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ARTICLE

Proof levels of graph theory students under the lens of the Van Hiele model

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ABSTRACT

This work is devoted to exploring proof abilities in Graph Theory of undergraduate students of the Degree in Computer Engineering and Technology of the University of Seville. To do this, we have designed a questionnaire consisting of five open-ended items that serve as instrument to collect data concerning their proof skills when dealing with graphs. We have thus analysed them adapting the methodology for computing the degrees of acquisition of the Van Hiele levels. Our analysis leads to different proof profiles of Graph Theory students whose characteristics provide empirical support to consider proof levels in Graph Theory from the perspective of the Van Hiele model.

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Levels of reasoning; proof; graph theory; Van Hiele model

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1. Introduction

Mathematics Education at the university level has emerged as a focus of research interest in the last decade (Durand-Guerrier et al., 2021). As a sign of this, some of the most relevant international conferences on Mathematics Education have dedicated specific working groups or specific meetings to this topic in recent years. Indeed, CERME[1] has a *Thematic Working Group* (*TWG*) on *University Mathematics Education* and ICME[2] on *Mathematics Education at Tertiary Level*. Likewise, new conferences publishing proceedings have recently emerged such as the congress of the INDRUM[3] or the *RUME*[4] *Conference in the United States*. Moreover, there are long-established journals that have University Mathematics Education among their priority objects of study, such as *The College Mathematics Journal* and *PRIMUS*[5], published since 1970 and 1991, respectively. In addition to these publications, since 2006 the Polish journal *Annales Universitates Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* publishes studies more focused on didactic aspects. All this educational research at tertiary level has helped to deepen into the study of the skills characterising the so called *Advanced Mathematical Thinking* (Tall, 1992), this is, precise mathematical definitions and logical deductions of theorems based upon them. This type of thinking is directly involved in our work since we are explicitly concerned with

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the ability of proof, whose relevance is reflected in a number of research papers focusing on this mathematical ability (Arnal-Bailera & Oller-Marcén, 2017; Demiray & Işiksal, 2017; Stylianides et al., 2007; Uğurel et al., 2015).

A relevant mathematical topic at tertiary level is Graph Theory because of its multiple applications in other areas such as Ecology, Chemistry, Transportation, Biology, Social Networks, Information and Communication, Circuits, Computer Networks, and Software Design; among others (Rosen, 2019). Furthermore, several studies in Mathematics Education (Cartier, 2008; DeBellis & Rosenstein, 2004; Grenier & Payan, 1999; Heinze et al., 2004; Leon et al., 2020) show the importance of the role that graphs can play in the acquisition of mathematical skills, specifically proof. Besides these works, much educational research has centred its attention in different aspects of the teaching and learning of Graph Theory such as task design (Niman, 1975; Santoso, 2018), resources for enhancing learning (Costa et al., 2014; Geschke et al., 2005) or the teaching of graphs in levels other than tertiary (Rosenstein, 2018), among others. However, different authors (Hazzan & Hadar, 2005; Medová et al., 2019) highlight the lack of research on the reasoning of Graph Theory students, which requires the development of adequate theoretical frameworks. This last issue is approached by Gavilán-Izquierdo and González (2016), who first point out the Van Hiele model to be applied in the teaching and learning of Graph Theory, and subsequently by Ferrarello and Mammana (2018), who present an experimental teaching activity which considers the nature of Van Hiele levels.

The Van Hiele model has mainly been applied to the field of Geometry (Burger & Shaughnessy, 1986; Chen et al., 2019; Gutiérrez & Jaime, 1995; Hoffer, 1983; Lee, 2015; Pandiscio & Knight, 2010; Perdikaris, 2004; Usiskin, 1982; Wang & Kinzel, 2014), specifically to undertake studies on the abilities of students in proof practices (Gutiérrez et al., 2004; Manero & Arnal-Bailera, 2021; Senk, 1989). In addition, this model has proved to be useful to describe the learning of other mathematical topics such as local approximation (Llorens & Pérez-Carreras, 1997), convergence of sequences (Navarro & Pérez-Carreras, 2006), convergence of series (Jaramillo, 2000) or functions (Nisawa, 2018). Thus, González et al. (2021) perform a theoretical analysis that produces a characterization of the learning of Graph Theory through four levels of reasoning under the lens of the Van Hiele model, whose validity must be tested in empirical studies as the present paper. This characterization is organised through the development of the processes of reasoning that students may activate when dealing with graphs: recognition, use and formulation of definitions, classification, and proof, thus providing descriptors of levels for each process.

The purpose of this work is to provide empirical support for the validity of the levels of proof in Graph Theory through the lens of the Van Hiele model proposed by González et al. (2021). Indeed, the results of González et al. (2021) are based on (1) prior research on students' mathematical thinking in areas different from Graph Theory, (2) their mathematical understanding of the concept of graph as researchers in Mathematics and Didactics of Mathematics, and (3) their experience as Graph Theory teachers. Thus, it is necessary to develop empirical studies as the present work that analyses students' answers that allow to evaluate the validity of this theoretical characterization. This is the natural process in the elaboration of an educational model (Jaime & Gutiérrez, 1990; O'Leary-Kelly & Vokurka, 1998) which requires empirical validation after the theoretical analysis. This implies the search for evidence of the indicators provided by these authors for the levels, as well as the study of their adequacy to the particularities of the Van Hiele model.

2. Theoretical considerations

2.1. The Van Hiele model for geometry

In the 1950s, Van Hiele (1957) and Van Hiele-Geldof (1957) propose a model that characterizes the development of geometric thinking through a sequence of five levels. However, we do not consider here the fifth level since it mainly concerns professional mathematicians' reasoning which are out of the scope of this study. Thus, students at level 1 (visualization) recognize geometric figures by their appearance and as a whole; level 2 (analysis) is characterised by the students' ability to handle the parts and mathematical properties of figures; the reasoning of level 3 (informal deduction) uses logical deductions, which enable students to interrelate properties of geometric figures; and students at level 4 (formal deduction) can produce formal proofs and deal with equivalent definitions of a concept. For a more detailed description of the levels, we refer the reader to the work of Van Hiele (1986).

These levels have a series of characteristics (Jaime & Gutiérrez, 1990) that differentiate them from levels proposed in other theoretical frameworks (Arnon et al., 2014; Biggs & Collis, 1982; Pirie & Kieren, 1989). Specifically, Van Hiele levels are (1) *hierarchical and sequential*, which means that for students to completely acquire a certain level it is necessary that they go across the preceding ones in a specific order (e.g. students cannot completely acquire level 3 before having completely acquired level 2); (2) *highly related with language*, this is, each level has specific vocabulary and different ways to understand mathematical concepts; and (3) *continuous*, which means that the acquisition of a certain level is not instantaneous and can start before the complete acquisition of a preceding level (e.g. students can start the acquisition of level 3 before completely having acquired level 2).

Gutiérrez and Jaime (1998) propose a way to regard the geometrical reasoning as decomposed into different processes of reasoning: recognition (i.e. identification of types of figures, as well as their components and properties), use of definitions (i.e. handling of geometrical concepts), formulation of definitions (i.e. elaboration of descriptions or characterizations of geometrical notions), classification (i.e. placement of geometrical objects into different families), and proof (i.e. explanation in some convincing way that a statement is true). Thus, these authors characterize the Van Hiele levels according to the degree of development of each of these processes. We next describe the development of the proof process, which is the focus of this paper.

Proof at level 1 is not considered by Gutiérrez and Jaime (1998). At level 2, proofs are characterized by verifications in particular cases. Students at level 3 can verify statements by using informal explanations based on mathematical properties. Moreover, they are able to understand formal proofs and even reproduce a few logical steps but they cannot produce themselves formal proofs, which characterizes level 4 students.

2.2. The Van Hiele model for graphs

González et al. (2021) provide a theoretical characterization of students' reasoning in Graph Theory based on Van Hiele levels. To present this characterisation, we first provide some fundamentals of Graph Theory for the sake of completeness. Indeed, a *graph* G consists of a pair (V, E) where V is any set, which is called the *vertex set*, and E is a set of pair of non-ordered pairs of elements from V, which is called *edge set*. The *pictorial representation*, which is the most common way to represent graphs, consists of drawing vertices as points in the plane which are joined by (non-necessarily straight) segments whenever the corresponding pairs of vertices form edges. Delving into the parts and properties of graphs, we say that two vertices are *adjacent* whenever they form an edge, and the *degree* of a vertex is the number of vertices adjacent to it. Finally, a graph is said to be *Eulerian* when it admits a sequence of adjacent vertices containing each of its edges exactly once and starting and finishing at the same vertex. Eulerian graphs are characterized by having all their vertices of even degree. Note that properties of graphs can be divided into *local* (i.e. associated with parts of the graph) and *global* (i.e. associated with the whole graph). Thus, the degree is local, while the Eulerian character is global. (We refer the reader to the book of Rosen (2019) for more information on Graph Theory.)

We can now describe the main indicators of the Van Hiele levels for Graph Theory (González et al., 2021). It is easy to see that they have the same nature as the Van Hiele levels for Geometry. Indeed, students at level 1 (visualization) have a visual type of recognition that limits them when identifying graphs and global properties; the reasoning at level 2 (analysis) is mainly supported by students' ability to identify global and local mathematical properties of graphs, which enables them to distinguish graphs independently from their representations; students at level 3 (informal deduction) can interrelate graph properties and provide logical arguments; and students at level 4 (formal deduction) perceive graphs as formal mathematical objects, and so they can work with equivalent definitions of the same concept and construct formal proofs of mathematical results. More details on the descriptors of each level can be found in the work of González et al. (2021).

Furthermore, these authors organize the descriptors of each level according to the processes proposed by Gutiérrez and Jaime (1998) adapted to the field of graphs. In particular, the evolution of the process of proof in Graph Theory and in Geometry are analogous from levels 2–4. However, the proof of level 1, which is not considered by Gutiérrez and Jaime (1998), is taken into account by González et al. (2021) due to the peculiarities of graphs. Indeed, proofs at level 2 are given by verifications in specific examples; students at level 3 are able to produce informal proofs to justify the truth of a statement, and they can understand the steps of a formal proof but they cannot write it themselves; and level 4 students can elaborate formal proofs, thus being able to perform classic techniques in Graph Theory such as induction, proof by contradiction, or proof by contraposition. Level 1 students just provide visual arguments to justify the truth of a statement or verify it in specific examples, as well as level 2 students do, but being very limited by the representations that they know for a graph.

3. Method

3.1. Data collection instrument

In order to evaluate the proof process in Graph Theory we have designed a five items questionnaire focused on detecting evidence of the indicators of the proof levels proposed by González et al. (2021). This instrument has been developed following the same ideas as the proof tasks proposed by Gutiérrez and Jaime (1998) to evaluate the proof process, which are open ended tasks that bring to light the reasoning of the students more clearly than multiple choice questions (Jaime & Gutiérrez, 1994). For more examples of this type of tasks, we refer the reader to the works of Aravena et al. (2016) and Burger and Shaughnessy (1986).

The first version of this questionnaire was validated by experts in Mathematics Education and experts in Graph Theory who are familiar with the Van Hiele model, none of them being involved in this work. Subsequently, this questionnaire was first administered to students enrolled in the course Logic and Discrete Mathematics of the Degree in Software Engineering of the Polytechnical University of Madrid, which allowed us to create an improved version of the questionnaire employed in this work. Moreover, we studied the scalability and internal coherence of the questionnaire, which have been measured through the Guttman and Cronbach's Alpha coefficients, respectively. The Guttman coefficient applied to our context (Mayberry, 1983) is given by $G = 1 - \frac{e}{l \cdot N}$, where e is the total number of errors, l is the number of levels and N is the number of participants in the study. This coefficient provides an idea of the extent to which an observed set of responses patterns agrees with that expected from a perfect scale (i.e. without errors), which reveals in our context to what extent the hierarchical nature of the Van Hiele levels appears in our results. We have obtained for the Guttman coefficient a value of 0.931, which is an indicator of the reliability of the questionnaire as it is greater than 0.9 (Torgerson, 1967). The Cronbach's Alpha coefficient is given by $\alpha = \left(\frac{k}{k-1}\right) \cdot \left(1 - \frac{\sum_{i=1}^{k} S_i^2}{S_t^2}\right)$, where k is the number of items, ${S_i}^2$ is the variance of item i, and ${S_t}^2$ is the variance of the total observed values. Thus, this coefficient shows to which extent the answers given by each student to the different tasks of the test manifest similar levels of reasoning. We have obtained a value of 0.874 for this parameter, which is another indicator of the reliability of the test since it is greater than 0.7 (Fraenkel & Wallen, 1996).

The questionnaire, that we next describe, consists of five items to evaluate students' proof skills in Graph Theory, which were answered independently. The tasks included in each item, all containing concepts familiar to the students, were selected in order to detect the different types of proof that can appear when dealing with graphs: visual (level 1), empirical (level 2), informal (level 3) and formal (level 4).

> Item 1. Remember that the degree of a vertex is the number of vertices adjacent to it (i.e. they are connected by an edge). What is the sum of the degrees of all the vertices of a graph? Give a proof of your answer.

The first item, which demands to prove that the sum of the degrees of the vertices of a graph is twice the number of its edges, assesses levels from 2 to 4. Indeed, the item does not serve to evaluate level 1 because the statement to be proved necessarily induces to explore a local property such as the degree of each vertex, which is not manageable at level 1. Thus, expected level 2 answers contain a specific value for the required sum and verifications in concrete examples. In contrast, level 3 answers, which may also contain verifications in specific examples, provide general mathematical reasoning but they do not explicitly show the double edge counting that appears in level 4 answers. Moreover, this last type of answers might be given using mathematical induction.

> Item 2. Fill in the gaps of the following sentences:
>
> For a complete graph with 4 vertices, the number of its edges is … and the sum of the degrees of all its vertices is …

For a cycle with 5 vertices, the number of its edges is … and the sum of the degrees of all its vertices is …

For a graph on m edges, the sum of the degrees of all vertices is … Justify your answer.

The second item provides a scaffolding for the proof of the preceding statement, thus demanding students to first count the sum of degrees and the number of edges of two specific graphs. Hence, this scaffolding helps students to find the general relation for any graph and deduce the corresponding proof, which precisely requires a double counting of its edges. Note that students could not go back to previous items during the replying of the questionnaire. Thus, this item does not assess level 1 for the same reasons provided for item 1 since it requires a proof of the same result. Level 2 expected answers, just like in item 1, are based on verifications in particular graphs. Also, any answers containing reasoning beyond verifications in examples reflect level 3, even if they explicitly contain the double edge counting idea. Indeed, this item does not assess level 4 because students have enough information to give the required proof just by linking a few logical steps, which is feasible by level 3 students.

Item 3. Remember that the complete bipartite graph $K_{n,m}$ is composed of two sets of vertices such that no edge has both vertices in the same set, and has every possible edge connecting vertices of both sets. Figure 1 shows the graph $K_{3,5}$.
Try to prove that no complete bipartite graph with an odd number of vertices is Eulerian.

The third item, which serves to assess all levels, asks for proving that no complete bipartite graph with an odd number of vertices is Eulerian. Answers with evidence of indicators of level 1 could contain visual arguments such as the impossibility to draw the graph without lifting the pencil from the paper, for instance in the example of graph provided in the questionnaire. Level 2 answers are limited to verifications in specific complete bipartite graphs, whilst level 3 answers contain reasoning made on mathematical properties and relations between these properties. This type of answers may contain gaps and provide conclusions obtained via a non-rigorous process, for instance, they could avoid a justification of the fact that a complete bipartite graph with an odd number of vertices must contain vertices of odd degree. Finally, level 4 answers are characterised by its degree of formality, showing proofs with a series of justified steps that logically lead us from the hypothesis to the thesis of the statement. Thus, these answers necessarily include the following ideas: (1) the degree of a vertex of a set of the complete bipartite graph equals the cardinality of the other set, and (2) in a complete bipartite graph with an odd number of vertices, exactly one of the two sets has odd cardinality.

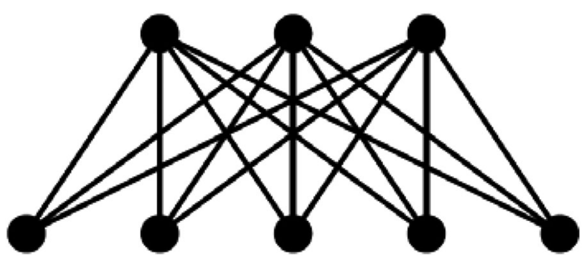

Figure 1. The complete bipartite graph $K_{3,5}$.

Item 4. Claim 1. If a graph has a vertex of odd degree, then that graph is not Eulerian.

Claim 2. If the complete bipartite graph $K_{n,m}$ has an odd number of vertices, then either n is odd and m is even or vice versa.

Considering the preceding claims, prove that no complete bipartite graph with an odd number of vertices is Eulerian.

The fourth item provides some hints to proof the statement of the preceding item. Thus, if this scaffolding does not work as a deterrent for students that use visual arguments, then they are assigned level 1. Level 2 expected answers, just like in item 3, lie in specific verifications in examples, in contrast with level 3 answers, which provide mathematical reasoning beyond examples. This item does not allow to assess level 4 of proof since students have enough information to construct the required proof just by properly relating the claims provided in the statement of the item.

Item 5. Here you have a proof of the fact that no complete bipartite graph with an odd number of vertices is Eulerian. Read it and try to understand it:

- The vertex set of the complete bipartite graph is partitioned into two sets, one having n vertices and the other having m vertices. Each vertex of a set is adjacent to all the vertices of the other set, and there are no adjacencies among vertices of the same set. (See Figure 2).
- Thus, the vertices of one set have degree m and the vertices of the other set have degree n.
- As the total number of vertices is $n + m$, which is an odd number, then either m is odd, and n is even or vice versa.
- Therefore, the graph has vertices of odd degree and so it is not Eulerian.

You have just seen a proof of the fact that no complete bipartite graph with an odd number of vertices is Eulerian. Give a **similar** proof for the following statement:

A complete bipartite graph $K_{n,m}$ is Eulerian if and only if n and m are both even.

The last item exhibits a formal proof of the result of item 3, and then it asks students to understand it and provide a similar proof for a different statement. Thus, this item does not assess the features of level 1 since it is required to be at level 2 to at least handle the mathematical properties appearing in the given proof. Again, level 2 answers should be made of specific verifications. Level 3 expected answers include general reasonings to prove the statement but only concerning one of the implications. Although the task demands the

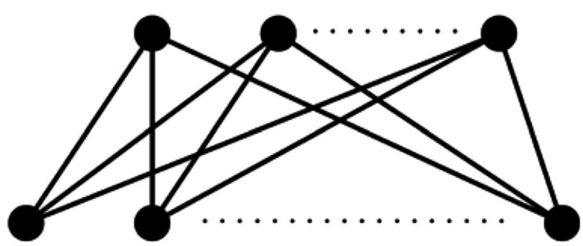

Figure 2. The complete bipartite graph $K_{n,m}$.

Table 1. Levels assessed by each item.

Item	Level			
1		2	3	4
2		2	3	
3	1	2	3	4
4	1	2	3	
5		2	3	4

replication of a proof, which could be done in principle by level 3 students, it is remarkable that the proof provided to students requires arguments for only one implication, while the proof demanded to students requires to consider two implications. Thus, it is necessary to possess sufficient formality to be aware of the necessity of proving both implications, which is characteristic of level 4 students.

We summarize the information provided in this section in Table 1 for the sake of readability.

3.2. Sample

The questionnaire was administered to 59 students (labelled from S1 to S59) enrolled in the course *Discrete Mathematics*, which corresponds to the second year of the Degree in Computer Engineering and Technology of the University of Seville. We point out that we have discarded the productions of 5 students (namely S1, S3, S7, S27, and S55) because they provided non-assessable answers to at least three items of the questionnaire (i.e. more than a half of the items).

3.3. Assessment of the proof process in graph theory

The data obtained in this study have been analysed by means of the method introduced by Gutiérrez et al. (1991) to compute the degrees of acquisition of the Van Hiele levels, which has been applied in several studies in the field of Geometry (Abdullah & Zakaria, 2013; Aravena et al., 2016; Gutiérrez & Jaime, 1995; Gutiérrez & Jaime, 1998; Huerta, 1999; Manero & Arnal-Bailera, 2021). This method, which provides a description of the development of the students' skills associated with each of the Van Hiele levels, is adequate in our study since the descriptors of the Van Hiele levels of graphs (González et al., 2021) have been mainly obtained by analogy with the descriptors of the Van Hiele levels for the geometrical case. We next summarize the steps of the procedure for applying such method.

Given the answers of a student to the five items, this method first assigns a *level* to each answer, according to the criteria developed in the subsection devoted to the data collection instrument, and also a *type* in accordance with the indicators described in the second column of Table 2.

Subsequently, each answer is marked with a percentage of acquisition for each of the levels evaluated by the corresponding item. Indeed, the percentage of acquisition of the level assigned to the answer corresponds with those given in the third column of Table 2; higher levels evaluated by the item are assigned 0%; and lower levels evaluated by the item

Table 2. Description of the types of answers and their corresponding weights (Gutiérrez et al., 1991, pp. 240–241).

Type	Description	Weight (%)
1	No reply or answers that cannot be codified or that indicate that the learner has not attained a given level but that give no information about any lower level.	0
2	Wrong and insufficiently worked out answers that give some indication of a given level of reasoning; answers that contain incorrect and reduced explanations, reasoning processes, or results.	20
3	Correct but insufficiently worked out answers that give some indication of a given level of reasoning; answers that contain very few explanations, inchoate reasoning processes, or very incomplete results.	25
4	Correct or incorrect answers that clearly reflect characteristic features of two consecutive Van Hiele levels and that contain clear reasoning processes and sufficient justifications.	50
5	Incorrect answers that clearly reflect a level of reasoning; answers that present reasoning processes that are complete but incorrect or answers that present correct reasoning processes that do not lead to the solution of the stated problem.	75
6	Correct answers that clearly reflect a given level of reasoning but that are incomplete or insufficiently justified.	80
7	Correct, complete, and sufficiently justified answers that clearly reflect a given level of reasoning.	100

are assigned 100%. This allows associating each answer with a 4-component vector corresponding with the four levels, being empty the components whose corresponding levels are not assessed by the item.

Once we have obtained the five vectors corresponding to the five items answered by a student, we consider the items evaluating each level and compute the arithmetical mean of the percentages of acquisition obtained. Therefore, each student is assigned a vector of 4 numerical components with the percentages of acquisition of each level (quantitative vector), which is assigned another vector in accordance with the terms of Table 3 (qualitative vector). The diagram provided in Figure 3 shows an example of application of the method that we have described in this subsection.

Note that, during the analysis of the answers given by the students to each item, we directly assigned a level and a type whenever the categorisations made individually by the

Table 3. Correspondence between quantitative and qualitative values of the degrees of acquisition (Gutiérrez et al., 1991, p. 238).

Quantitative acquisition	Qualitative acquisition
[0, 15]	No acquisition
(15, 40)	Low
[40, 60]	Intermediate
(60, 85)	High
[85, 100]	Complete

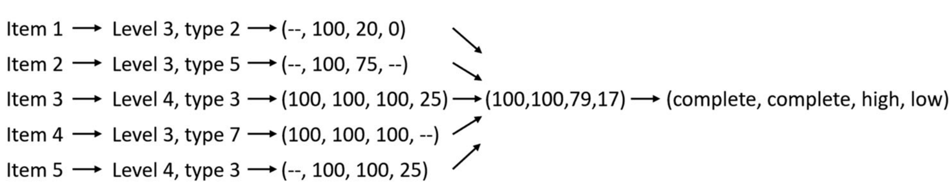

Figure 3. Scheme of application of the method to obtain the degrees of acquisition in a specific example.

four researchers agreed. In case of discrepancies, the researchers opened a discussion to finally reach a consensus.

4. Results

4.1. Levels and types of answers obtained for each item

We next show examples of answers for each level of proof to illustrate the diversity found in the students' productions. Due to space limitations, we cannot show examples of each type for all levels, but we provide four answers given by students (one per level) of different type to the same item. Concretely we have chosen item 3 since it assesses the four levels, allowing the reader to compare the features of the different levels and types while keeping the underlying content.

An example of level 1 answer to item 3 was given by student S23 (see Figure 4) since the lack of Eulerian character was justified using a particular representation and visual arguments through a non-mathematical language: '*you could lift the pencil from the paper [...] there is a moment when*'. The answer is assigned type 6 since it is correct in the sense of level 1 because Eulerian character can be recognised in visual terms as the possibility to draw the edges of the graph without lifting the pencil from the paper and without repeating visited edges. Also, the features observed in the answer clearly reflect level 1 of proof. We have considered it as insufficiently justified since it remains to mention the fact that the Eulerian character requires starting and finishing at the same place.

A level 2 answer to the same item is provided by student S24 (see Figure 5) since it contains a verification in a specific graph using mathematical vocabulary instead of visual arguments (typically found in level 1 answers). Indeed, the student considers the number of vertices and edges thus exhibiting the relation between them: '*It should fulfil the following characteristic: $A = 2V\text{-}1$*'. This answer has been labelled with type 5 since it is incorrect, as the student has checked a property that do not characterize Eulerian graphs, and it clearly shows level 2 analytic characteristics.

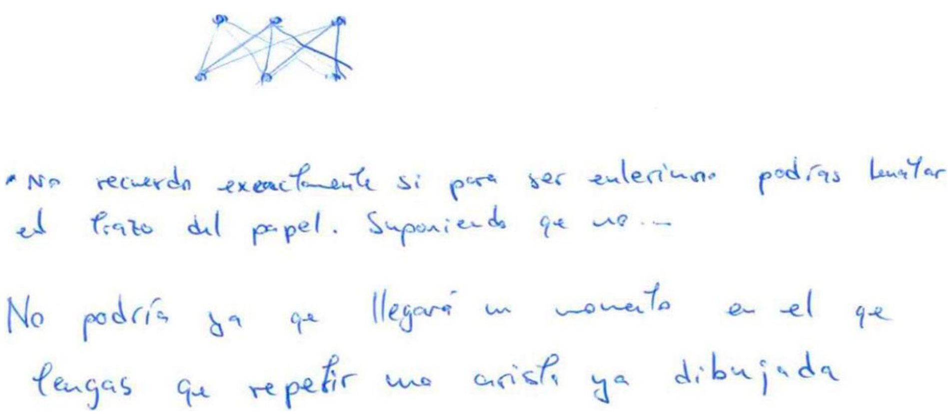

Figure 4. Example of level 1 and type 6 answer, given by student S23. Translation into English: I don't remember exactly if to be Eulerian you could lift the pencil from the paper. Assuming that you cannot ... You couldn't as there is a moment when you have to repeat an already drawn Edge.

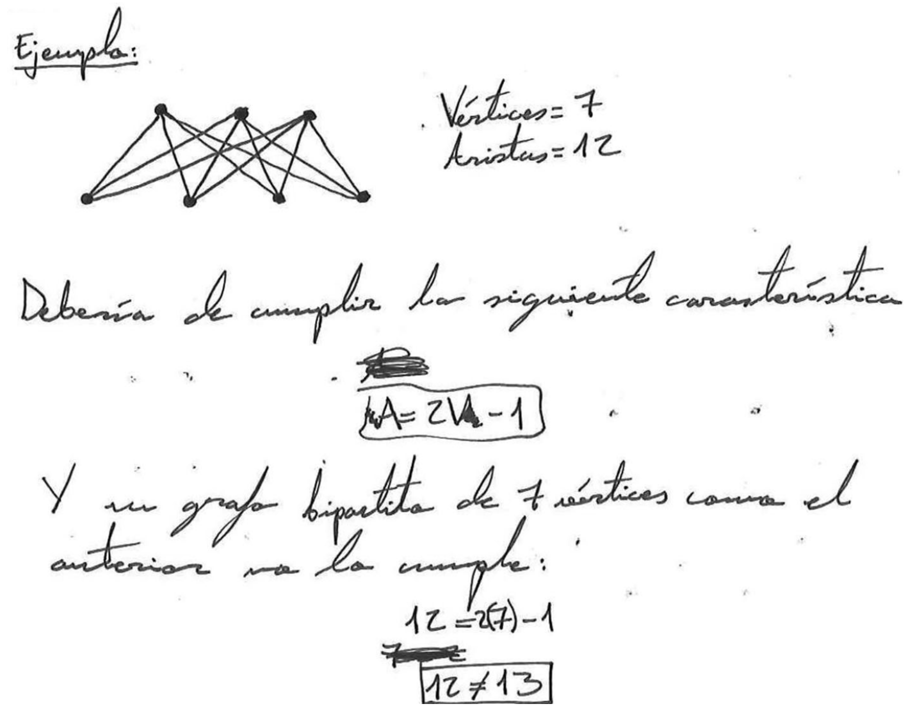

Figure 5. Example of level 2 and type 5 answer, given by student S24. Translation into English: Example: Vertices $= 7$; Edges $= 12$; It should fulfil the following characteristic; $A = 2V - 1$; And a bipartite graph of 7 vertices as the preceding does not: $12 = 2(7) - 1$; $12 \neq 13$.

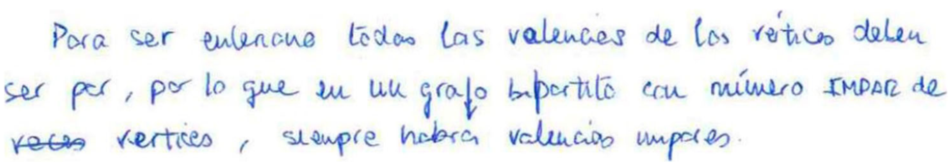

Figure 6. Example of level 3 and type 2 answer, given by student S4. Translation into English: In order to be Eulerian the degrees of all vertices should be even, then in a bipartite graph with an odd number of vertices, there are always odd vertices.

Student S4 gives a level 3 answer (see Figure 6) since it exhibits a certain use of propositional logic and its associated vocabulary (‘*all vertices ... , then in a ... , there are always*’), which is inner to the informal proofs that students at this level usually produce. It is a type 3 answer because it is wrong, as the second claim provided by the student is not a consequence of the first one, and insufficiently worked out (reduced explanations), thus reflecting only some indicators of the level.

Finally, a level 4 proof is given by student S49 (see Figure 7), who starts from the hypothesis of the statement to be proved and, through a series of logical steps, deduces the thesis. Note the vocabulary of formal proofs: ‘*Let G(V,A) bipartite ... for any* $v_1 \in V_1$’. Also, this is a type 7 answer because it is complete, as it clearly displays the formality that

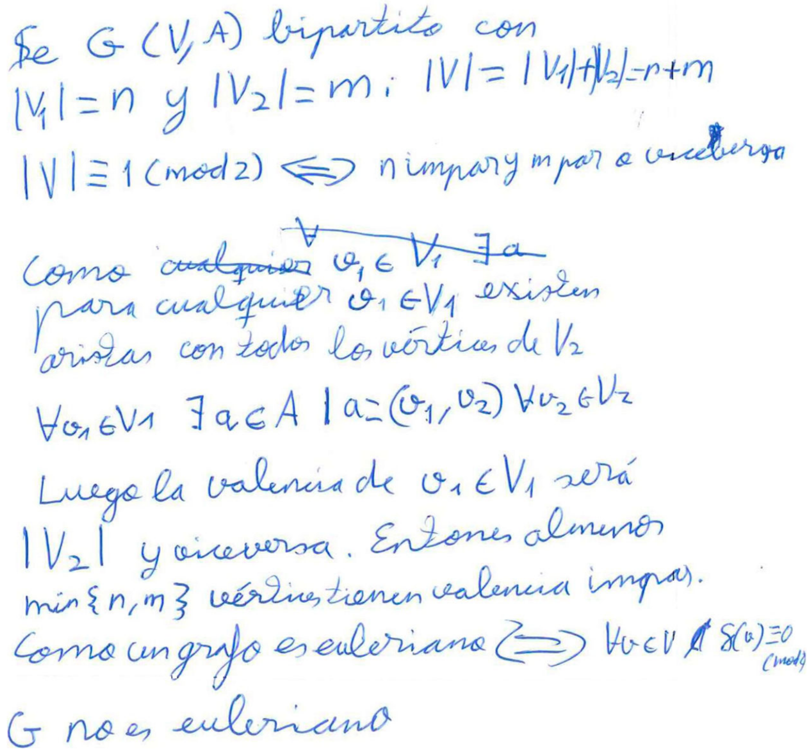
Se $G(V,A)$ bipartito con
$|V_1|=n$ y $|V_2|=m$; $|V|=|V_1|+|V_2|=n+m$
$|V|\equiv 1 (mod 2) \Leftrightarrow$ n impar y m par o viceversa
Como ~~cualquier $v_1 \in V_1$ $\exists a$~~
para cualquier $v_1 \in V_1$ existen
aristas con todos los vértices de V_2
$\forall v_1 \in V_1$ $\exists a \in A$ | $a=(v_1, v_2)$ $\forall v_2 \in V_2$
Luego la valencia de $v_1 \in V_1$ será
$|V_2|$ y viceversa. Entones almenos
$min\{n,m\}$ vértices tienen valencia impar.
Como un grafo es euleriano $\Leftrightarrow$ $\forall v \in V$ $\delta(v) \equiv 0$ (mod)
G no es euleriano

Figure 7. Example of level 4 and type 7 answer, given by student S49. Translation into English: Let $G(V, A)$ bipartite with $|V_1| = n$ and $|V_2| = m$; $|V| = |V_1| + |V_2| = n + m$; $|V| \equiv 1$ $(mod\ 2) \Leftrightarrow n$ is odd and m is even or vice versa; Since for any $v_1 \in V_1$ there are edges with all vertices of V_2; $\forall v_1 \in V_1 \exists a \in A | a = (v_1, v_2) \forall v_2 \in V_2$; Hence the degree of $v_1 \in V_1$ is $|V_2|$ and vice versa. Therefore at least $\min\{n, m\}$ vertices have odd degree. Since a graph is Eulerian $\Leftrightarrow \forall v \in V\ \delta(v) \equiv 0 \pmod 2$; G is not Eulerian.

Table 4. Distribution of levels assigned to each item.

	Number of students (Percentage)				
	Item 1	Item 2	Item 3	Item 4	Item 5
Non-classifiable	5 (9.26%)	5 (9.26%)	15 (27.78%)	4 (7.41%)	3 (5.56%)
Level 1	–	–	3 (5.56%)	0 (0.00%)	–
Level 2	18 (33.33%)	15 (27.78%)	4 (7.41%)	11 (20.37%)	5 (9.26%)
Level 3	12 (22.22%)	34 (62.96%)	18 (33.33%)	39 (72.22%)	37 (68.51%)
Level 4	19 (35.19%)	–	14 (25.92%)	–	9 (16.67%)

this level requires, and correct, since it contains an argumentation that properly links each of the steps conforming the proof.

We now present the results obtained for each item in Table 4, where the row with non-classifiable answers corresponds with those of type 1, which are not assigned to any level. We point out that we have found answers of each level of proof that is measured by the instrument. Also, note that the lowest percentages appear at level 1, while the highest values correspond with level 3 in most of the items, specifically, all except for item 1.

4.2. Levels of proof obtained by students

The degrees of acquisition of each level obtained by the students of the sample (see Table 5) show that most of them have high or complete acquisition of levels 1 (70.37%) and 2 (74.08%). This evinces sufficient proof skills to provide visual arguments and check statements in specific examples. Concerning level 3, there is at least 14% of the students in each of the five intervals of acquisition considered. Also, 64.81% of the sample shows at most intermediate acquisition of level 3, which indicates some difficulties of these students when trying to prove the truthfulness of a statement using generic arguments instead of checking in examples. Finally, the vast majority of students (92.59%) feature at most intermediate acquisition of level 4, which is insufficient to perform formal proofs of mathematical results.

Concerning the quantitative vectors of the degrees of acquisition of the students, we observe that 72.22% of them have their components in decreasing order for the four levels of proof. With regard to the 15 remaining vectors (27.78%), 13 of them have less acquisition of level 1 than level 2, while the other two present less acquisition of level 3 than level 4. Specifically, students S14 and S32 have respectively (100, 70, 55, 60) and (100, 80, 50, 60) as quantitative vectors. However, both vectors are associated to the same qualitative vector (complete, high, intermediate, intermediate), whose components are in decreasing order.

We have obtained 6 profiles of students according to their qualitative vectors (see Table 6). To do this, we have grouped the students first according to the highest level of proof acquired by the student (high or complete acquisition), and then depending on whether they have some acquisition (low or intermediate) or not (no acquisition) of the higher levels. Indeed, profile P1 contains students showing the maximum development of

Table 5. Distribution of the degrees of acquisition of each level of proof obtained for the sample.

	Number of students (Percentage)				
	No acquisition	Low	Intermediate	High	Complete
Level 1	3 (5.56%)	0 (0.00%)	13 (24.07%)	0 (0.00%)	38 (70.37%)
Level 2	0 (0.00%)	2 (3.70%)	12 (22.22%)	18 (33.33%)	22 (40.75%)
Level 3	11 (20.37%)	16 (29.63%)	8 (14.81%)	10 (18.52%)	9 (16.67%)
Level 4	32 (59.26%)	6 (11.11%)	12 (22.22%)	3 (5.56%)	1 (1.85%)

Table 6. Proof profiles obtained in the study, considering the order No acquisition $\leq$ Low Intermediate $\leq$ High $\leq$ Complete.

	Degrees of acquisition				
	Level 1	Level 2	Level 3	Level 4	Number of students (Percentage)
Profile P1	High or Complete	High or Complete	High or Complete	High or Complete	4[a] (7.41%)
Profile P2	High or Complete	High or Complete	High or Complete	Low or Intermediate	15 (27.78%)
Profile P3	High or Complete	High or Complete	Low or Intermediate	$\leq$ Intermediate	14[b] (25.92%)
Profile P4	High or Complete	High or Complete	No acquisition	No acquisition	7[c] (12.96%)
Profile P5	Complete	Intermediate	$\leq$ Intermediate	No acquisition	5 (9.26%)
Profile P6	Intermediate	Low or Intermediate	$\leq$ Low	No acquisition	9[d] (16.67%)

[a]One student does not satisfy the level 1 condition.
[b]Four students do not satisfy the level 1 condition.
[c]Two students do not satisfy the level 1 condition.
[d]Three students do not satisfy the level 1 condition.

proof skills, thus featuring most of the abilities that characterize level 4, and profile P2 has level 3 students who are acquiring level 4. Profiles P3 and P4 are made up of level 2 students, differing in the fact that the former are acquiring higher levels and the latter do not show any acquisition of them. Finally, profile P5 contains level 1 students featuring some level 2 and 3 proof skills, while students of profile P6, who also handle some level 2 and 3 abilities, have not attained level 1.

5. Discussion and conclusions

The results presented in the preceding section give empirical support to the validity of the proposal of proof levels in Graph Theory through the lens of the Van Hiele model (González et al., 2021), thus reaching the objective proposed in the beginning of this study. Indeed, all descriptors for every level of proof have been observed in our data since we have categorised students' answers with each of these levels. Furthermore, we have obtained evidence for the properties of the Van Hiele levels, as we next discuss.

The diversity of answers has allowed to verify the specificity of language displayed at each level. Indeed, we have shown four particular answers that clearly illustrate the nature of each type of vocabulary. Thus, level 1 students use everyday language to provide visual arguments, in contrast to the analytical language employed by level 2 students to check mathematical properties in concrete examples. At level 3, we can observe the use of some words typical of propositional logic that students use to make informal proofs. Finally, level 4 students use a more precise language than in the previous level, which is necessary to produce formal proofs with the rigor that this level demands.

Regarding the hierarchical and sequential character of Van Hiele levels, the degrees of acquisition of each level obtained by students reflect this property since the higher the level, the lower the degree of acquisition. Indeed, on the one hand, the most frequent degree of acquisition for each level decreases: complete for levels 1 and 2, low for level 3, and no acquisition for level 4. Furthermore, we find that the frequency of complete acquisition decreases with respect to the levels. Thus, these facts are global validation factors of this characteristic of the Van Hiele levels (Gutiérrez et al., 1991). On the other hand, if we individually examine the quantitative vectors of the degrees of acquisition assigned to each student, the high percentage of them whose components appear in decreasing order is a local validation factor. Also, considering the six profiles obtained in the study, it is remarkable that they fit the expected hierarchical character for the Van Hiele levels. Specifically, the qualitative vectors associated with profiles P1, P2 and P4 strictly agree with this characteristic, since those featuring low or intermediate acquisition of a certain level, have high or complete acquisition of the previous levels; the qualitative vectors of the profiles P3, P5 and P6 are not strictly adapted to this characteristic because they have low or intermediate acquisition of at least two levels, although they also have their components in decreasing order.

Looking now into the transition from a level to another, we can see from our results that this occurs continuously, which is also observed in works exploring geometrical reasoning (Burger & Shaughnessy, 1986; Gutiérrez et al., 1991; Perdikaris, 2011; Voskoglou, 2017). This is, students who have attained a certain level (i.e. with high or complete acquisition) have already begun the acquisition of the next level. Thus, the profiles obtained in our study reinforce continuity since all but profile P4 show students in transition between levels. (Obviously, this does not have sense for students of profile P1 since they have reached the maximum degree of reasoning in the proof process). Indeed, we have only found level 2 students who have not started the acquisition of higher levels, which are precisely those of P4 profile. This points out the fact that the transition between levels 2 and 3 could occur less gradually than in other levels, at least for the proof process. This is not surprising since achieving level 3 implies to understand what a proof really is, as remarked by Senk (1989) for the geometrical context, who points out that this implies students to move from verifications in concrete examples to proofs based on general mathematical arguments.

We have detected certain anomalies in the results whose possible causes could be explored in future works. Indeed, we have obtained a remarkable number of students with more acquisition of a certain level than the previous one, mostly students with more acquisition of level 2 than 1, in contrast to the hierarchical nature expected for the levels of proof under study. This could be due to two main reasons, both of methodological nature. The first reason could be related to our questionnaire because it only contains two items evaluating level 1, which might generate unreliable values. This could be fixed by increasing the number of tasks in the corresponding questionnaire that serve to assess level 1. The second, according to the methodology of Gutiérrez et al. (1991), refers to non-assessable responses (type 1), which are assigned no acquisition for all levels evaluated by the corresponding task. Thus, these responses are weighted with 0 at level 1 (whenever it is assessed by the task) even for students who could have attained this level. We could explore possible modifications of this methodology in future studies in order to solve this issue. In addition, we could increase the number of respondents considering several University degrees from different countries.

Our work has shown empirical evidence of the suitability of employing the Van Hiele model to analyse students' development in the proof process for Graph Theory. Thus, we contribute to the literature on the reasoning in Graph Theory, which is scarce as noticed by authors like Hazzan and Hadar (2005) and Medová et al. (2019). In future works, we could search for empirical support for the rest of the processes undertaken in this theoretical model: recognition, use and formulation of definitions, and classification. Regarding teaching issues, even though our results already serve to identify difficulties of Graph Theory students, it would be also interesting to explore whether the instructional aspects of the Van Hiele model, such as the sequencing according to its phases, produce a better acquisition of the levels in Graph Theory.

Notes

1. Congress of the European Society for Research in Mathematics Education.
2. International Congress on Mathematical Education.
3. International Network for Didactic Research in University Mathematics.
4. Research in Undergraduate Mathematics Education.
5. Problems, Resources, and Issues in Mathematics Undergraduate Studies.

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