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Backstepping Control of a Hyperbolic PDE System with Zero Characteristic Speed

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Abstract: In this paper, we study the single-input boundary feedback stabilization of 3×3 linear hyperbolic partial differential equations (PDEs) with two counterconvecting PDEs and the third one with zero characteristic speed. We design a full-state backstepping controller which exponentially stabilizes the origin in the L^2 sense. The zero transport velocity makes the previous backstepping designs inapplicable (their application would result in a controller with infinite gains). To employ backstepping in the presence of zero speed, we use an invertible Volterra transformation only for the PDEs with nonzero speeds, leaving the state of the zero-speed PDE unaltered in the target system, but making the target zero-speed PDE input-to-state stable with respect to the decoupled and stable counterconvecting nonzero-speed PDEs. In addition to achieving stabilization, we produce an explicit bound on the rate of convergence of the target system by a method of successive approximations and the use of Laplace transform. Simulation results are presented to illustrate the effectiveness of the proposed control design.

Keywords: Backstepping, Hyperbolic Systems, Linear Control, Partial Differential Equations, Stabilization, Zero Transport Speed.

1. INTRODUCTION

In this work, we study the boundary feedback stabilization of a certain class of 3×3 hyperbolic partial differential equations (PDEs) with two states convecting in opposite directions and the third one with zero characteristic speed, with actuation only at one boundary. For some values of the coupling coefficients, the system may become unstable, and the goal of this work is the design of stabilizing boundary feedback laws.

In the last years, boundary control of hyperbolic PDEs has been widely studied in the literature; in particular, the backstepping method has emerged as a powerful tool to design control laws and observers for such systems. However, systems with zero speed states have been neglected in previous backstepping designs. For instance, the result in Hu et al. (2016) covers the stabilization of $(n+m) \times (n+m)$ hyperbolic systems, consisting of n equations convecting in one direction and m controlled equations counter-convecting in the opposite direction. The extension of this approach for the output feedback regulation with additional disturbances is proposed in Deutscher and Gabriel (2018). An adaptive observer design for $(n+1) \times (n+1)$ hyperbolic systems can be found in Anfinsen et al. (2016), where the methodology can be used even for the case of unknown or incorrect parameters. Recently, the control design of hyperbolic systems with nonstrict-feedback couplings with ODEs has also been studied, see e.g. de Andrade et al. (2018) and references therein. All these developments assume nonzero characteristic speeds, becoming inapplicable otherwise.

Outside of the backstepping literature, there exists a few studies on exact boundary controllability for very specific hyperbolic systems with identically zero or vanishing characteristic speeds, as can be seen in Coron et al. (2009) or Li and Rao (2009). An approach based on static output feedback controllers was proposed in (Yong, 2019) but requires the so-called structural stability conditions which are rather strict on the coefficients of the system.

The disregard in the control literature for hyperbolic systems containing states with zero velocity does not imply

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they are not of practical interest; on the contrary, multiple applications do exist. A first example is a model of heat transfer dynamics in solar thermal plants based on direct steam generation technology (Guo et al., 2017). In this application, solar radiation heats the water to generate superheated steam which is used by a turbine generator to convert the thermal energy into electricity. The pipe temperature dynamics is usually described by a hyperbolic PDE with zero characteristic speed, which is coupled with the multiphase flow equations. The exponential stabilization of the states and reference tracking is crucial to correct and safe operation of the system, and can be performed by manipulating the boundary mass flow rate, using a pump placed at the inlet of the solar field.

A second application that fits into the zero speed framework is the photobioreactor for biomass production, where pH reference tracking must be achieved in presence of disturbances (Fernández et al., 2016). In this plant, the output can be manipulated by injecting carbon dioxide; the use of advanced control strategies helps to avoid carbon limitations, enhancing the performance of the culture, and consequently, reducing the amount of greenhouse gases released into the atmosphere.

A few more examples include models with thermoacoustic instabilities—a zero transport velocity in thermoacoustics is a direct consequence of the second law of thermodynamics—, double-pass laser amplifiers (Ren et al., 2012) and neurofilament transport in axons (Craciun et al., 2005).

Motivated by these applications, this paper aims to extend the infinite-dimensional backstepping methodology to a 3×3 system containing a state with zero velocity speed. As explained, previous results such as Hu et al. (2016) or Di Meglio et al. (2013) become inapplicable since they would result in a controller with infinite gain. Our idea is to apply the backstepping transformation only to the PDEs with nonzero speeds, leaving the state of the zero-speed PDE unaltered in the target system, but making the target zero-speed PDE input-to-state stable with respect to the decoupled and stable counterconvecting nonzero-speed target PDEs. In addition to achieving stabilization, we produce an explicit bound on the rate of convergence of the target system by a method of successive approximations and the use of Laplace transform. Compared with other results in the literature for hyperbolic PDEs containing states with zero characteristic speeds, such as Li and Rao (2009), our approach can be applied to a richer family of hypebolic systems that can be unstable in the nonzero speed part of the plant. In particular we provide numerical simulations to show the effectiveness of the method for an open-loop unstable case.

The rest of the paper is organized as follows. In Section 2, we present the model and the problem statement. In Section 3, we design a stabilizing control law using the backstepping methodology. The results are illustrated using numerical simulations in Section 4. Finally, Section 5 provides some concluding remarks and directions of future work.

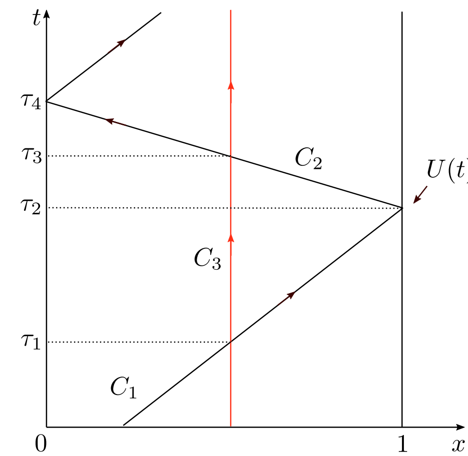

Fig. 1. Characteristic lines of system. The characteristic lines C_1 (with slope λ_1) and C_2 (with slope $-\lambda_2$) correspond to (1) and (2), respectively, whereas C_3 corresponds to (3). The reflection mechanism is illustrated at the points $x = 0$ and $x = 1$ at the time instants τ_2 and τ_4, respectively.

2. PROBLEM DESCRIPTION

In this work, we consider the following hyperbolic system:

$$\partial_t u(t,x) = -\lambda_1 \partial_x u(t,x) + c_{12} p(t,x) + c_{13} v(t,x), \quad (1)$$

$$\partial_t p(t,x) = \lambda_2 \partial_x p(t,x) + c_{21} u(t,x) + c_{23} v(t,x), \quad (2)$$

$$\partial_t v(t,x) = c_{31} u(t,x) + c_{32} p(t,x) - c_{33} v(t,x), \quad (3)$$

$$p(t,1) = U(t) + \rho u(t,1), \quad (4)$$

$$u(t,0) = q p(t,0), \quad (5)$$

where $t \in [0,\infty)$ is time, $x \in [0,1]$ is the space coordinate, λ_1 and λ_2 are positive constants, c_{12}, c_{13},c_{21}, c_{23}, c_{31}, c_{32}, c_{33} are real constants, ρ and q are reflection coefficients, with $q \neq 0$, and U is the control variable.

The initial condition of (1)-(5) is

$$u(0,x) = u_0(x), \quad p(0,x) = p_0(x), \quad v(0,x) = v_0(x),$$

where u_0, p_0 and v_0 are given functions in $L^2(0,1)$.

System (1)-(5) is hyperbolic, with two characteristic speeds with opposite signs, associated with (1) and (2), respectively, and one identically zero speed in (3). The latter means that the characteristic corresponding to (3) is vertical on the (x, t) plane (see Figure 1). As shown in Li and Rao (2009), the exact controllability of (1)-(5) can be obtained (without any constraints on the parameters) by imposing boundary controllers and an in-domain controller for the equation with zero characteristic speed. However, the question of whether the system (1)-(5) is stabilizable or not has not yet been pursued; we leave it for future research and simply state the following assumption that guarantees stabilizability, as shown in Section 3.

Assumption 2.1. It is assumed that $c_{33} > 0$.

All engineering applications presented in Section 1 satisfy the aforementioned assumption. The interested reader is referred to Fernández et al. (2016); Ren et al. (2012); Craciun et al. (2005) and references therein for more details. This assumption means that (3) would be exponentially stable in the absence of the coupling with (1)-(2). As we shall see in the following example, which is a simplified version of system (1)-(5), zero characteristic speeds may result in essential singularities in the transfer functions of the system.

Example 2.1. Consider the following linear hyperbolic system

$$\partial_t p(t,x) = \lambda_2 \partial_x p(t,x) + c_{23} v(t,x), \tag{6}$$
$$\partial_t v(t,x) = c_{32} p(t,x) - c_{33} v(t,x), \tag{7}$$
$$p(t,1) = U(t). \tag{8}$$

For simplicity, assume that the initial condition of (6)-(8) is zero. Then, applying the Laplace transform into (6)-(8), yields the following boundary value problem:

$$\lambda_2 \frac{d\hat{p}}{dx}(s,x) = \left(s - \frac{c_{23}c_{32}}{s+c_{33}}\right)\hat{p}(s,x),$$
$$\hat{v}(s,x) = \frac{c_{32}}{s+c_{33}}\hat{p}(s,x),$$
$$\hat{p}(s,1) = \hat{U}(s)$$

where $\hat{p}$, $\hat{v}$ and $\hat{U}$ are the Laplace transform of p, v and U, respectively.

This expression allows us to derive the distributed transfer matrix expressing the states $\hat{p}$ and $\hat{v}$ at each point $x \in [0,1]$ as a function of the input $\hat{U}$:

$$\hat{p}(s,x) = G(s,x)\hat{U}(s),$$
$$\hat{v}(s,x) = \frac{c_{32}}{s+c_{33}} G(s,x)\hat{U}(s),$$

with

$$G(s,x) = \mathrm{e}^{-\frac{1}{\lambda_2}\left(s - \frac{c_{23}c_{32}}{s+c_{33}}\right)(1-x)}.$$

For any $x \in [0,1)$, $G(s,x)$ has an essential singularity at $s = -c_{33}$. In fact, by contradiction assume that $s = -c_{33}$ is not an essential singularity. Then, there exists $n \in \mathbb{N}$ such that

$$\lim_{s \to -c_{33}} (s + c_{33})^n G(s,x) = 0.$$

Define the sequence $s_k = -\frac{c_{23}c_{32}}{k} - c_{33}$. Note that $s_k \to -c_{33}$ as $k \to \infty$, however,

$$|(s_k + c_{33})^n G(s_k,x)| = \frac{|c_{23}c_{32}|^n}{k^n} \mathrm{e}^{-\frac{c_{33}}{\lambda_2}(1-x)} \times \mathrm{e}^{-\frac{1}{\lambda_2}\left(\frac{c_{23}c_{32}}{k}\right)(1-x)} \mathrm{e}^{-\frac{1}{\lambda_2}k(1-x)}.$$

Therefore, for any $n \in \mathbb{N}$ and $x \in [0,1)$,

$$\lim_{k\to\infty} |(s_k + c_{33})^n G(s_k,x)| \neq 0,$$

which contradicts the assumption that $s = -c_{33}$ is not an essential singularity.

Although the existence of the essential singularity was shown for a simplified problem, it is expected that it is also present in (1)-(5), but accompanied by a much more complex expression. If Assumption 2.1 holds, then the essential singularity is located in the left-hand side of the complex plane and a boundary control law can be designed to achieve exponential stability as we shall see in the next sections. On the other hand, if $c_{33} \leq 0$ one can expect that boundary controllability cannot be obtained and an in-domain controller in (3) would be required for stabilization.

In the next section, we design a backstepping controller so that the null solution of (1)-(5) becomes stable. In this study, it is assumed that full-state measurements are available to use in the control law.

3. CONTROL DESIGN

Consider the following backstepping transformation of Volterra type:

$$\alpha = u - \int_0^x K_1(x,\xi)u(t,\xi)d\xi - \int_0^x K_2(x,\xi)p(t,\xi)d\xi - \int_0^x K_3(x,\xi)v(t,\xi)d\xi, \tag{9}$$

$$\beta = p - \int_0^x Q_1(x,\xi)u(t,\xi)d\xi - \int_0^x Q_2(x,\xi)p(t,\xi)d\xi - \int_0^x Q_3(x,\xi)v(t,\xi)d\xi, \tag{10}$$

where the kernels K_i and Q_i, for $i \in \{1,2,3\}$, satisfy the following PDEs:

$$\lambda_1\partial_x K_1 + \lambda_1 \partial_\xi K_1 = -c_{21}K_2 - c_{31}K_3, \tag{11}$$
$$\lambda_1\partial_x K_2 - \lambda_2 \partial_\xi K_2 = -c_{12}K_1 - c_{32}K_3, \tag{12}$$
$$\lambda_1\partial_x K_3 = -c_{13}K_1 - c_{23}K_2 + c_{33}K_3, \tag{13}$$
$$\lambda_2\partial_x Q_1 - \lambda_1 \partial_\xi Q_1 = c_{21}Q_2 + c_{31}Q_3, \tag{14}$$
$$\lambda_2\partial_x Q_2 + \lambda_2 \partial_\xi Q_2 = c_{12}Q_1 + c_{32}Q_3, \tag{15}$$
$$\lambda_2\partial_x Q_3 = c_{13}Q_1 + c_{23}Q_2 - c_{33}Q_3, \tag{16}$$

in the triangular domain

$$\mathcal{T} = \{(x,\xi) \in \mathbb{R}^2 : \ 0 \leq \xi \leq x \leq 1\},$$

with boundary conditions

$$K_1(x,0) = \frac{\lambda_2}{q\lambda_1}K_2(x,0), \quad Q_1(x,x) = -\frac{c_{21}}{\lambda_1+\lambda_2}, \tag{17}$$
$$K_2(x,x) = \frac{c_{12}}{\lambda_1+\lambda_2}, \quad Q_2(x,0) = \frac{\lambda_1 q}{\lambda_2}Q_1(x,0), \tag{18}$$
$$K_3(x,x) = \frac{c_{13}}{\lambda_1}, \quad Q_3(x,x) = -\frac{c_{23}}{\lambda_2}. \tag{19}$$

The kernel equations can be written as two separate PDE systems, one for K_i, for $i \in \{1,2,3\}$, and another for Q_i, for $i \in \{1,2,3\}$. The kernel PDEs (11)-(16) are transformed (with the help of the boundary conditions (17)-(19)) to integral equations. The solution of the integral equations is continuous and in fact differentiable w.r.t. some of the variables (e.g., Q_3 is $\mathcal{C}^1$ w.r.t. x).

From the boundedness of the kernels and the theory of Volterra integral equations, it follows that the inverse of transformation (9)-(10) always exists and can be defined as

$$u = \alpha + \int_0^x L_1(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L_2(x,\xi)\beta(t,\xi)d\xi + \int_0^x L_3(x,\xi)v(t,\xi)d\xi, \tag{20}$$

$$p = \beta + \int_0^x M_1(x,\xi)\alpha(t,\xi)d\xi + \int_0^x M_2(x,\xi)\beta(t,\xi)d\xi + \int_0^x M_3(x,\xi)v(t,\xi)dx, \tag{21}$$

where L_i and M_i, for $i \in \{1,2,3\}$ are the inverse kernels, which verify similar equations to (11)–(19).

For the transformation (9)–(10), we state the following lemma:

Lemma 3.1. Let U be given by the following control law

$$U(t) = -\rho u(t,1) + \int_0^1 Q_1(1,\xi)u(t,\xi)d\xi + \int_0^1 Q_2(1,\xi)p(t,\xi)d\xi + \int_0^1 Q_3(1,\xi)v(t,\xi)d\xi. \tag{22}$$

Then the transformation (9)–(10) maps (1)–(5) into the following target system:

$$\partial_t\alpha(t,x) = -\lambda_1\partial_x\alpha(t,x), \tag{23}$$
$$\partial_t\beta(t,x) = \lambda_2\partial_x\beta(t,x), \tag{24}$$
$$\partial_t v(t,x) = c_{31}\alpha(t,x) + c_{32}\beta(t,x) - c_{33}v(t,x) + \int_0^x N_1(x,\xi)\alpha(t,\xi)d\xi + \int_0^x N_2(x,\xi)\beta(t,\xi)d\xi + \int_0^x N_3(x,\xi)v(t,\xi)d\xi, \tag{25}$$
$$\beta(t,1) = 0, \tag{26}$$
$$\alpha(t,0) = q\beta(t,0), \tag{27}$$

with

$$N_1(x,\xi) = c_{31}L_1(x,\xi) + c_{32}M_1(x,\xi),$$
$$N_2(x,\xi) = c_{31}L_2(x,\xi) + c_{32}M_2(x,\xi),$$
$$N_3(x,\xi) = c_{31}L_3(x,\xi) + c_{32}M_3(x,\xi).$$

3.1 *Stability of the target system*

In this section we study the stability of the target system (23)-(27) with two different approaches. In the first one, a Lyapunov functional is proposed to show the exponential stability of the equilibrium at the origin of (23)-(27) in the L^2 norm. The second approach is based on the analysis of the explicit solution of (23)-(27), which allows us to bound the long-term behavior in the general case, thus characterizing the exponential decay of the system.

Lyapunov stability: Consider the following Lyapunov functional:

$$V(t) = \int_0^1 \left(\frac{A}{2\lambda_1}\mathrm{e}^{-\sigma x}\alpha^2(t,x) + \frac{B}{2\lambda_2}\mathrm{e}^{\sigma x}\beta^2(t,x)\right)dx + \frac{1}{2}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx, \tag{28}$$

where A, B, σ and δ are positive constants to be defined.

The time derivative of (28), using (23)-(27), is

$$\dot V = \int_0^1 \left(-A\mathrm{e}^{-\sigma x}\alpha(t,x)\alpha_x(t,x) + B\mathrm{e}^{\sigma x}\beta(t,x)\beta_x(t,x)\right)dx + \int_0^1 v(t,x)\mathrm{e}^{-\delta x}\left(c_{31}\alpha(t,x) + c_{32}\beta(t,x)\right)dx - c_{33}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx + \int_0^1\int_0^x \mathrm{e}^{-\delta x}(N_1(x,s)\alpha(t,s) + N_2(x,s)\beta(t,s) + N_3(x,s)v(t,s))\,v(t,x)dsdx. \tag{29}$$

Integrating by parts the first two terms of (29), and plugging the boundary conditions (26)-(27), yields

$$\int_0^1 \left(-A\mathrm{e}^{-\sigma x}\alpha(t,x)\partial_x\alpha(t,x) + B\mathrm{e}^{\sigma x}\beta(t,x)\partial_x\beta(t,x)\right)dx = -\sigma\int_0^1\Big(A\mathrm{e}^{-\sigma x}\alpha^2(t,x) + B\mathrm{e}^{\sigma x}\beta^2(t,x)\Big)dx - A\alpha^2(t,1)\mathrm{e}^{-\sigma} + (Aq^2 - B)\beta^2(t,0). \tag{30}$$

Now, turning our attention into the second line of (29), we get the following upper bound, after applying Young's inequality and using $\mathrm{e}^{-\delta x} \le 1 \le \mathrm{e}^{\sigma}\mathrm{e}^{-\sigma x}$ and $\mathrm{e}^{-\delta x} \le 1 \le \mathrm{e}^{\sigma x}$,

$$\int_0^1 v(t,x)\mathrm{e}^{-\delta x}\left(c_{31}\alpha(t,x) + c_{32}\beta(t,x)\right)dx \le \frac{2c_{31}^2\mathrm{e}^{\sigma}}{c_{33}}\int_0^1 \mathrm{e}^{-\sigma x}\alpha^2(t,x)dx + \frac{2c_{32}^2}{c_{33}}\int_0^1 \mathrm{e}^{\sigma x}\beta^2(t,x)dx + \frac{c_{33}}{4}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx. \tag{31}$$

Define now $\overline{N}_i = \max_{(x,y)\in\mathcal{T}}|N_i(x,y)|$, for $i \in \{1,2,3\}$. Then, applying the Cauchy-Schwarz inequality and Young's inequality,

$$\int_0^1\int_0^x \mathrm{e}^{-\delta x}\left(N_1(x,s)\alpha(t,s) + N_2(x,s)\beta(t,s)\right)v(t,x)dsdx \le \frac{2\overline{N}_1^2\mathrm{e}^{\sigma}}{c_{33}}\int_0^1 \mathrm{e}^{-\sigma x}\alpha^2(t,x)dx + \frac{2\overline{N}_2^2}{c_{33}}\int_0^1 \mathrm{e}^{\sigma x}\beta^2(t,x)dx + \frac{c_{33}}{4}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx. \tag{32}$$

Finally,

$$\int_0^x |N_3(x,s)v(t,s)|ds \le \overline{N}_3\frac{\mathrm{e}^{\delta x/2}}{\sqrt{\delta}}\sqrt{\int_0^1 \mathrm{e}^{-\delta s}v^2(t,s)ds},$$

and thus,

$$\int_0^1\int_0^x \mathrm{e}^{-\delta x}N_3(x,s)v(t,s)v(t,x)dsdx \le \frac{\overline{N}_3}{\sqrt{\delta}}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx$$

Therefore we reach

$$\begin{aligned}\dot V \leq &-\left(A\sigma - \frac{2(c_{31}^2+\overline{N}_1^2)\mathrm{e}^{\sigma}}{c_{33}}\right)\int_0^1 \mathrm{e}^{-\sigma x}\alpha^2(t,x)dx\\ &-\left(B\sigma - \frac{2(c_{32}^2+\overline{N}_2^2)}{c_{33}}\right)\int_0^1 \mathrm{e}^{\sigma x}\beta^2(t,x)dx\\ &-\left(\frac{c_{33}}{2} - \frac{\overline{N}_3}{\sqrt{\delta}}\right)\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx\\ &+(Aq^2-B)\beta^2(t,0) \end{aligned} \tag{33}$$

Choosing

$$B = Aq^2 + 1, \tag{34}$$

$$A = \mathrm{e}^{\sigma}, \tag{35}$$

$$\sigma = \max\left\{\frac{2(c_{31}^2+\overline{N}_1^2)}{c_{33}}, \frac{2(c_{32}^2+\overline{N}_2^2)}{c_{33}}\right\} + 1, \tag{36}$$

$$\delta = \frac{16\overline{N}_3^2}{c_{33}^2}, \tag{37}$$

we get

$$\begin{aligned}\dot V \leq &-\int_0^1 \mathrm{e}^{-\sigma x}\alpha^2(t,x)dx - \int_0^1 \mathrm{e}^{\sigma x}\beta^2(t,x)dx\\ &-\frac{c_{33}}{4}\int_0^1 \mathrm{e}^{-\delta x}v^2(t,x)dx\\ \leq &-KV \end{aligned} \tag{38}$$

for $K = \min\left\{\frac{2\lambda_1}{A}, \frac{2\lambda_2}{B}, \frac{c_{33}}{2}\right\} > 0$, thus proving exponential stability of the equilibrium $\alpha \equiv \beta \equiv v \equiv 0$. By norm equivalences and considering the direct and inverse backstepping transformations, the exponential stability of the equilibrium $u \equiv p \equiv v \equiv 0$ in the L^2 norm follows. Notice that this proof shows that the target zero-speed PDE (25) is input-to-state stable with respect to the other two (stable) PDEs (23)–(24).

An explicit bound on the rate of convergence. By inspecting the target system (23)-(27) and using the method of characteristics, one can notice that $\beta(t,x) = 0\ \forall x \in [0,1]$ and $t \geq \frac{1}{\lambda_2}$. Then, by the cascade structure of (23)-(27) it follows that $\alpha(t,x) = 0\ \forall x \in [0,1]$ and $t \geq \frac{1}{\lambda_2} + \frac{1}{\lambda_1}$.

Thus, from the time $t \geq \frac{1}{\lambda_2} + \frac{1}{\lambda_1}$ onwards we can rewrite (25) as

$$\partial_t v(t,x) = -c_{33}v(t,x) + \int_0^x N_3(x,\xi)v(t,\xi)d\xi. \tag{39}$$

Applying the Laplace transform to (39) and denoting as before the Laplace transform of v by $\hat v$, we have that

$$\hat v(s,x) = \frac{v_0(x)}{s+c_{33}} + \frac{1}{s+c_{33}}\int_0^x N_3(x,\xi)\hat v(s,\xi)d\xi,$$

which is a Volterra-type integral equation for v in the Laplace domain.

The solution of this integral equation will be generated using the method of successive approximation as

$$\hat v(s,x) = \sum_{n=0}^{\infty} \hat V^n(s,x),$$

with

$$\begin{aligned}\hat V^0(s,x) =& \frac{v_0(x)}{s+c_{33}},\\ \hat V^n(s,x) =& \int_0^x N_3(x,\xi)\hat V^{n-1}(s,\xi)d\xi, \quad \text{for } n \geq 1.\end{aligned}$$

The explicit expression the general term of the power series can be obtained after computing the first few terms (this formula can be verified by induction):

$$\begin{aligned}\hat V^n(s,x) = &\frac{1}{(s+c_{33})^{n+1}}\int_0^x N_3(x,\tau_1)\int_0^{\tau_1} N_3(\tau_1,\tau_2)\ldots\\ &\times \int_0^{\tau_{n-1}} N_3(\tau_{n-1},\tau_n)v_0(\tau_n)d\tau_n \ldots d_{\tau_1}.\end{aligned} \tag{40}$$

Then, using the inverse Laplace transform, from (40) it follows that

$$\begin{aligned}V^n(t,x) =& \frac{t^n \mathrm{e}^{-c_{33}t}}{n!}\int_0^x N_3(x,\tau_1)\int_0^{\tau_1} N_3(\tau_1,\tau_2)\ldots\\ &\times \int_0^{\tau_{n-1}} N_3(\tau_{n-1},\tau_n)v_0(\tau_n)d\tau_n \ldots d\tau_1,\end{aligned} \tag{41}$$

and

$$v(t,x) = \sum_{n=0}^{\infty} V^n(t,x).$$

Now we study the convergence of the series. Assume for simplicity that the initial condition v_0 is continuous. Define

$$\overline{v}_0 = \max_{x\in[0,1]} |v_0(x)|.$$

Then,

$$|v(t,x)| \leq \overline{v}_0 \mathrm{e}^{-c_{33}t} I_0\left(2\sqrt{\overline{N}_3\, t\, x}\right), \tag{42}$$

where $I_0(\cdot)$ is the zero-order modified Bessel function of the first kind.

Since $c_{33} > 0$, we have that

$$\lim_{t\to\infty} \overline{v}_0 \mathrm{e}^{-c_{33}t} I_0\left(2\sqrt{\overline{N}_3\, t\, x}\right) = 0,$$

for any $x \in [0,1]$. Then, it follows that the solutions of the target system (23)–(27) decay to zero exponentially, and its temporal behaviour becomes quite clear. Finally, considering the direct and inverse backstepping transformations, the exponential decay of the states (u,p,v) follows.

Remark 3.1. Inequality (42) represents an explicit upper-bound on $|v(t,x)|$ after time $t = 1/\lambda_1 + 1/\lambda_2$.

4. SIMULATION RESULTS

In this section, we present numerical simulations of system (1)-(4) with the proposed control law (22). The finite differences method was used to compute the states of the system and solve the kernel PDEs. The parameters were chosen as $q = -0.7$, $\rho = 0.5$, $c_{12} = 2.3$, $c_{13} = 1$, $c_{21} = -2.15$, $c_{23} = 1$, $c_{31} = 1$, $c_{32} = 1$, $c_{33} = 0.5$, $\lambda_1 = 1.25$ and $\lambda_2 = 0.9$, which corresponds to an open-loop unstable system.

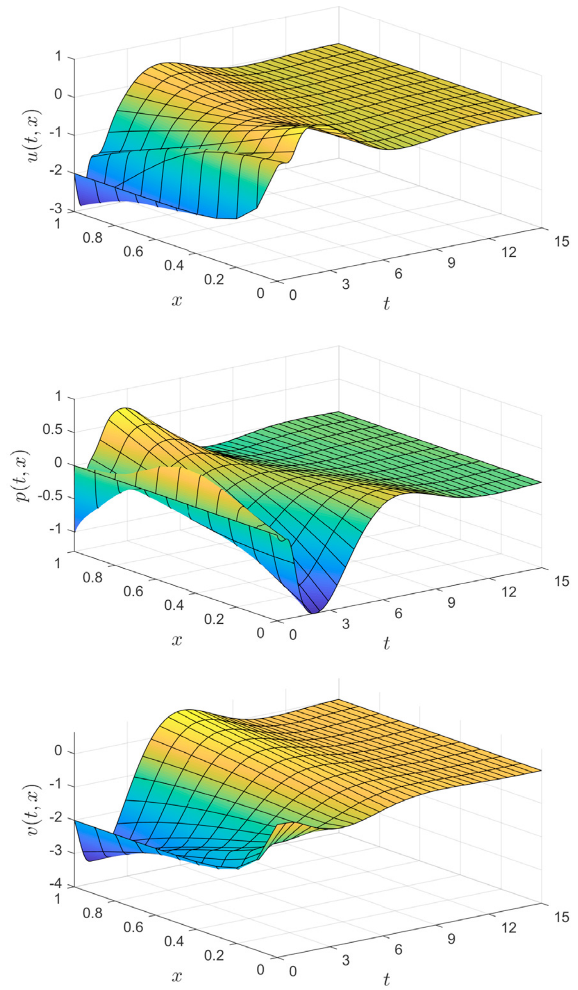

Fig. 2. Closed-loop states as a function of x and t.

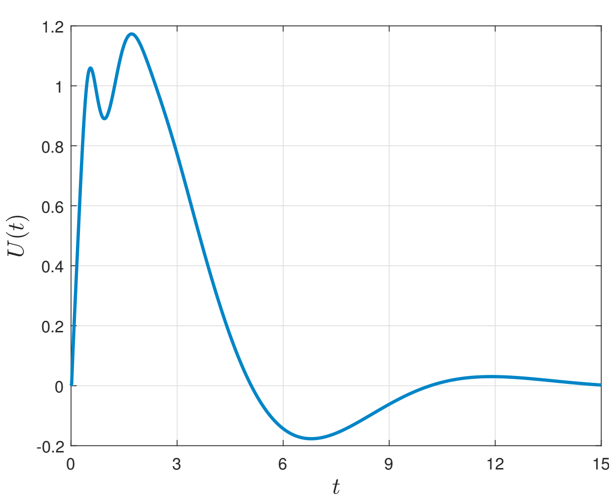

Fig. 3. Control signal as a function of t.

Figures 2 and 3, show the closed-loop states and the control signal, respectively. As can be seen in Figures 2 and 3, the system states decay to zero after an initial transient.

5. CONCLUSIONS

This paper dealt with single-input boundary feedback stabilization of 3×3 linear hyperbolic partial differential equations (PDEs) with two counterconvecting PDEs and the third one with zero characteristic speed. The model was motivated by a number of engineering applications including solar plants, photobioreactors or lasers. Exponential stability of the null equilibrium of the system in the L^2 norm was obtained and tested numerically.

It is quite clear that the considered method can be extended to an $(n+m+r) \times (n+m+r)$ hyperbolic system, consisting of n equations convecting in one direction, m controlled equations counter-convecting in the opposite direction and r zero-speed equations. This will be the topic of an upcoming journal publication.

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