Qualitative modelling in ecology

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1 Introduction

In this example a non-linear system is examined. It can be used as a model of the growth of two species. The model tries to capture, in the simplest way, dependencies of growth rate on food supply and negative effects of over crowding. In the same way, dependencies of the growth rate of predators on the quantity of preys are captured.

Instead of explicit formulas for the equations, certain qualitative assumptions are made regarding the form of the equations. These assumptions show the qualitative knowledge about the system that should be taken. Qualitative analysis and bifurcation theory can be applied to obtain the long-run qualitative behaviour of the trajectories of the system.

2 Specifying a model

The system to model, is composed by preys and predators. The preys depend on the environment and predators depend on the preys. Every population has a birth rate and a death rate. The growth rate is defined as the birth rate minus the death rate. In the following x stands for prey and y stands for predators. The knowledge about the variables in the model and relations between them are given by the following assumptions:

The growth rate of preys depends only on the population of preys. Let f1(x) be the growth rate of the prey. It is plausible to assume that there is a limiting population x1 such that f1(x1)=0, f1(x)<0, x>x1. The growth rate is assumed to be positive for small populations f1(0)=a>0. Take into account positive social phenomena, for example, a medium size population may be better organised to obtain food than a small one. This implies that there is a value x0 of the population where f1 has a local maximum. So see that f1(x0)=b, 0<x0<x1, b>a. Let us study the consequences in the behaviour of the aspect of f1. The properties of f1 are:

$$f1 \equiv \begin{cases} 0 < x0 < x1, 0 < a < b \\ f1(0) = a, f1(x1) = 0 \\ f1(x0) = b, f1'(x0) = 0 \end{cases}$$

The prey population is the total food supply for the predator. The growth rate of the predators depends on the prey populations. Let f2(x) be the growth rate of predators. Let us assume that there is a value x2 of x such as f2(x2) =0 and that f2 is a monotone increasing function. It is known that for large sizes of prey population, the growth rate has a value b. So f2(∞)=b. The properties of f2 are:

$$f2 \equiv \begin{cases} x2 > 0, b > 0 \\ f2(x2) = 0 \\ f2(\infty) = b \end{cases}$$

Let us assume that the number of prey killed by each predator is a monotone increasing function of prey number. Let f3(x) be this number. Then f3(0)=0, f3(∞)=d and f3(x) ≅ e x for small and medium values of x.

$$f3(x) \equiv \begin{cases} d > 0, e > 0 \\ f3(0) = 0, f3(\infty) = d \\ f3(x) \cong ex \text{ for small } x \end{cases}$$

Other details regarding ecological model can be found in [ART93].

3 Qualitative model

Ideas from [KUI86] are here used to represent the qualitative knowledge of the systems. This knowledge is represented by a set of variables, a set of important values of these variables and set of functional constraints between them. Monotone and nonmonotone functional constraints are used, and the derivative operator and arithmetic operators do. A list of couples of points to specify the properties of functional constraints is here given. The qualitative model of the system is

$$a > 0$$

$$b > a, c > 0, d > 0$$

$$0 < x0, x0 < x1$$

$$f1: (-\infty,-\infty), (0,a), (x0,b), (x1,0), (+\infty,-\infty)$$

$$f2: (-\infty,-\infty), (x2,0), (+\infty,c)$$

$$f3: (-\infty,-\infty), (0,0), (+\infty,d)$$

$$\frac{dx}{dt} = x\, f1(x) - f3(x)\, y$$

$$\frac{dy}{dx} = f2(x)\, y$$

Represent our knowledge of the system in the qualitative model. In this model the derivative operator and the arithmetic operators, *, -, appear. Properties of the functions f1, f2, f3 are represented by a list of couples of points that are in their graph. Let us assume these functions are continuous. To use the knowledge regarding the relations between f3(x) and x we put the model in the following form

$$\frac{dx}{dy} = f3(x)\,(g(x) - y)$$

$$\frac{dy}{dx} = f2(x)\,y$$

where $g = \dfrac{x\,f1(x)}{f3(x)}$

is a new operator defined by the list of couples of points

$$g: (-\infty, -\infty), (0, a/e), (xi0, b/e), (x1, 0), (+\infty, -\infty)$$

and xi0 is a point near x0.

The properties of g can be easily obtained by its definition. Qualitative analysis to obtain the long-run behaviour of the model is used. This analysis look for the equilibria of the systems and then it studies their stabilities. The system has three equilibrium. These equilibrium verifies

- E1 : x=0, y=0
- E2 : x=x1, y=0
- E3 : x=x2, y2=g(x2)

The stability of equilibria are obtained from the jacobian matrix

$$J = \begin{pmatrix} f3'\,(g - y) + f3\,g' & -f3 \\ f2'\,y & f2 \end{pmatrix}$$

This matrix in the equilibrium E3 is

$$J3 = \begin{pmatrix} f3(x2)\,g'(x2) & -f3(x2) \\ f2'(x2)\,y2 & 0 \end{pmatrix}$$

and the characteristic polynomial is $P(\lambda) = \lambda^2 + a1\lambda + a2 = 0$ with

$$a1 = -f3(x2)\,g'(x2)$$

$$a2 = f3(x2)\,f2'(x2)\,y2$$

The equilibrium E3 is stable if a1>0, a2>0. So E3 is stable if xi0<x2<x1.If x2>x1 then E3 is unstable but equilibrium E2 is stable. If x2 < xi0 then all equilibrium are unstable. In this case bifurcation theory of dynamical systems can help to find new attractors.
Hoft bifurcation says that if a stable equilibrium loses its stability, because a couple of eigenvalues of jacobian matrix change the sign of imaginary part, then a limit cycle appears. This condition occurs when the coefficient a1 changes from minus to plus and a2 remain positive. This bifurcation appears when the equilibrium x2 cross the point xi0. So we can conclude that if x2 < xi0 all the equilibria are unstable but a limit cycle exists.
The analysis above shows that the attractor of the system depends on the relative position of points x2, x1 and xi0. There are three possibilities:

1. xi0<x2<x1: The system has a point attractor where x and y are positive.
2. 0<x2<xi0: The system has a cycle limit attractor
3. x1<x2: The system has a point attractor where x is positive and y=0.

In all cases the system has only one attractor.
More detail about dynamical systems and bifurcations theory can be found in[ART93] and [HIS75].

4. Qualitative Trajectories

From qualitative analysis of dynamical systems we know that every trajectory in a system ends in a attractor. So in case 1 the trajectories will reach an equilibrium in the long run. In case 3 it is the same, but in this equilibrium there are no predators. Case 2 is different because the trajectories will oscillate in the long run.
To obtain the short run of the trajectories we have used ideas from [SAC90]. The phase space is show in figures 1 and 2 for the cases 1 and 2 above.

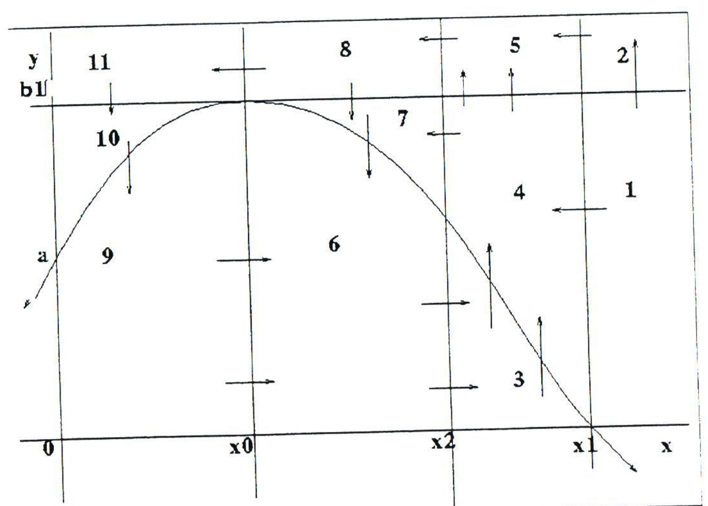

Figure 1

The phase space is divided into regions by curves. So g(x)-y=0 is between regions 3 and 4, 6 and 7, 9 and 10 in figure 1. Other curves are x=0, x=xi0, x=x2, x=x1, y=0 and y=b/eb.

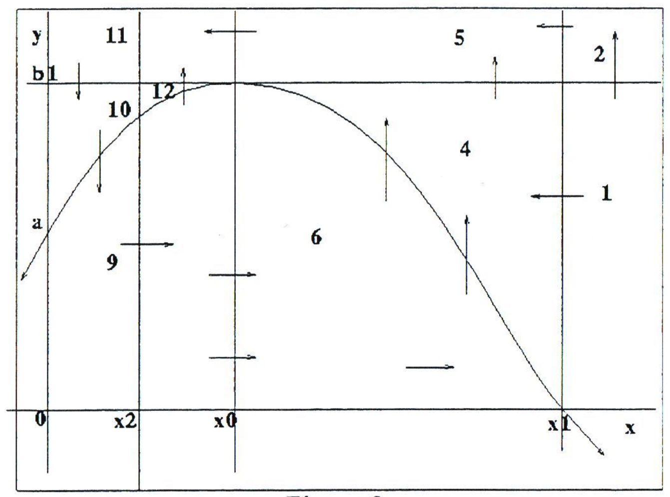

Figure 2

The trajectories of the system cross from a region to other neighbouring region as is shown in the figure. It is obtained as in [SAC90]. For a trajectorie cross from a region r1 to other r2 via boundary u, its tangent t at the intersection point with u must form an acute angle with the normal n to u. In algebraic terms, the inner product t.n must be positive. In our case it is posible to conclude the sign of this product between every two regions.

From regions in the phase space we can obtain a directec graph. The vertexes of this graph are the regions and the stable equilibria. The graph associated to figure 1 is is figure 3.

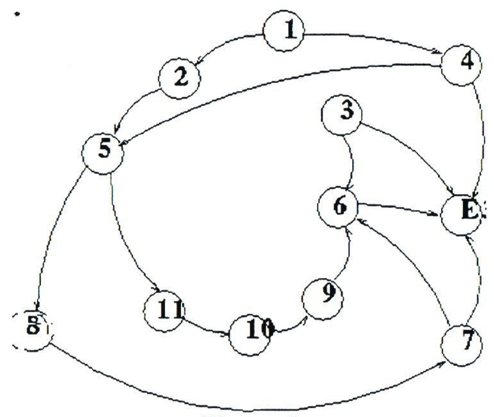

Figure 3

Qualitative trajectories are defined by this graph. So a possible trajectorie is 1,2,5,8,11, 10,9,6,3,4,7,6,3,E3. The graph associated to case 2 do not have vertexes associated to stable equilibria. Other properties of dynamical system can be used to conclude that in this case the system has an stable oscillation.

If the function f1 above does not have maximum the case 2 do no appear . So the system do not have a stable oscillation. The qualitative properties of f1 determine if the system will reach an equilibrium or an stable oscillation in the long-run behaviour.

5 Conclusions

Different methodologies can be used to represent qualitative knowledge regarding a system. In this case we have used qualitative modelling. We have applied dynamical

systems theory to obtain conclusion about the behaviour of the systems. Some results of this analysis can be obtained in a automated way, but more research is needed.

References

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