Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 55-36 (2022) 175–180

Order of Legendre-LMI conditions to assess stability of time-delay systems

Mathieu Bajodek *, **Alexandre Seuret** **, **Frédéric Gouaisbaut** *.

* *LAAS-CNRS, Université de Toulouse, UPS, 7 avenue du Colonel Roche, 31077 Toulouse, France*
** *Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain*

Abstract: This paper investigates the stability analysis of time-delay systems through Lyapunov arguments. Using the existence of a complete Lyapunov-Krasovskii functional and relying on the polynomial approximation theory, our main goal is to approximate the complete Lyapunov functional and to take profit of the supergeometric convergence rate of the truncated error part. Necessary and sufficient conditions in the linear matrix inequality (LMI) framework for sufficiently large approximated orders are consequently proposed. Moreover, an estimation of the necessary order is provided analytically with respect to system parameters.

Keywords: Time-delay systems; Stability conditions; LMI; Converse Lyapunov theorem; Polynomial approximation

1. INTRODUCTION

An extension of the Lyapunov necessary and sufficient theorem has been introduced in Datko (1970) to analyze the stability of linear infinite-dimensional systems. This contribution ensures the existence of a complete quadratic Lyapunov functional without giving more details on its design. Focusing now on time-delay systems, such a complete Lyapunov functional has been constructed in Kharitonov (2013). Nevertheless, the Lyapunov inequalities conditions stay theoretical and cannot be implemented. In parallel, infinite-dimensional systems are often modeled by approximated methods to be analyzed numerically. It is the case of Padé approximations in the frequency domain (see Golub and Van Loan (1989)) or pseudo-spectral methods in the time domain (see Gottlieb and Orszag (1977)). Within this approximation context, most of the results release in the convergence of the solution towards the expected one. However, few results are able to provide a sufficiently high approximation index enough to establish the stability properties using the convergence of the error part.

In the wide variety of pseudo-spectral methods, two categories stand out. From one side, collocation methods amount to interpolate the state. The idea is to create a discretization map such that the error is exactly zero at the interpolated points. In between these points, the approximation functions can be selected to be constant (see Gu et al. (2003)), linear (see Gu (2013); Medvedeva and Zhabko (2015)), polynomial (see spline methods) or exponential (see Egorov and Mondié (2014)). Applying an interpolation leads to necessary conditions of stability as in Egorov and Mondié (2014); Medvedeva and Zhabko (2019) or to sufficient conditions of stability based on discretized Lyapunov functional as in Gu et al. (2003). Recently, these results have been proven to converge asymptotically using some convergence properties and deliver inner and outer estimations of the stability regions (see respectively Gu (2013) and Gomez et al. (2021)). From the other side, *tau* methods amount to approximate the state. The idea is to take support of a complete Hilbert basis in order to proceed an approximation through Galerkin-like methods using, for instance, Fourier trigonometric basis or Legendre polynomials. Simple Lyapunov functionals based on the first coefficient (i.e. the mean value of the state) of such basis have led to linear matrix inequality (LMI) conditions exposed in Fridman (2014), which are, however, quite conservative. To reduce this conservatism, hierarchic Lyapunov functionals taking support on the n first Legendre coefficients have been proposed in Seuret and Gouaisbaut (2013, 2015). It is only recently that the gap between these approximated functionals and the original complete functional has been closed. Indeed, the authors of Bajodek et al. (2022) haven proven that sufficient Legendre-LMI conditions converge and become necessary for sufficiently large orders.

This paper explains the methodology, which allows to obtain the necessary and sufficient stability conditions developed in Bajodek et al. (2022), and improves the estimation of the necessary order n^*. After recalling the complete Lyapunov functional of time-delay systems, an intermediate result quantifies the quality of the approximation of the complete Lyapunov functional so that the Lyapunov converse theorem holds. Based on the selection of a Legendre approximation, which ensures exponentially fast uniform convergence, the necessary and sufficient condition of stability for time-delay systems resumes to an LMI of size n^* expressed with respect to system parameters. Finally, the LMI test as well as the order n^* are computed on an academic example.

Notations: Throughout the paper, $\mathbb{N}$, $\mathbb{R}^{m\times p}$, $\mathbb{S}^m$ and $\mathbb{S}^m_+$ denote the sets of natural numbers, real matrices of size $m \times p$, symmetric matrices of size m and positive symmet-

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2022.11.353

ric matrices of size m, respectively. For any square real matrix M, $M^\top$ denotes the transpose of M, $\mathcal{H}(M)$ stands for $M+M^\top$ and tril(M) stands for the triangular lower part of M. Furthermore, for any $M \in \mathbb{S}^m$, its minimal and maximal eigenvalues are denoted $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$. The 2-norm of matrix M in $\mathbb{R}^{m\times p}$ is $|M| = \sqrt{\bar{\sigma}(M^\top M)}$. The vector $u = \mathrm{vec}(M)$ in $\mathbb{R}^{mp\times 1}$ collocates the columns of matrix M and 'vec^{-1}' denotes the inverse operator. Notation $\otimes$ refers to the matrix Kronecker product. The set of square-integrable functions from $(-h,0)$ to $\mathbb{R}^{m\times p}$ is noted $L^2(-h,0;\mathbb{R}^{m\times p})$. Let also $H^1(-h,0;\mathbb{R}^{m\times p})$ be the set of functions F, such that F and F' are in $L^2(-h,0;\mathbb{R}^{m\times p})$. For any φ in $\mathbb{R}^{n_x}\times L^2(-h,0;\mathbb{R}^{n_z})$, the norm is given by $\|\varphi\| = \sqrt{|\varphi(0)|^2+\int_{-h}^0 |\varphi(\theta)|^2\,\mathrm{d}\theta}$. Lastly, denote x_t such that $x_t(\theta)=x(t+\theta)$, for any θ in $[-h,0]$.

2. CONVERSE LYAPUNOV THEOREM FOR TIME-DELAY SYSTEMS

2.1 Presentation of the system

Consider a linear time-delay system given by

$$\begin{cases} \dot{x}(t) = Ax(t)+A_d x(t-h), & \forall t\geq 0,\\ x(\theta) = \varphi(\theta), & \forall \theta\in[-h,0], \end{cases} \tag{1}$$

where the delay $h>0$ and matrices A, A_d in $\mathbb{R}^{n_x\times n_x}$ are constant and known. The initial condition φ belongs to

$$\mathcal{D} = \left\{ \begin{matrix} \varphi(\theta) = \begin{cases}\varphi_0 & \text{if } \theta=0,\\ \varphi_1(\theta) & \text{if } \theta\in[-h,0],\end{cases} \text{ such that} \\ \begin{bmatrix}\varphi_0\\ \varphi_1\end{bmatrix} \in \mathbb{R}^{n_x}\times H^1(-h,0;\mathbb{R}^{n_x}),\ \varphi_1(0)=\varphi_0 \end{matrix} \right\}.$$

Definition 1. (GES). The trivial solution to (1) is said globally exponentially stable (GES), if there exist $\kappa\geq 1$ and $\mu>0$ such that x_t, solution to (1) generated by any initial condition φ in $\mathcal{D}$, verifies

$$\|x_t\| \leq \kappa\, \mathrm{e}^{-\mu t}\|\varphi\|, \quad \forall t\geq 0. \tag{2}$$

Remark 1. It is worth noticing that the regularity properties of the solutions to time-delay systems ensure that such a definition in the sense of $\|\cdot\|$-norm also implies that there exist $\bar\kappa\geq 1$ and $\bar\mu>0$ such that

$$|x(t)| \leq \bar\kappa\, \mathrm{e}^{-\bar\mu t}\sup_{[-h,0]}|\varphi(\theta)|, \quad \forall t\geq 0. \tag{3}$$

2.2 Existence of a complete Lyapunov functional

Since Kharitonov and Zhabko (2003); Kharitonov (2013), it is well known that a necessary and sufficient stability condition of system (1) is the existence of a complete Lyapunov functional, which is recalled below.

For any W_1, W_2 and W_3 in $\mathbb{S}_+^{n_x}$ and for any x_t in $\mathcal{D}$, define the following Lyapunov functional

$$\begin{aligned}\mathcal{V}(x_t) =\, & x_t^\top(0)U(0)x_t(0) + 2x_t^\top(0)\int_{-h}^0 U(\theta+h)A_d x_t(\theta)\mathrm{d}\theta \\ &+ \int_{-h}^0\int_{-h}^0 x_t^\top(\theta_1)A_d^\top U(\theta_2-\theta_1)A_d x_t(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &+ \int_{-h}^0 x_t^\top(\theta)\left(\frac{\theta+h}{h}W_2+W_3\right)x_t(\theta)\mathrm{d}\theta,\end{aligned} \tag{4}$$

where U is a matrix function from $[-h,h]$ to $\mathbb{R}^{n_x\times n_x}$, called the Lyapunov matrix. Interestingly, referring to (Kharitonov, 2013, Section 2.10), a method is provided to build analytically this matrix. Its expression is given by

$$U(\theta) = \begin{cases} \mathrm{vec}^{-1}\left(\begin{bmatrix} I_{n_x^2} & 0\end{bmatrix}\mathrm{e}^{\theta M}N^{-1}\begin{bmatrix}-\mathrm{vec}(W)\\ 0\end{bmatrix}\right) & \text{if } \theta\leq 0,\\ U^\top(-\theta) & \text{if } \theta>0, \end{cases} \tag{5}$$

where matrices

$$\begin{aligned} W &= W_1+W_2+W_3, \quad M = \begin{bmatrix} -A^\top\otimes I_{n_x} & -A_d^\top\otimes I_{n_x}\\ I_{n_x}\otimes A_d^\top & I_{n_x}\otimes A^\top\end{bmatrix},\\ N &= \begin{bmatrix} A^\top\otimes I_{n_x}+I_{n_x}\otimes A^\top & A_d^\top\otimes I_{n_x}\\ I_{n_x^2} & 0\end{bmatrix} + \begin{bmatrix} I_{n_x}\otimes A_d^\top & 0\\ 0 & -I_{n_x^2}\end{bmatrix}\mathrm{e}^{-hM}. \end{aligned} \tag{6}$$

Such a complete Lyapunov functional $\mathcal{V}$ has been built so that its time-derivative along the trajectories of the system yields

$$\dot{\mathcal{V}}(x_t) = -\frac{1}{h}\int_{-h}^0 \begin{bmatrix} x_t(0)\\ hx_t(\theta)\\ x_t(-h)\end{bmatrix}^\top \begin{bmatrix} W_1 & 0 & 0\\ * & \frac{1}{h}W_2 & 0\\ * & * & W_3\end{bmatrix}\begin{bmatrix} x_t(0)\\ hx_t(\theta)\\ x_t(-h)\end{bmatrix}\mathrm{d}\theta. \tag{7}$$

A necessary and sufficient condition of stability for system (1) based on this functional is formulated below.

2.3 Necessary and sufficient condition of stability

The following theorem is obtained by application of the Lyapunov theorem. The proposed formulation is issued from (Gu et al., 2003, Theorem 5.19).

Theorem 1. System (1) is GES if and only if there exist positive scalars $\alpha_1,\alpha_2,\alpha_3$ such that $\mathcal{V}$ defined by (4) satisfies, for any x_t in $\mathcal{D}$, inequalities

$$\alpha_1\|x_t\|^2 + \int_{-h}^0 x_t^\top(\theta)\left(\frac{\theta+h}{h}W_2\right)x_t(\theta)\mathrm{d}\theta \leq \mathcal{V}(x_t), \tag{8a}$$

$$\mathcal{V}(x_t) \leq \alpha_2\|x_t\|^2, \tag{8b}$$

$$\dot{\mathcal{V}}(x_t) \leq -\alpha_3\|x_t\|^2, \tag{8c}$$

where $\dot{\mathcal{V}}$ denotes the time-derivative of $\mathcal{V}$ along (1).

Remark 2. It is worth noticing that in the sense of the stability definition given by (3), the Lyapunov-Krasovskii version of this theorem also holds replacing (8) by

$$\alpha_1|x_t(0)|^2 \leq \mathcal{V}(x_t) \leq \alpha_2\sup_{[-h,0]}|x_t(\theta)|^2, \tag{9a}$$

$$\dot{\mathcal{V}}(x_t) \leq -\alpha_3|x_t(0)|^2. \tag{9b}$$

Refer to Chaillet et al. (2019) for further details.

Proof : To begin with, the sufficiency refers to the application of the Lyapunov theorem. The proof of necessity starts by noting that (8c) already holds by construction of the Lyapunov matrix (7). Indeed, expression (7) leads to $\alpha_3 = \max(\underline{\sigma}(W_1), h\underline{\sigma}(W_2))$. Equation (8b) is also easy to prove since $\mathcal{V}$ is quadratic with respect to x_t. Let focus now on (8a) and introduce a functional $\mathcal{W}$ as follows

$$\mathcal{W}(x_t) = \mathcal{V}(x_t) - \alpha_1\|x_t\|^2 - \int_{-h}^0 x_t^\top(\theta)\left(\frac{\theta+h}{h}W_2\right)x_t(\theta)\mathrm{d}\theta.$$

According to (7), differentiating $\mathcal{W}$ along the trajectories of (1) leads to

$$\begin{aligned}\dot{\mathcal{W}}(x_t) =& -\begin{bmatrix} x_t(0)\\ x_t(-h)\end{bmatrix}^\top\begin{bmatrix} W_1 & 0\\ 0 & W_3\end{bmatrix}\begin{bmatrix} x_t(0)\\ x_t(-h)\end{bmatrix}\\ & -\alpha_1\begin{bmatrix} x_t(0)\\ x_t(-h)\end{bmatrix}^\top\begin{bmatrix}\mathcal{H}(A)+I_{n_x} & A_d\\ A_d^\top & -I_{n_x}\end{bmatrix}\begin{bmatrix} x_t(0)\\ x_t(-h)\end{bmatrix}.\end{aligned}$$

Then, there exists a sufficiently small $\alpha_1 > 0$ such that $\mathcal{W}$ is negative. Integrating the above inequality from t to ∞, we obtain

$$\lim_{T\to\infty} \mathcal{W}(x_T) - \mathcal{W}(x_t) \leq 0, \ \forall t \geq 0.$$

As $x_T \underset{T\to\infty}{\longrightarrow} 0$ because system (1) is assumed to be GES, inequality $\mathcal{W}(x_t) \geq 0$ holds, which concludes the proof. □

A modified version of Theorem 1 will be exposed in the next section. Indeed, in practice, the closed-form of the Lyapunov matrix U cannot be calculated precisely enough and the necessary and sufficient LMI conditions (8) need to be simplified to be tested. Then, it will rely on an approximation of the complete Lyapunov functional.

3. MODIFIED CONVERSE LYAPUNOV THEOREM

3.1 Legendre polynomials approximation

The Lyapunov matrix U is composed of a complex combination of exponential functions. In order to derive tractable numerical tests, the idea is to introduce approximations of U, which satisfy convergence properties to ease this purpose. Define the Legendre polynomials approximations at any order n in $\mathbb{N}^*$ as

$$\begin{aligned} U(\theta+h) &= \underbrace{\mathbf{U}_{1,n}\ell_n\left(\frac{\theta+h}{h}\right)}_{U_{1,n}(\theta+h)} + \tilde{U}_{1,n}(\theta), \ \forall\theta\in[-h,0], \\ U(\theta) &= \underbrace{\mathbf{U}_{2,n}\ell_n\left(\frac{\theta+h}{2h}\right)}_{U_{2,n}(\theta)} + \tilde{U}_{2,n}(\theta), \ \forall\theta\in[-h,h], \end{aligned} \tag{10}$$

where matrix ℓ_n in $\mathbb{R}^{nn_x\times n_x}$ collocates the n first Legendre polynomials scaled on $[0,1]$

$$\ell_n(\theta) = [l_0(\theta) \ \dots \ l_{n-1}(\theta)]^\top \otimes I_{n_x}, \quad \forall\theta\in[0,1]. \tag{11}$$

In this decomposition, the constant matrices $\mathbf{U}_{1,n}$ and $\mathbf{U}_{2,n}$ have been selected as the orthogonal projection of $U(\theta+h)$ and $U(\theta)$ on the interval $[-h,0]$ and $[-h,h]$ respectively, and their expressions are given by

$$\begin{aligned} \mathbf{U}_{1,n} &= \left(\int_{-h}^{0} U(\theta+h)\ell_n^\top\left(\frac{\theta+h}{h}\right)\mathrm{d}\theta\right)\mathcal{I}_n, \\ \mathbf{U}_{2,n} &= \frac{1}{2}\left(\int_{-h}^{h} U(\theta)\ell_n^\top\left(\frac{\theta+h}{2h}\right)\mathrm{d}\theta\right)\mathcal{I}_n, \end{aligned} \tag{12}$$

where the normalization matrix of Legendre polynomials is equal to $\mathcal{I}_n = \mathcal{I}_n^{I_{n_x}}$ where, for all $S\in\mathbb{S}^{n_x}$, we define

$$\mathcal{I}_n^S = \frac{1}{h}\left(\int_0^1 \ell_n(\theta)\ell_n^\top(\theta)\mathrm{d}\theta\right)^{-1}\otimes S = \mathrm{diag}\left(\frac{1}{h},\dots,\frac{2n-1}{h}\right)\otimes S. \tag{13}$$

For any W_1, W_2 and W_3 in $\mathbb{S}_+^{n_x}$ and x_t in $\mathcal{D}$, we introduce a new functional

$$\begin{aligned} \mathcal{V}_n(x_t) &= x_t^\top(0)U(0)x_t(0) + 2x_t^\top(0)\int_{-h}^{0} U_{1,n}(\theta+h)A_d x_t(\theta)\mathrm{d}\theta \\ &+ \int_{-h}^{0}\int_{-h}^{0} x_t^\top(\theta_1)A_d^\top U_{2,n}(\theta_2-\theta_1)A_d x_t(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &+ \int_{-h}^{0} x_t^\top(\theta)\left(\frac{\theta+h}{h}W_2 + W_3\right)x_t(\theta)\mathrm{d}\theta. \end{aligned} \tag{14}$$

Such a functional will be considered as an approximation of the original $\mathcal{V}$. Denoting the error $\tilde{\mathcal{V}}_n = \mathcal{V} - \mathcal{V}_n$, we obtain

$$\begin{aligned} \tilde{\mathcal{V}}_n(x_t) &= 2x_t^\top(0)\int_{-h}^{0} \tilde{U}_{1,n}(\theta)A_d x_t(\theta)\mathrm{d}\theta \\ &+ \int_{-h}^{0}\int_{-h}^{0} x_t^\top(\theta_1)A_d^\top \tilde{U}_{2,n}(\theta_2-\theta_1)A_d x_t(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \end{aligned} \tag{15}$$

where $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$ refer to the errors expressed in (10).

Nevertheless, the price to pay is that the functional $\mathcal{V}_n$ generated by this approximation at order n does not necessarily verify the condition (8) of the complete one. In this sequel, we provide a new theorem stating a necessary and sufficient condition of stability even though the complete Lyapunov functional is not built by U but by its approximation $U_{1,n}$ and $U_{2,n}$. This theorem naturally relies on $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$, which are seen as an approximation error.

3.2 A new necessary and sufficient condition of stability

The main result of the paper is presented below.

Theorem 2. Let W_1, W_2 and W_3 in $\mathbb{S}_+^{n_x}$ and assume that the following inequality holds

$$\Psi_n(\theta) := \begin{bmatrix} W_1 + \mathcal{H}(\tilde{U}_{1,n}(0)A_d) & \Psi_n^1(\theta) & -\tilde{U}_{1,n}(-h)A_d \\ * & \frac{1}{h}W_2 & \Psi_n^2(\theta) \\ * & * & W_3 \end{bmatrix} \succ 0, \tag{16}$$

for a given n in $\mathbb{N}^*$ and for all θ in $[-h,0]$ with

$$\begin{aligned} \Psi_n^1(\theta) &= A^\top\tilde{U}_{1,n}(\theta)A_d - \tilde{U}_{1,n}'(\theta)A_d + A_d^\top\tilde{U}_{2,n}(\theta)A_d, \\ \Psi_n^2(\theta) &= A_d^\top\tilde{U}_{1,n}^\top(\theta)A_d - A_d^\top\tilde{U}_{2,n}^\top(\theta+h)A_d. \end{aligned}$$

System (1) is GES if and only if there exist positive scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\mathcal{V}_n$ defined by (14) satisfies, for any x_t in $\mathcal{D}$, inequalities

$$\alpha_1\|x_t\|^2 \leq \mathcal{V}_n(x_t) \leq \alpha_2\|x_t\|^2, \tag{17a}$$

$$\dot{\mathcal{V}}_n(x_t) \leq -\alpha_3\|x_t\|^2. \tag{17b}$$

Proof : Trivially, the sufficiency of the statement is verified by application of the Lyapunov theorem. For the necessity of the proposed statement, let us first note that the right-hand part of (17a) is directly obtained because the Lyapunov functional is quadratic with respect to x_t. Let now focus on the two remained inequalities. Concerning (17b), the time-derivative of $\mathcal{V}_n = \mathcal{V} - \tilde{\mathcal{V}}_n$ along (1) given by (7) and (15) leads to

$$\begin{aligned} \dot{\mathcal{V}}_n(x_t) &= -\frac{1}{h}\int_{-h}^{0}\begin{bmatrix} x_t(0) \\ x_t(\theta) \\ x_t(-h)\end{bmatrix}^\top \begin{bmatrix} W_1 & 0 & 0 \\ * & \frac{1}{h}W_2 & 0 \\ * & * & W_3\end{bmatrix}\begin{bmatrix} x_t(0) \\ x_t(\theta) \\ x_t(-h)\end{bmatrix}\mathrm{d}\theta \\ &-2(Ax_t(0)+A_d x_t(-h))^\top\int_{-h}^{0}\tilde{U}_{1,n}(\theta)A_d x_t(\theta)\mathrm{d}\theta \\ &+2x_t^\top(0)\left(\left[\tilde{U}_{1,n}(\theta)A_d x_t(\theta)\right]_{-h}^{0} + \int_{-h}^{0}\tilde{U}_{1,n}'(\theta)A_d x_t(\theta)\mathrm{d}\theta\right) \\ &-2\int_{-h}^{0}\left[x_t^\top(\theta_1)A_d^\top\tilde{U}_{2,n}(\theta_2-\theta_1)A_d x_t(\theta_2)\right]_{\theta_1=-h}^{\theta_1=0}\mathrm{d}\theta_2 \\ &+\int_{-h}^{0}\int_{-h}^{0} x_t^\top(\theta_1)A_d^\top(\partial_{\theta_1}+\partial_{\theta_2})\tilde{U}_{2,n}(\theta_2-\theta_1)A_d x_t(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2. \end{aligned}$$

The last term vanishes and the previous expression boils down to

$$\dot{\mathcal{V}}_n(x_t) = -\frac{1}{h}\int_{-h}^{0}\begin{bmatrix} x_t(0) \\ hx_t(\theta) \\ x_t(-h)\end{bmatrix}^\top \Psi_n(\theta)\begin{bmatrix} x_t(0) \\ hx_t(\theta) \\ x_t(-h)\end{bmatrix}\mathrm{d}\theta, \tag{18}$$

where Ψ_n is given in (16). Therefore if matrix Ψ_n is positive definite there exists a sufficiently small $\alpha_3 > 0$ such that $\Psi_n(\theta) \succ \alpha_3 I_{3n_z}$ which yields to (17b).
In order to ensure the left-hand part of (17a), let us introduce the following functional

$$\mathcal{W}_n(x_t) = \mathcal{V}_n(x_t) - \alpha_1 \|x_t\|^2. \tag{19}$$

According to the previous calculations (18), differentiating $\mathcal{W}_n$ along the trajectories of (1) leads to

$$\begin{aligned}\dot{\mathcal{W}}_n(x_t) = &-\frac{1}{h}\int_{-h}^{0}\begin{bmatrix} x_t(0) \\ hx_t(\theta) \\ x_t(-h)\end{bmatrix}^\top \Psi_n(\theta)\begin{bmatrix} x_t(0) \\ hx_t(\theta) \\ x_t(-h)\end{bmatrix}\mathrm{d}\theta \\ &-\alpha_1\begin{bmatrix} x_t(0) \\ x_t(-h)\end{bmatrix}^\top\begin{bmatrix}\mathcal{H}(A)+I_{n_x} & A_d \\ A_d^\top & -I_{n_z}\end{bmatrix}\begin{bmatrix} x_t(0) \\ x_t(-h)\end{bmatrix}.\end{aligned} \tag{20}$$

Therefore, if Ψ_n is positive definite, then there exists a sufficiently small $\alpha_1 > 0$ such that $\dot{\mathcal{W}}_n(x_t)$ is negative definite. Integrating then $\dot{\mathcal{W}}_n(x_t)$ from t to infinity yields

$$\lim_{T\to\infty} \mathcal{W}_n(x_T) - \mathcal{W}_n(x_t) = -\mathcal{W}_n(x_t) \leq 0,$$

where we have used the assumption that system (1) is GES meaning that $\mathcal{W}_n(x_T) \underset{T\to\infty}{\longrightarrow} 0$. Hence, (19) ensures that there exists a sufficiently small $\alpha_1 > 0$ such that the right-hand part of inequality (17a) holds. □

In the following, inequality (8a) in Theorem 1 and inequality (17b) in Theorem 2 are translated into a numerical condition.

3.3 Necessary and sufficient LMI condition of stability

For any $(\mathbf{P}_n, R, S) \in \mathbb{S}^{(n+1)n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_x}$, introduce the Lyapunov candidate functional considered in Seuret and Gouaisbaut (2014, 2015)

$$\begin{aligned}\mathcal{V}_n(x_t) = &\begin{bmatrix} x_t(0) \\ \int_{-h}^{0}\ell_n(\frac{\theta+h}{h})x_t(\theta)\mathrm{d}\theta\end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x_t(0) \\ \int_{-h}^{0}\ell_n(\frac{\theta+h}{h})x_t(\theta)\mathrm{d}\theta\end{bmatrix} \\ &+\int_{-h}^{0} x_t^\top(\theta)\left(\frac{\theta+h}{h}R + S\right)x_t(\theta)\mathrm{d}\theta.\end{aligned} \tag{21}$$

Using an integration by parts, the time-derivative of $\mathcal{V}_n$ along the trajectories of system (1) is given by

$$\begin{aligned}\dot{\mathcal{V}}_n(x_t) = &\,2\begin{bmatrix} x_t(0) \\ \int_{-h}^{0}\ell_n(\frac{\theta+h}{h})x_t(\theta)\mathrm{d}\theta\end{bmatrix}^\top \mathbf{P}_n\begin{bmatrix} A_d \\ -\ell_n(0)\end{bmatrix}x_t(-h) \\ &+2\begin{bmatrix} x_t(0) \\ \int_{-h}^{0}\ell_n(\frac{\theta+h}{h})x_t(\theta)\mathrm{d}\theta\end{bmatrix}^\top \mathbf{P}_n\begin{bmatrix} A & 0 \\ \ell_n(1) & \mathcal{L}_n\end{bmatrix}\begin{bmatrix} x_t(0) \\ \int_{-h}^{0}\ell_n(\frac{\theta+h}{h})x_t(\theta)\mathrm{d}\theta\end{bmatrix} \\ &+\begin{bmatrix} x_t(0) \\ x_t(-h)\end{bmatrix}^\top\begin{bmatrix} R+S & 0 \\ 0 & -S\end{bmatrix}\begin{bmatrix} x_t(0) \\ x_t(-h)\end{bmatrix} \\ &-\frac{1}{h}\int_{-h}^{0} x_t^\top(\theta)Rx_t(\theta)\mathrm{d}\theta,\end{aligned} \tag{22}$$

where the derivation matrix of Legendre polynomials is

$$\mathcal{L}_n = -\mathrm{tril}(\ell_n(1)\ell_n^\top(1) - \ell_n(0)\ell_n^\top(0))\mathcal{I}_n, \tag{23}$$

In this context, the following theorem can be established.

Theorem 3. For a given n in $\mathbb{N}^*$, assume that inequality (16) holds (i.e. $\Psi_n(\theta) \succ 0$, for all θ in $[-h, 0]$).
System (1) is GES if and only if there exist matrices $(\mathbf{P}_n, S, R) \in \mathbb{S}^{(n+1)n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_x}$ the two following linear matrix inequalities hold

$$\begin{aligned}\Xi_n^+ &= \mathbf{P}_n + \mathbf{I}_n^S \succ 0, \\ \Xi_n^- &= \begin{bmatrix}\mathcal{H}(\mathbf{P}_n\mathbf{A}_n) + \mathbf{C}_n^\top(R+S)\mathbf{C}_n - \frac{1}{h}\mathbf{I}_n^R & \mathbf{P}_n\mathbf{B}_n \\ * & -S\end{bmatrix} \prec 0,\end{aligned} \tag{24}$$

where matrices are defined by

$\mathbf{I}_n^S = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{I}_n^S\end{bmatrix}$, $\mathbf{A}_n = \begin{bmatrix} A & 0 \\ \ell_n(1) & \mathcal{L}_n\end{bmatrix}$, $\mathbf{B}_n = \begin{bmatrix} A_d \\ -\ell_n(0)\end{bmatrix}$, $\mathbf{C}_n = \begin{bmatrix} I_{n_x} \\ 0\end{bmatrix}^\top$.

Proof : Firstly, the proof of the sufficiency is provided in Seuret and Gouaisbaut (2015) using for a given order n in $\mathbb{N}^*$ the Lyapunov candidate functional given by (21) and applying the Lyapunov theorem.
Secondly, to prove the necessity, we take support on Theorem 2. For all W_1, W_2 and W_3 in $\mathbb{S}_+^{n_x}$, let matrices

$$\begin{aligned}\mathbf{P}_n &= \begin{bmatrix} U(0) & \mathcal{Q}_n \\ * & \mathcal{T}_n\end{bmatrix}, \quad R = W_2, \quad S = W_3, \\ \mathcal{Q}_n &= \int_{-h}^{0} U(\theta+h)A_d\ell_n^\top(\theta)\mathcal{I}_n\mathrm{d}\theta, \\ \mathcal{T}_n &= \int_{-h}^{0}\int_{-h}^{0}\mathcal{I}_n\ell_n(\theta_1)A_d^\top U(\theta_2-\theta_1)A_d\ell_n^\top(\theta_2)\mathcal{I}_n\mathrm{d}\theta_1\mathrm{d}\theta_2.\end{aligned} \tag{25}$$

Note that $(\mathbf{P}_n, R, S)$ belongs to $\mathbb{S}^{(n+1)n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_x}$ using the symmetric property of U. Moreover, the Lyapunov candidate functional $\mathcal{V}_n$ given by (21) is the Legendre approximation (14) of the Lyapunov converse functional $\mathcal{V}$. Assume that inequality (16) holds at order n and that system (1) is GES. Applying (8a) in Theorem 1 and (17b) in Theorem 2, there exist positive scalars α_1, α_3 such for the following inequalities hold

$$\alpha_1\|x_t\|^2 + \int_{-h}^{0} x_t^\top(\theta)\left(\frac{\theta+h}{h}R\right)x_t(\theta)\mathrm{d}\theta \leq \mathcal{V}(x_t), \tag{26a}$$

$$\dot{\mathcal{V}}_n(x_t) \leq -\alpha_3\|x_t\|^2. \tag{26b}$$

Then, these inequalities are applied for x_t in $\mathcal{D}_n \subset \mathcal{D}$ given by

$$\mathcal{D}_n = \left\{\begin{array}{l}\varphi(\theta) = \begin{cases}\varphi_0 & \text{if } \theta = 0, \\ \ell_n^\top\left(\frac{\theta+h}{h}\right)\zeta_n & \text{if } \theta \in (-h, 0), \\ \varphi_{-h} & \text{if } \theta = -h,\end{cases} \\ \text{such that } \begin{bmatrix}\varphi_0 & \zeta_n & \varphi_{-h}\end{bmatrix}^\top \in \mathbb{R}^{(n+2)n_x}.\end{array}\right\}.$$

Thus, inequalities (26a),(26b) lead to $\Xi_n^+ \succ 0$ and $\Xi_n^- \prec 0$, respectively, which concludes the proof. □

Theorem 3 provides a necessary and sufficient LMI condition for the stability of system (1). Nonetheless, it does not guarantee the existence of an order n such that the LMI must be satisfied. The estimation of this order is reserved to the next section.

4. ESTIMATION OF THE NECESSARY ORDER

4.1 Convergence properties

In the light of Dunkl and Xu (2001), we provide convergence properties satisfied by Legendre polynomial approximation in order to estimate an order from which the necessary and sufficient stability condition holds.

Firstly, the Lyapunov matrix U associated to W defined in (5) is continuous on $[-h, h]$ and satisfies

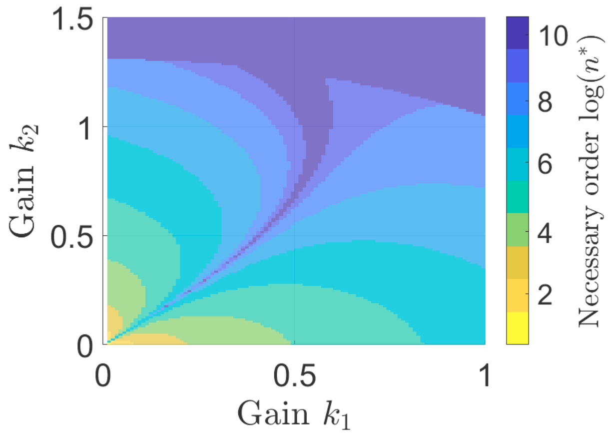

Fig. 1. Estimation of the necessary order n^* for example (36).

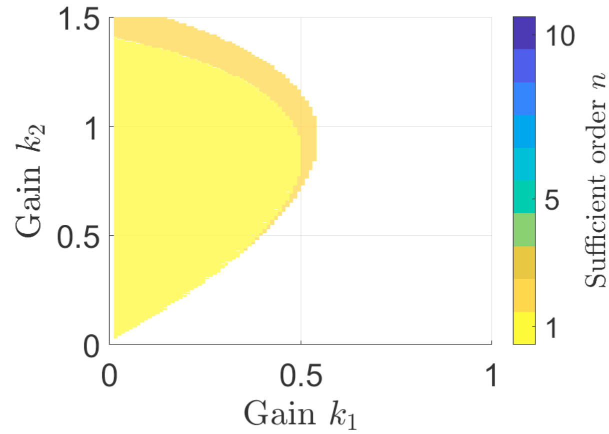

Fig. 2. Sufficient order n to satisfy LMI (24) for example (36).

$$\sup_{[-h,0)} \left| U^{(d)} \right| \leq |M|^d \rho |W|, \quad \forall d \in \mathbb{N}, \tag{27}$$

where $U^{(d)}$ is the d^{th} derivative of U, the parameter

$$\rho = \sqrt{n_x} \, e^{h|M|} \left| N^{-1} \right|, \tag{28}$$

and the matrices M, N and W are given by (6). Thanks to this smoothness, the uniform convergence of $\tilde{U}_{1,n}$ towards zero is supergeometric as proposed in Property 1 below.

Property 1. For all $n \geq 6$, the Legendre approximation error $\tilde{U}_{1,n}$ satisfies the inequalities

$$\sup_{[-h,0]} \left| \tilde{U}_{1,n} \right| \leq \frac{\mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{3}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} |W|, \tag{29a}$$

$$h \sup_{[-h,0]} \left| \tilde{U}'_{1,n} \right| \leq \frac{\mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} |W|, \tag{29b}$$

where

$$\mu = \frac{h|M|}{2}, \quad \rho_1 = \sqrt{\frac{2\pi^3}{\lceil \mu \rceil}} \rho, \tag{30}$$

and matrices M, N and W are given by (6).

Proof : The proof can be found in (Wang and Xiang, 2012, Theorem 2.5) using Legendre polynomials properties and $d = n - 1 - \lceil \mu \rceil$ successive integrations by parts. □

Property 2. For any $n \geq 4$, the Legendre approximation error $\tilde{U}_{2,n}$ satisfied the inequality

$$\sup_{\theta \in [-h,h]} \left| \tilde{U}_{2,n} \right| \leq \frac{\rho_2}{\sqrt{n-3}} |W|, \tag{31}$$

where

$$\rho_2 = \sqrt{2\pi} h (1 + \sqrt{n_x} \pi h \, e^{h|M|} \left| M^2 N^{-1} \right|), \tag{32}$$

and matrices M, N and W are given by (6).

Proof : The proof is provided in Bajodek (2022). □

The convergence properties of Legendre polynomial approximation ensures the existence of an order n^* from which Theorem 3 holds, as explained below.

4.2 Estimation of the orders

The following theorem estimates the orders for which Theorem 3 can be applied.

Theorem 4. Theorem 3 holds for orders $n \geq 6$ for which the following inequality holds

$$1 - \frac{\mu^{n-\lceil \mu \rceil} \rho'_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} - \frac{\rho'_2}{\sqrt{n-3}} \geq 0. \tag{33}$$

where scalars ρ'_1, ρ'_2 are equal to

$$\begin{cases} \rho'_1 = (3 + |A| + \frac{1}{h} + |A_d|)(2+h) |A_d| \rho_1, \\ \rho'_2 = 2(2+h) |A_d|^2 \rho_2, \end{cases} \tag{34}$$

Proof : Choosing matrices $W_1 = I_{n_x}$, $W_2 = hI_{n_z}$ and $W_3 = I_{n_z}$, it is worth noticing that Ψ_n^1 and Ψ_n^2 given by (16) are upper bounded by

$$\left| \Psi_n^1 \right| \leq \left(\frac{(|A| + \frac{1}{h}) \mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} + \frac{|A_d| \rho_2}{\sqrt{n-3}} \right) (2+h) |A_d|,$$

$$\left| \Psi_n^2 \right| \leq \left(\frac{\mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} + \frac{\rho_2}{\sqrt{n-3}} \right) (2+h) |A_d|^2.$$

From Schur complement, inequality (16) is satisfied under the conditions

$$\begin{cases} 1 - \left(\frac{(3 + |A| + \frac{1}{h}) \mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} + \frac{|A_d| \rho_2}{\sqrt{n-3}} \right) (2+h) |A_d| \geq 0, \\ 1 - \left(\frac{(|A| + \frac{1}{h} + |A_d|) \mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} + \frac{2|A_d| \rho_2}{\sqrt{n-3}} \right) (2+h) |A_d| \geq 0, \\ 1 - \left(\frac{(1 + |A_d|) \mu^{n-\lceil \mu \rceil} \rho_1}{(n-\frac{7}{2}) \dots (\lceil \mu \rceil + \frac{3}{2})} + \frac{|A_d| \rho_2}{\sqrt{n-3}} \right) (2+h) |A_d| \geq 0, \end{cases} \tag{35}$$

Consequently, the estimated order n^* satisfy (33), which concludes the proof. □

Thanks to the approximation of U by Legendre polynomials and leaning on the supergeometric convergence rate of error $\tilde{U}_{1,n}$, we greatly reduce the estimate of the order n^* proposed in Bajodek et al. (2022). These estimated orders are shown in the last example section.

5. NUMERICAL EXAMPLE

Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix}, \quad h = 1, \qquad (36)$$

for proportional gains $k_1 > 0$ and $k_2 \geq 0$.

Figure 1 reports the necessary orders n^* given by (33) that corresponds to system (36) for several gains (k_1, k_2) in $[0, 1] \times [0, 1.5]$. These orders n^* still very large to test the necessary and sufficient LMI conditions (24).

As a complement, Figure 2 reports the sufficient order n such that our sufficient condition of stability holds. Increasing the order n from 1 to n^*, we are looking for the first order $\check{n}$ from which the LMI condition (24) is true. One remarks that there are lower than the necessary order, which leaves a room for forthcoming improvements.

6. CONCLUSIONS

In this work, three steps leading to necessary and sufficient LMI conditions of stability are highlighted. First, the complete Lyapunov functional is presented. Second, from Theorem 2, we are now able to quantify the degree of conservatism introduced by an approximation process. Finally, the necessary and sufficient Legendre-LMI test provided by Theorem 3 is established. The size of the LMI test can be calculated thanks to Theorem 4.

Following the same methodology, this work paves the way to develop new LMI conditions for larger classes of systems. Future directions of research concern time-delay systems with multiple delays (see Alexandrova and Zhabko (2021)), distributed delays (see Egorov et al. (2017)) or non linear delay systems (see Di Ferdinando et al. (2021)). More generally, an extension to linear ordinary-partial interconnected differential equations would be promising to regard.

Future works could also be dedicated to the reduction of the necessary order n^* from which the necessary and sufficient condition holds. The goal would be to get closer to the sufficient order $\check{n}$. To do so, we could consider other pair of matrices W_1, W_2 and W_3 and treat directly the necessary condition $\Psi_n \succ 0$ or investigate other pseudo-spectral methods.

REFERENCES

Alexandrova, I.V. and Zhabko, A.P. (2021). Lyapunov -Krasovskii functionals for homogeneous systems with multiple delays . *Vestnik of Saint Petersburg University*, 17(10), 183—195.

Bajodek, M. (2022). *Stability analysis of linear ODE-PDE interconnected systems*. Ph.D. thesis, Université Paul Sabatier, Toulouse, France.

Bajodek, M., Seuret, A., and Gouaisbaut, F. (2022). On the necessity of sufficient LMI conditions for time-delay systems arising from Legendre approximation. hal-03435008v2. Submitted to Automatica.

Chaillet, A., Orlowski, J., and Pepe, P. (2019). A relaxed lyapunov-krasovskii condition for global exponential stability of lipschitz time-delay systems. In *Proc. 58th IEEE Conference on Decision and Control (CDC)*, 43–48.

Datko, R. (1970). Extending a theorem of A. M. Liapunov to Hilbert space. *Journal of Mathematical Analysis and Applications*, 32(3), 610–616.

Di Ferdinando, M., Pepe, P., and Di Gennaro, S. (2021). A Converse Lyapunov-Krasovskii Theorem for the Global Asymptotic Local Exponential Stability of Nonlinear Time-Delay Systems. *IEEE Control Systems Letters*, 5(1), 7–12.

Dunkl, C. and Xu, Y. (2001). *Orthogonal polynomials of several variables*. Encyclopedia of Mathematics ands its applications, 2 edition.

Egorov, A., Cuvas, C., and Mondié, S. (2017). Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays. *Automatica*, 80(6), 118–224.

Egorov, A. and Mondié, S. (2014). Necessary stability conditions for linear delay systems. *Automatica*, 50(12), 3204–3208.

Fridman, E. (2014). Tutorial on Lyapunov-based methods for time-delay systems. *European Journal of Control*, 20, 271–283.

Golub, G. and Van Loan, C. (1989). *Matrix Computations*. Johns Hopkins University Press, Baltimore.

Gomez, M., Egorov, A., and Mondié, S. (2021). Necessary and sufficient stability condition by finite number of mathematical operations for time-delay systems of neutral type. *IEEE Transactions on Automatic Control*, 66(6), 2802–2808.

Gottlieb, D. and Orszag, S. (1977). *Numerical analysis of spectral methods: theory and applications*. SIAM.

Gu, K. (2013). *Complete quadratic Lyapunov-Krasovskii functional: limitations, computational efficiency, and convergence*. In Advances in Analysis and Control of Time-Delayed Dynamical Systems. World Scientific.

Gu, K., Kharitonov, V., and Chen, J. (2003). *Stability of Time-Delay Systems*. Birkhäuser, Boston, USA.

Kharitonov, V. and Zhabko, A. (2003). Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39(1), 15–20.

Kharitonov, V. (2013). *Time-Delay Systems: Lyapunov Functionals and Matrices*. Control engineering. Birkhäuser.

Medvedeva, I. and Zhabko, A. (2015). Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems. *Automatica*, 51, 372–377.

Medvedeva, I. and Zhabko, A. (2019). Stability of neutral type delay systems: A joint Lyapunov-Krasovskii and Razumikhin approach. *Automatica*, 106, 83–90.

Seuret, A. and Gouaisbaut, F. (2013). Wirtinger-based integral inequality: application to time-delay systems. *Automatica*, 49(9), 2860–2866.

Seuret, A. and Gouaisbaut, F. (2014). Complete quadratic Lyapunov functionals using Bessel-Legendre inequality. In *European Control Conference (ECC)*, 448–453.

Seuret, A. and Gouaisbaut, F. (2015). Hierarchy of LMI conditions for the stability analysis of time-delay systems. *Systems and Control Letters*, 81, 1–7.

Wang, H. and Xiang, S. (2012). On the convergence rates of Legendre Approximation. *Mathematics of computation*, 81(278), 861–877.