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ORIGINAL ARTICLE

Risk analysis sampling methods in terrorist networks based on the Banzhaf value

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This article introduces the Banzhaf and the Banzhaf–Owen values as novel measures of risk analysis of a terrorist attack, determining the most dangerous terrorists in a network. This new approach counts with the advantage of integrating at the same time the complete topology (i.e., nodes and edges) of the network and a coalitional structure on the nodes of the network. More precisely, the characteristics of the nodes (e.g., terrorists) of the network and their possible relationships (e.g., types of communication links), as well as coalitional information (e.g., level of hierarchies) independent of the network. First, for these two new measures of risk analysis, we provide and implement approximation algorithms. Second, as illustration, we rank the members of the Zerkani network, responsible for the attacks in Paris (2015) and Brussels (2016). Finally, we give a comparison between the rankings established by the Banzhaf and the Banzhaf–Owen values as measures of risk analysis.

KEYWORDS
Banzhaf value, cooperative games, networks, ranking, sampling

1 | INTRODUCTION

Over recent years, new jihadist cells have formed relatively easily in the Western World with the aim of committing terrorist acts. One of the recruitment organizations most active and dangerous is known as the Zerkani network.

The Zerkani network was named after Khalid Zerkani, a Moroccan who was living in the Brussels municipality of Molenbeek and that was introduced into the network by Reda Kriket. On November 13, 2015, the Islamic State carried out simultaneous attacks in several places in France, such as the Bataclan concert hall, killing more than a hundred people and wounding hundreds. Few months later, on March 22, 2016, part of those who were involved behind of the Paris attacks managed to launch another massive attack in Brussels, detonating suicide bombs at Zaventem International Airport and in Maelbeek subway station, killing 32 people and injuring many more. The main figures responsible for the tactical operations of the Paris and Brussels' attacks were Abdelhamid Abaaoud and jihadist recruiter Khalid Zerkani. The Zerkani network provided personnel, training, planning, attack, escape, and evasion. Although the individual exact role of Zerkani is not entirely clear, he was found guilty of being one of those responsible for perpetrating both attacks mentioned above.

Due to the increase in both frequency and intensity of these attacks in recent years, as well as their damage to society, it is essential to enhance investigations to neutralize potential attack attempts. For this purpose, one of the key aspects is to evaluate the risk analysis of a potential terrorist attack identifying and, immediately thereafter, determining the relevance of each of the key members of the network. However, since the resources of agencies (for instance, the police or the national intelligence services, among others) are limited, they must be allocated in an optimal way in order to stop potential terrorist attacks before they occur. Among many other things, a key issue is to identify the main leaders of a network. It is well-known that breaking up such a network, by identifying its members, immediately neutralizes its criminal activity.

From a purely mathematical point of view, the data of a terrorist network can be studied through the construction of a graph. The terrorists become the nodes and the edges represent the interaction between each pair of these individuals. Once the network structure is built, there are several ways to rank their members by their relevance in the organization. The most used methods for ranking are based on standard network measures, such as degree centrality, betweenness centrality, and closeness centrality (see, for details, Koschade, 2006). However, the main drawbacks of this kind of measures focus on the fact that they only consider the structure of the network. In this line of research, other approaches have been considered in literature as alternative. For instance, the standard social network centrality approach has been also used in analyzing criminal and terror networks. See, for instance, Sparrow (1991), Peterson (1994), Klerks (2001), Carley et al. (2003), or Farley (2003). Over the years, the interest in the subject is evident, as it is shown in Calderoni (2012), Berlusconi et al. (2016), Calderoni et al. (2017, 2020), Grassi et al. (2019), Catino et al. (2022), and Bertoni et al. (2022), among others, to analyze the creation of networks as power or crimes organizational instruments.

One common issue to all these proposals is the fact that not considering valuable information about the communication between the members of the network. These deficiencies are solved in Lindelauf et al. (2013) and in Husslage et al. (2015), by taking into account the heterogeneity of links and nodes through the use of cooperative game theory. Risk analysis and game theory are strongly connected (Cox, 2009). Noncooperative game-theoretical security models have been used, for example, to improve process plant protection from terrorist attacks (Zhang & Reniers, 2016). Instead in this article, we focus on the perspective of cooperative games with transferable utility, called TU-games to introduce new risk analysis measures in networks, in particular, when dealing with terrorist networks. Usually, TU-games are considered for the modeling of those multiagent interactive situations, in which collaboration of involved individuals in groups is a key aspect to achieve a common goal. When applying them on terrorist networks, a ranking of the members of the network according to its relevance can be determined. For instance, the members of the Zerkani network are ranked in Hamers et al. (2019) accordingly to the Shapley value (Shapley, 1953) (see Algaba et al., 2019b, for an updating of results about this value) for the TU-games considered in Husslage et al. (2015). However, other coalitional values, as the well-known Banzhaf value (Banzhaf, 1964) for TU-games, can be used as a criterion. Although it is originally considered for voting situations, its usage is extended to general TU-games in Owen (1975). Besides, it is used in Owen (1986) when restrictions of communication are represented by graphs, being characterized by Alonso-Meijide and Fiestras-Janeiro (2006), in this context. Applications of the Banzhaf value also arise in electrical engineering (Chow, 1961), in computation (Ben-Or & Linial, 1985), in genetic (Lucchetti et al., 2010), and even, it can be used as a design tool in coalitional control (Muros et al., 2017). However, the use of the Banzhaf value, that at first assume that all coalitions can be formed, can certainly be limited because there exist some situations in real world in which cooperation among players may be restricted. Namely, transferable utility games with *a priori* unions (or TU-games with *a priori* unions) are introduced to model these situations, with multiple applications in fields as political science, logistics, or cost allocation problems, among others. In fact, in this setting, the Banzhaf–Owen value (Owen, 1982) is introduced generalizing the Banzhaf value for TU-games with *a priori* unions.

In practice, the main drawback of these values is computational (see Deng & Papadimitriou, 1994, for details). However, such solutions can be obtained in polynomial time for certain classical applications of cooperative game theory. See, for instance, Littlechild and Owen (1973) for the case of the Shapley value in airport games; the exact expression the Banzhaf value for microarray games is obtained in Lucchetti et al. (2010); and Leech (2002), Algaba et al. (2003), Alonso-Meijide and Bowles (2005), or Algaba et al. (2007) compute exactly coalitional values in voting situations. The development of heuristics and exact solutions to find voting systems that generate a power distribution can be found in Kurz and Napel (2014). For a more general setting, sampling methodologies (Cochran, 2007) have become increasingly important as an alternative solution to the computational issues raised above. We refer to Fernández-García and Puerto-Albandoz (2006) and Castro et al. (2009) for the Shapley value estimation, and to Bachrach et al. (2010) for the Banzhaf value estimation. In settings with *a priori* unions, Saavedra-Nieves and Fiestras-Janeiro (2021) use sampling for estimating the Banzhaf–Owen value. As immediate applications, Saavedra-Nieves and Saavedra-Nieves (2020) estimate the random arrival rule (the Shapley value) for bankruptcy games, or Hamers et al. (2019) rank the members of a terrorist network by its relevance, estimating the Shapley value.

In this article, we focus on ranking the members of the Zerkani network considering the existence of different degrees of relationships among them. With this aim, we focus on those coalitional values inspired by the approach of the Banzhaf value (Banzhaf, 1964). Given the natural interpretation they present in terms of the average marginal contribution, their application in contexts other than voting systems is of interest. First, we focus on the Banzhaf value, following the results obtained by Hamers et al. (2019) for the Shapley value. In other direction, the possible affinities among the members of a network can be naturally described in terms of an *a priori* coalitional structure, which makes possible the extension of the Banzhaf value through the definition of TU-games with *a priori* unions. This strongly justifies the introduction of the Banzhaf–Owen value in this context, as mechanisms of ranking since let enrich the information about the terrorist network for obtaining the rankings. Due to the difficulties in computing, in an exact way, both of them, we make use of sampling techniques for approximating these results. Specifically, we consider the procedure in Bachrach et al. (2010) for estimating the Banzhaf value, and we vary the proposal in Saavedra-Nieves and Fiestras-Janeiro (2021), by adding the hypothesis of replacement in sampling, for approximating the Banzhaf–Owen value. Once they are obtained, it is possible to rank the terrorists from the Zerkani network according to the decreasing order of both estimations with the purpose of achieving a risk analysis on a potential terrorist attack.

The work is organized as follows. Section 2 introduces the basic notation on TU-games and on coalitional values. Section 3 deals with the sampling methodologies for estimating the Banzhaf value and the Banzhaf–Owen value as new measures of risk analysis. Section 4 focuses on ranking the members of the Zerkani network through the approximations of the Banzhaf value and the Banzhaf–Owen value. Four appendices are included in the Online Resource Section. Appendix A numerically details the ranking obtained under the estimations of the Banzhaf value. Appendix B depicts the one obtained under the Banzhaf–Owen value estimation. Appendix C contains a comparison of several possibilities of *a priori* unions system. Appendix D includes the R code used for obtaining the results.

2 | RISK ANALYSIS MEASURES FOR NETWORKS BASED ON SOLUTIONS FOR TU-GAMES

This section introduces some basic notions on cooperative game theory and on graph theory to provide a better understanding of the rest of the manuscript.

Formally, a TU-game is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players, and $v : 2^n \longrightarrow \mathbb{R}$ is a map satisfying that $v(\emptyset) = 0$. A coalition is a subset of players $S \subseteq N$ and $v(S)$ denotes the maximum value that players in S can receive by their cooperation. From now on, $\mathcal{G}^N$ denotes the set of all TU-games with set of players N.

Here, we mainly follow the ideas in Lindelauf et al. (2013), through the usage of two specific classes of games, which take into account both the structure of the network and the relational and the individual strength of the members of a network. These two games have already been used to analyze the Zerkani network, although the first of them was slightly modified with respect to the original TU-game introduced in Husslage et al. (2015).

Formally, a network can be represented by an undirected graph $G = (N, E)$, where N denotes the node set of the graph that represents the set of members of the network and E is the set of links, that describes all relationships that exist between these members. A relationship between member i and j is denoted by ij, with $ij \in E$. Notice that $ij \in E$ if and only if $ji \in E$.

Thus, if a coalition $S \subseteq N$ forms, the subnetwork G_S is defined by the members of S and its links in E, that is, $G_S = (S, E_S)$ where $E_S = \{ij \in E : i, j \in S\}$. A coalition $S \subseteq N$ is said to be a connected coalition, if the network G_S is connected, otherwise, S is called disconnected.

Associated to any network $G = (N, E)$, the influence and the relations of individuals in a network G can be modeled through two parameters. First, considering the influence of individuals in $G = (N, E)$, represented by a set of weights on player set N, that is, $\mathcal{I} = \{w_i\}_{i \in N}$ with $w_i \geq 0$. Second, taking into account the relational strength between members of the network, given by a set of weights on the edges E, that is, $\mathcal{R} = \{k_{ij}\}_{ij \in E}$ with $k_{ij} \geq 0$.

The *weighted connectivity TU-game (wconn)* (N, v^{wconn}), and the *additive weighted connectivity TU-game (awconn)* (N, v^{awconn}), with respect to $G = (N, E)$ based on $\mathcal{I}$ and $\mathcal{R}$ were introduced in Husslage et al. (2015). Let f be a non-negative function depending on coalition S, the influence of individuals represented by $\mathcal{I}$, and the strength between them given by $\mathcal{R}$. This function is usually a measure of the effectiveness of coalitions in the network which reflects the situation and information at hand. An example of function f used in this context is the one introduced in Lindelauf et al. (2013), and that is shown below. That is, for each $S \subseteq N$ in a given network G, we have that

$$f(S, \mathcal{I}, \mathcal{R}) = \begin{cases} \left(\sum_{j \in S} w_j\right) \cdot \max_{lh \in E_S} k_{lh}, & \text{if } |S| > 1, \\ w_S, & \text{if } |S| = 1. \end{cases} \quad (1)$$

This map assigns to each possible coalition S the sum of the weights of their members multiplied by the maximum weight on the set of relationships connecting the subnetwork induced by S.

Namely, for each $S \subseteq N$, let Σ_S the set of components (maximal connected coalitions) in G_S, the weighted connectivity game, (N, v^{wconn}), and the additive weighted connectivity game, (N, v^{awconn}), are, respectively, given by the following expressions in a general framework:

$$v^{\text{wconn}}(S) = \begin{cases} f(S, \mathcal{I}, \mathcal{R}), & \text{if } S \quad \text{connected}, \\ \max_{T \in \sum_S} v^{\text{wconn}}(T), & \text{if } S \quad \text{disconnected}, \end{cases} \quad (2)$$

$$v^{\text{awconn}}(S). = \begin{cases} f(S, \mathcal{I}, \mathcal{R}), & \text{if } S \quad \text{connected}, \\ \sum_{T \in \sum_S} v^{\text{awconn}}(T), & \text{if } S \quad \text{disconnected}. \end{cases} \quad (3)$$

Observe that the worth of each disconnected coalition, in the weighted connectivity game, is based on the most effective component of this coalition whereas in the additive weighted connectivity game[1] all components or maximal connected subsets of a disconnected coalition S are effective.

[1] It is important to emphasize that this definition of game is consistent and it has been widely used to analyze networks derived from graphs as in communication situations (Myerson, 1977) or in more general settings reflecting communication properties as hypergraphs communication situations (van den Nouweland et al., 1992), union stable systems (Algaba et al., 2001a, Algaba et al., 2001b), or voting structures to describe problems in which there exists a feedback between the economic influence and the

A payoff vector $z = (z_i)_{i \in N} \in \mathbb{R}^n$ is a vector where z_i represents the payoff associated to player i by its collaboration in a given TU-game (N, v). In general, a solution concept (a solution) is a map $\phi : \mathcal{G}^N \to \mathbb{R}^n$ that assigns to each TU-game (N, v) a payoff vector.

A well-known solution concept for TU-games is the Banzhaf value. It was introduced in Banzhaf (1964) for simple games and later extended to general TU-games (Owen, 1975). Let $(N, v) \in \mathcal{G}^N$, the Banzhaf value is formally defined for all $i \in N$ as

$$Bz_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)). \tag{4}$$

This solution can be interpreted as the average of marginal contributions of a player to those coalitions that not contain it. In fact, the average of the marginal contributions of each player is indicative of its influence on the overall set of players, as it is the case in voting situations.

A TU-game with *a priori* unions is a triplet (N, v, P) where $(N, v) \in \mathcal{G}^N$ and $P = \{P_1, \dots, P_m\}$ is a partition of N. In this case, we assume that P is a coalition structure that restricts the cooperation among the players in N. The set of all TU-games with *a priori* unions with set of players N will be denoted by $\mathcal{U}^N$.

The Banzhaf–Owen value (Owen, 1982) is an extension of the Banzhaf value for TU-games with *a priori* unions. Let $(N, v, P) \in \mathcal{U}^N$, for all $i \in N$, the Banzhaf–Owen value is given by

$$BzO_i(N, v, P) = \sum_{R \subseteq P \setminus P_i} \frac{1}{2^{m-1}} \sum_{S \subseteq P_i \setminus \{i\}} \frac{1}{2^{p_i - 1}} (v(\cup_{P_l \in R} P_l \cup S \cup \{i\}) - v(\cup_{P_l \in R} P_l \cup S)), \tag{5}$$

where $P_i \in P$ such that $i \in P_i$ and $p_i = |P_i|$. Besides, a coalition $T \subseteq N \setminus \{i\}$ is compatible with partition P for a player $i \in N$, if $T = \cup_{P_l \in R} P_l \cup S$ for a coalition of unions $R \subseteq P \setminus P_i$ and a coalition of players $S \subseteq P_i \setminus \{i\}$.

In general, solutions for TU-games with *a priori* unions assume that the players in an union act jointly, so only contributions of a player to the coalitions formed by a subset of full unions and the players in its own union are averaged. However, if the coalition structure is formed by unitary unions or only by the grand coalition, the Banzhaf–Owen value and the Banzhaf value prescribe the same allocation.

Once both approaches of TU-games for modeling networks are formally introduced, we note that, in line with Husslage et al. (2015), solution concepts can be applied for the two games for providing a game-theoretic centrality measure. Let $G = (N, E)$ be a network based on $\mathcal{I}$ and $\mathcal{R}$,and let (N, v^{wconn}) and (N, v^{awconn}) be the TU-games associated to G. From now on, we will focus on two new centrality measures arisen of considering the Banzhaf and Banzhaf–Owen values for the two TU-games (N, v^{wconn}) and (N, v^{awconn}).

political power (Algaba et al., 2019a). Another framework is to analyze networks with hierarchical and communication features, simultaneously, as in Algaba et al. (2018).

3 | SAMPLING PROCEDURES TO ESTIMATE THE BANZHAF VALUE AND THE BANZHAF–OWEN VALUE

Although the notion of marginal contribution of a player is intuitively clear, computing the Banzhaf and the Banzhaf–Owen values becomes a computationally difficult task when the amount of players involved in the TU-game substantially increases. This fact justifies the needing of searching alternatives for providing good approximations of both solutions.

Along this section, we formally present those algorithms used for estimating the two above-mentioned values.

3.1 | A sampling procedure to estimate the Banzhaf value

We want to estimate the Banzhaf value of a TU-game (N, v). Following Bachrach et al. (2010), we formalize a procedure for estimating the Banzhaf value when the number of players involved is sufficiently large. The steps of the sampling procedure are the ones depicted below:

- The sampling population is the set of all coalitions of $N \setminus \{i\}$.
- The parameter to be estimated is $Bz_i(N, v)$, that is, the player i's Banzhaf value.
- The characteristic to study in each sampling unit, $T \subseteq N \setminus \{i\}$, is the player i's marginal contribution to coalition T. That is,

$$x(T)_i = v(T \cup \{i\}) - v(T).$$

- We take with replacement a sample of ℓ coalitions in $N \setminus \{i\}$, that is, $\mathcal{T} = \{T_1, \dots, T_\ell\}$, with $T_j \subseteq N \setminus \{i\}$ for all $j = 1, \dots, \ell$ and $1 < \ell \leq 2^{n-1}$.
- The estimation of $Bz_i(N, v)$, for every $i \in N$, is the mean of the marginal contributions over the sample, that is, $\overline{Bz}_i = \frac{1}{\ell} \sum_{j=1}^{\ell} x(T_j)_i$ where ℓ denotes the sampling size.

Once we apply this procedure for each player, the vector $\overline{Bz} = (\overline{Bz}_1, \dots, \overline{Bz}_n)$ corresponds to the estimation of the Banzhaf value of all players in (N, v). A fundamental issue in the problem focuses in bounding the error in the estimation, which is often not possible to be measured in practice. For this reason, the following probabilistic bound can be theoretically provided instead:

$$\mathbb{P}(|\overline{Bz}_i - Bz_i| \geq \varepsilon) \leq \alpha, \text{ with } \varepsilon > 0 \text{ and } \alpha \in (0, 1].$$

Let $\varepsilon > 0$, $\alpha \in (0,1)$, and (N, v) be a TU-game. If $r_i = \max_{R,R' \subseteq N\setminus\{i\}} (x(R)_i - x(R')_i)$, then it is satisfied that

$$\ell \geq \min\left\{\frac{1}{4\alpha\varepsilon^2}, \frac{\ln(2/\alpha)}{2\varepsilon^2}\right\} r_i^2 \text{ implies that } \mathbb{P}(|\overline{Bz}_i - Bz_i| \geq \varepsilon) \leq \alpha. \quad (6)$$

Thus, the estimated Banzhaf value usually becomes a good approximation of the real one when sampling sizes sufficiently enlarge (see, for details, Bachrach et al., 2010).

Thus, determining the range of the marginal contributions in this setting plays a fundamental role in the analysis of the error. Specifically, for the TU-games in (2) and (3), such range fundamentally depends on the function f considered to measure the effectiveness of coalitions.

In what follows, we provide results to bound the error in estimating centrality measures in networks when considering the effectiveness function f for coalitions given in (1).

Proposition 1. *Let (N, v^{wconn}), (N, v^{awconn}) be the weighted connectivity and the additive weighted connectivity TU-games associated to a given network G by using the function in (1). For every $i \in N$, it is satisfied that*

$$r_i = \left(\sum_{j\in N} w_j\right) \cdot max_{lh\in E_N} k_{lh}. \quad (7)$$

Proof. This result readily follows from the fact that the marginal contributions satisfies, for both approaches of TU-games considered, that

$$0 \leq x(S)_i \leq \left(\sum_{j\in N} w_j\right) \cdot \max_{lh\in E_N} k_{lh}$$

for any agent i and for any coalition S in N. □

The following corollary can be immediately obtained.

Corollary 1. *Let $\varepsilon > 0$, $\alpha \in (0,1)$, and (N, v^{wconn}) or (N, v^{awconn}) the weighted connectivity game or the additive weighted connectivity game associated to a given network G by using the function in (1). If ℓ satisfies that*

$$\ell \geq \min\left\{\frac{1}{4\alpha\varepsilon^2}, \frac{\ln(2/\alpha)}{2\varepsilon^2}\right\}\left(\left(\sum_{j\in N} w_j\right) \cdot max_{lh\in E_N} k_{lh}\right)^2, \quad (8)$$

then, $\mathbb{P}(|\overline{Bz}_i - Bz_i| \geq \varepsilon) \leq \alpha$, for every $i \in N$.

3.1.1 | A two-stage sampling procedure to estimate the Banzhaf–Owen value

Now, we follow the ideas of the two-stage procedure in Saavedra-Nieves and Fiestras-Janeiro (2021) to approximate the Banzhaf–Owen value. However, unlike the approach given in Saavedra-Nieves and Fiestras-Janeiro (2021), we add the hypothesis of replacement at both steps. Thus, given a TU-game with an *a priori* coalitional structure (N, v, P), with $P = \{P_1, \dots, P_m\}$ and a fixed arbitrary player $i \in N$, the procedure for estimating the Banzhaf–Owen value is described below:

- The sampling population is the set of all compatible coalitions with P for player i.
- The parameter to be estimated is $BzO_i(N, v, P)$, for all $i \in N$.
- The characteristic to be studied in each sampling unit corresponds to i's marginal contribution for each coalition T that is compatible with P for i. If we consider $T \subseteq N \setminus \{i\}$ in terms of $R = \{P_k : P_k \subset T\}$ and $S = T \cap P_i$, then

$$x(R, S)_i = v(\cup_{P_l\in R} P_l \cup S \cup \{i\}) - v(\cup_{P_l\in R} P_l \cup S).$$

- The sampling procedure involves two steps:
  - First, we take with replacement a sample $\mathcal{R} = \{R_1, \dots, R_{\ell_r}\}$ of ℓ_r coalitions $R_j \subseteq P \setminus P_i$.
  - After, for every $R_j \in \mathcal{R}$, we choose with replacement a sample $\mathcal{S}_{R_j} = \{S_{j1}, \dots, S_{j\ell_s}\}$ of ℓ_s coalitions $S_{jk} \subseteq P_i \setminus \{i\}$.

  As a result, we obtain a sample of $\ell_r\ell_s$ compatible coalitions, where each element takes the form $\cup_{P_l\in R} P_l \cup S_{jk}$ for $j = 1, \dots, \ell_r$, with $1 \leq \ell_r \leq 2^{m-1}$, and $k = 1, \dots, \ell_s$, with $1 \leq l_s \leq 2^{p_i-1}$.
- The mean of the marginal contribution vectors over the sample corresponds to the estimation of the Banzhaf–Owen value. That is,

$$\overline{BzO}_i = \frac{1}{\ell_r}\sum_{j=1}^{\ell_r}\left(\frac{1}{\ell_s}\sum_{k=1}^{\ell_s} x(R_j, S_{jk})_i\right),$$

  where ℓ_r and ℓ_s are the sampling sizes.

By applying this procedure for all $i \in N$, we obtain $\overline{BzO} = (\overline{BzO}_1, \dots, \overline{BzO}_n)$ which corresponds to the estimation of the Banzhaf–Owen value for (N, v, P). Besides, we consider $\overline{BzO}_i^{R_j} = \frac{1}{\ell_s}\sum_{k=1}^{\ell_s} x(R_j, S_{jk})_i$ as the unbiased estimator of $BzO_i^{R_j} = \frac{1}{2^{p_i-1}}\sum_{S\subseteq P_i\setminus\{i\}} x(R_j, S)$, that is, the theoretical mean of player i's marginal contributions using all compatible coalitions with coalition of unions R_j.

Below, we focus on analyzing the properties of the estimator of the Banzhaf–Owen value of player i, $\overline{BzO}_i$, from a statistical perspective. First, according to the definition of this estimator, it is unbiased because

$$\mathbb{E}(\overline{BzO}_i) = \mathbb{E}_1(\mathbb{E}_2(\overline{BzO}_i)) = BzO_i,$$

being $\mathbb{E}_1(\cdot)$ and $\mathbb{E}_2(\cdot)$ the mean operators in coalitions of unions and in coalitions into the union which i belongs to, respectively.

Besides, if $\text{var}_1(\cdot)$ and $\text{var}_2(\cdot)$ are the operators for variances with respect to the coalitions of unions and with respect to the coalitions into the unions which i belongs to, respectively, then the variance of $\overline{BzO}_i$ is

$$\text{var}(\overline{BzO}_i) = \text{var}_1(\mathbb{E}_2(\overline{BzO}_i)) + \mathbb{E}_1(\text{var}_2(\overline{BzO}_i)), \quad (9)$$

or, equivalently,

$$\text{var}(\overline{BzO}_i) = \frac{1}{\ell_r}\left(\theta_a^2 + \frac{\theta_b^2}{\ell_s}\right), \quad (10)$$

where

$$\theta_a^2 = \frac{1}{2^{m-1}-1}\sum_{R\subseteq P\setminus P_i}\left(BzO_i^R - BzO_i\right)^2, \text{ and}$$

$$\theta_b^2 = \frac{1}{2^{m-1}}\sum_{R\subseteq P\setminus P_i}\left(\frac{1}{2^{p_i-1}-1}\sum_{S\subseteq P_i\setminus\{i\}}\left(x(R,S)_i - BzO_i^R\right)^2\right).$$

Roughly speaking, θ_a^2 refers to the variability of the means of unions with respect to BzO_i, θ_b^2 denotes the average of the variances of the marginal contributions with respect to its theoretical mean BzO_i^R, for each $R \subseteq P \setminus P_i$.

Notice that the hypothesis of nonreplacement is analyzed in Saavedra-Nieves and Fiestras-Janeiro (2021). However, the main properties on the estimator are still satisfied as well as the main conclusions about the difficulties in obtaining probabilistic bounds of the incurred error.

4 | COMPUTATIONAL ANALYSIS OF THE RISK IN ZERKANI NETWORK

In this section, we analyze, from a purely computational point of view, the ranking problem of the terrorists within the Zerkani network, applying both the approximation of the Banzhaf value and the one of the Banzhaf–Owen value, which have not been never used as centrality measures. For this purpose, we will use the *wconn* and *awconn* TU-games, given in (2) and (3), respectively. In addition to the numerical results associated to these approximations, Appendix D in the Online Resource Section describes the R code required for obtaining the results under both approaches.

In particular, for the case of the Zerkani network, the function f considered is specifically given by the function in (1). In order to get the rankings, it has been necessary to use the approximating procedures considered in Section 3, since the Zerkani network contains 47 individuals. We compare the rankings obtained and show the analysis computational of the process.

4.1 | The Zerkani network

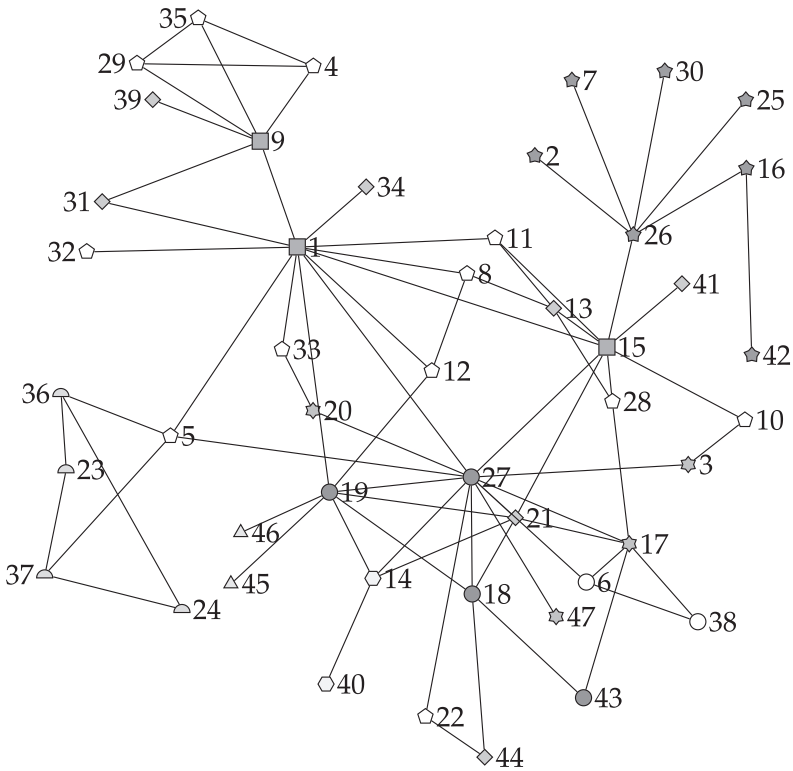
Now, we briefly describe the structural organization of the Zerkani network. Figure 1 shows its associated graph, that is built by using the information of the 47 individuals belonging to it.

By means of 11 possible relationships, different extra weights are assigned to the edges of the associated network, since initially they all have value 1. The same happens with the members, by adding a weight to their nodes according to their influence in the network. We consider the weights given in Hamers et al. (2019) on the links (edges) and on the members (nodes) of the network that appears in Table 1.

Since there are several individuals who are linked through two of these links, there are a total of 13 weights corresponding to the different connections between terrorists. For instance, Abdelhamid Abaaoud (weight equal to 4), Fabien Clain (4), Khalid Zerkani (5), Miloud F. (2), and Mohamed Belkaid (3) have associated a weight larger than one.

In our risk analysis, we focus on the top-10 of the rankings, although we completely rank all 47 individuals. As mentioned, the most important task consists of identifying the most dangerous terrorists.

In order to establish the rankings of terrorists, we consider two quite different schemes, although both of them are based on the Banzhaf value. First, we studied the problem from a myopic perspective without taking into account the possible affinities among members of the network. After this, we will introduce these features of the terrorists in the Zerkani network, in terms of an adequate partition, since adding this interesting information enriches the input data and it may be fundamental. As the TU-games, *wconn* and *awconn*, already consider network connectivity, as well as the weights of individuals and their relationships, we have taken into account in this article the partition describing the features of the terrorists in the network. More specifically, the considered partition has 10 unions, $P = \{P_1, P_2, \ldots, P_{10}\}$. The first union, P_1, groups the high ranks of the network, that is, majors and those devoted to recruit terrorists. Union P_2 corresponds to the associated to the upper-level charges. Next, those who have been recruited or who are under the authority of a major lead to union P_3. One of the relationships to take into account is to travel with, since during travels can be created strong relationships or it is assumed that can be discovered hide intentions, it gives way to unions P_4, P_5, and P_6. Moreover, inside the Zerkani network, there are several terrorists who also belong to another network, called Kriket and they form another union, P_7. In the same way, there are also two individuals associated with *Forging Ring*, grouped in P_8. The two last groups are due to that two individuals were arrested simultaneously in *Forest* and they are associated without having more details about it, giving way to P_9 and P_{10}. Table2 specifies the members of each union as a



(1) Abdelhamid Abaaoud
(2) Abderrahmane Ameroud
(3) Abid Aberkan
(4) Adrien Guihal
(5) Ahmed Dahmani
(6) Ali Oulkadi
(7) Anis Bari
(8) Chakib Akrouh
(9) Fabien Clain
(10) Fatima Aberkan
(11) Gelel Attar
(12) Hasna Ait Boulahcen
(13) Ilias Mohammadi
(14) Khaled Ledjeradi
(15) Khalid Zerkani
(16) Miloud F.
(17) Mohamed Abrini
(18) Mohamed Bakkali
(19) Mohamed Belkaid
(20) Mohammed Amri
(21) Najim Laachraoui
(22) Osama Krayem
(23) Paris Attacker A
(24) Paris Attacker B
(25) Rabah M.
(26) Reda Kriket
(27) Salah Abdeslam
(28) Souleymane Abrini
(29) Thomas Mayet
(30) Y. A.
(31) Sid Ahmed Ghlam
(32) Ayoub el Khazzani
(33) Mehdi Nemmouche
(34) Reda Hame
(35) Macreme Abrougui
(36) Salzburg Refugee A
(37) Salzburg Refugee B
(38) Ibrahim Abdeslam
(39) AQI
(40) Djamal Eddine Ouali
(41) Soufiane Alilou
(42) AQIM
(43) Ibrahim El Bakraoui
(44) Khalid El Bakraoui
(45) Tawfik A.
(46) Identity Unknown
(47) Hamza Attou

FIGURE 1 Graph of the Zerkani network.

summary. Different geometrical figures and colors allow for identifying members of the Zerkani network who belong to a same element of the partition P in Figure 1.

4.2 | The Banzhaf value approximation in the Zerkani network

As above-mentioned, one of the objectives is to analyze the *top-10* of the ranking according to the risk of the terrorists imposed by the decreasing order of the components of the estimated Banzhaf value in the Zerkani network. In general, the usage of some properties of (N, v^{wconn}) and (N, v^{awconn}), that would reduce the computational complexity, is not possible in this setting. Appendix D.3 describes the R code that specifically implements the procedure in Section 3.1 on the Zerkani network.

By simplicity, we obtain 1000 estimations of the Banzhaf value by using the sampling procedure described in Section 3.1 with a sample size equal to $\ell = 1000$ for the TU-games (N, v^{wconn}) and (N, v^{awconn}), respectively. Besides, we use the theoretical properties that satisfy the resulting estimator to obtain a more exact estimation. By the Central Limit Theorem, when averaging all of these 1000 approximations, the final result is equivalent to obtain a only estimation with $\ell = 10^6$, in both cases. Table 3 shows the theoretical errors provided in Corollary 1 for the problem of estimating the Banzhaf value when $r_i = 300$, as Proposition 1 ensures for the case of the Zerkani network.

Table 4 depicts the top-10 terrorists belonging to the Zerkani network according to the Banzhaf value and the corresponding results. More details can be found in Appendix A that enumerates the overall list of the members of Zerkani network and the Banzhaf value estimations. We remark the

TABLE 1 List of relationships, weights for links, and weights for starting nodes.

Relationships	Weights on links	Extra weight for starting nodes
"Associate of"	2	0
"Brother of"	1	0
"Commander of"	2	2
"Family relationship"	1	0
"Funded"	1	2
"Lived with"	2	0
"Nephew of"	1	0
"Recruiter of"	1	1
"Supporter of"	1	1
"Traveled to Syria with"	2	0
"Traveled with"	2	0
"Associate and traveled with"	4	0
"Traveled and lived with"	4	0

TABLE 2 Members of unions of the partition P considered.

Union	Members
P_1	Abdelhamid Abaaoud, Fabien Clain, and Khalid Zerkani.
P_2	Chakib Akrouh, Gelel Attar, Hasna Ait Boulahcen, Fatima Aberkan, Osama Krayem, Souleymane Abrini, Ayoub el Khazzani, Mehdi Nemmouche, Thomas Mayet, Macreme Abrougui, Ahmed Dahmani, and Adrien Guihal.
P_3	Sid Ahmed Ghlam, Reda Hame, AQI, Ilias Mohammadi, Soufiane Alilou, Najim Laachraoui, and Khalid El Bakraoui.
P_4	Paris Attacker A, Paris Attacker B, Salzburg Refugee A, and Salzburg Refugee B.
P_5	Mohammed Amri, Hamza Attou, Mohamed Abrini, and Abid Aberkan.
P_6	Mohamed Belkaid, Salah Abdeslam, Mohamed Bakkali, and Ibrahim El Bakraoui.
P_7	Reda Kriket, Rabah M., Y. A., Abderrahmane Ameroud, Miloud F., Anis Bari, and AQIM.
P_8	Khaled Ledjeradi and Djamal Eddine Ouali.
P_9	Unknown identity and Tawfik A.
P_{10}	Ibrahim Abdeslam and Ali Oulkadi.

TABLE 3 Theoretical errors (ε) for $\ell = 10^3$ and $\ell = 10^6$.

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
$\ell = 10^3$	1.34069	1.48773	1.78297
$\ell = 10^6$	0.04240	0.04705	0.05638

first 10 positions and we check some differences when the Banzhaf value is applied to the weighted connectivity game (N, v^{wconn}) and the additive weighted connectivity game (N, v^{awconn}). See more details in Table A.1. Notice that Khalid Zerkani, who is considered to be the leader of the network,

TABLE 4 Top-10 of the ranking of terrorists in the Zerkani network, according to the estimated Banzhaf value for (N, v^{wconn}) and (N, v^{awconn}) with $\ell = 10^6$.

	Ranking R_{wconn}		Ranking R_{awconn}	
Pos.	Terrorist	$\overline{Bz}$	Terrorist	$\overline{Bz}$
1	Abdelhamid Abaaoud	38.326053	Mohamed Belkaid	31.411917
2	Salah Abdeslam	35.073561	Salah Abdeslam	26.167231
3	Khalid Zerkani	33.930235	Khalid Zerkani	25.716073
4	Mohamed Belkaid	33.267557	Abdelhamid Abaaoud	24.155854
5	Najim Laachraoui	18.774721	Mohamed Bakkali	18.297970
6	Mohamed Bakkali	18.307468	Najim Laachraoui	17.311352
7	Fabien Clain	11.891287	Fabien Clain	16.512582
8	Reda Kriket	8.316217	Reda Kriket	10.625338
9	Ahmed Dahmani	8.129191	Mohamed Abrini	6.257625
10	Mohamed Abrini	6.94882	Miloud F.	5.832491

goes to the third position under both approaches. However, Abdelhamid Abaaoud occupies the first position under the weighted connectivity game whereas, he moves to the fourth position under the additive weighted connectivity game. With respect to Mohamed Belkaid, we note that he moves up from the fourth position in the weighted connectivity game to the first position when considering the additive weighted connectivity game. Analogous comments can be extracted from the remainder of the list of members in the top-10. However, a thorough analysis of the similarity between the two rankings can be numerically given by the Spearman's correlation coefficient (0.8848) or by the Kendall's correlation coefficient (0.7243) on the positions of the terrorists. In both cases, the correlations are large enough to indicate a high degree of similarity of the rankings.

Once the Banzhaf value is estimated, we check how the sampling proposal for approximating the Banzhaf value performs in terms of variability. By construction, we have obtained the results shown in Table 4, by averaging 1000 estimations of the Banzhaf value for the Zerkani network by using sample sizes equal to $\ell = 10^3$.

Table 5 summarizes, from a purely statistical point of view, the 1000 obtained results for the 10 more relevant terrorists in Zerkani network by using the estimated Banzhaf value with $\ell = 10^3$ in the weighted connectivity game. Notice that the order established for the top-10 in Table 4 can be also maintained when using as criteria the main statistical measures.

Analogous conclusions can be obtained from the case of the additive weighted connectivity game, in view of the statistical summary for the 1000 estimations of the Banzhaf value for the 10 terrorists in the top-10 in Table 4. The numerical results are included in Table 6.

TABLE 5 Statistical summary of the 1000 estimations of the Banzhaf value for the *wconn* game.

	Terrorist	Min.	First Qu.	Median	Mean	Third Qu.	Max.	CV
1	Abdelhamid Abaaoud	36.741	37.990	38.323	38.326	38.693	40.209	0.014
2	Salah Abdeslam	32.228	34.490	35.092	35.074	35.637	37.687	0.024
3	Khalid Zerkani	32.044	33.527	33.904	33.9305	34.326	35.867	0.018
4	Mohamed Belkaid	30.890	32.744	33.260	33.268	33.790	35.804	0.024
5	Najim Laachraoui	16.250	18.347	18.794	18.775	19.239	20.773	0.037
6	Mohamed Bakkali	16.361	17.883	18.305	18.308	18.744	20.740	0.036
7	Fabien Clain	10.578	11.648	11.902	11.891	12.132	13.186	0.032
8	Reda Kriket	7.349	8.115	8.315	8.316	8.528	9.269	0.037
9	Ahmed Dahmani	7.390	7.982	8.133	8.129	8.274	8.922	0.027
10	Mohamed Abrini	6.242	6.810	6.943	6.949	7.084	7.730	0.029

Abbreviation: CV, coefficient of variation.

TABLE 6 Statistical summary of the 1000 estimations of the Banzhaf value for the *awconn* game.

	Terrorist	Min.	First Qu.	Median	Mean	Third Qu.	Max.	CV
1	Mohamed Belkaid	29.127	30.907	31.386	31.412	31.896	33.714	0.023
2	Salah Abdeslam	23.860	25.678	26.182	26.167	26.679	28.434	0.028
3	Khalid Zerkani	24.599	25.440	25.727	25.716	25.956	26.857	0.014
4	Abdelhamid Abaaoud	23.080	23.952	24.157	24.156	24.363	25.172	0.013
5	Mohamed Bakkali	16.420	17.873	18.294	18.298	18.726	20.494	0.034
6	Najim Laachraoui	14.940	16.931	17.300	17.311	17.713	19.119	0.037
7	Fabien Clain	15.875	16.395	16.514	16.513	16.619	17.042	0.010
8	Reda Kriket	10.200	10.519	10.624	10.625	10.733	11.132	0.015
9	Mohamed Abrini	5.836	6.186	6.256	6.258	6.324	6.682	0.018
10	Miloud F.	5.492	5.765	5.833	5.833	5.894	6.136	0.017

Abbreviation: CV, coefficient of variation.

TABLE 7 Statistical summary of the 1000 processing times (in seconds) for the Banzhaf value estimations.

	Min.	First Qu.	Median	Mean	Third Qu.	Max.
User time	5385.142	8715.153	9481.655	9538.363	10,498.958	13,820.818
System time	4.745	6.393	7.842	23.041	14.366	330.296
Elapsed time	5390.902	8721.884	9490.329	9562.124	10,547.094	13,848.808

From these results, we also check the variability of rankings. From a purely quantitative approach, we compute the coefficient of variation (CV), that is, the ratio of the standard deviation of the estimations and its mean. The proximity of their values to zero ensures the low numerical variability of the results, as well as the representativeness of the mean value. From this point of view, Najim Laachraoui and Mohamed Bakkali, and Reda Kriket and Ahmed Dahmani, respectively, reverse their positions under the weighted connectivity game when considering the minimum estimations. Besides, Fabien Clain and Najim Laachraoui also exchange their positions when considering the minimum values for the game (N, v^{awconn}).

We complete this analysis measuring the computational effort required in obtaining these 1000 estimations. Table 7 includes a summary of the processing times (in seconds) for each of these 1000 repetitions. We distinguish between the User time, the System time, and the Elapsed time. Focusing on the User time, we see that more than 75% of the estimations have been obtained in less than 3 hours of real computing time.

4.3 | The Banzhaf–Owen value approximation in the Zerkani network

Along this subsection, we will determine the *top-10* of the ranking according to the risk of the terrorists given by the estimation of the Banzhaf–Owen value in the Zerkani network. Appendix D.4 in the Online Resource Section describes the

TABLE 8 Top-10 of the ranking of terrorists in the Zerkani network, according to the average of 1000 estimations of the Banzhaf–Owen value for (N, v^{wconn}) and (N, v^{awconn}) with $\ell_r = 10^2$ and $\ell_s = 10$.

	Ranking R_{wconn}		Ranking R_{awconn}	
Pos.	Terrorist	$\overline{BzO}$	Terrorist	$\overline{BzO}$
1	Khalid Zerkani	39.498143	Mohamed Belkaid	32.192285
2	Abdelhamid Abaaoud	35.775328	Khalid Zerkani	27.932848
3	Salah Abdeslam	33.400368	Salah Abdeslam	26.959565
4	Mohamed Belkaid	33.380580	Abdelhamid Abaaoud	25.322645
5	Mohamed Bakkali	22.755968	Mohamed Bakkali	22.471120
6	Fabien Clain	12.285880	Fabien Clain	15.886888
7	Ahmed Dahmani	9.879967	Reda Kriket	10.833910
8	Reda Kriket	9.176184	Miloud F.	5.906707
9	Najim Laachraoui	4.701998	Ahmed Dahmani	5.735320
10	Mohamed Abrini	4.589148	Khaled Ledjeradi	5.374710

R code specifically built for applying the second procedure, presented in Section 3.2 and applied to the specific case of the Zerkani network.

From the data about the members of the Zerkani network, we recall that the terrorists of the network have been grouped according to their rank, their function, their direct relationship with some terrorists, among others, in such a way that it gives rise to the creation of the partition above described with 10 unions, $P = \{P_1, P_2, \ldots, P_{10}\}$, as presented in Table 2.

In accordance with the scheme followed for the estimation of the Banzhaf value, we rank the members of the Zerkani network according to the decreasing order of the estimated Banzhaf–Owen value. By simplicity, we obtain 1000 estimations of the Banzhaf–Owen value for the TU-games (N, v^{wconn}) and (N, v^{awconn}), by using the two-stage sampling procedure, described in Section 3.1, with $\ell_r = \min\{10^2, 2^{m-1}\}$ and $\ell_s = \min\{10, 2^{p_i-1}\}$. Note that sample sizes are taken to ensure that they are always smaller than the corresponding population sizes, that is, $2^{m-1}2^{p_i-1}$, for each $i \in N$. Despite dealing with finite populations, we estimate the Banzhaf–Owen value as the average of these 1000 approximations. Table 8 enumerates the top-10 of the most relevant members of the Zerkani network, when ordering according to the estimated Banzhaf–Owen value. A detailed list containing all ranked members is depicted in Table B.2 of Appendix B in the Online Resource Section. Note that Khalid Zerkani is the most influential terrorist under the weighted connectivity approach, when applying the Banzhaf–Owen value, and the second one under the additive weighted connectivity approach. Abdelhamid Abaaoud occupies the second position under the weighted connectivity approach and he goes to the fourth one under the additive one. Now, Salah Abdeslam is in the third position under both approaches. Similar conclusions can be obtained for the remainder of terrorists. From a purely numerical comparative approach, the Spearman's correlation coefficient (0.9092) and the Kendall's correlation coefficient (0.7502) ensure a high degree of similarity between the positions obtained for each terrorist in each of the two rankings considered.

After the Banzhaf–Owen value estimation, we check the variability of our sampling proposal, in this practical situation, through a small simulation study. By focusing only on the 10 most influential network members given by Table 8, we analyze the 1000 obtained results for their estimated Banzhaf–Owen value. Table 9 summarizes these 1000 results for the estimated Banzhaf–Owen value for the case of the weighted connectivity game.

Table 10 summarizes, from a statistical point of view, the 1000 estimations of the Banzhaf–Owen value for the individuals in the top-10 when considering the additive approach.

In view of the results, the rankings obtained generally hold even over the maximum and minimum values of the estimated components of the Banzhaf–Owen value under the two approaches considered. The exception is the exchange of Abdelhamid Abaaoud and Mohamed Bakkali's positions when considering the maximum estimations of the Banzhaf–Owen value for the game (N, v^{awconn}). Regarding to the CV, we observe smaller values, in general, under the additive approach.

Similar as for the Banzhaf value, we analyze the processing times (in seconds) required in the Banzhaf–Owen value estimation as a measure of the effort for this purpose. They are described in Table 11 which includes a summary of these amounts.

4.4 | Assessing the influence of the *a priori* unions system

Hereby, we make a brief discussion on the rankings obtained under the estimations of the Banzhaf value and the Banzhaf–Owen value. To this purpose, we use the sampling procedures considered. For the case of the Banzhaf value, we consider the average of 100 estimations by taking $\ell = 1000$ coalitions of $N \setminus \{i\}$, for each $i \in N$, and the average of 100 estimations of the Banzhaf–Owen value with $\ell_r = 10^2$ and $\ell_s = 10$. Table 12 illustrates the top-10 of the rankings obtained.

Below, we briefly comment the resulting rankings of the members of the Zerkani network. In general, no drastic changes are remarkable in the list of individuals belonging to the top-10 in the obtained rankings, which indicates that both are useful tools for this purpose. However, we can emphasize the fact that most of the positions change when using the different approaches considered in this article. Probably, this may be due to the organizational and logistical role played by such terrorists, in such a way that their weight will change with the approach under consideration. Notice that, others, as the ones who carry out the suicide and, therefore, the action (Salzburg Refugee A or B, among others) usually appear from 11th position onward. Again, the Spearman's

TABLE 9 Statistical summary of the 1000 estimations of the Banzhaf–Owen value for the *wconn* game.

	Terrorist	Min.	First Qu.	Median	Mean	Third Qu.	Max.	CV
1	Khalid Zerkani	33.050	38.069	39.425	39.498	40.956	47.903	0.056
2	Abdelhamid Abaaoud	30.618	34.709	35.761	35.775	36.833	41.350	0.045
3	Salah Abdeslam	27.026	32.232	33.344	33.400	34.560	39.090	0.054
4	Mohamed Belkaid	27.204	32.280	33.386	33.381	34.467	38.894	0.049
5	Mohamed Bakkali	14.74	21.052	22.688	22.756	24.503	32.908	0.117
6	Fabien Clain	9.54	11.753	12.278	12.286	12.803	15.095	0.064
7	Ahmed Dahmani	6.385	9.192	9.861	9.880	10.518	12.970	0.100
8	Reda Kriket	5.792	8.658	9.161	9.176	9.734	11.516	0.092
9	Najim Laachraoui	3.659	4.478	4.708	4.702	4.925	6.117	0.071
10	Mohamed Abrini	3.660	4.370	4.580	4.589	4.814	5.610	0.073

Abbreviation: CV, coefficient of variation.

TABLE 10 Statistical summary of the 1000 estimations of the Banzhaf–Owen value for the *awconn* game.

	Terrorist	Min.	First Qu.	Median	Mean	Third Qu.	Max.	CV
1	Mohamed Belkaid	28.020	31.270	32.131	32.192	33.158	37.035	0.043
2	Khalid Zerkani	23.765	27.077	27.896	27.933	28.790	32.045	0.045
3	Salah Abdeslam	22.468	25.942	26.945	26.960	27.949	31.808	0.053
4	Abdelhamid Abaaoud	22.575	24.647	25.353	25.323	25.976	28.408	0.039
5	Mohamed Bakkali	15.015	20.814	22.424	22.471	24.100	31.750	0.112
6	Fabien Clain	14.413	15.548	15.889	15.887	16.183	17.803	0.031
7	Reda Kriket	9.648	10.579	10.835	10.834	11.077	12.056	0.035
8	Miloud F.	5.501	5.813	5.906	5.907	5.998	6.297	0.023
9	Ahmed Dahmani	4.200	5.435	5.720	5.735	6.040	7.380	0.079
10	Khaled Ledjeradi	4.640	5.240	5.370	5.375	5.510	6.050	0.040

Abbreviation: CV, coefficient of variation.

TABLE 11 Statistical summary of the 1000 processing times (in seconds) for the Banzhaf–Owen value estimations.

	Min.	First Qu.	Median	Mean	Third Qu.	Max.
User time	4610.515	7658.268	8223.145	8359.013	9203.457	11,959.200
System time	2.790	4.208	5.298	17.124	8.905	260.98
Elapsed time	4614.003	7662.244	8235.254	8376.096	9217.770	12,170.890

TABLE 12 Top-10 of terrorists in Zerkani network, according to the average of 100 estimations of the Banzhaf value with $\ell = 1000$, and of the average of 100 estimations of the Banzhaf–Owen value with $\ell_r = 10^2$ and $\ell_s = 10$.

Ranking R_{wconn}				Ranking R_{awconn}			
Terrorist	$\overline{Bz}$	Terrorist	$\overline{BzO}$	Terrorist	$\overline{Bz}$	Terrorist	$\overline{BzO}$
Ab. Abaaoud	38.372	Khalid Zerkani	39.328	Mohamed Belkaid	31.333	Mohamed Belkaid	32.274
Salah Abdeslam	34.993	Ab. Abaaoud	35.702	Salah Abdeslam	26.114	Khalid Zerkani	27.854
Khalid Zerkani	33.992	Salah Abdeslam	33.639	Khalid Zerkani	25.752	Salah Abdeslam	27.010
Mohamed Belkaid	33.144	Mohamed Belkaid	33.400	Ab. Abaaoud	24.206	Ab. Abaaoud	25.426
Najim Laachraoui	18.827	Mohamed Bakkali	22.473	Mohamed Bakkali	18.360	Mohamed Bakkali	22.181
Mohamed Bakkali	18.367	Fabien Clain	12.298	Najim Laachraoui	17.381	Fabien Clain	15.892
Fabien Clain	11.903	Ahmed Dahmani	9.900	Fabien Clain	16.538	Reda Kriket	10.830
Reda Kriket	8.316	Reda Kriket	9.166	Reda Kriket	10.620	Miloud F.	5.916
Ahmed Dahmani	8.111	Najim Laachraoui	4.740	Mohamed Abrini	6.242	Ahmed Dahmani	5.712
Mohamed Abrini	6.924	Mohamed Abrini	4.621	Miloud F.	5.833	Khaled Ledjeradi	5.379

correlation coefficient and the Kendall's correlation coefficient can be used numerically to summarize the degree of similarity between the positions of terrorists in each pair of rankings. The first coefficient is 0.9547 and 0.9609 when, respectively, comparing the rankings based on the Banzhaf value and the Banzhaf–Owen value for the TU-games (N, v^{wconn}) and (N, v^{awconn}). With respect to the Kendall's correlation coefficient is equal to 0.8390 for the TU-game (N, v^{wconn}), and equal to 0.8504 when using the TU-game (N, v^{awconn}).

Khalid Zerkani is leading the ranking of risk when an *a priori* unions system exists under the weighted connectivity approach. Recall that this man was who directed a recruitment network in the Brussels area. He was not present or coordinated the attacks of Paris and Brussels, but he had a high influence on all those who were related to him. He is currently imprisoned on terrorism-related charges. In general, we check that Khalid Zerkani moves up positions in the top-10 under the presence of an *a priori* union system. In the additive approach, the person who always occupies the first position in all rankings is Mohamed Belkaid, even without an *a priori* union system. Najim Laachraoui did not belong to the top-10 under the usage of the Banzhaf–Owen value when the additive approach is assumed.

In general, these results are in line with the reality of the Zerkani network. Along with Khalid Zerkani and Abdelhamid Abaaoud, the role of Mohamed Bakkali is also important since he was alleged intellectual author of the attack of Paris. It is believed that he selected those who were going to be in the war zones or in Europe. He died in a police raid, when Chakib Akrouh detonated his explosives belt. Another individual in the ranking is Salah Abdeslam. He was the most wanted man in Europe after the Paris attack. Fabien Clain was one planner of Paris attack and explored the different places where to perform the blows. Then, Reda Kriket was a recruiter for the network and provided money to it. Meanwhile, Khaled Ledjeradi was someone very required in the network, since he was the leader of an organization that created fake documents for the members of the network, allowing them to travel. Finally, about Miloud F. not much information is available, but he was arrested in Turkey in 2005 and this allowed for arresting Reda Kriket later. About Salzburg Refugee A and Salzburg Refugee B, barely there is information about them, but they are the points of union of the attackers Paris A and B to the network, so the supervision of the last two subjects mentioned may have been key to the cessation of the attack. The decision not to increase police surveillance on them, even if it was not a good one, it can be justified from our results since only under the additive perspective, these individuals move high in the ranking.

4.5 | On the robustness of the Banzhaf–Owen value

From the definition of the Banzhaf–Owen value, it seems clear that the choice of the partition to be considered for its computation will largely condition the final ranking. Moreover, when dealing with networks, it is clear that the task of obtaining information about the affinities between agents is not trivial, as they usually refer to illicit activities that are unknown to the vast majority. This section aims to analyze the effect of a system of *a priori* unions on the final rankings of terrorists belonging to the Zerkani network under the Banzhaf–Owen value.

Following the topology of the graph of the Zerkani network in Figure 1, we consider those partitions for the Zerkani network P^a, P^b, and P^c that are, respectively, represented in Figure 2. Again, we use as criterion different geometrical figures and colors to identify those members of the Zerkani network who belong to a same element of each partition considered.

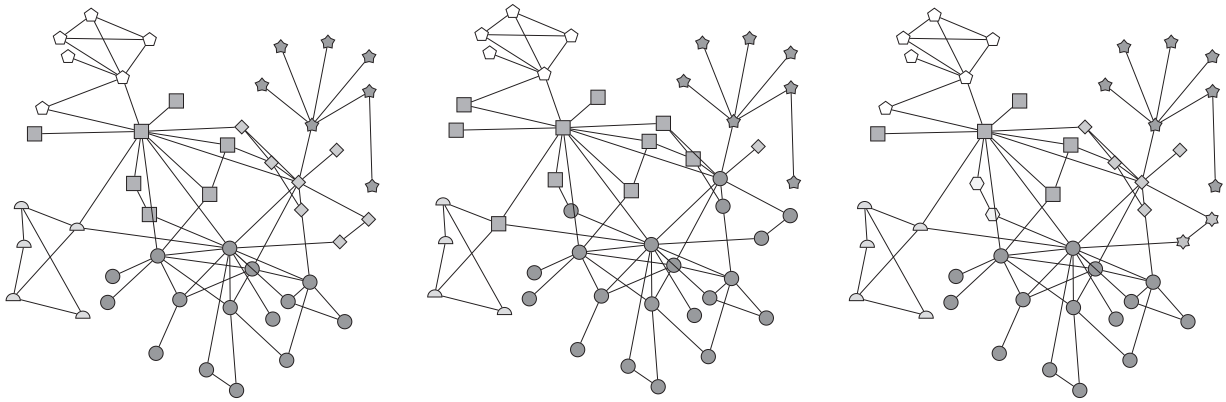
Finally, we make a brief discussion on the rankings obtained by considering a single estimation of the Banzhaf value and the Banzhaf–Owen value with P, P^a, P^b, and P^c for (N, v^{wconn}) and (N, v^{awconn}), respectively. For both analysis, we take $\ell = 1000$, $\ell_r = 10^2$, and $\ell_s = 10$. The overall rankings based on these estimations are included in Appendix C in the Online Resource Section.

First, we comment some relevant issues for the ranking based on (N, v^{wconn}). As above, Table 13 only shows the top-10 of the rankings. In view of these results, we see that the 10 most influential terrorists in Zerkani network mostly coincide, although the fact of considering different partitions makes their positions vary according to their affinities. Recall that the Banzhaf–Owen value is reduced to the Banzhaf value, when the partition is formed only by the individual elements. The main difference among the rankings lies in Najim Laachraoui, who is the only one that does not appear in the ranking based on partition P. Although they are less influential, in the rest of the ranking, there are more marked differences between the terrorists' positions.

Similar conclusions can be extracted from the rankings based on (N, v^{awconn}). Again, we only consider the top-10 of the rankings, that are shown in Table 14. As before, most of the individuals in these lists coincide, although, with the partition under consideration, their positions change. The main difference is the absence of Ahmed Dahmani when considering the Banzhaf value, which makes Mohamed Abrini appears in such ranking. Besides, Najim Laachraoui also does not appear when using the Banzhaf–Owen value with P, and Khaled Ledjeradi enters the top-10.

Although we have stuck to the 10 most influential terrorists under the different criteria used, the comparison between rankings should be done in a more comprehensive and not merely in a visual way. Thus, the degree of similarity of all terrorist rankings is studied through the computation of Spearman's correlation coefficient and of Kendall's correlation coefficient on their positions.

Tables 15 (upper triangular matrix) shows the Spearman's correlation coefficients obtained when the weighted connectivity approach is considered. In view of these results, we observe that the estimation of Banzhaf–Owen value by using P provides the most different ranking for the members of



(A) P^a $(m = 6)$. (B) P^b $(m = 6)$. (c) P^c $(m = 8)$.

FIGURE 2 Elements $P_1, \ldots, P_m$ of partitions P^a, P^b, and P^c.

TABLE 13 Top-10 of terrorists in Zerkani network, according to an estimation of the Banzhaf value with $\ell = 1000$ for (N, v^{wconn}), and of an estimation of the Banzhaf–Owen value for (N, v^{wconn}), with $\ell_r = 10^2$ and $\ell_s = 10$ by using P, P^a, P^b, and P^c.

Terrorist	$\overline{Bz}$	Terrorist	P $\overline{BzO}$	Terrorist	P^a $\overline{BzO}$	Terrorist	P^b $\overline{BzO}$	Terrorist	P^c $\overline{BzO}$
Ab. Abaaoud	37.837	Khalid Zerkani	39.475	Khalid Zerkani	40.400	Mohamed Belkaid	39.288	Ab. Abaaoud	40.499
Salah Abdeslam	35.412	Ab. Abaaoud	36.583	Mohamed Belkaid	38.700	Ab. Abaaoud	37.150	Khalid Zerkani	35.529
Khalid Zerkani	33.564	Mohamed Belkaid	34.493	Salah Abdeslam	36.200	Salah Abdeslam	34.550	Salah Abdeslam	35.492
Mohamed Belkaid	32.873	Salah Abdeslam	34.253	Ab. Abaaoud	33.909	Khalid Zerkani	31.453	Mohamed Belkaid	33.013
Najim Laachraoui	19.387	Mohamed Bakkali	21.141	Mohamed Bakkali	25.563	Najim Laachraoui	23.269	Najim Laachraoui	19.937
Mohamed Bakkali	19.254	Fabien Clain	11.155	Najim Laachraoui	25.413	Mohamed Bakkali	21.534	Mohamed Bakkali	19.388
Fabien Clain	11.659	Ahmed Dahmani	10.200	Ahmed Dahmani	9.925	Fabien Clain	13.931	Fabien Clain	11.128
Reda Kriket	8.758	Reda Kriket	9.342	Fabien Clain	7.938	Reda Kriket	11.506	Reda Kriket	9.482
Ahmed Dahmani	8.009	Ilias Mohammadi	4.651	Reda Kriket	7.469	Ahmed Dahmani	8.553	Ahmed Dahmani	8.013
Mohamed Abrini	6.712	Mohamed Abrini	4.610	Mohamed Abrini	6.047	Mohamed Abrini	6.144	Mohamed Abrini	5.783
Khaled Ledjeradi	5.012	Khaled Ledjeradi	4.490	Khaled Ledjeradi	5.122	Khaled Ledjeradi	4.763	Khaled Ledjeradi	4.513

the Zerkani network since it has associated the shortest coefficients. We can even assert that the ranking based on the estimation of the Banzhaf value is more similar to those based on P^a, P^b, and P^c (it has slightly larger correlations). Since these partitions are clearly different, this shows the robustness of our methodology to identify the most influential terrorists. Similar conclusions can be extracted from the consideration of the Kendall's correlation coefficients (see Table 16).

Besides, Table 15 (lower triangular matrix) shows the Spearman's correlation coefficients obtained when the additive approach of the TU-game is considered. Similarly, we can extract some conclusions. With its lower Spearman's correlation values, we can be sure that the ranking based on the Banzhaf–Owen value with P is the most different of all, even more than the ranking itself provided by the Banzhaf value of the considered TU-game. Even so, the values of the correlations are larger than 0.9 in all the comparisons made, indicating that, despite considering very different partitions, the rankings they induce are very similar. In this line, the same conclusions can be drawn from Table 16, when considering the Kendall's correlation coefficients in the lower triangular matrix.

TABLE 14 Top-10 of terrorists in Zerkani network, according to an estimation of the Banzhaf value with $\ell = 1000$ for (N, v^{awconn}), and of an estimation of the Banzhaf–Owen value for (N, v^{awconn}), with $\ell_r = 10^2$ and $\ell_s = 10$ by using P, P^a, P^b, and P^c.

			P		P^a		P^b		P^c
Terrorist	$\overline{Bz}$	**Terrorist**	$\overline{BzO}$	**Terrorist**	$\overline{BzO}$	**Terrorist**	$\overline{BzO}$	**Terrorist**	$\overline{BzO}$
Mohamed Belkaid	31.098	Mohamed Belkaid	33.008	Mohamed Belkaid	38.675	Mohamed Belkaid	38.456	Mohamed Belkaid	33.266
Salah Abdeslam	26.797	Salah Abdeslam	27.130	Khalid Zerkani	29.641	Salah Abdeslam	28.006	Khalid Zerkani	27.148
Khalid Zerkani	25.672	Khalid Zerkani	27.020	Salah Abdeslam	28.606	Ab. Abaaoud	26.028	Salah Abdeslam	25.672
Ab. Abaaoud	23.994	Ab. Abaaoud	24.870	Mohamed Bakkali	25.719	Khalid Zerkani	23.678	Ab. Abaaoud	24.687
Mohamed Bakkali	19.210	Mohamed Bakkali	20.805	Najim Laachraoui	24.006	Najim Laachraoui	22.969	Mohamed Bakkali	20.282
Najim Laachraoui	17.776	Fabien Clain	15.445	Ab. Abaaoud	22.656	Mohamed Bakkali	22.038	Najim Laachraoui	19.546
Fabien Clain	16.454	Reda Kriket	10.854	Fabien Clain	15.084	Fabien Clain	17.841	Fabien Clain	16.640
Reda Kriket	10.698	Ahmed Dahmani	6.200	Reda Kriket	11.103	Reda Kriket	11.272	Reda Kriket	11.145
Mohamed Abrini	6.306	Miloud F.	6.115	Ahmed Dahmani	6.831	Ahmed Dahmani	6.056	Miloud F.	5.800
Miloud F.	5.639	Khaled Ledjeradi	5.300	Miloud F.	5.934	Miloud F.	5.997	Ahmed Dahmani	5.792

TABLE 15 Spearman's correlation matrix for the rankings of the Zerkani network under the weighted connectivity (upper triangular matrix) and additive (lower triangular matrix) approaches.

	$\overline{Bz}$	$\overline{BzO}, P$	$\overline{BzO}, P^a$	$\overline{BzO}, P^b$	$\overline{BzO}, P^c$
$\overline{Bz}$	-	0.9431	0.9725	0.9702	0.9813
$\overline{BzO}, P$	0.9624	-	0.9226	0.9137	0.9176
$\overline{BzO}, P^a$	0.9844	0.9529	-	0.9757	0.9762
$\overline{BzO}, P^b$	0.9816	0.9433	0.9899	-	0.9775
$\overline{BzO}, P^c$	0.9838	0.9452	0.9864	0.9768	-

TABLE 16 Kendall's correlation matrix for the rankings of the Zerkani network under the weighted connectivity (upper triangular matrix) and additive (lower triangular matrix) approaches.

	$\overline{Bz}$	$\overline{BzO}, P$	$\overline{BzO}, P^a$	$\overline{BzO}, P^b$	$\overline{BzO}, P^c$
$\overline{Bz}$	-	0.9075	0.8205	0.9130	0.8945
$\overline{BzO}, P$	0.8538	-	0.7817	0.7687	0.7724
$\overline{BzO}, P^a$	0.8982	0.8335	-	0.8797	0.8723
$\overline{BzO}, P^b$	0.8908	0.8150	0.9334	-	0.8853
$\overline{BzO}, P^c$	0.9075	0.8205	0.9130	0.8945	-

5 | CONCLUSIONS

Until now, the usage of game-theoretical solutions for the risk analysis of terrorist networks was limited to the approach given by the Shapley value (Shapley, 1953). Another very interesting and well-known value in the literature is the Banzhaf value (Banzhaf, 1964). In this article, we focus on the Banzhaf value and its extension for the case in which a coalitional structure, that restricts the affinities among their members, is given: the Banzhaf–Owen value (Owen, 1982). These values cannot be obtained in an exact way when the number of players increases, so sampling methods have been provided and implemented for their computation. Then, these values have been innovatively applied to obtain rankings of the most important terrorist of the Zerkani network. In fact, these measures applied in these networks may offer new tools to analyze the risk of an attack to the intelligence services.

As in any multiagent organization, the members of a terrorist network are hierarchical within. Although all are part of the same network, each of them will perform a function, and it will be related to nearby members of those of the same rank. It makes sense, therefore, to consider the members of the network in groups according to their features. This directly implies the existence of a partition, as the one considered along this article. Thus, the construction of two rankings has been carried out through the approximation of the Banzhaf–Owen value, for the two approaches of TU-games under study. Both take into account the characteristics of the network, the individuals as well as their relationships. In order to choose the proper partition, a deep analysis of each terrorist's contributions to the network has been made, having divided them according to their facets. In this sense, the characteristics of the most realistic partition can be enhanced by those responsible of the security services, since they usually manage relevant information on the organization of this kind of terrorist structures.

Focus on the partition considered, notice that the commanders and recruiters within the network will be those who manage, order, and coordinate all the movements that the group carries out. However, they will avoid getting involved in compromised issues due to their status. For this situation, they have associates or recruits who will obey them and serve as a connection with the attackers (which would be in a lowest level in the network). Meanwhile, lower-level groupings will carry out the hard work of exposing themselves to intelligence agencies, traveling throughout Europe with false documentation or carrying out the killing actions of hundreds of people. Unlike of the Banzhaf value, all this information about the partition is taken into account in the Banzhaf–Owen value. Therefore, the results integrate this additional information, which can be key in determining who are the most dangerous or influential members within each union or group considered in the partition. Specifically, the Banzhaf–Owen value introduces a coalitional structure that, respectively, enriches the results provided by the Banzhaf value when extra information about relationships is considered. It distributes first among unions (teams) and then among the members of each team. Therefore, when considering the terrorists according to their facets, the most dangerous in each team may be obtained and, in particular, not only the most influential of the organization or recruitment as the Banzhaf value does, but also the most important among those who carry out the action, which may be essential to neutralize it. The choice of the most suitable coalitional structure describing the reality of the network ensures that the rankings obtained accurately reflect the hierarchy of its members. As empirically checked, an appropriate choice, while not leading to major changes in overall ranking trends, does give way to small changes in the attackers' positions, which can be key to make an efficient use of the usually scarce surveillance resources. Further research will include the usage of more technical mechanisms to determine the partition used in the Banzhaf–Owen value as well as the analysis of the Zerkani Network with other risk analysis measures as the position value used as a solution concept in the literature of TU-games (see, for instance, Borm et al., 1992; van den Nouweland et al., 1992; or Algaba et al., 2000).

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SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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