itô's calculus for SDEs driven by G-Brownian motion as well as Gronwall inequalities.

The content of this paper is organized as follows: In Section 2, we recall some necessary preliminaries and results in G-framework. In Section 3, we establish sufficient conditions to ensure pth moment exponential stability and quasi sure exponential stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion by using the G-Lyapunov techniques. In Section 4, we give sufficient conditions of quasi sure exponential stability with respect to a part of the variables of G-stochastic perturbed systems based on Gronwall's inequalities and recently developed Itô's calculus for GSDEs. In Section 5, the obtained results are used to analyze the stability with respect to a part of the variables of G-stochastic systems and spring-mass-damper system. Finally, in Section 6, some conclusions are included.

2 Preliminaries

In this section, we briefly recall some notations and preliminaries about sublinear expectations and G-Brownian motion. For more details, one can refer to see [13, 20, 22, 23].

Notations on G-stochastic calculus

- $\mathbb{R}^n$: the space of $n-$dimensional real column vectors.
- $\langle x, y\rangle$: the scalar product of two vectors $x, y \in \mathbb{R}^n$.
- If $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.
- $\mathcal{B}(\Omega)$: the Borel $\sigma-$algebra of Ω.
- $\mathcal{C}_{b,Lip(\mathbb{R}^n)}$: the space of all bounded real-valued Lipschitz continuous functions.
- $\mathbb{L}^0$: the space of all $\mathcal{B}(\Omega)-$measurable real functions.
- $B_0(\omega)$: all bounded elements in $\mathbb{L}^0(\omega)$.
- $\mathbb{L}^p_G(\Omega)$: Banach space under the natural norm $||X||^p = \widehat{\mathbb{E}}(|X|^p)^{\frac{1}{p}}$.

- $M_G^{p,0}(0,T) = \{\zeta := \zeta_t(\omega) = \sum_{j=0}^{N-1} \zeta_j \mathbb{1}_{[t_j,t_{j+1})}(t), \quad \forall N > 0,$
$$0 = t_0 < ... < t_N = T,\ \zeta_j \in \mathbb{L}_G^p(\omega_{t_j}),\ j = 0,1,2,..,N-1\}.$$

- $M_G^p(0,T)$: the completion of $M_G^{p,0}$ under $||\eta||_{M_G^p} = |\int_0^T \widehat{\mathbb{E}}[|\eta(t)|^p]dt|^{\frac{1}{p}}$.

- $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$.

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We further suppose that $\mathcal{H}$ satisfies $a \in \mathcal{H}$ for each constant a and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$.

Definition 2.1. A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,
i) Monotonicity: if $X \geq Y$, then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$.
ii) Constant preserving: $\widehat{\mathbb{E}}[a] = a,\ \forall\ a \in \mathbb{R}$.
iii) Sub-additivity: $\widehat{\mathbb{E}}[X+Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$.
iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. $Y = (Y_1, ..., Y_n)$, where $Y_i \in \mathcal{H}$ is called a n-dimensional random vector in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.

Definition 2.2. Weakly compact sets are defined by sets which are compact with respect to the weak topology of a Banach space.

The representation of a sublinear expectation can be expressed as a supermum of linear expectations.

Theorem 2.1. *[22] There exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$, such that*
$$\widehat{\mathbb{E}}(X) = \sup_{p \in \mathcal{P}} \mathbb{E}_p(X), \quad X \in \mathbb{L}_G^1(\Omega).$$

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, an n-dimensional random vector $Y = (Y_1, .., Y_n) \in \mathcal{H}$ is said to be independent from an m-dimensional random vector $X = (X_1, .., X_m) \in \mathcal{H}$ under the sublinear expectation $\widehat{\mathbb{E}}$, if for any test function $\varphi \in \mathcal{C}_{b,Lip}(\mathbb{R}^{m+n})$
$$\widehat{\mathbb{E}}(\varphi(X,Y)) = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x,Y)]|_{x=X}].$$

Definition 2.4. Let X_1 and X_2 be two n-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if
$$\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)], \quad \forall \varphi \in \mathcal{C}_{b,Lip(\mathbb{R}^n)}.$$

$\bar{X}$ is said to be an independent copy of X if $\bar{X} \overset{d}{=} X$ and $\bar{X}$ is independent from X.

Definition 2.5. (G-Normal Distribution) A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called G-normal distributed, denoted by $X \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, for a given pair $0 \leq \bar{\sigma} \leq \bar{\sigma}$, if for any $a, b \geq 0$

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of X.

Let Ω be the space of all $\mathbb{R}^d-$valued continuous paths $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$. We assume moreover that Ω is a metric space equipped with the following distance:

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} \left(\max_{0\leq t\leq N} (|\omega_t^1 - \omega_t^2|) \wedge 1 \right),$$

and consider the canonical process $B_t(\omega) = \omega_t,\ t \in [0, +\infty),$ for $\omega \in \Omega$; then for each fixed $T \in [0, +\infty)$, we have

$$L_{ip}^0(\Omega_T) := \{\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}) :\ n \geq 1,\ 0 \leq t_1 \leq ... \leq t_n \leq T,\ \varphi \in C_{b,lip}(\mathbb{R}^{d\times n})\}.$$

Definition 2.6. On the sublinear expectation space $(\Omega, L_{ip}^0(\Omega_T), \widehat{\mathbb{E}})$, the canonical process $(B_t)_{t\geq 0}$ is called a G-Brownian motion, if the following properties are satisfied:
(i) $B_0 = 0$.
(ii) for $t, s \geq 0$, the increment $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$, where X is G-normal distributed.
(iii) for $t, s \geq 0$, the increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ for each $n \in \mathbb{N}$, and $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$.
Moreover, the sublinear expectation $\widehat{\mathbb{E}}[.]$ is called G-expectation.

Remark 2.2. The standard Brownian motion is a particular case of G-Brownian motion. Then, for $\bar{\sigma}^2 = \underline{\sigma}^2 = 1$ the G-Brownian motion is degenerate to the standard Brownian motion.

For simplicity, let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion. The letter G denotes the function

$$G(a) := \frac{1}{2}\widehat{\mathbb{E}}[aB_1^2] = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R},$$

with $\underline{\sigma}^2 := -\widehat{\mathbb{E}}[-B_1^2] \leq \widehat{\mathbb{E}}[B_1^2] := \overline{\sigma}^2,\ 0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$. Recall that $a^+ = \max\{0, a\}$ and $a^- = -\min\{0, a\}$.

Now, we introduce the natural choquet capacity.

Definition 2.7. Let $\mathcal{B}(\Omega)$ the Borel $\sigma-$algebra and $\mathcal{P}$ be a weakly compact collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$, then the capacity $\hat{C}(.)$ associated to $\mathcal{P}$ is defined by

$$\hat{C}(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 2.8. A set $A \subset \mathcal{B}(\Omega)$ is polar if $\hat{C}(A) = 0$. A property holds "quasi-surely" (q.s.) if it holds outside a polar set.

Now, we recall the following lemmas, which will be crucial in our stability analysis.

Lemma 2.3. *[8] Let $\{A_k\} \subset \mathcal{B}(\Omega)$ such that*

$$\sum_{k=1}^{\infty} \hat{C}(A_k) < \infty.$$

Then, $\lim_{k\to\infty} \sup A_k$ is polar.

Lemma 2.4. *[8] Let $X \in \mathbb{L}^0(\Omega)$ satisfying $\widehat{\mathbb{E}}|X|^p < \infty$, for $p > 0$. Then, for each $M > 0$,*

$$\hat{C}(|X| > M) \leq \frac{\widehat{\mathbb{E}}|X|^p}{M^p}.$$

Lemma 2.5. *[9] For each $p \geq 1$, $\eta \in M_G^p(0,T)$, and $0 \leq s \leq t \leq T$.*

$$\widehat{\mathbb{E}}\left[\sup_{s\leq u\leq t} \left|\int_s^u \eta_r d\langle B^a, B^{\bar{a}}\rangle_r\right|^p\right] \leq \left(\frac{\sigma_{(a+\bar{a})+(a+\bar{a})^T} + \sigma_{(a-\bar{a})+(a-\bar{a})^T}}{4}\right)^p |t-s|^{p-1} \int_s^t \widehat{\mathbb{E}}[|\eta_u|^p]du.$$

Lemma 2.6. *[9] Let $p \geq 2$, $\eta \in M_G^p(0,T)$ and $0 \leq s \leq t \leq T$. Then, there exists some constant C_p depending only on p, such that*

$$\widehat{\mathbb{E}}\left[\sup_{s\leq u\leq t} \left|\int_s^u \eta_r dB_r\right|^p\right] \leq C_p|t-s|^{\frac{p}{2}-1} \int_s^t \widehat{\mathbb{E}}[|\eta_u|^p]du.$$

3 P-th moment exponential stability of G-stochastic differential equations with respect to a part of the variables

Several authors dealt with the problem of stability with respect to a part of variables of stochastic differential equations driven by standard Brownian motion in the sense of Lyapunov, see for instance [6, 11, 12, 31]. The novelty in our work is to tackle the problem of stability with respect to a part of variables of stochastic differential equations driven by G-Brownian motion by using the method of G-Lyapunov's function and recently developed Itô's calculus for SDE driven by G-Brownian motion.

Consider the following SDE driven by an m-dimensional G-Brownian motion:

$$dx(t) = f(t, x(t))dt + h(t, x(t))d\langle B\rangle_t + g(t, x(t))dB_t, \quad \forall x \in \mathbb{R}^n, \quad t \geq 0, \tag{3.1}$$

where $B_t = (B_1(t), ..., B_m(t))^T$ is an m-dimensional G-Brownian motion, and $(\langle B\rangle)_{t\geq 0}$ is the quadratic variation process of B.
For the well-posedness of system (3.1), we assume that f, h and $g \in M_G^2([0,T], \mathbb{R}^n)$ satisfy the following global Lipschitz condition:

$$|\phi(t,x) - \phi(t,\widetilde{x})| \leq K|x - \widetilde{x}|, \quad \forall t \in [0,T], \quad x, \widetilde{x} \in \mathbb{R}^n,$$

$\phi = f,\ h,\ g$ respectively, and K is a positive constant. For the purpose of stability, we further assume that $f(t,0) = h(t,0) = g(t,0) = 0,\ \forall t \geq 0$.

Denote $x = (x_1, x_2)^T$ where $x = (x_{1.1}, ..., x_{1.n_1})^T \in \mathbb{R}^{n_1}$, $x_2 = (x_{2.1}, ..., x_{2.n_2})^T \in \mathbb{R}^{n_2}$, $n_1 > 0,\ n_2 \geq 0,\ n_1 + n_2 = n$;

$$|x_1| = \sqrt{x_{1.1}^2 + ... + x_{1.n_1}^2}, \quad |x_2| = \sqrt{x_{2.1}^2 + ... + x_{2.n_2}^2}, \quad |x| = \left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}}.$$

Under the precedent assumptions, there exists a unique global solution:

$$x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$$

corresponding to the initial condition $x(t_0) = x_0 = (x_{1_0}, x_{2_0}) \in \mathbb{R}^n$, see [16, 20, 27]. In what follows we use $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$, or simply $x(t) = (x_1(t), x_2(t))$ to denote a solution of our system on some small interval.

Definition 3.1. The equilibrium point $x = 0$ of the G-SDE (3.1) is said to be
(i) Pth moment exponentially stable with respect to x_1, if there exist positive constants λ_1, λ_2, and $p > 0$ such that for all $x_0 \in \mathbb{R}^n$, the following inequalities are satisfied:

$$\widehat{\mathbb{E}}(|x_1(t; t_0, x_0)|^p) \leq \lambda_1 |x_0|^p e^{-\lambda_2(t-t_0)}, \quad \forall t \geq t_0 \geq 0.$$

In particular, for $p = 2$ we said that the G-stochastic system (3.1) is exponentially stable in mean square with respect to x_1.
(ii) Quasi surely exponentially stable with respect to x_1, if

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln\left(|x_1(t, t_0, x_0)|\right) < 0, \text{ q.s.}, \tag{3.2}$$

for all $x_0 \in \mathbb{R}^n$.

Definition 3.2. The solution of the sub-system with respect to the variable x_2 is said to be quasi surely globally uniformly bounded, if for each $\alpha > 0$, there exists $c = c(\alpha) > 0$ (independent of t_0), such that

$$\text{for every } t_0 \geq 0, \text{ and all } x_{2_0} \in \mathbb{R}^{n_2} \text{ with } \quad |x_{2_0}| \leq \alpha, \quad \sup_{t\geq t_0} |x_2(t, t_0, x_0)| \leq c(\alpha), \quad \text{q.s.}, \tag{3.3}$$

where $x_{2_0} = x_2(t_0; t_0, x_0)$.

Definition 3.3. Denote by $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ the family of all real-valued functions $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t.
If $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$, we define an operator L (called as G-Lyapunov function) from $\mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ as follows:

$$LV(t,x) := V_t(t,x) + V_x f(t,x) + G\Big(\langle V_x(t,x), 2h(t,x)\rangle + \langle V_{xx}(t,x)g(t,x), g(t,x)\rangle\Big),$$

where
$V_t(t,x) = \dfrac{\partial V}{\partial t}(t,x)$; $V_x(t,x) = (\dfrac{\partial V}{\partial x_1}(t,x), \dfrac{\partial V}{\partial x_2}(t,x))$; $V_{xx}(t,x) = \Big(\dfrac{\partial^2 V}{\partial x_i \partial x_j}(t,x)\Big)_{n\times n}$.

By G-Itô's formula [23], it follows

$$dV(t,x(t)) = LV(t,x(t))dt + V_x(t,x(t))g(t,x(t))dB_t.$$

Now, we aim to prove the pth moment exponential stability of SDEs driven by G-Brownian motion (3.1) with respect to a part of the variables, via the G-Lyapunov functions.

Theorem 3.1. *Assume that there exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and positive constants c_i $(i = 1,2,3)$, p such that for all $t \geq t_0 \geq 0$, and all $x = (x_1, x_2) \in \mathbb{R}^n$,*

$$c_1|x_1|^p \leq V(t,x) \leq c_2|x_1|^p, \tag{3.4}$$

$$LV(t,x) \leq -c_3|x_1|^p. \tag{3.5}$$

Then, the trivial solution of the G-stochastic system (3.1) is pth moment exponentially stable with respect to x_1.

In order to prove the previous theorem, we need to recall an important lemma.

Lemma 3.2. *[18] Let $\eta \in M_G^1(0,T)$ and $M_t = \int_0^t \eta(s)d\langle B\rangle_s - \int_0^t 2G(\eta(s))ds$.*
Then, for each $t \in [0,T]$, we have $\widehat{\mathbb{E}}(M_t) \leq 0$.

Proof of Theorem 3.1. Applying G-Itô's formula to $e^{\frac{c_3}{c_2}t}V(t,x(t))$, we obtain

$$\begin{aligned}
&d(e^{\frac{c_3}{c_2}t}V(t,x(t))) \\
&= e^{\frac{c_3}{c_2}t}\Big(\frac{c_3}{c_2}V(t,x(t)) + V_t(t,x(t)) + \langle V_x(t,x(t)), f(t,x(t))\rangle\Big)dt \\
&+ e^{\frac{c_3}{c_2}t}\langle V_x(t,x(t)), h(t,x(t))\rangle d\langle B\rangle_t + e^{\frac{c_3}{c_2}t}\langle V_x(t,x(t)), g(t,x(t))\rangle dB_t \\
&+ \frac{1}{2}e^{\frac{c_3}{c_2}t}\langle V_{xx}(t,x(t))g(t,x(t)), g(t,x(t))\rangle d\langle B\rangle_t.
\end{aligned}$$

That is,

$$
\begin{aligned}
&e^{\frac{c_3}{c_2}t}V(t,x(t))\\
=&e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\Big[\frac{c_3}{c_2}V(s,x(s))+V_s(s,x(s))+\langle V_x(s,x(s)),f(s,x(s))\rangle\\
&+G\left(\langle V_x(s,x(s)),2h(s,x(s))\rangle+\langle V_{xx}(s,x(s))g(s,x(s)),g(s,x(s))\rangle\right)\Big]ds\\
&-\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}G\left(\langle V_x(s,x(s)),2h(s,x(s))\rangle+\langle V_{xx}(s,x(s))g(s,x(s)),g(s,x(s))\rangle\right)ds\\
&+\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),h(s,x(s))\rangle d\langle B\rangle_s+\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),g(s,x(s))\rangle dB_s\\
&+\frac{1}{2}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_{xx}(s,x(s))g(s,x(s)),g(s,x(s))\rangle d\langle B\rangle_s\\
&=e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s))+LV(s,x(s))\right)ds+M_t^{t_0}\\
&+\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),g(s,x(s))\rangle dB_s.
\end{aligned}
$$

Where,

$$
\begin{aligned}
M_t^{t_0}=&\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),h(s,x(s))\rangle d\langle B\rangle_s\\
&+\frac{1}{2}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_{xx}(s,x(s))g(s,x(s)),g(s,x(s))\rangle d\langle B\rangle_s\\
&-\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}G\left(\langle V_x(s,x(s)),2h(s,x(s))\rangle+\langle V_{xx}(s,x(s))g(s,x(s)),g(s,x(s))\rangle\right)ds.
\end{aligned}
$$

Taking, G-expectation on both sides, it yields that

$$
\begin{aligned}
&\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t)))\\
&\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\widehat{\mathbb{E}}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\Big(\frac{c_3}{c_2}V(s,x(s))+LV(s,x(s))\Big)ds+\widehat{\mathbb{E}}(M_t^{t_0})\\
&+\widehat{\mathbb{E}}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),g(s,x(s))\rangle dB_s.
\end{aligned}
$$

Since, $V_x(t,x)$ is uniformly continuous in t uniformly Lipschitz in x on $[0,T]\times B_0(\Omega)$. Thus, based upon Proposition 3.11 in [16], we observed that $\exp(\frac{c_3}{c_2}(\cdot))\langle V_x(\cdot,x(\cdot)),g(\cdot,x(\cdot))\rangle\in M_G^p(0,T)$, for any $p\geq 2$. Then, we obtain

$$
\widehat{\mathbb{E}}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\langle V_x(s,x(s)),g(s,x(s))\rangle dB_s=0.
$$

Using Lemma 3.2, we have $\widehat{\mathbb{E}}(M_t^{t_0}) \leq 0$. Consequently, we obtain

$$\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t))) \leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) + LV(s,x(s))\right)ds.$$

This together with (3.4) and (3.5), implies

$$\begin{aligned}\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t))) &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) - c_3|x_1(s)|^p\right)ds\\ &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) - \frac{c_3}{c_2}V(s,x(s))\right)ds\\ &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))\\ &\leq c_2e^{\frac{c_3}{c_2}t_0}|x_{1_0}|^p.\end{aligned}$$

Thus, we obtain

$$\widehat{\mathbb{E}}(V(t,x(t))) \leq c_2e^{-\frac{c_3}{c_2}(t-t_0)}|x_{1_0}|^p.$$

Due to (3.4) again and the fact that $|x_{1_0}| \leq |x_0|$, we deduce that

$$\widehat{\mathbb{E}}(|x_1(t,t_0,x_0)|^p) \leq \frac{\widehat{\mathbb{E}}(V(t,x(t)))}{c_1} \leq \frac{c_2}{c_1}e^{-\frac{c_3}{c_2}(t-t_0)}|x_0|^p.$$

Therefore, the G-stochastic system (3.1) is pth moment exponentially stable with respect to x_1. □

Theorem 3.3. *Assume that there exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and positive constants c_i $(i = 1,2,3)$, p, such that for all $t \geq t_0 \geq 0$, and all $x = (x_1, x_2) \in \mathbb{R}^n$,*

$$c_1|x_1|^p \leq V(t,x) \leq c_2|x_1|^p, \tag{3.6}$$

$$LV(t,x) \leq (-c_3 + \varphi(t))|x_1|^p, \tag{3.7}$$

where $\varphi(t)$ is a continuous nonnegative function, with

$$\int_0^{+\infty} \varphi(t)dt \leq M < +\infty. \tag{3.8}$$

Then, the trivial solution of the G-stochastic system (3.1) is pth moment exponentially stable with respect to x_1.

Proof of Theorem 3.3. By using similar reasoning as above, we obtain

$$\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t))) \leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0)) + \widehat{\mathbb{E}}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s)) + LV(s,x(s))\right)ds.$$

On account of (3.6) and (3.7), we obtain

$$\begin{aligned}\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t))) &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\widehat{\mathbb{E}}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s,x(s))+(-c_3+\varphi(s))|x_1(s)|^p\right)ds\\ &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\widehat{\mathbb{E}}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\Big(\frac{c_3}{c_2}V(s,x(s))-\frac{c_3}{c_2}V(s,x(s))\\ &+\frac{\varphi(s)}{c_1}V(s,x(s))\Big)ds.\end{aligned}$$

That is,

$$\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t)))\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))+\widehat{\mathbb{E}}\int_{t_0}^{t}\frac{\varphi(s)}{c_1}e^{\frac{c_3}{c_2}s}V(s,x(s))ds. \tag{3.9}$$

Applying now the Gronwall lemma [7], and condition (3.6), we obtain

$$\begin{aligned}\widehat{\mathbb{E}}(e^{\frac{c_3}{c_2}t}V(t,x(t))) &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))e^{\frac{1}{c_1}\int_{t_0}^{t}\varphi(s)ds}\\ &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))e^{\frac{1}{c_1}\int_{0}^{+\infty}\varphi(s)ds}\\ &\leq e^{\frac{c_3}{c_2}t_0}V(t_0,x(t_0))e^{\frac{M}{c_1}}\\ &\leq e^{\frac{c_3}{c_2}t_0}c_2|x_{1_0}|^pe^{\frac{M}{c_1}}.\end{aligned}$$

Hence, we see that

$$\widehat{\mathbb{E}}(V(t,x(t)))\leq c_2e^{\frac{M}{c_1}}|x_{1_0}|^pe^{-\frac{c_3}{c_2}(t-t_0)}.$$

Due to the fact that $|x_{1_0}|\leq|x_0|$ and condition (3.6), we deduce from the last inequality that

$$\widehat{\mathbb{E}}(|x_1(t)|^p)\leq\frac{c_2}{c_1}e^{\frac{M}{c_1}}|x_0|^pe^{-\frac{c_3}{c_2}(t-t_0)}.$$

Setting $\lambda_1=\frac{c_2}{c_1}e^{\frac{M}{c_1}}$ and $\lambda_2=\frac{c_3}{c_2}$, we conclude that the G-stochastic system (3.1) is pth moment exponentially stable with respect to x_1. □

Mao [19] developed the condition under which the pth moment exponential stability implies the almost sure exponential stability of stochastic differential equations driven by standard Brownian motion. In what follows, our target is to involve conditions under which the pth moment exponential stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion implies the quasi sure exponential stability with respect to a part of the variables.

The system (3.1) might be regarded as the following form:

$$\begin{cases}dx_1(t)=f_1(t,x_1(t),x_2(t))dt+h_1(t,x_1(t),x_2(t))d\langle B\rangle_t+g_1(t,x_1(t),x_2(t))dB_t\\ \\ dx_2(t)=f_2(t,x_1(t),x_2(t))dt+h_2(t,x_1(t),x_2(t))d\langle B\rangle_t+g_2(t,x_1(t),x_2(t))dB_t.\end{cases}$$

with the same initial condition $x(t_0) = x_0 = (x_{1_0}, x_{2_0}),\ f := (f_1, f_2),\ h = (h_1, h_2),\ g := (g_1, g_2)$, and

- $f_1, h_1, g_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$.
- $f_2, h_2, g_2 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$.

Theorem 3.4. *Consider the G-stochastic system (3.1), assume that there exists a positive constant η such that*

$$\widehat{\mathbb{E}}\Big(|f_1(t,x_1,x_2)|^p + |h_1(t,x_1,x_2)|^p + |g_1(t,x_1,x_2)|^p\Big) < \eta\widehat{\mathbb{E}}(|x_1|^p),\ \forall x_1 \in \mathbb{R}^{n_1}, \forall x_2 \in \mathbb{R}^{n_2}, \forall t \geq t_0. \tag{3.10}$$

Then, the pth moment exponential stability with respect to x_1 of the G-stochastic system (3.1), implies the quasi sure exponential stability.

Proof. By the definition of the pth moment exponential stability with respect to x_1, there exist a pair of positive constants λ_1 and λ_2, such that

$$\widehat{\mathbb{E}}(|x_1(t;t_0,x_0)|^p) \leq \lambda_1|x_0|^p e^{-\lambda_2(t-t_0)}, \quad \forall t \geq t_0 \geq 0. \tag{3.11}$$

Furthermore, we have

$$x_1(t+s) = x_1(t) + \int_t^{t+s} f_1(u,x_1(u),x_2(u))du + \int_t^{t+s} h_1(u,x_1(u),x_2(u))d\langle B\rangle_u + \int_t^{t+s} g_1(u,x_1(u),x_2(u))dB_u,$$

which implies,

$$\begin{aligned}|x_1(t+s)|^p \leq 4^{p-1}\Big(|x_1(t)|^p + |\int_t^{t+s} f_1(u,x_1(u),x_2(u))du|^p \\ + |\int_t^{t+s} h_1(u,x_1(u),x_2(u))d\langle B\rangle_u|^p + |\int_t^{t+s} g_1(u,x_1(u),x_2(u))dB_u|^p\Big).\end{aligned}$$

Using the sub-additivity of G-expectation, we obtain

$$\begin{aligned}&\widehat{\mathbb{E}}(\sup_{0\leq s\leq\tau}|x_1(t+s)|^p) \\ &\leq 4^{p-1}\Big[\widehat{\mathbb{E}}|x_1(t)|^p + \widehat{\mathbb{E}}\left(\int_t^{t+s}|f_1(u,x_1(u),x_2(u))|du\right)^p + \widehat{\mathbb{E}}\left(\sup_{0\leq s\leq\tau}|\int_t^{t+s} g_1(u,x_1(u),x_2(u))dB_u|^p\right) \\ &+ \widehat{\mathbb{E}}\left(\sup_{0\leq s\leq\tau}|\int_t^{t+s} h_1(u,x_1(u),x_2(u))d\langle B\rangle_u|^p\right)\Big].\end{aligned} \tag{3.12}$$

On account of (3.10), (3.11) and by Hölder's inequality [8], we obtain

$$\begin{aligned}\widehat{\mathbb{E}}\left(\int_t^{t+\tau}|f_1(u,x_1(u),x_2(u))|du\right)^p &\leq \tau^p\int_t^{t+\tau}\widehat{\mathbb{E}}|f_1(u,x_1(u),x_2(u))|^p du \\ &\leq \frac{\lambda_1}{\lambda_2}\eta\tau^p|x_0|^p e^{-\lambda_2(t-t_0)}.\end{aligned} \tag{3.13}$$

On the other side, based on (3.10), (3.11) and Lemma 2.6, we obtain

$$\begin{aligned}\widehat{\mathbb{E}}\left(\sup_{0\le s\le\tau}|\int_t^{t+\tau}g_1(u,x_1(u),x_2(u))dB_u|^p\right)&\le C_p\tau^{\frac{p}{2}-1}\int_t^{t+\tau}\widehat{\mathbb{E}}|g_1(u,x_1(u),x_2(u))|^p du\\&\le\frac{\lambda_1}{\lambda_2}C_p\eta\tau^{\frac{p}{2}}|x_0|^p e^{-\lambda_2(t-t_0)}.\end{aligned}\tag{3.14}$$

Likewise, by Lemma 2.5 we obtain

$$\begin{aligned}\widehat{\mathbb{E}}\left(\sup_{0\le s\le\tau}|\int_t^{t+\tau}h_1(u,x_1(u),x_2(u))d\langle B\rangle_u|^p\right)&\le C'_p\tau^{p-1}\int_t^{t+\tau}\widehat{\mathbb{E}}|h_1(u,x_1(u),x_2(u))|^p du\\&\le\frac{\lambda_1}{\lambda_2}C'_p\eta\tau^p|x_0|^p e^{-\lambda_2(t-t_0)},\end{aligned}\tag{3.15}$$

where C'_p is a positive constant dependent only on p.
Then, we conclude from the above inequalities (3.13), (3.14), and (3.15) that

$$\widehat{\mathbb{E}}(\sup_{0\le s\le\tau}|x_1(t+s)|^p)\le Re^{-\lambda_2 t},$$

with $R=4^{p-1}\frac{\lambda_1}{\lambda_2}|x_0|^p\Big(\lambda_2+\eta\tau^p(1+C_p\tau^{-\frac{p}{2}}+C'_p)\Big)$.

Now, let $\varepsilon\in(0,\lambda_2)$ be arbitrary, then thanks to Lemma 2.4, we obtain

$$\begin{aligned}&\hat{C}\left(\sup_{0\le s\le\tau}|x_1(n\tau+s)|^p>e^{-(\lambda_2-\varepsilon)n\tau}\right)\\&\le e^{(\lambda_2-\varepsilon)n\tau}\widehat{\mathbb{E}}(\sup_{0\le s\le\tau}|x_1(n\tau+s)|^p)\\&\le Re^{-\varepsilon n\tau}.\end{aligned}$$

By the Borel-Cantelli lemma for the capacity 2.3, we see that there exists $n_0:=n_0(\omega)$, such that for almost all $\omega\in\Omega$, $n>n_0(\omega)$,

$$\sup_{0\le s\le\tau}|x_1(t+s)|^p\le e^{-(\lambda_2-\varepsilon)n\tau},\ \text{q.s.},$$

where $t\in[n\tau,(n+1)\tau]$. Thus, we obtain

$$\frac{1}{t}\log(|x_1(t)|)=\frac{1}{pt}\log(|x_1(t)|^p)\le-\frac{(\lambda_2-\varepsilon)n\tau}{pn\tau},\ \text{q.s.}$$

Hence, $\lim_{t\longrightarrow\infty}\sup\frac{1}{t}\log(|x_1(t)|)\le-\frac{(\lambda_2-\varepsilon)}{p},\quad\text{q.s.}$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result

$$\lim_{t\longrightarrow\infty}\sup\frac{1}{t}\log(|x_1(t)|)\leq -\frac{\lambda_2}{p},\quad \text{q.s.}$$

That is, the G-stochastic (3.1) system is quasi surely exponentially stable with respect to x_1.
□

4 Exponential stability of G-stochastic perturbed differential equations with respect to a part of the variables

In this section, we consider the following linear stochastic system:

$$dx(t) = Ax(t)dt,\quad \forall t\geq t_0\geq 0, \tag{4.1}$$

where,

$$A=\begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix},\quad x:=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\ n_1>0,\ n_2\geq 0,\ n_1+n_2=n.$$

- A_1 is a constant $n_1\times n_1$ matrix.
- A_2 is a constant $n_2\times n_2$ matrix.

The above system (4.1) might be regarded as the following system:

$$\begin{cases} dx_1(t) = A_1x_1(t)dt\\ dx_2(t) = A_2x_2(t)dt,\end{cases} \tag{4.2}$$

with initial condition $x(t_0):=x_0:=(x_{1_0},x_{2_0})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Assume that some parameters are excited or perturbed by G-Brownian motion, and the perturbed system has the form:

$$\begin{cases} dx_1(t) = A_1x_1(t)dt + g(t,x_1(t),x_2(t))dB_t\\ dx_2(t) = A_2x_2(t)dt,\end{cases} \tag{4.3}$$

with the same initial conditions, where $B_t=(B_1(t),....,B_m(t))^T$ is an m-dimensional G-Brownian motion, and $g:\mathbb{R}^+\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\longrightarrow\mathbb{R}^{n_1\times m}$.
Assume that conditions of existence and uniqueness of solutions are satisfied, see [22], and $x(t,t_0,x_0)=(x_1(t;t_0,x_0),x_2(t;t_0,x_0))$ is the solution of system (4.3).
Suppose that the origin of the linear stochastic system (4.2) is quasi surely exponentially stable with respect to x_1. Further, we assume that the origin $x=(0,0)$ is an equilibrium point of the

G-stochastic perturbed system (4.3), that is $g(t,0,0)=0$ for all $t\geq 0$.

The objective of this section is to state sufficient conditions under which the G-stochastic perturbed system (4.3) is still quasi surely exponentially stable with respect to x_1 by using recently developed Itô's calculus for SDEs driven by G-Brownian motion and Gronwall inequalities.

Theorem 4.1. *Let λ_1 be the maximum of the real parts of all eigenvalues of $-A_1$, we suppose that there exist a constant $c_1 \geq 0$, and a polynomial $p_1(t)$ such that for all $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, and sufficiently large t,*

$$|g(t,x_1(t),x_2(t))|^2 \leq p_1(t)e^{(-2\lambda_1+c_1)t}, \quad q.s. \tag{4.4}$$

Furthermore, we assume that $\lim_{t\longrightarrow\infty}\sup \dfrac{\log|e^{A_1t}|^2}{t} \leq -c_2$, where c_2 is a positive constant. Then, we have

$$\lim_{t\longrightarrow\infty}\sup \frac{\log|x_1(t;t_0,x_0)|^2}{t} \leq -(c_2-c_1), \ q.s.$$

for all $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$.
In particular, if $c_2 > c_1$, then the G-stochastic perturbed system (4.3) is said to be quasi surely exponentially stable with respect to x_1.

In order to prove this theorem, let us start by recalling an important Gronwall lemma [7], which will be very useful later on.

Lemma 4.2. *Let $b(t)$, $c(t)$, and $u(t)$ be continuous functions for $t \geq t_0 \geq 0$, and $b(t)$ be nonnegative for $t \geq t_0 \geq 0$, ϕ is a constant and suppose*

$$u(t) \leq \phi + \int_{t_0}^{t} [b(s)u(s)+c(s)]\, ds, \quad t \geq t_0 \geq 0.$$

Then,

$$u(t) \leq \phi \exp\left(\int_{t_0}^{t} b(\tau)d\tau\right) + \int_{t_0}^{t} c(s)\exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds, \quad t \geq t_0 \geq 0.$$

We need to recall the following lemma, which is essential in our analysis.

Lemma 4.3. *[20] Let B_t be a one-dimensional G-Brownian motion, suppose that there exist constants $\varepsilon > 0$ and $\alpha > 0$, such that*

$$\widehat{\mathbb{E}}\left(\exp\left[\frac{\alpha^2}{2}(1+\varepsilon)\int_0^T g^2(s)d\langle B\rangle_s\right]\right) < \infty, \quad \forall g \in M_G^2(0,T).$$

Then, for any $T>0$ and $\beta > 0$,

$$\hat{C}\left(\sup_{0\leq t\leq T}\left[\int_0^t g(s)dB_s - \frac{\alpha}{2}\int_0^t g^2(s)d\langle B\rangle_s\right] > \beta\right) \leq \exp(-\alpha\beta).$$

Now, we are in a position to prove our main result in this section.

Proof of Theorem 4.1. Fix $\epsilon > 0$ arbitrarly, and there exists $\rho = \rho(\epsilon)$ such that

$$|e^{-A_1 t}|^2 \le \rho e^{(2\lambda_1+\epsilon)t}, \quad p_1(t) \le \rho e^{\epsilon t}, \quad t > 0.$$

Applying G-Itô's formula, we obtain

$$d(e^{-A_1 t} x_1(t)) = e^{-A_1 t} g(t, x_1(t), x_2(t)) dB_t.$$

Define $W(t) = |e^{-A_1 t} x_1(t)|^2$, based on the G-Itô's formula again it yields that

$$W(t) = W(t_0) + 2\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) dB_s$$
$$+ \int_{t_0}^t \text{trace}\left(e^{-A_1 s} g(s, x_1(s), x_2(s)) g_1^T(s, x_1(s), x_2(s)) e^{-A_1^T s}\right) d\langle B\rangle_s. \tag{4.5}$$

From Lemma 4.3, it follows that for any $\alpha > 0,\ \beta > 0$, and $\tau > t_0$,

$$\hat{C}\Big(\sup_{t_0\le t\le\tau} \big[\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) dB_s - \frac{\alpha}{2}$$
$$\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) g_1^T(s, x_1(s), x_2(s)) e^{-A_1^T s} e^{-A_1 s} x_1(s) d\langle B\rangle_s\big] > \beta\Big) \le \exp(-\alpha\beta).$$

Choose an arbitrary $\theta > 1$, and let k be an integer large enough so that $k > t_0$, and set

$$\alpha = e^{-c_1 k}, \quad \beta = \theta e^{c_1 k}\log k, \quad \tau = k.$$

Then, we obtain

$$\hat{C}\Big(\sup_{t_0\le t\le k} \big[\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) dB_s - \frac{e^{-c_1 k}}{2}$$
$$\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) g^T(s, x_1(s), x_2(s)) e^{-A_1^T s} e^{-A_1 s} x_1(s) d\langle B\rangle_s\big] > \theta e^{c_1 k}\log k\Big)$$
$$\le \frac{1}{k^\theta}.$$

Applying the Borel-Cantelli lemma for the capacity 2.3, we see that for almost all $\omega \in \Omega$, there exists $k_0 = k_0(\omega)$, such that

$$\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) dB_s$$
$$\le \frac{e^{-c_1 k}}{2}\int_{t_0}^t x_1^T(s) e^{-A_1^T s} e^{-A_1 s} g(s, x_1(s), x_2(s)) g^T(s, x_1(s), x_2(s)) e^{-A_1^T s} e^{-A_1 s} x_1(s) d\langle B\rangle_s$$
$$+ \theta e^{c_1 k}\log k, \quad \text{for all } k > k_0,\ t_0 \le t \le k.$$

Using condition (4.4), it follows that

$$\begin{aligned}
&\int_{t_0}^{t} x_1^T(s)e^{-A_1^T s}e^{-A_1 s}g(s,x_1(s),x_2(s))dB_s \\
&\leq \frac{e^{-c_1 k}}{2}\int_{t_0}^{t} x_1^T(s)e^{-A_1^T s}e^{-A_1 s}g(s,x_1(s),x_2(s))g^T(s,x_1(s),x_2(s))e^{-A_1^T s}e^{-A_1 s}x_1(s)d\langle B\rangle_s \\
&+\theta e^{c_1 k}\log k \\
&\leq \frac{e^{-c_1 k}}{2}\rho^2\int_{t_0}^{t} W(s)e^{c_1 s}d\langle B\rangle_s + \theta e^{c_1 k}\log k.
\end{aligned}$$

This together with (4.5), we obtain

$$\begin{aligned}
W(t) &\leq W(t_0) + e^{-c_1 k}\rho^2\int_{t_0}^{t} W(s)e^{c_1 s}d\langle B\rangle_s + 2\theta e^{c_1 k}\log k \\
&+ \int_{t_0}^{t}\text{trace}\left(e^{-A_1 s}g(s,x_1(s),x_2(s))g^T(s,x_1(s),x_2(s))e^{-A_1^T s}\right)d\langle B\rangle_s \\
&\leq W(t_0) + e^{-c_1 k}\rho^2\int_{t_0}^{t} W(s)e^{c_1 s}d\langle B\rangle_s + 2\theta e^{c_1 k}\log k + b\rho^2\int_{t_0}^{t} e^{c_1 s}d\langle B\rangle_s. \qquad (4.6)
\end{aligned}$$

By Peng [20, Chapter III], we have for each $0 \leq s \leq t \leq T$,

$$\langle B\rangle_t - \langle B\rangle_s \leq \bar{\sigma}^2(t-s).$$

Based on this fact and the inequality (4.6), it follows that

$$W(t) \leq W(t_0) + e^{-c_1 k}\rho^2\overline{\sigma}^2\int_{t_0}^{t} W(s)e^{c_1 s}ds + 2\theta e^{c_1 k}\log k + b\rho^2\overline{\sigma}^2\int_{t_0}^{t} e^{c_1 s}ds.$$

Applying the Lemma 4.2, we obtain

$$\begin{aligned}
W(t) &\leq \left(W(t_0) + 2\theta e^{c_1 k}\log k\right)\exp\left(e^{-c_1 k}\rho^2\overline{\sigma}^2\int_{t_0}^{t} e^{c_1 s}ds\right) \\
&+ b\rho^2\overline{\sigma}^2\int_{t_0}^{t}\exp\left(\rho^2\overline{\sigma}^2 e^{-c_1 k}\int_{s}^{t} e^{c_1 r}dr\right)e^{c_1 s}ds \\
&\leq \left(W(t_0) + 2\theta e^{c_1 k}\log k + \frac{b\rho^2\overline{\sigma}^2}{c_1}e^{c_1 k}\right)\exp\left(e^{-c_1 k}\rho^2\overline{\sigma}^2\int_{t_0}^{t} e^{c_1 s}ds\right) \\
&\leq \left(W(t_0) + 2\theta e^{c_1 k}\log k + \frac{b\rho^2\overline{\sigma}^2}{c_1}e^{c_1 k}\right)\exp\left(\frac{\rho^2\overline{\sigma}^2}{c_1}\right), \quad t_0 \leq t \leq k, \quad k \geq k_0, \quad q.s.
\end{aligned}$$

That is,

$$W(t) \leq \left(W(t_0) + 2\theta e^{c_1 k}\log k + \frac{b\rho^2\overline{\sigma}^2}{c_1}e^{c_1 k}\right)\exp\left(\frac{\rho^2\overline{\sigma}^2}{c_1}\right), \quad t_0 \leq t \leq k, \quad k \geq k_0, \quad q.s. \quad (4.7)$$

Since $\theta > 1$ is arbitrary, and $\dfrac{W(t)}{e^{c_1 t}\log t} \leq \dfrac{W(t)}{e^{c_1(k-1)}\log(k-1)}, \quad k-1 \leq t \leq k.$

From (4.7), we see immediately that

$$\begin{aligned}\lim_{t\longrightarrow\infty}\sup \frac{W(t)}{e^{c_1 t}\log(t)} &\leq \lim_{k\longrightarrow\infty}\sup \frac{\left(W(t_0) + 2\theta e^{c_1 k}\log k + \frac{b\rho^2\overline{\sigma}^2}{c_1}e^{c_1 k}\right)\exp\left(\dfrac{\rho^2\overline{\sigma}^2}{c_1}\right)}{e^{c_1(k-1)}\log(k-1)} \\ &\leq 2\exp\left(c_1 + \frac{\rho^2\overline{\sigma}^2}{c_1}\right) \quad q.s.\end{aligned}$$

Since,

$$\lim_{t\longrightarrow\infty}\sup \frac{\log|x_1(t;t_0,x_0)|^2}{t} \leq \lim_{t\longrightarrow\infty}\sup \frac{\log|e^{At}|^2}{t} + \lim_{t\longrightarrow\infty}\sup \frac{\log|e^{-At}x_1(t;t_0,x_0)|^2}{t}.$$

Consequently, we obtain

$$\lim_{t\longrightarrow\infty}\sup \frac{\log|x_1(t;t_0,x_0)|^2}{t} \leq -c_2 + c_1 = -(c_2 - c_1), \quad \text{q.s.}$$

Then, if the inequality $c_2 > c_1$ is satisfied, we deduce that the G-stochastic perturbed system (4.3) is quasi surely exponentially stable with respect to x_1. □

5 Examples

Some illustrative examples are given to demonstrate the validity and accuracy of our results.

Example 5.1. *To highlight the utility and the importance of our found out result, we consider the example of spring-mass model with viscous damping (see Fig.1), whose equation is given by:*

$$\ddot{q}(t) + c(t)\dot{q}(t) + kq(t) = 0, \quad t \geq 0, \tag{5.1}$$

$$q(0) = q_0,\ \dot{q}(0) = \dot{q}_0,$$

where $c(t)$ is a time-varying damping coefficient. This damping corresponds to the type of resistance to motion and energy dissipation that is encountered when a piston with perforations is moved through a cylinder filled with a viscous fluid, for example oil, and k is the stiffness of the spring coefficient.

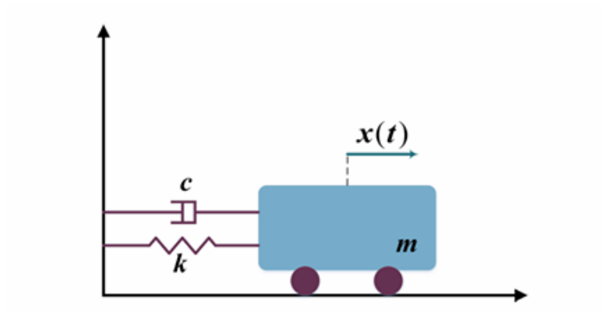

Figure 1: Spring-mass-damper system.

The spring-mass-damper system is an impressive system to analyze it, since the physical intuition conducts one to surmise that if $c(t) \geq \beta \geq 0,\ t \geq 0$, *for that reason the zero solution* $(q(t), \dot{q}(t)) = (0,0)$ *is asymptotically stable, due to the fact that the constant of energy dissipated. Nevertheless, this is not the case, a simple counterexample (see [3]) is* $c(t) = 2 + e^t,\ k = 1,\ q(0) = 2$ *and* $\dot{q}(0) = -1$, *which yields that* $q(t) = 1 + e^{-t},\ t \geq 0$. *Thus,* $q(t) \to 1$ *as* $t \to \infty$. *This is due to the fact that damping increases so fast that the system arrests at* $q = 1$.

The system (5.1) *can be equivalently written as*

$$\begin{cases} dz_1(t) = z_2(t)dt, \quad z_1(0) = q_0,\ t \geq 0 \\ dz_2(t) = (-kz_1(t) - c(t)z_2(t))dt, \quad z_2(0) = \dot{q}_0, \end{cases} \tag{5.2}$$

where $z_1 := q$ *and* $z_2 = \dot{q}$.
The solutions $(z_1(t), z_2(t))$ *to system* (5.2) *are equivalently characterized by the solution* $x_1(t), t \geq 0$, *of the following system:*

$$\begin{cases} dx_1(t) = f_1(x_1(t), x_2(t))dt, \ x_1(0) = (q_0, \dot{q}_0),\ t \geq 0 \\ dx_2(t) = 1dt, \quad x_2(0) = 0, \end{cases} \tag{5.3}$$

where $x_1 = (z_1, z_2) \in \mathbb{R}^2, x_2 = t \in \mathbb{R}_+$, *and*

$$f_1 = \begin{pmatrix} x_1 v \\ -x_1 L(x_2) \end{pmatrix}, \quad L(x_2) = \begin{pmatrix} k \\ c(x_2) \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_2(x_1, x_2) = 1.$$

Assume that some parameters of system (5.1) are excited or perturbed by a G-Brownian motion, and the perturbed system has the following form:

$$\begin{cases} dx_1(t) = f_1(x_1(t), x_2(t))dt + g(t, x_1(t), x_2(t))dB_t, \ x_1(0) = (q_0, \dot{q}_0), \quad t \geq 0 \\ dx_2(t) = 1dt, \quad x_2(0) = 0, \end{cases} \tag{5.4}$$

where B *is a one-dimension G-Brownian motion,* $B = N(0 \times [\frac{1}{2}, 1])$, *and* $g(t,x) = \begin{pmatrix} g_1(t,x) \\ g_2(t,x) \end{pmatrix}$, *with*

$$\begin{cases} g_1(t,x) = \alpha z_2, \quad \alpha > 0, \\ \\ g_2(t,x) = \dfrac{\alpha}{\sqrt{2}}\sqrt{\cos(x_2)}z_2 z_1 - \alpha\sqrt{2\sin(x_2)}z_1 z_2. \end{cases}$$

It is clear that the G-stochastic system (5.4) *is unstable since the state* x_2*, representing the time, is unbounded. Consequently, it is very interesting to analyze if it is still possible to prove some stability properties with respect to some of the variables in the problem.*

Let $c(t) = 4 + cos(t)$*,* $k = 2$*, and consider the following Lyapunov-like function:*

$$V(t, x_1, x_2) = x_1^T P(x_2) x_1,$$

where $P(x_2) = \begin{pmatrix} 6+\cos(x_2) & 1 \\ 1 & 1 \end{pmatrix}$.

It is clear that, $x_1^T P_1 x_1 \leq V(t, x_1, x_2) \leq x_1^T P_2 x_1$*, where*

$$P_1 = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that P_1 *and* P_2 *have the eigenvalues* $0.76, 5.24$ *and* $0.84, 7.16$*, respectively.*

Therefore,

$$0.76|x_1|^2 \leq V(t, x_1, x_2) \leq 7.16|x_1|^2.$$

Applying the G-Itô formula to V*, one has*

$$\begin{aligned} LV(t, x_1, x_2) &= -\sin(x_2)z_1^2 - 4z_1^2 - 2\cos(x_2)z_2^2 - 6z_2^2 + G(12\alpha^2 z_2^2 + 2\alpha^2\cos(x_2)z_2^2) \\ &\leq -3z_1^2 - 4z_2^2 + G(12\alpha^2 z_2^2) + G(2\alpha^2\cos(x_2)z_2^2) \\ &\leq -3z_1^2 - 4z_2^2 + 6\alpha^2 z_2^2 + \alpha^2 z_2^2 \\ &\leq -3z_1^2 - 4z_2^2 + 7\alpha^2 z_2^2. \end{aligned}$$

For $\alpha = \dfrac{1}{\sqrt{7}}$ *we then obtain*

$$\begin{aligned} LV(t, x_1, x_2) &\leq -3z_1^2 - 4z_2^2 + z_2^2 \\ &\leq -3z_1^2 - 3z_2^2 \\ &= -3|x_1|^2. \end{aligned}$$

Thus, constants in Theorem 3.1 become $c_1 = 0.76$*,* $c_2 = 7.16$*,* $c_3 = 3$*,* $p = 2$*, and we conclude that the G-stochastic system* (5.4) *is exponentially stable in mean square with respect to* x_1*.*

Example 5.2. *Consider the following G-stochastic system:*

$$\begin{cases} dx_1(t) = -2x_1 dt - \dfrac{1}{2} sin^2(x_2)e^{-2t}x_1 d\langle B\rangle_t + (1 + e^{-t}|sin(x_2)|)x_1 dB_t \\ dx_2(t) = 2\cos(t)x_2 dt, \end{cases} \tag{5.5}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ *,* B *is a one-dimension G-Brownian motion and* $B = N(0 \times [\frac{1}{2}, 1])$*, with initial value* $x_0 = (x_{1_0}, x_{2_0})$*.*
The solution of the sub-system with respect to the variable x_2 *is quasi surely globally uniformly bounded. In fact, for all* $t \geq t_0 \geq 0$*, and all* $x_{2_0} \in \mathbb{R}$ *with* $|x_{2_0}| \leq \beta$*, we have* $|x_2(t)| \leq \beta e^{2\sin(t)}$ *q.s.*

Denote $V = x_1^2$*, then we obtain*

$$\begin{aligned} LV(t,x) &= V_x(t,x)f_1(t,x) + G(2V_x(t,x)h_1(t,x) + V_{xx}g_1^2(t,x)) \\ &= -4x_1^2 + G\left(-2x_1^2\sin^2(x_2)e^{-2t} + 2(1 + e^{-t}|\sin(x_2)|)^2 x_1^2\right) \\ &= -4x_1^2 + G\left(2x_1^2 + 4e^{-t}|\sin(x_2)|x_1^2\right). \end{aligned}$$

By the sub-additivity of the function G, we obtain

$$\begin{aligned} LV(t,x) &\leq -4x_1^2 + G(2x_1^2) + G(4e^{-t}|\sin(x_2)|x_1^2) \\ &\leq -4x_1^2 + x_1^2 + 2e^{-t}|\sin(x_2)|x_1^2 \\ &\leq -3x_1^2 + 2e^{-t}x_1^2. \end{aligned}$$

Thus,

$$LV(t,x) \leq (-3 + \varphi(t))x_1^2,$$

where $\varphi(t) = 2e^{-t}$*, which satisfies condition (3.8) of Theorem 3.3.*
Hence, all conditions of Theorem 3.3 are fulfilled with $p = 2$ *and then the G-stochastic system (5.5) is exponentially stable in mean square with respect to* x_1*.*
Furthermore, we have

1. $|f_1(t,x)|^2 = 4|x_1|^2$,
2. $|h_1(t,x)|^2 = |\frac{1}{2}sin^2(x_2)e^{-2t}x_1|^2 \leq |x_1|^2$,
3. $|g_1(t,x)|^2 = |(1 + e^{-t}|sin(x_2)|)x_1|^2 \leq 2(1 + e^{-2t}|\sin(x_2)|^2)|x_1|^2 \leq 4|x_1|^2$.

Then, we obtain

$$\hat{\mathbb{E}}\left(|f_1(t,x)|^2 + |h_1(t,x)|^2 + |g_1(t,x)|^2\right) \leq 9\hat{\mathbb{E}}\left(|x_1|^2\right).$$

Hence, based on Theorem 3.4, with $p = 2$ *and* $\eta = 9$*, one can deduce that the G-stochastic system (5.5) is quasi sure exponentially stable with respect to* x_1*.*

Remark 5.3. Note that, we cannot establish the exponential stability in all variables for the above example because the solution of the sub-system with respect to the variable x_2 is quasi surely globally uniformly bounded but not attractive.

Remark 5.4. The sub-system with respect to the variable x_2 can be taken in such a way that all trajectories are quasi surely globally uniformly bounded but not attractive.

Example 5.5. *Consider the following G-stochastic system*

$$\begin{cases} dx_1(t) = -2x_1(t)dt + (1+t^2)e^{-4.5t}\dfrac{x(t)}{1+|x(t)|}dB_t \\ dx_2(t) = A_2x_2(t)dt, \end{cases} \tag{5.6}$$

where $x = (x_1, x_2) \in \mathbb{R}^3$, $x_2 = (z_1, z_2) \in \mathbb{R}^2$ *and* B *is a one-dimensional G-Brownian motion,*

$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With initial value $x_0 = (x_{1_0}, x_{2_0})$, *and* $x_{2_0} = (z_{1_0}, z_{2_0})$.

By simple resolution, we obtain

$$x_2(t) = \begin{pmatrix} z_{1_0}(t,w)\cos(t) - z_{2_0}(t,w)\sin(t) \\ z_{1_0}(t,w)\sin(t) + z_{2_0}(t,w)\cos(t) \end{pmatrix}.$$

Using the Euclidean norm, we deduce

$$|x_2(t)| = |x_{2_0}|.$$

The solution of the sub-system with respect to the variable x_2 *is quasi surely globally uniformly bounded. In fact, for all* $t \geq t_0 \geq 0$, *and all* $x_{2_0} \in \mathbb{R}^2$ *with* $|x_{2_0}| \leq \alpha'$, *we have* $|x_2(t)| \leq \alpha' \quad q.s.$, *and the constants of Theorem 4.1 are* $c_1 = 0.5$, *and* $c_2 = 2$.
Hence, by Theorem 4.1 we deduce that $\limsup_{t\to\infty} \dfrac{\log|x_1(t,t_0,x_0)|^2}{t} \leq -1.5$ *q.s. for all* $t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^2$. *Indeed, the G-stochastic perturbed system (5.6) is quasi sure exponentially stable with respect to* x_1.

6 Conclusion

In this paper, we dealt with the analysis problem of stability of stochastic differential equations driven by G-Brownian motion. Some stability criteria for the pth moment exponential stability and quasi sure exponential stability with respect to a part of the variables are established. The main technical tools for deriving stability results are G-Lyapunov functions and recently developed Itô's calculus for G-SDEs. Some examples have been investigated to validate the developed methods.

Acknowledgement

The authors would like to thank the editor and the anonymous reviewer for valuable comments and suggestions, which allowed us to improve the paper.

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