Programa de Doctorado "Matemáticas"

PhD DISSERTATION

GEOMETRY IN METRIC SPACES AND FIXED POINT THEOREMS FOR CLASSES OF NONEXPANSIVE TYPE MAPPINGS

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To my parents

To my wife Joana Luísa

Contents

Introduction

The main purpose of this work is to present recent results obtained on the existence of fixed points for nonexpansive mappings and orbit-nonexpansive mappings in the general context of metric spaces. Additionally, our techniques will allow us to deduce the existence of common fixed points for groups of such mappings based on features of the closed balls of the metric space. In order to do that, the concepts of normal structure and uniform normal structure will be analyzed and extended from the Banach space framework to the more general environment of metric spaces. Applications to important families of metric spaces without linear structure will be displayed.

Fixed Point Theory is a wide field of Mathematics which has three major branches, namely, Topological Fixed Point Theory, Metric Fixed Point Theory and Discrete Fixed Point Theory. This work is focused on Metric Fixed Point Theory which is a branch whose starting point is considered to be the publishing of [4] by S. Banach. In this article we can see for the first time the famous Banach's Contraction Principle which states that every contraction mapping from a complete metric space into itself has a unique fixed point.

Studying the aforementioned Banach's result leads us naturally to the study of the extreme case where the Lipschitz constant is replaced by 1. Such mappings are called nonexpansive mappings and there are many famous results which guarantee the existence of fixed points of these mappings under certain conditions, see for example [10],[23],[34] and [11].

The work is divided into three chapters which are subdivided into sections. Most of the non-original results are presented in Chapter 1. Some non-original concepts and results are presented in the other chapters in order to give more logical coherence to the text. We try to present references and authorship to all such results.

As already mentioned, in Chapter 1 we present the basic concepts and results that we believe are necessary for reading and understanding the other chapters. It is in this chapter that we introduce most of the notation used throughout the work. Whenever possible, we have tried to provide examples and some of the historical context of what is presented. The chapter is divided into six sections.

In the first section we introduce some basic definitions and general results regarding metric spaces. In the second section we define the concept of fixed point of a self-mapping and also what it means for a class of self-mappings to have the fixed point property. Finally, we present some classes of self-mappings which we will study in this work, the most important of them being the class of nonexpansive mappings. In the third section we introduce the concept of normal structure in the context of metric spaces and present some examples in the context of Banach spaces. Then, we present in the context of metric spaces a famous result by Kirk which states that on a bounded metric space, normal structure and compactness of the family of admissible sets imply the fixed point property for nonexpansive mappings. The proof we present will give us a model for other proofs presented in this work. We finish the section providing some examples in the context of Banach spaces of sets which satisfy the requirements of Kirk's theorem. In the fourth section we introduce the concept of uniform normal structure in the context of metric spaces. Even though every metric space with uniform normal structure has normal structure, the study of this structure on its own is justified by the fact that it is easier to work with. In the fifth section we give a brief presentation of hyperconvex metric spaces which are a heavily studied special case of metric spaces with uniform normal structure. In sixth section we give a brief presentation of CAT(0) spaces which are another case example of metric spaces with uniform normal structure. In the seventh section we introduce the concept of uniform relative normal structure in the context of metric spaces, a concept which we will use and extend in Chapter 2. We also present an equivalent characterization of this concept and state a metric version of a theorem by Soardi which says that on a bounded metric space, uniform relative normal structure and compactness of the family of admissible sets imply the fixed point property for nonexpansive mappings.

In Chapter 2 and Chapter 3 we will present some recent results in Metric Fixed Point Theory most of which can be found in the articles [19] and [20] by Rafael Espínola García, María Japón and myself.

Chapter 2 is divided into four sections. In the first section we make a brief presentation of some sequence spaces which will be used in the following sections. In the second section we present some subsets of c with the fixed point property for nonexpansive mappings. Two of the examples we present namely, Example 2.2.1 and Example 2.2.2 give a positive answer to a question posed in [22] on whether it was possible to find a closed, bounded and convex subset of $(c, \|\cdot\|_\infty)$ which is non-weakly compact and has the fixed point property for nonexpansive mappings without being hyperconvex. In the third section we introduce the concept of (p, q)-uniform relative normal structure which formally extends the concept of relative normal structure presented in Chapter 1. We then show that on a bounded metric space, (p, q)-uniform relative normal structure and compactness of the family of admissible sets imply the fixed point property for nonexpansive mappings.

Moreover, we show that on a hyperconvex (M, d), taking an equivalent metric close enough to d, leads to (p, q)-uniform relative normal structure. In the fourth section we present some examples of sets with the fixed point property for nonexpansive mappings which illustrate what we have built in the previous section.

Chapter 3 is divided into three sections. In the first section we present the concept of orbit of a self-mapping and the class of orbit-nonexpansive mappings over a metric space. We then show some properties of this class of mappings and present examples so the concepts introduced can be grasped more easily. In the second section we introduce the concept of a family of interlaced orbit-nonexpansive mappings and study how this concept together with normal structure and compactness of the family of admissible sets leads to the existence of common fixed points for such families. We also define what it means for a group of self-mappings to act on a metric space. We then show that our fixed point result also applies to the class of orbit-nonexpansive mappings and to a group of orbit-nonexpansive mappings acting on a metric space. In the third section we again study fixed points of interlaced orbit-nonexpansive mappings but this time under the assumption of (p, q)-uniform relative normal structure and compactness of the family of admissible sets. In both Sections 2 and 3 we show that our results extend previous results found in the literature. In the fourth and final section, we present some open questions which arose from our research.

Chapter 1

Preliminaries

In this chapter we will present some basic definitions and results which we believe are necessary for the understanding of what will be presented in the following chapters. The first two sections consist in introducing notations, definitions and general results regarding metric spaces, classes of self-mappings and fixed points of such mappings.

In the third and fourth sections we introduce the concepts of normal structure and uniform normal structure in the context of metric spaces. Kirk's Theorem and its proof are presented in this context.

In the fifth and sixth section we give a brief presentations of hyperconvex metric spaces and CAT(0) spaces.

Finally, in the seventh section we introduce the concept of uniform relative normal structure in the context of metric spaces.

1.1 Basic definitions and results

In what follows we will introduce some basic definitions and results which will be used throughout this work.

Definition 1.1.1. *Let (M, d) be a metric space, A a nonempty bounded subset of M, $x_0 \in M$. We define:*

$$D(x_0, A) = \sup_{y \in A} \{d(x_0, y)\};$$
$$r(A) = \inf_{x \in A} \{D(x, A)\};$$
$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\} = \sup_{x \in A} \{D(x, A)\};$$
$$B(x_0, r) = \{y \in M : d(x_0, y) \leq r\};$$
$$B[A, r] = \bigcap_{x \in A} B(x, r) = \{y \in M : A \subset B(y, r)\}.$$

We have that $r(A)$ is called the **Chebyshev radius of** A while $\operatorname{diam}(A)$ is called the **diameter of** A. Also, $B(x_0, r)$ is called the **closed ball centered at** x_0 **of radius** r.

The notation $B[A,r]$ to denote the set $\bigcap_{x \in A} B(x,r)$ was introduced in [19] and, as we will see throughout this work, makes proofs and definitions much clearer.

We will now present two lemmas which will be needed further. Their proofs make use of many of the concepts we have defined above.

Lemma 1.1.1. *Let (M,d) be a metric space and A and A' be nonempty bounded subsets of M. Then:*

i) *$x_0 \in B[A,r]$ if and only if $D(x_0,A) \leq r$;*

ii) *$0 < r \leq r'$ implies $B[A,r] \subset B[A,r']$;*

iii) *$A \subset A'$ implies $B[A',r] \subset B[A,r]$.*

Proof: Let (M,d), A and A' be as above

i) $(\Longrightarrow)$ Given $x_0 \in B[A,r]$ then, $x_0 \in B(x,r)$ for all $x \in A$ which implies that $d(x_0,x) \leq r$ for all $x \in A$ and therefore, $D(x_0,A) = \sup_{x \in A}\{d(x_0,x)\} \leq r$.

$(\Longleftarrow)$ Given $x_0 \in M$ such that $D(x_0,A) = \sup_{x \in A}\{d(x_0,x)\} \leq r$ then, $d(x_0,x) \leq r$ for all $x \in A$ which implies that $x_0 \in B(x,r)$ for all $x \in A$ and therefore, $x_0 \in B[A,r]$.

ii) If $0 < r \leq r'$ then, for any $x_0 \in B[A,r]$ we have that $x_0 \in B(x,r) \subset B(x,r')$ for all $x \in A$ and therefore, $x_0 \in B[A,r']$.

iii) If $A \subset A'$ then, for any $x_0 \in B[A',r]$ we have that $x_0 \in B(x,r)$ for all $x \in A'$. In particular, since $A \subset A'$, we have that $x_0 \in B(x,r)$ for all $x \in A$ which implies that $x_0 \in B[A,r]$. Therefore, $B[A',r] \subset B[A,r]$.

■

Lemma 1.1.2. *Let (M,d) be a metric space and A a nonempty bounded subset of M. Then, $\frac{1}{2}\operatorname{diam}(A) \leq r(A) \leq \operatorname{diam}(A)$. Moreover, if $\operatorname{diam}(A) > 0$ and $r(A) < \operatorname{diam}(A)$ there exists $0 < r < \operatorname{diam}(A)$ such that $A \cap B[A,r] \neq \emptyset$.*

Proof: Let (M,d) be a metric space and let A be a nonempty bounded subset of M.

Fix $z \in A$. For any $x,y \in A$ we have that

$$d(x,y) \leq d(x,z) + d(z,y) \leq D(z,A) + D(z,A) = 2D(z,A)$$

which implies that

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\} \leq 2D(z,A)$$

and therefore, $\frac{1}{2}\operatorname{diam}(A) \leq D(z,A) \leq \operatorname{diam}(A)$.

Since z is an arbitrary element of A, it follows that for all $z \in A$ we have that

$$\frac{1}{2}\operatorname{diam}(A) \leq D(z,A) \leq \operatorname{diam}(A)$$

which implies that

$$\frac{1}{2}\operatorname{diam}(A) \leq \inf_{z\in A}\{D(z,A)\} = r(A) \leq \operatorname{diam}(A).$$

Suppose now that $\operatorname{diam}(A) > 0$ and $r(A) < \operatorname{diam}(A)$. Then, since $0 < \frac{1}{2}\operatorname{diam}(A) \leq r(A)$, we have that $r(A) > 0$.

Given $x \in A$, for any $a \in A$ we have that

$$d(x,a) \leq \sup_{a\in A}\{d(x,a)\} = D(x,A)$$

which implies that $A \subset B(x, D(x,A))$.

Then, since x is an arbitrary element of A, we have that $A \subset B(x, D(x,A))$ for all $x \in A$.

Now, since $\inf_{x\in A}\{D(x,A)\} = r(A) < \operatorname{diam}(A)$ there exists $x_0 \in A$ such that $D(x_0,A) < \operatorname{diam}(A)$. Moreover, since $0 < r(A) \leq D(x_0,A)$ we have that $0 < D(x_0,A)$ and therefore, taking $r = D(x_0,A)$ it follows that $A \subset B(x_0,r)$ and $0 < r < \operatorname{diam}(A)$ which implies that $x_0 \in A \cap B[A,r]$.

■

We will now introduce the concept of convexity structure which will allow us, in some sense, to extend to metric spaces the concept of convexity used in vector spaces. This concept was first introduced in [28] for general sets and the definition we present for metric spaces is a bit different from the original one.

Definition 1.1.2. *Let (M,d) be a metric space. A nonempty family $\mathcal{F}$ of subsets of M is said to be a* ***convexity structure on*** (M,d) *if it has the following properties:*

i) $\mathcal{F}$ is closed under intersections;

ii) $\mathcal{F}$ contains the closed balls of M.

Looking at property ii) above we can immediately see that being a convexity structure on (M,d) depends on the metric.

In the literature the concept of convexity structure is defined in many different ways, all of them aiming to emulate the idea of convexity in a given context. All definitions in the literature require property i) to hold. In the original abstract definition presented in [28], property ii) is replaced by requiring that $\emptyset$ and M belong to $\mathcal{F}$. We will show below that, in the cases which we will be interested in this work, the latter property follows from i) and ii).

- If M has more than one point then, since every metric space is Hausdorff, by taking $x, y \in M$ with $x \neq y$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$ and therefore i) and ii) imply that $\emptyset \in \mathcal{F}$.
- If (M, d) is bounded then, $M = B(x, \operatorname{diam}(M))$ for any $x \in M$ and therefore, if property ii) holds we have that $M \in \mathcal{F}$.

In the following definition we introduce the convexity structure which will be used the most in this work.

Definition 1.1.3. *Let (M, d) be a metric space. A subset A of M is said to be **admissible** if it is an intersection of closed balls of (M, d). The set of admissible subsets of (M, d) is denoted by $\mathcal{A}_d(M)$.*

It is easy to see that $\mathcal{A}_d(M)$ is a convexity structure on M. Also, since any convexity structure $\mathcal{F}$ on M is closed under intersections and contains the closed balls of M, it follows that $\mathcal{F}$ must contain $\mathcal{A}_d(M)$. Thus, $\mathcal{A}_d(M)$ is the smallest family which defines a convexity structure on (M, d).

Moreover, given $r > 0$ and A a nonempty bounded subset of (M, d) then $B[A, r] \in \mathcal{A}_d(M)$.

Whenever there is no risk of confusion, we will drop the subscript d and simply write $\mathcal{A}(M)$.

Example 1.1.1. *Let $(X, \|\cdot\|)$ be a normed vector space and let τ be a topology on X for which the closed balls of $(X, \|\cdot\|)$ are τ-closed. Then, it is easy to see that the family the family of τ-closed, bounded and convex subsets of X forms a convexity structure.*

Some important particular cases of the previous example which are heavily used in the study of Fixed Point Theory in Banach spaces are listed below.

- If τ is the topology induced by the norm, we get that the family of closed (with respect to the norm), bounded and convex subsets of X forms a convexity structure.
- If $(X, \|\cdot\|)$ is a Banach space and τ is the weak topology or the weak*-topology (in case of a dual Banach space).

- If τ is the closed in measure topology on the Lebesgue space $L_1[0,1]$.

It is worth mentioning that sometimes, when we work with a normed vector space $(X, \|\cdot\|)$ (or any subset of it with the induced norm), instead of saying that $\left(X, d_{\|\cdot\|}\right)$ has a certain property, we will simply say that $(X, \|\cdot\|)$ has it.

Now, we will define a few concepts which will be needed in some proofs.

Definition 1.1.4. *Let (M,d) be a metric space and let $\mathcal{F}$ be a family of self-mappings on M we define*

$$\mathcal{A}_{\mathcal{F}}(M) = \{A \in \mathcal{A}(M),\ A \neq \emptyset,\ T(A) \subset A \text{ for all } T \in \mathcal{F}\}.$$

Given $X \subset M$, we define

$$\operatorname{cov}(X) = \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}(M)}} A \text{ and } \operatorname{cov}_{\mathcal{F}}(X) = \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}_{\mathcal{F}}(M)}} A,$$

as, respectively, the admissible and $\mathcal{F}$-admissible covers of X in M.

Since $\mathcal{A}(M)$, by definition, is closed under intersections, both hulls in the previous definition are elements of $\mathcal{A}(M)$. Also, if $\mathcal{F} = \{T\}$ we will write $\mathcal{A}_T(M)$ instead of $\mathcal{A}_{\mathcal{F}}(M)$. In the following proposition we present basic properties of the sets we defined above which will be used throughout the text.

Proposition 1.1.1. *Let (M,d) be a metric space, $\mathcal{F}$ a family of self-mappings on M and X a nonempty subset of M. Then, we have that*

i) $X \subset \operatorname{cov}(X) \subset \operatorname{cov}_{\mathcal{F}}(X)$;

ii) $T(\operatorname{cov}_{\mathcal{F}}(X)) \subset \operatorname{cov}_{\mathcal{F}}(X)$ *for all* $T \in \mathcal{F}$;

iii) $B[X,r] = B[\operatorname{cov}(X), r]$ *for all* $r > 0$;

iv) $X \in \mathcal{A}(M)$ *if and only of* $X = \operatorname{cov}(X)$.

Proof: Let (M,d), $\mathcal{F}$ and X be as above.

i) Since every element of the set $\{A \in \mathcal{A}(M) : A \subset X\}$ contains X, it follows that the intersection of all such sets contain X and therefore, $X \subset \operatorname{cov}(X)$.

   Since every element of the set $\{\mathcal{A}_{\mathcal{F}}(M) : A \subset X\}$ is admissible and contains X, it follows that each such set contains $\operatorname{cov}(X)$ which implies that the intersection of all such sets contain $\operatorname{cov}(X)$ and therefore, $\operatorname{cov}(X) \subset \operatorname{cov}_{\mathcal{F}}(X)$.

   Hence, $X \subset \operatorname{cov}(X) \subset \operatorname{cov}_{\mathcal{F}}(X)$.

ii) Let $T \in \mathcal{F}$. For all $L \in \{\mathcal{A}_{\mathcal{F}}(M) : A \subset X\}$, we have that $L \in \mathcal{A}_{\mathcal{F}}(M)$ which implies that $T(L) \subset L$ for all such L. Thus,

$$T(\operatorname{cov}_{\mathcal{F}}(X)) = T\left(\bigcap_{\substack{X \subset A \\ A \in \mathcal{A}_{\mathcal{F}}(M)}} A\right) \subset \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}_{\mathcal{F}}(M)}} T(A) \subset \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}_{\mathcal{F}}(M)}} A = \operatorname{cov}_{\mathcal{F}}(X).$$

iii) Let $r > 0$. By item i) we have that $X \subset \operatorname{cov}(X)$ which implies that $B[\operatorname{cov}(X), r] \subset B[X, r]$. Thus, if $B[X, r] = \emptyset$ it follows that $B[\operatorname{cov}(X), r] = B[X, r]$.

Now, suppose that $B[X, r] \neq \emptyset$. Given $z \in B[X, r]$ we have that $d(z, x) \leq r$ for all $x \in X$ which implies that $X \subset B(z, r)$. Since $B(z, r)$ is an admissible set which contains X, it follows from the definition of $\operatorname{cov}(X)$ that $\operatorname{cov}(X) \subset B(z, r)$.

Thus, $d(z, y) \leq r$ for all $y \in \operatorname{cov}(X)$ which implies that $z \in B(y, r)$ for all $y \in \operatorname{cov}(X)$ and therefore, $z \in B[\operatorname{cov}(X), r]$.

Hence, $B[X, r] \subset B[\operatorname{cov}(X), r]$ and then it follows that $B[X, r] = B[\operatorname{cov}(X), r]$.

iv) ($\Longrightarrow$) If $X \in \mathcal{A}(M)$ then, since $X \subset X$, it follows from the definition of $\operatorname{cov}(X)$ that $\operatorname{cov}(X) \subset X$. Thus, since we know by item i) that $X \subset \operatorname{cov}(X)$ it follows that $X = \operatorname{cov}(X)$.

($\Longleftarrow$) Since $\operatorname{cov}(X) \in \mathcal{A}(M)$ for all $X \subset M$, it follows that if $X = \operatorname{cov}(X)$ then, $X \in \mathcal{A}(M)$.

■

The following proposition tells us a simple way to write any admissible set on a metric space.

Proposition 1.1.2. *Let (M, d) be a metric space. Then, for every $A \in \mathcal{A}(M)$ we have that $A = \bigcap_{x \in M} B(x, D(x, A))$.*

Proof: Given a fixed $y \in A$, for every $x \in M$ we have that $d(x, y) \leq \sup\{d(x, z) : z \in A\} = D(x, A)$ which implies that $y \in \bigcap_{x \in M} B(x, D(x, A))$. Thus, $A \subset \bigcap_{x \in M} B(x, D(x, A))$. In particular, $\bigcap_{x \in M} B(x, D(x, A)) \neq \emptyset$.

Since $A \in \mathcal{A}(M)$ there exists a family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \Gamma}$ such that $A = \bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha)$.

Given $\alpha \in \Gamma$, since $A \subset B(x_\alpha, r_\alpha)$, we have that $d(x_\alpha, y) \leq r_\alpha$ for all $y \in A$ which implies that $D(x_\alpha, A) \leq r_\alpha$ and therefore,

$$\bigcap_{x\in M} B\left(x, D\left(x, A\right)\right) \subset B\left(x_\alpha, D\left(x_\alpha, A\right)\right) \subset B\left(x_\alpha, r_\alpha\right).$$

Hence, $\bigcap\limits_{x\in M} B\left(x, D\left(x, A\right)\right) \subset \bigcap\limits_{\alpha\in\Gamma} B\left(x_\alpha, r_\alpha\right) = A$ which implies that

$$A = \bigcap_{x\in M} B\left(x, D\left(x, A\right)\right).$$

■

Definition 1.1.5. *Let M be a set. A family $\{X_\lambda\}_{\lambda\in\Gamma}$ of subsets of M is said to have the* ***finite intersection property*** *if $\bigcap\limits_{\lambda\in\Gamma_f} X_\lambda \neq \emptyset$ for all finite $\Gamma_f \subset \Gamma$.*

Definition 1.1.6. *A family $\mathcal{L}$ of subsets of a metric space (M, d) is said to be* ***compact*** *if every subset of it which has the finite intersection property has nonempty intersection.*

If in the previous definition we restrict ourselves to only taking countable subsets of $\mathcal{L}$ with the finite intersection property then, we get the notion of a **countably compact** family.

Example 1.1.2. *It is a well known fact from general topology that a topological space is compact if and only if the family of its closed subsets is compact.*

Another well known result from general topology, tells us that every compact metric space is complete. The next proposition tells us that a similar result holds when we assume that the family of admissible sets is compact.

Proposition 1.1.3. *Let (M, d) be a metric space such that the family $\mathcal{A}(M)$ is compact. Then (M, d) is complete.*

Proof: Let (M, d) be as above and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M.

If $(x_n)_{n\in\mathbb{N}}$ is eventually constant then it obviously converges.

Suppose that $(x_n)_{n\in\mathbb{N}}$ is not eventually constant. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that the set $\{d(x_n, x_m) : m, n \in \mathbb{N}\}$ is bounded. Thus, for each $n \in \mathbb{N}$ the number $r_n = \sup\limits_{m\geq n} \{d(x_n, x_m)\}$ always exists and also, since $(x_n)_{n\in\mathbb{N}}$ is not eventually constant we have that $r_n > 0$ for all $n \in \mathbb{N}$.

Moreover, given $n, n' \in \mathbb{N}$ with $n \leq n'$ we have that

$$d\left(x_n, x_{n'}\right) \leq \sup_{m\geq n} \{d\left(x_n, x_m\right)\} = r_n.$$

Now, consider the family of closed balls $\{B(x_n, r_n)\}_{n\in\mathbb{N}}$.

Given $\{n_1, \ldots, n_k\} \subset \mathbb{N}$ with $n_i < n_{i+1}$ for all $1 \leq i \leq k-1$, it follows from what we have seen above that $d(x_{n_i}, x_{n_k}) \leq r_i$ for all $1 \leq i \leq k$ which gives us that $x_{n_k} \in \bigcap_{i=1}^{k} B(x_{n_i}, r_{n_i})$.

Thus, for any finite nonempty $F \subset \mathbb{N}$ we have that $\bigcap_{n\in F} B(x_n, r_n) \neq \emptyset$ which implies that $\{B(x_n, r_n)\}_{n\in\mathbb{N}}$ has the finite intersection property.

Now, since every closed ball is in $\mathcal{A}(M)$ and $\mathcal{A}(M)$ is compact, it follows that there exists $x \in \bigcap_{n\in\mathbb{N}} B(x_n, r_n)$.

Let $\epsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \dfrac{\epsilon}{2}$ for all $m, n \in \mathbb{N}$ with $n_0 \leq n \leq m$. Thus, given any fixed $n \geq n_0$ we have that $d(x_n, x_m) < \dfrac{\epsilon}{2}$ for all $m \geq n$ which implies that $r_n \leq \dfrac{\epsilon}{2} < \epsilon$ and therefore, $r_n < \epsilon$ for all $n \geq n_0$. Since $d(x, x_n) < r_n$ for all $n \in \mathbb{N}$ we have that $d(x, x_n) < \epsilon$ for all $n \geq n_0$. Therefore, $\lim_{n\to\infty} x_n = x$.

Hence, (M, d) is complete.

■

Now, we will show that the compactness of the family of admissible sets of a metric space is inherited by the family of admissible sets of an admissible subset of the metric space.

Proposition 1.1.4. *Let (M, d) be a metric space, let X be a nonempty element of $\mathcal{A}(M)$ and consider the metric space $\left(X, d_{|X}\right)$. If $\mathcal{A}(M)$ is compact then, $\mathcal{A}(X)$ is compact.*

Proof: Let (M, d) and X be as above and suppose that $\mathcal{A}(M)$ is compact.

Observe first that given any $x \in X$ and $r > 0$, the closed ball in X centered at x of radius r is the set $X \cap B(x, r)$.

Let $\{A_i\}_{i\in I}$ be a family of elements of $\mathcal{A}(X)$ which has the finite intersection property.

For each $i \in I$ since $A_i \in \mathcal{A}(X)$, it follows from Proposition 1.1.2 that

$$A_i = \bigcap_{x\in X} \left(X \cap B(x, D(x, A_i))\right) = X \cap \bigcap_{x\in X} B(x, D(x, A_i)).$$

Now, since $\bigcap_{x\in X} B(x, D(x, A_i))$ and X are elements of $\mathcal{A}(M)$ which is a convexity structure, we have that $A_i \in \mathcal{A}(M)$ for all $i \in I$ and therefore, $\{A_i\}_{i\in I}$ is a family of elements of $\mathcal{A}(M)$ which has finite intersection property.

Thus, since $\mathcal{A}(M)$ is compact we have that

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I} \left(X \cap \bigcap_{x\in X} B(x, D(x, A_i))\right) = X \cap \bigcap_{i\in I} \left(\bigcap_{x\in X} B(x, D(x, A_i))\right) \neq \emptyset.$$

Hence, $\mathcal{A}(X)$ is compact.

■

We finish this section by presenting a lemma which will be invoked several times in this work.

Lemma 1.1.3. *Let (M,d) be a bounded metric with more than one element space such that $\mathcal{A}(M)$ is compact and let $\mathcal{F}$ be a family of self-mappings on M. Then, $\mathcal{A}_{\mathcal{F}}(M)$ has a minimal element with respect to set inclusion.*

Proof: Let (M,d), $\mathcal{A}(M)$ and $\mathcal{F}$ be as above and recall that

$$\mathcal{A}_{\mathcal{F}}(M) = \{A \in \mathcal{A}(M),\ A \neq \emptyset,\ T(A) \subset A \text{ for all } T \in \mathcal{F}\}.$$

For every $T \in \mathcal{F}$ we have that $T(M) \subset M$. Since M is bounded with more than one element, we have that its diameter is a positive real number and then, given any $x \in M$ we have that $M = B(x, \operatorname{diam}(M))$. Thus, M is a closed ball in M which tells us that $M \in \mathcal{A}(M)$ and therefore, $\mathcal{A}_{\mathcal{F}}(M) \neq \emptyset$.

Now, consider the inclusion partial order over $\mathcal{A}_{\mathcal{F}}(M)$.

Let $\mathcal{C} = \{A_\alpha\}_{\alpha\in\Gamma}$ be a nonempty totally ordered subset of $\mathcal{A}_{\mathcal{F}}(M)$. Then, given a nonempty finite $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\} \subset \mathcal{C}$ there exists $1 \leq i_0 \leq n$ such that $A_{\alpha_{i_0}} \subset A_{\alpha_i}$ for all $1 \leq i \leq n$ which implies that $A_{\alpha_{i_0}} \subset \bigcap\limits_{i=1}^{n} A_{\alpha_i}$ (actually, the two sets are equal). Since $A_{\alpha_{i_0}} \in \mathcal{A}_{\mathcal{F}}(M)$ we have that $A_{\alpha_{i_0}} \neq \emptyset$ and therefore, $\bigcap\limits_{i=1}^{n} A_{\gamma_i} \neq \emptyset$.

Thus, $\bigcap\limits_{\alpha\in\Gamma_f} A_\alpha \neq \emptyset$ for all $\Gamma_f \subset \Gamma$ finite and nonempty, that is, $\mathcal{C}$ has the finite intersection property.

Now, since $\mathcal{A}(M)$ is compact and $\mathcal{C} \subset \mathcal{A}_{\mathcal{F}}(M) \subset \mathcal{A}(M)$, it follows that $\bigcap\limits_{\alpha\in\Gamma} A_\alpha \neq \emptyset$.

Finally, given $y \in T\left(\bigcap\limits_{\alpha\in\Gamma} A_\alpha\right)$, there exists $x \in \bigcap\limits_{\alpha\in\Gamma} A_\alpha$ such that $T(x) = y$. Then, since $x \in A_\alpha$ and $T(A_\alpha) \subset A_\alpha$ for all $\alpha \in \Gamma$ we have that $y = T(x) \in A_\alpha$ for all $\alpha \in \Gamma$, that is, $y \in \bigcap\limits_{\alpha\in\Gamma} A_\alpha$. Thus, $T\left(\bigcap\limits_{\alpha\in\Gamma} A_\alpha\right) \subset \bigcap\limits_{\alpha\in\Gamma} A_\alpha$ and therefore, $\bigcap\limits_{\alpha\in\Gamma} A_\alpha \in \mathcal{A}_{\mathcal{F}}(M)$.

Hence, any nonempty totally ordered subset of $\mathcal{A}_{\mathcal{F}}(M)$ has a lower bound in $\mathcal{A}_{\mathcal{F}}(M)$ and therefore it follows from Zorn's lemma that $\mathcal{A}_{\mathcal{F}}(M)$ has a minimal element.

■

1.2 Fixed points and fixed point property

In this section we present the main concepts around which this work revolves, namely, the concepts of fixed point of a self-mapping and the fixed point property.

Definition 1.2.1. *Let X be a nonempty set and $T : X \to X$ a mapping. We say that $x \in X$ is a **fixed point of** T if $Tx = x$. The set of all fixed points of T will be denoted by $Fix(T)$.*

In the following we present a class of mappings each of which has at least one fixed point.

Example 1.2.1. *Let X be a vector space whose zero vector is $\mathbf{0}$. It is known from Linear Algebra that for any given linear mapping $T : X \to X$ we have that $T(\mathbf{0}) = \mathbf{0}$ which gives us that $\mathbf{0}$ is a fixed point of T.*

Observe that in the example above, $\mathbf{0}$ is a common fixed point of the class of linear mappings from X into X. We will study common fixed points of some other classes of mappings in Chapter 3.

Next we present a very simple example of a mapping without fixed points.

Example 1.2.2. *Consider the mapping $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2 + 1$. We have that T has no fixed points since the equation $x^2 + 1 = x$ has no real solutions.*

Now, we will introduce some classes of mappings which we will be interested in this work. We will also give a bit of historical context.

Definition 1.2.2. *Let (M, d_M) and (N, d_N) be metric spaces. A mapping $T : M \to N$ is said to be **lipschitzian** if there exists $k \geq 0$ such that $d_N(Tx, Ty) \leq k d_M(x, y)$ for all $x, y \in M$. The smallest k for which the previous inequality holds is called the Lipschitz constant of T and for this k, we say that T is k-lipschitzian. Moreover, if T is k-lipschitzian with $k < 1$ then T is said to be a **contraction** and if $d_N(Tx, Ty) = d_M(x, y)$ for all $x, y \in M$, T is called an isometry.*

In [4] Banach presented his fixed point theorem which we state below.

Theorem 1.2.1 (Banach)**.** *Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping. Then, T has a unique fixed point x_T. Moreover, given any $x_0 \in M$ the sequence $(T^n x_0)_{n \in \mathbb{N}}$ converges to x_T.*

The previous theorem, also known in the literature as Banach's Contraction Principle, is really amazing since it guarantees not only the existence but also the uniqueness of fixed points of contraction self-mappings on a complete metric. Also, it led to further studies of fixed points of self-mappings giving rise to what is known today as **Metric Fixed Point Theory**.

After Banach's result was presented, mathematicians started to study other classes of mappings in order to see if they could obtain similar results, this led to the study of the class of nonexpansive mappings which we introduce below.

Definition 1.2.3. *Let (M, d_M) and (N, d_N) be metric spaces. A mapping $T : M \to N$ is said to be **nonexpansive** if $d_N (Tx, Ty) \leq d_M (x, y)$ for all $x, y \in M$.*

It is important to mention that in this work, whenever we talk about a self-mapping, we mean a mapping $T : (M, d) \to (M, d)$, that is, we always consider the same metric on the domain and codomain of T. Although the two previous definitions were presented in the most general form, in this work we will only deal with self-mappings.

We can immediately see that contraction mappings and isometries are nonexpansive mappings. The class of nonexpansive mappings will be the most important class studied in this work and we will present some extensions of it in Chapter 3.

The next two examples show that, unlike contraction mappings, general nonexpansive mappings defined on complete metrics spaces do not always have fixed points. Our examples will be defined respectively over $\mathbb{R}^n$ and $[1, +\infty)$ with their usual metrics.

Example 1.2.3. *Consider the normed space $(\mathbb{R}^n, \|\cdot\|_2)$ where $\|\cdot\|_2$ is the euclidean norm. Let x_0 be a nonzero vector of $\mathbb{R}^n$ and consider the mapping $T_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_{x_0}x = x + x_0$ for all $x \in \mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ we have that*

$$\|T_{x_0}x - T_{x_0}y\|_2 = \|x + x_0 - (y + x_0)\|_2 = \|x + x_0 - y - x_0\|_2 = \|x - y\|_2$$

which tells us that T_{x_0} is an isometry and therefore, a nonexpansive mapping.

Now, observe that since x_0 is not the zero vector, the equation $x_0 + x = x$ has no solution in $\mathbb{R}^n$ which implies that T_{x_0} has no fixed points.

Example 1.2.4. *Consider the metric space $([1, +\infty), |\cdot|)$ where $|\cdot|$ is the usual norm on $\mathbb{R}$ and consider the mapping $T : [1, +\infty) \to [1, +\infty)$ defined by $Tx = x + \dfrac{1}{x}$ for all $x \in [1, +\infty)$. For any $x, y \in [1, +\infty)$ with $x \neq y$ we have that $1 < xy$ which implies that*

$$|Tx - Ty| = \left| x + \frac{1}{x} - \left(y + \frac{1}{y} \right) \right| = \left| x - y - \frac{x - y}{xy} \right| =$$

$$\left| (x - y) \left(1 - \frac{1}{xy} \right) \right| = |x - y| \left| 1 - \frac{1}{xy} \right| = |x - y| \left(1 - \frac{1}{xy} \right) < |x - y| .$$

and therefore, not only we obtain that T is a nonexpansive mapping but we also have that the inequality is strict whenever $x \neq y$.

Now, observe the equation $x + \frac{1}{x} = x$ has no solution in $[1, +\infty)$ which implies that T has no fixed points.

A nonexpansive mapping satisfying the strict inequality whenever $x \neq y$ as in the previous example is called a **weak contractive** mapping.

As we mentioned earlier, Theorem 1.2.1 is an amazing result and the next example gives us a glimpse of its power.

Example 1.2.5. *Let $\lambda \neq 0$, $K : [a,b] \times [a,b] \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ continuous functions and consider the following integro-differential equation*

$$f(x) = h(x) + \lambda \int_a^b K(x,y) f(y)\, dy$$

*which is called a **Fredholm integral equation of second kind**.*

If we consider the metric on $C[a,b]$ given by $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$ then, we know from Analysis that $(C[a,b], d)$ is a complete metric space and therefore, if we define the operator $T : (C[a,b], d) \to (C[a,b], d)$ by

$$(Tf)(x) = h(x) + \lambda \int_a^b K(x,y) f(y)\, dy,$$

we can write the Fredholm equation above as $Tf = f$. Thus, f is a solution of the Fredholm equation if and only if f is a fixed point of T.

Now, let $M = \sup_{(x,y)\in[a,b]\times[a,b]} |K(x,y)|$ and observe that for any $f, g \in C[a,b]$ we have that

$$d((Tf),(Tg)) = \sup_{x\in[a,b]} |(Tf)(x) - (Tg)(x)| =$$

$$\sup_{x\in[a,b]} \left| h(x) + \lambda \int_a^b K(x,y) f(y)\, dy - \left(h(x) + \lambda \int_a^b K(x,y) g(y)\, dy \right) \right| =$$

$$\sup_{x\in[a,b]} \left| \lambda \int_a^b K(x,y) f(y)\, dy - \lambda \int_a^b K(x,y) g(y)\, dy \right| =$$

$$\sup_{x\in[a,b]} |\lambda| \left| \int_a^b K(x,y) (f(y) - g(y))\, dy \right| \leq$$

$$|\lambda| \sup_{x\in[a,b]} \int_a^b |K(x,y)| \, |f(y) - g(y)|\, dy \leq$$

$$|\lambda| M d(d,f) \int_a^b 1 dy = |\lambda| M (b-a) d(f,g).$$

Hence, if $|\lambda| < \dfrac{1}{M(b-a)}$ we have that T is a contraction mapping on $(C[a,b], d)$ and then it follows from Theorem 1.2.1 that the Fredholm integral equation above has a unique solution on $(C[a,b], d)$. Moreover, if *we take any $f_0 \in C[a,b]$, the sequence $(T^n f_0)_{n\in\mathbb{N}}$ converges to the solution of the equation and therefore, we know (at least theoretically) how to find the solution of the equation.*

Once it was established that Theorem 1.2.1 does not work for nonexpansive mappings it became clear that it was necessary to study which conditions we need to impose on the metric space in order to guarantee the existence of fixed points for nonexpansive. The first result in this direction was presented by Browder in [10] where he showed that nonexpansive self-mappings defined on a bounded, closed, convex subset of a Hilbert space have fixed points. Improvements of Browder's result were presented in that same year assuming weaker conditions, one of this results (Kirk's Theorem) will be studied and extended in this work.

Definition 1.2.4. *Let $\mathcal{C}$ be a class of self-mappings on a metric space (M,d). We say that (M,d) has the **fixed point property** (FPP for short) for the elements of $\mathcal{C}$ if every $T \in \mathcal{C}$ has a fixed point.*

Example 1.2.6. *If we take a complete metric space (M,d) and let*

$$\mathcal{C} = \{T : M \to M \mid T \text{ is a contraction}\},$$

then what Banach's result tells us is that (M,d) has the FPP for the elements of $\mathcal{C}$.

Two interesting questions arise from what we have seen above, namely:

- Given a class $\mathcal{C}$ of self-mappings on a metric space (M,d), is there any condition that we can impose on (M,d) in order to guarantee that it has the FPP for the elements of $\mathcal{C}$?
- Given a metric space (M,d) satisfying some condition(s), what classes of self-mappings on (M,d) will have the FPP?

For example, in Banach's result we only ask the metric space to be complete in order to get the FPP for the class of contraction mappings. In this work we will explore both these questions.

1.3 Normal structure

The concept of normal structure was first introduced by Brodskiĭ and Mil'man in [9] to study fixed point of isometries. In [34] Kirk used this structure to study fixed points of nonexpansive mappings in reflexive Banach spaces. The concept was further extended to the metric space context by Kijima and Takahashi in [32] where they also extended Kirk's result to this context.

Definition 1.3.1. *A metric space (M, d) is said to have* ***normal structure*** *if for every $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that $r(A) < \operatorname{diam}(A)$.*

Observe that given a metric space (M, d) and a convexity structure $\mathcal{F}$ on it, since $\mathcal{A}(M) \subset \mathcal{F}$, if for every $A \in \mathcal{F}$ with $0 < \operatorname{diam}(A)$ we have that $r(A) < \operatorname{diam}(A)$ then, Definition 1.3.1 is satisfied and therefore, (M, d) already has normal structure.

When working on a Banach space $(X, \|\cdot\|)$ and considering the metric induced by the norm, one usually replaces $\mathcal{A}(X)$ in Definition 1.1 by the family of closed, bounded and convex subsets of $(X, \|\cdot\|)$ and when $(X, \|\cdot\|)$ is said to have normal structure this is what is meant. By applying the same idea to the family of weakly compact (weak*-compact, in the case of a dual Banach space) and convex subsets of X when endowed with the weak topology (weak*-topology) we obtain what is called **weak normal structure**.

Example 1.3.1. *It follows from Theorem 4.1 of [6] that every uniformly convex Banach space has normal structure.*

The following classes of spaces are known to be uniformly convex Banach spaces (see [5, Part 3, Chapter II]) and therefore have normal structure:

- *Hilbert spaces.*
- *Closed subspaces of a uniformly convex Banach space.*
- *$L_p(X, \sigma, \mu)$ (and therefore, ℓ_p) for $1 < p < +\infty$.*

Since the previous example only dealt with reflexive Banach spaces we now present examples of nonreflexive spaces with normal structure.

Example 1.3.2. *The space ℓ_1 (seen as the dual space of c_0) is a Banach space with weak normal structure.*

Example 1.3.3. *The space $L_1[0, 1]$ with the convexity structure formed by the sets which are convex and compact in measure has normal structure.*

Now, we will show that normal structure is inherited by admissible sets with positive diameter.

Proposition 1.3.1. *Let (M, d) be a metric space which has normal structure. Given any $X \in \mathcal{A}(M)$ such that $\operatorname{diam}(X) > 0$, we have that $\left(X, d_{|X}\right)$ has normal structure.*

Proof: Let (M, d) be as above, let $X \in \mathcal{A}(M)$ such that $\operatorname{diam}(X) > 0$ and consider the metric space $\left(X, d_{|X}\right)$.

If we take $A \in \mathcal{A}(X)$ with $\operatorname{diam}(A) > 0$ and proceed as in the proof of Proposition 1.1.4, we obtain that $A \in \mathcal{A}(M)$ and then, since (M, d) has normal structure, we have that $r(A) < \operatorname{diam}(A)$.

Thus, since $r(A)$ and $\operatorname{diam}(A)$ are intrinsic to the set A (with respect to the metric d), it follows that $\left(X, d_{|X}\right)$ has normal structure.

■

The following proposition gives us a characterization of normal structure which makes use of the notation $B[A, r]$. This characterization is the one we will use throughout this work since it makes the study of extensions of the concept easier.

Proposition 1.3.2. *A metric space (M, d) has normal structure if and only if for each $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ there exists $0 < r < \operatorname{diam}(A)$ such that*

$$A \cap B[A, r] \neq \emptyset.$$

Proof: Let (M, d) be a metric space.
($\Longrightarrow$) If (M, d) has normal structure then, given $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that $r(A) < \operatorname{diam}(A)$ and therefore, it follows from Lemma 1.1.2 that there exists $0 < r < \operatorname{diam}(A)$ such that $A \cap B[A, r] \neq \emptyset$.

($\Longleftarrow$) Suppose that for each $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ there exists $0 < r < \operatorname{diam}(A)$ such that $A \cap B[A, r] \neq \emptyset$.

Let $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$. Then, there exists x_A and $0 < r_A < \operatorname{diam}(A)$ such that $x_A \in A \cap B[A, r]$ which implies that $d(x_A, x) \leq r$ for all $x \in A$ and therefore, $D(x_A, A) = \sup\{d(x_A, x) : x \in A\} \leq r$.

Hence, $r(A) = \inf\{D(x, A) : x \in A\} \leq D(x_A, A) \leq r_A < \operatorname{diam}(A)$ and then, since A was an arbitrary element of $\mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ it follows that $r(A) < \operatorname{diam}(A)$ for any such set and therefore, (M, d) has normal structure.

■

In the following theorem we can see that when some conditions are imposed on a metric space, we obtain the FPP for the class of nonexpansive mappings.

Theorem 1.3.1 (Kirk's Theorem)**.** *Let (M, d) be a bounded metric space such that which has normal structure and such that $\mathcal{A}(M)$ is compact. Then, (M, d) has the FPP for nonexpansive mappings.*

Proof: Let (M, d) be as above and let $T : M \to M$ be a nonexpansive mapping.

Since $\mathcal{A}(M)$ is compact, it follows from Lemma 1.1.3 that we can take a minimal element A_0 of $\mathcal{A}_T(M)$ with respect to inclusion.

Since $T(A_0) \subset A_0$ and $A_0 \in \mathcal{A}(M)$ we have that $\operatorname{cov}(T(A_0)) \subset A_0$ which gives us that $T(\operatorname{cov}(T(A_0))) \subset T(A_0) \subset \operatorname{cov}(T(A_0))$ and therefore,

$\text{cov}\,(T\,(A_0)) \in \mathcal{A}_T\,(M)$. Thus, it follows from the minimality of A_0 that $A_0 = \text{cov}\,(T\,(A_0))$.

Since $A_0 \neq \emptyset$, there exists $x_0 \in A_0$. We affirm that $A_0 = \{x_0\}$.

Suppose that A_0 has more than one element. Then, it follows that $0 < \text{diam}\,(A_0)$ and therefore, since (M, d) has normal structure, there exists $0 < r < \text{diam}\,(A_0)$ such that $A_0 \cap B\,[A_0, r] \neq \emptyset$.

Now, take $a_0 \in A_0 \cap B\,[A_0, r]$.

Since $a_0 \in B\,[A_0, r]$ we have that $d\,(a_0, a) \leq r$ for all $a \in A_0$ and then since T is nonexpansive, it follows that $d\,(Ta_0, Ta) \leq d\,(a_0, a) \leq r$ for all $a \in A_0$. Thus, $T\,(A_0) \subset B\,(Ta_0, r)$.

Since $B\,(Ta_0, r) \in \mathcal{A}\,(M)$ it follows that $A_0 = \text{cov}\,(T\,(A_0)) \subset B\,(Ta_0, r)$which implies that $d\,(Ta_0, a) \leq r$ for all $a \in A_0$ and therefore, $Ta_0 \in B\,[A_0, r]$.

Thus, since $Ta_0 \in A_0$ we have that $Ta_0 \in A_0 \cap B\,[A_0, r]$ and then since a_0 was an arbitrary element of $A_0 \cap B\,[A_0, r]$, we have that

$$T\,(A_0 \cap B\,[A_0, r]) \subset A_0 \cap B\,[A_0, r]\,.$$

Therefore, $A_0 \cap B\,[A_0, r] \in \mathcal{A}_T\,(M)$.

Now, observe that given $x, y \in A_0 \cap B\,[A_0, r]$, since $x \in A_0$ and $y \in B\,[A_0, r]$ we have that $d\,(x, y) \leq r < \text{diam}\,(A_0)$ which implies that

$$\text{diam}\,(A_0 \cap B\,[A_0, r]) < \text{diam}\,(A_0)$$

and therefore, $A_0 \cap B\,[A_0, r]$ is a proper subset of A_0 which contradicts the minimality of A_0.

Hence, $A_0 = \{x_0\}$ for some $x_0 \in M$ and therefore, since $T\,(A_0) \subset A_0$ we have that x_0 is a fixed point of T.

Thus, since $T : M \to M$ was an arbitrary nonexpansive mapping, we have that (M, d) has the FPP for nonexpansive mappings.

■

It is worth mentioning that Kirk showed in [35] that the compactness of $\mathcal{A}\,(M)$ can be replaced by countably compactness in the previous theorem.

As examples of metric spaces satisfying the requirements of Theorem 1.3.1 we can mention the following:

- Closed bounded and convex subsets of a Banach space.
- Weak*-compact and convex subsets of ℓ_1.
- Convex and compact in measure subsets of $L_1\,[0, 1]$.

1.4 Uniform normal structure

The concept of Uniform normal structure was first introduced by Gillespie and Williams in [24]. It is a structure which implies uniform structure and is widely explored and applied in metric fixed point theory since it is easier to handle than plain uniform structure.

Definition 1.4.1. *A metric space (M, d) is said to have* ***uniform normal structure*** *if there exists some $c \in (0, 1)$ such that $r(A) \leq c \cdot \operatorname{diam}(A)$ for every $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$. When we need to emphasize the constant c, we say that (M, d) has c-uniform normal structure.*

The next proposition which can be found in [2], tells us that finite dimensional normed spaces have uniform normal structure. We believe it is worth presenting the proof of the proposition here since it uses several concepts presented in this chapter and also because it is a nice application of Helly's Theorem which we state before the proposition.

Theorem 1.4.1 (Helly's Theorem)**.** *Let $(X, \|\cdot\|)$ be a normed space of dimension n and consider a finite collection $A_1, \ldots, A_m$ of convex subsets of X where, $m \geq n+1$. If the intersection of every $n+1$ of these sets is nonempty, then $\bigcap_{i=1}^{m} A_i \neq \emptyset$.*

Proposition 1.4.1. *Let $(X, \|\cdot\|)$ be a normed space of dimension n. Then, $\left(X, d_{\|\cdot\|}\right)$ has $\dfrac{n}{n+1}-$uniform normal structure.*

Proof: Let A be an admissible subset of $\left(X, d_{\|\cdot\|}\right)$ such that $0 < \operatorname{diam}(A)$. It follows from Lemma 1.1.2 that $0 < r(A)$.

Let $0 < r < r(A)$.

Observe that $A \cap B[A, r] = \emptyset$ because if this were not the case, there would exist $x \in A$ such that $\|x - a\| \leq r$ for all $a \in A$ which would imply $D(x, A) \leq r$ contradicting the fact that $r(A) \leq D(x, A)$. Thus, it follows from Helly's theorem that there exists $x_1, \ldots, x_{n+1}$ in A such that

$$\bigcap_{i=1}^{n+1} B(x_i, r) \cap A = \emptyset.$$

Now, since A is admissible, we have that it is convex we which implies that $\overline{x} = \dfrac{1}{n+1} \sum_{i=1}^{n} x_n \in A$. Then, there exists $1 \leq j \leq n+1$ such that $r < \|\overline{x} - x_j\|$. We have also that

$$\|\overline{x} - x_j\| = \left\| \frac{1}{n+1} \sum_{i=1}^{n+1} x_i - x_j \right\| = \left\| \frac{1}{n+1} \sum_{i=1}^{n+1} x_i - \frac{1}{n+1}(n+1)\, x_j \right\| =$$

$$\left\| \frac{1}{n+1} \left(\sum_{i=1}^{n+1} x_i - (n+1)\, x_j \right) \right\| = \frac{1}{n+1} \left\| \sum_{i=1}^{n+1} x_i - (n+1)\, x_j \right\| =$$

$$\frac{1}{n+1}\left\|\sum_{\substack{i=1\\ i\neq j}}^{n+1}(x_i-x_j)\right\| \leq \frac{1}{n+1}\sum_{\substack{i=1\\ i\neq j}}^{n+1}\|x_i-x_j\| \leq$$

$$\frac{1}{n+1}\left(n\max_{\substack{1\leq i\leq n+1\\ i\neq j}}\|x_i-x_j\|\right) \leq \frac{n}{n+1}\operatorname{diam}(A)$$

which gives us that $r < \frac{n}{n+1}\operatorname{diam}(A)$. Thus, $r < \frac{n}{n+1}\operatorname{diam}(A)$ for all $0 < r < r(A)$ and therefore, $r(A) \leq \frac{n}{n+1}\operatorname{diam}(A)$.

Hence, $r(A) \leq \frac{n}{n+1}\operatorname{diam}(A)$ for all $A \in \mathcal{A}(X)$ subset of $\left(X, d_{\|\cdot\|}\right)$ with $0 < \operatorname{diam}(A)$ and therefore, $\left(X, d_{\|\cdot\|}\right)$ has $\frac{n}{n+1}-$uniform normal structure.

■

Proposition 1.4.2. *A metric space* (M,d) *has* c*-uniform normal structure if and only if for each* $A \in \mathcal{A}(M)$ *with* $0 < \operatorname{diam}(A)$ *we have that*

$$A \cap B[A, c\cdot\operatorname{diam}(A)] \neq \emptyset.$$

Proof: Let (M,d) be a metric space.
$(\Longrightarrow)$ Suppose (M,d) has c-uniform normal structure. Then, given $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that $r(A) = \inf\{D(x,A) : x \in A\} \leq c\cdot\operatorname{diam}(A)$ which implies that there exists $x_A \in A$ such that $D(x_A, A) = \sup\{d(x_A, x) : x \in A\} \leq c\cdot\operatorname{diam}(A)$ and therefore, $x_A \in B[A, c\cdot\operatorname{diam}(A)]$.

Hence, $A \cap B[A, c\cdot\operatorname{diam}(A)] \neq \emptyset$ and then, since A was an arbitrary element of $\mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ it follows that $A \cap B[A, c\cdot\operatorname{diam}(A)] \neq \emptyset$ for any such set.

$(\Longleftarrow)$ Suppose that for each $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that $A \cap B[A, c\cdot\operatorname{diam}(A)] \neq \emptyset$.

Let $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ and take $x_A \in A \cap B[A, c\cdot\operatorname{diam}(A)]$. Then, we have that $d(x_A, x) \leq c\cdot\operatorname{diam}(A)$ for all $x \in A$ and therefore, $D(x_A, A) = \sup\{d(x_A, x) : x \in A\} \leq c\cdot\operatorname{diam}(A)$.

Hence, $r(A) = \inf\{D(x,A) : x \in A\} \leq D(x_A, A) \leq c\cdot\operatorname{diam}(A)$ and then, since A was an arbitrary element of $\mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ it follows that $r(A) \leq c\cdot\operatorname{diam}(A)$ for any such set and therefore, (M,d) has c-uniform normal structure.

■

As we have seen in Proposition 1.1.3, given a metric space (M,d), the compactness of $\mathcal{A}(M)$ implies the completeness of (M,d). It is worth mentioning that Khamsi showed in [30] that if a complete metric space (M,d) has uniform normal structure then, $\mathcal{A}(M)$ is countably compact. Kulesza and Lim showed in [36] that under the same hypotheses, compactness and countably compactness of $\mathcal{A}(M)$ are equivalent. The results of Khamsi, Kulesza and Lim toghether result in the following lemma.

Lemma 1.4.1. *Let (M,d) be a bounded complete metric space with uniform normal structure. Then, $\mathcal{A}(M)$ is compact.*

Also, Theorem 1.3.1 and Lemma 1.4.1 imply the following theorem which was first stated in this form by Khamsi in [30].

Theorem 1.4.2. *Let (M,d) be a bounded complete metric space. If (M,d) has uniform normal structure then, it has the FPP for nonexpansive mappings.*

We will now state a few definitions and results which will give us a tool to construct sets with uniform normal structure from a family of sets with uniform normal structure.

Definition 1.4.2. *Let $\{(M_i,d_i)\}_{1\leq i\leq n}$ be a family of metric spaces and let $M=\prod_{i=1}^{n} M_i$. We define $d_\infty : M\times M\to\mathbb{R}$ by $d_\infty(x,y)=\max_{1\leq i\leq n}\{d_i(x_i,y_i)\}$ for all $x=(x_1,\dots,x_n)$ and $y=(x_1,\dots,x_n)\in M$. It can be shown that d_∞ is a metric on M.*

Lemma 1.4.2. *Let $\{(M_i,d_i)\}_{1\leq i\leq n}$ be a family of metric spaces and let (M,d_∞) be as in Definition 1.4.2. For each $1\leq i\leq n$ let A_i be a nonempty bounded subset of M_i and let $A=\prod_{i=1}^{n} A_i$. Then we have that:*

(i) $\operatorname{diam}(A)=\max_{1\leq i\leq n}\{\operatorname{diam}(A_i)\}$ *which gives us in particular that A is bounded subset of M;*

(ii) for all $x=(x_1,\dots,x_n)\in A$, $D(x,A)=\max_{1\leq i\leq n}\{D(x_i,A_i)\}$;

(iii) $r(A)=\max_{1\leq i\leq n}\{r(A_i)\}$.

(iv) Also, $\mathcal{A}(M)=\left\{\prod_{i=1}^{n} A_i : A_i\in\mathcal{A}(M_i)\right\}$.

Proposition 1.4.3. *Let $\{(M_i,d_i)\}_{1\leq i\leq n}$ be a family of metric spaces such that (M_i,d_i) has c_i-uniform normal structure for each $1\leq i\leq n$, let $c=\max_{1\leq i\leq n}\{c_i\}$ and let (M,d_∞) be as in Definition 1.4.2. Then, (M,d_∞) has c-uniform normal structure.*

Proof: Let $\{(M_i, d_i)\}_{1\leq i\leq n}$, (M, d_∞) and c be as above.

Given $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$, for each $1 \leq i \leq n$ there exists $A_i \in \mathcal{A}(M_i)$ such that $A = \prod_{i=1}^{n} A_i$.

By Lemma 1.4.2 we have that

$$\operatorname{diam}(A_i) = \max_{1\leq i\leq n} \{\operatorname{diam}(A_i)\} \text{ and } r(A) = \max_{1\leq i\leq n} \{r(A_i)\}.$$

For each $1 \leq i \leq n$, since (M_i, d_i) has c_i-uniform normal structure we have that $r(A_i) \leq c_i \cdot \operatorname{diam}(A_i)$ and then it follows that given $1 \leq i \leq n$ we have that

$$r(A_i) \leq \max_{1\leq j\leq n} \{c_j\} \operatorname{diam}(A_i) \leq \max_{1\leq j\leq n} \{c_j\} \max_{1\leq j\leq n} \{r(A_j)\} = c \cdot \operatorname{diam}(A)$$

and therefore, $r(A) \leq c \cdot \operatorname{diam}(A)$.

Hence, for all $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that $r(A) \leq c \cdot \operatorname{diam}(A)$ and therefore, (M, d_∞) has $c-$uniform normal structure.

■

Definition 1.4.2, Lemma 1.4.2 and Proposition 1.4.3, with proper adaptations, have infinite versions as we can see for example in Proposition 12 of [30].

Using Propositions 1.4.1 and 1.4.3 we can present the following example.

Example 1.4.1. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be finite dimensional normed spaces of dimensions n_1 and n_2 respectively. Then $(X_1 \oplus X_2, \|\cdot\|_\infty)$ has $c-$uniform normal structure where, $c = \max\left\{\frac{n_1}{n_1+1}, \frac{n_2}{n_2+1}\right\}$ and for each $(x_1, x_2) \in X_1 \oplus X_2$, $\|(x_1, x_2)\|_\infty = \max\{\|x_1\|_1, \|x_2\|_2\}$.*

It is easy to see that Example 1.4.1 can be easily extended to any finite direct sum.

1.5 Hyperconvex spaces

The concept of hyperconvexity was introduced in [3] by Aronszajn and Panitchpakdi in an attempt to extend the Hanh-Banach theorem to the context of metric spaces. Since its first appearance, hyperconvex spaces have been shown to be very useful in Metric Fixed Point theory. In this section we will make a brief presentation of those spaces with emphasis in the properties we will use in further chapters. A very thorough exposition on the subject can be found in [16].

Definition 1.5.1. *A metric space (M, d) is said to be* ***hyperconvex*** *if for every family $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq$*

$r_\alpha + r_\beta$ *for all* $\alpha, \beta \in \Gamma$, *it follows that* $\bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$.

As our first example, we will show that the real line is a hyperconvex space.

Example 1.5.1. *The metric space* $(\mathbb{R}, |\cdot|)$ *is hyperconvex.*

To see why this is true, let $\{I_\alpha\}_{\alpha \in \Gamma} = \{[x_\alpha - r_\alpha, x_\alpha + r_\alpha]\}_{\alpha \in \Gamma}$ *be a family of closed nondegenerate intervals such that* $|x_\alpha - x_\beta| \leq r_\alpha + r_\beta$ *for all* $\alpha, \beta \in \Gamma$ *and consider the sets* $A = \{x_\alpha - r_\alpha : \alpha \in \Gamma\}$ *and* $B = \{x_\alpha + r_\alpha : \alpha \in \Gamma\}$.

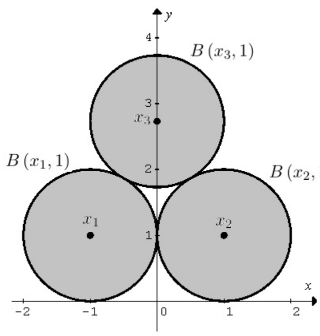
It is easy to see that $x_\beta - r_\beta \leq x_\alpha + r_\alpha$ *for all* $\alpha, \beta \in \Gamma$ *which implies that* A *is bounded above and* B *is bounded below and therefore, there exist* $u = \sup A$ *and* $v = \inf B$.

Given $x \in [u, v]$ *we have that* $u \leq x \leq v$ *which gives us that* $a \leq x \leq b$ *for all* $a \in A$ *and* $b \in B$ *and therefore,* $x \in I_\alpha$ *for all* $\alpha \in \Gamma$. *Thus,* $[u, v] \subset \bigcap_{\alpha \in \Gamma} I_\alpha$ *and in particular,* $\bigcap_{\alpha \in \Gamma} I_\alpha \neq \emptyset$. *This already tells us that* $(\mathbb{R}, |\cdot|)$ *is a hyperconvex metric space.*

Moreover, given $x \in \bigcap_{\alpha \in \Gamma} I_\alpha$ *we have that* $x \in I_\alpha$ *for all* $\alpha \in \Gamma$ *which gives us that* $x_\alpha - r_\alpha \leq x \leq x_\alpha - r_\alpha$ *for all* $\alpha \in \Gamma$ *and therefore,* $\sup A \leq x \leq \inf B$, *that is,* $x \in [u, v]$. *Thus,* $\bigcap_{\alpha \in \Gamma} I_\alpha \subset [u, v]$ *which implies that* $\bigcap_{\alpha \in \Gamma} I_\alpha = [u, v]$.

In what follows we present a simple example of a metric space which is not hyperconvex.

Example 1.5.2. *The metric space* $\left(\mathbb{R}^2, d_2\right)$ *where* d_2 *is the euclidean metric, is **not** hyperconvex as it can be seen from the picture below, where* $x_1 = (-1, 1)$, $x_2 = (1, 1)$ *and* $x_3 = \left(0, 1 + \sqrt{3}\right)$.



The previous example can be extended to show that $(\mathbb{R}^n, d_2)$ is not hyperconvex.

We will now state some results which will give us tools to identify hyperconvex spaces and also construct hyperconvex spaces from a family of hyperconvex spaces. The proofs of the results can all be found in [16].

Proposition 1.5.1. *Let $\{(M_i, d_i)\}_{1\leq i\leq n}$ be a family of hyperconvex spaces and let (M, d_∞) be as in Definition 1.4.2. Then, (M, d_∞) is a hypeconvex space.*

Proposition 1.5.2. *Let $\{(M_\alpha, d_\alpha)\}_{\alpha\in\Gamma}$ be a family of hyperconvex spaces and let $\mathcal{M} = \prod_{\alpha\in\Gamma} M_\alpha$. Fix $a = (a_\alpha)_{\alpha\in\Gamma} \in \mathcal{M}$ and define*

$$M = \left\{ (x_\alpha)_{\alpha\in\Gamma} \in \mathcal{M} : \sup_{\alpha\in\Gamma} d_\alpha (x_\alpha, a_\alpha) < +\infty \right\}.$$

Then, (M, d_∞) is a hyperconvex metric space where d_∞ is defined by $d_\infty ((x_\alpha), (y_\alpha)) = \sup_{\alpha\in\Gamma} \{d_\alpha (x_\alpha, y_\alpha)\}$ for all $(x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma} \in M$.

Proposition 1.5.3. *Let (M, d) be a hyperconvex space. If A is a nonempty admissible subset of M then, $\left(A, d_{|A}\right)$ is a hyperconvex metric subspace of (M, d).*

Example 1.5.3. *It follows from Example 1.5.1 and Proposition 1.5.1 that the metric space $(\mathbb{R}^n, \|\cdot\|_\infty)$ is hyperconvex, where*

$$\|(x_1, \dots, x_n)\|_\infty = \max_{1\leq i\leq n} \{|x_i|\}.$$

Comparing the previous example with Example 1.5.2 we can see that being a hyperconvex metric space depends on the metric we defined on the set.

The next example and the proposition that follows it (whose proof can also be found in [16]) give us the prototypical hyperconvex metric space.

Example 1.5.4. *Let I be a nonempty set of indices and consider the hyperconvex space $(\mathbb{R}, |\cdot|)$. By letting $\mathcal{M} = \mathbb{R}^I$ (the set of all functions from I to $\mathbb{R}$), taking $a = (a_\alpha)_{\alpha\in I} \in \mathcal{M}$ for which $a_\alpha = 0$ for all $\alpha \in I$ and proceeding as in Proposition 1.5.2, we obtain that $(\ell_\infty (I), d_\infty)$ is a hyperconvex space where $\ell_\infty (I)$ is the set of all bounded functions from I to $\mathbb{R}$.*

Proposition 1.5.4. *Any metric space can be isometrically embedded in a hyperconvex metric space of the form $(\ell_\infty (I), d_\infty)$ for some nonempty index set I.*

The following results will tell us that bounded hyperconvex spaces have the FPP for nonexpansive mappings. We chose to present the proofs of the results so the reader can appreciate some features of hyperconvexity which will be used further.

Proposition 1.5.5. *Let (M, d) be a hyperconvex metric space. Then, $\mathcal{A}(M)$ is a compact family.*

Proof: Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a family of sets in $\mathcal{A}(M)$ which has the finite intersection property. By Proposition 1.1.2, for every $\alpha \in \Gamma$ we have that

$$A_\alpha = \bigcap_{x\in M} B\left(x, D\left(x, A_\alpha\right)\right).$$

Consider the family of closed balls $\mathcal{B} = \{B\left(x, D\left(x, A_\alpha\right)\right)\}_{x\in M, \alpha\in\Gamma}$.

Given $x, x' \in M$ and $\alpha, \beta \in \Gamma$, the finite intersection property tells us that $A_\alpha \cap A_\beta = \left(\bigcap_{x\in M} B\left(x, D\left(x, A_\alpha\right)\right)\right) \cap \left(\bigcap_{x\in M} B\left(x, D\left(x, A_\beta\right)\right)\right) \neq \emptyset$ which implies that $B\left(x, D\left(x, A_\alpha\right)\right) \cap B\left(x', D\left(x', A_\beta\right)\right) \neq \emptyset$ and therefore,

$$d\left(x, x'\right) \leq D\left(x, A_\alpha\right) + D\left(x', A_\beta\right).$$

The hiperconvexity of (M, d) implies that $\bigcap_{\alpha\in\Gamma} A_\alpha = \bigcap_{B\in\mathcal{B}} B \neq \emptyset$.

Hence, $\mathcal{A}(M)$ is compact.

■

Corollary 1.5.1. *Every hyperconvex metric space is complete.*

Proof: If (M, d) is a hyperconvex metric space then, Proposition 1.5.5 tells us that $\mathcal{A}(M)$ is compact. Thus, it follows from Proposition 1.1.3 that (M, d) is complete.

■

Proposition 1.5.6. *Let (M, d) be a hyperconvex metric space. Then, for every nonempty $A \in \mathcal{A}(M)$ we have that $r(A) = \frac{1}{2}\operatorname{diam}(A)$. In particular, every hyperconvex space has $\frac{1}{2}$-uniform normal structure.*

Proof: Let (M, d) be a hyperconvex metric space, let A be a nonempty element of $\mathcal{A}(M)$ and let $\delta = \operatorname{diam}(A)$. By Lemma 1.1.2 we already know that $\frac{\delta}{2} \leq r(A)$.

By Proposition 1.1.2, we have that $A = \bigcap_{x\in M} B\left(x, D\left(x, A\right)\right)$.

Now, consider the family of closed balls

$$\mathcal{B} = \{B\left(x, D\left(x, A\right)\right)\}_{x\in M} \cup \left\{B\left(a, \frac{\delta}{2}\right)\right\}_{a\in A}.$$

Given $x, x' \in M$, if we take $a \in A$, we have that

$$d\left(x, x'\right) \leq d\left(x, a\right) + d\left(x' a\right) \leq D\left(x, A\right) + D\left(x', A\right).$$

Given $a, a' \in M$, we have that

$$d\left(a,a'\right) \leq \delta = \frac{\delta}{2} + \frac{\delta}{2}.$$

Given $x \in M$ and $a \in A$ we have that

$$d\left(x,a\right) \leq D\left(x,A\right) \leq D\left(x,A\right) + \frac{\delta}{2}.$$

Thus, it follows from the hiperconvexity of (M,d) that

$$A \cap \bigcap_{a\in A} B\left(a, \frac{\delta}{2}\right) = \bigcap_{B\in\mathcal{B}} B \neq \emptyset.$$

Now, if we take $x_0 \in \bigcap_{B\in\mathcal{B}} B$, since $x_0 \in \bigcap_{a\in A} B\left(a, \frac{\delta}{2}\right)$ we have that $d\left(x_0,a\right) \leq \frac{\delta}{2}$ for all $a \in A$ which implies that $D\left(x_0,A\right) \leq \frac{\delta}{2}$ and therefore,

$$r\left(A\right) = \inf\left\{d\left(x,y\right) : x \in A\right\} \leq D\left(x_0,A\right) \leq \frac{\delta}{2}.$$

Hence, $r\left(A\right) = \frac{\delta}{2}$.

■

Theorem 1.5.1. *Every bounded hyperconvex metric space has the FPP for nonexpansive mappings*

Proof: If (M,d) is a bounded hyperconvex metric space then, Corollary 1.5.5 tells us that (M,d) is complete and Proposition 1.5.6 tells us that it has $\frac{1}{2}$-uniform normal structure.

Hence, it follows from Theorem 1.4.2 that (M,d) has the FPP for nonexpansive mappings.

■

1.6 CAT(0) spaces

In this section we will make a brief presentation of CAT(0) spaces. These spaces have been studied in the context of Metric Fixed Point Theory (see for example [33] and [17]) and they are also known to have uniform normal structure (see [38] and references therein). Most facts about CAT(0) spaces presented in this section can be found in [8].

Definition 1.6.1. *Given a metric space (M,d) and $x,y \in M$, a* ***geodesic path joining*** *x and y is a mapping $\gamma : [0,d\left(x,y\right)] \subset \mathbb{R} \to M$ such that $\gamma\left(0\right) = x, \gamma\left(d\left(x,y\right)\right) = y$ and $d\left(\gamma\left(s\right),\gamma\left(t\right)\right) = |s-t|$ for all $s,t \in [0,d\left(x,y\right)]$. The set $\gamma\left([0,d\left(x,y\right)]\right)$ is said to a be* ***geodesic joining*** *x and y.*

Observe that any geodesic path joining x and y is an isometry and therefore continuous. Moreover, for any given geodesic path γ joining x and y we have that the path

$$\begin{array}{rcl}\delta : [0, d(x,y)] & \to & X \\ t & \to & \gamma(d(x,y) - t)\end{array}$$

is a geodesic path joining y and x such that $\gamma([0, d(x,y)]) = \delta([0, d(x,y)])$.

Definition 1.6.2. *A metric space (M,d) is said to be a **geodesic space** if for any $x, y \in M$ there exists a geodesic joining x and y. Moreover, if for each $x, y \in X$ there is a unique geodesic joining them, then the space is said to be **uniquely geodesic**.*

Example 1.6.1. *Any normed space $(X, \|\cdot\|)$ is a geodesic space since for any given $x, y \in X$ the mapping*

$$\begin{array}{rcl}\gamma : [0, \|x-y\|] & \to & X \\ t & \to & \left(1 - \dfrac{t}{\|x-y\|}\right)x + \dfrac{t}{\|x-y\|}y\end{array}$$

is a geodesic joining x and y. Actually, $\gamma([0, \|x-y\|])$ is the line segment joining x and y.

Next we define a notion of convexity in geodesic spaces.

Definition 1.6.3. *Given a geodesic space (M,d) a subset C of M is said to be convex if for every $x, y \in C$ any geodesic joining x and y is contained in C.*

Example 1.6.2. *Given a geodesic space (M,d) we have that the empty set is convex by vacuity. Also, by definition, given any $x, y \in M$ we have that any geodesic joining x and y is contained in M which implies that M is convex.*

Example 1.6.3. *It is well known ([8, see Chapter I.4]) that inner product spaces are uniquely geodesic and that any closed ball in those spaces are convex.*

Proposition 1.6.1. *Given a geodesic space (M,d) and $\{C_i\}_{i\in I}$ a family of convex subsets of M. We have that $\bigcap\limits_{i\in I} C_i$ is a convex subset of M.*

Proof: Let (M,d) and $\{C_i\}_{i\in I}$ be as above and let $C = \bigcap\limits_{i\in I} C_i$. If $C = \emptyset$ then, as we have already seen in Example 1.6.2, C is convex.

Suppose that $C \neq \emptyset$ and let $x, y \in C$.

Given any $i \in I$, since C_i is convex, we have that C_i contains any geodesic joining x and y. Thus, any geodesic joining x and y is contained in C and therefore, C is convex.

■

Corollary 1.6.1. *Given a geodesic space (M, d), if all the closed balls of M are convex, then any $A \in \mathcal{A}(M)$ is convex and the family of all convex subsets of (M, d) is a convexity structure on (M, d).*

Proof: Let (M, d) be as above and let $\mathcal{C}$ be the family of convex subsets of (M, d). Since any $A \in \mathcal{A}(M)$ is an intersection of closed balls of M and each such ball is in $\mathcal{C}$, if follows from Proposition 1.6.1 that $\mathcal{A}(M) \subset \mathcal{C}$.

It follows from Example 1.6.2 that $\emptyset, M \in \mathcal{C}$. Also, Proposition 1.6.1 tells us that $\mathcal{C}$ is closed under intersections.

Hence, $\mathcal{C}$ is a convexity structure on (M, d).

■

Next we present the model spaces of constant negative and zero curvature. A thorough presentation of these spaces can be found in [8, see Chapter I.2].

Let $\mathbb{E}^n$ denote the Euclidean space $(\mathbb{R}^n, d_2)$ and let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $\mathbf{u} = (u_1, \ldots, u_n, u_{n+1})$ and $\mathbf{v} = (v_1, \ldots, v_n, v_{n+1})$ the real number $\langle \mathbf{u}, \mathbf{v} \rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i$.

Definition 1.6.4. *Then the **real hyperbolic** n**-space** $\mathbb{H}^n$ is the set*

$$\left\{ \boldsymbol{u} = (u_1, \ldots, u_n, u_{n+1}) \in \mathbb{E}^{n,1} : \langle \boldsymbol{u}, \boldsymbol{u} \rangle = -1, u_{n+1} \geq 1 \right\}.$$

Proposition 1.6.2. *[8, Proposition 2.6] Let $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the function that assigns to each pair $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique nonnegative number $d(\boldsymbol{u}, \boldsymbol{v})$ such that*

$$\cosh d(\boldsymbol{u}, \boldsymbol{v}) = -\langle \boldsymbol{u}, \boldsymbol{v} \rangle .$$

Then, d is a metric.

Now, we are ready to define the model spaces M_κ^2 for $\kappa \leq 0$.

Definition 1.6.5. *Given $\kappa \in (-\infty, 0]$ we define M_κ^2 as:*

i) the euclidean space $\mathbb{E}^2$ if $\kappa = 0$;

ii) the space obtained from the hyperbolic space $\mathbb{H}^2$ by multiplying the distance function (as in Proposition 1.6.2) by the constant $\dfrac{1}{\sqrt{-\kappa}}$.

Definition 1.6.6. *Let (M, d) be a geodesic space, a **geodesic triangle** $\Delta(p, q, r)$ consists of three points p, q and r in M (**the vertices** of the triangle) and three geodesics joining each pair of vertices (**the edges** of the triangle). For the geodesic triangle $\Delta(p, q, r)$ a **comparison triangle** is a triangle $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ in the Euclidean plane $\mathbb{E}^2$ such that $d(p, q) = d_2(\bar{p}, \bar{q}), d(q, r) = d_2(\bar{q}, \bar{r})$ and $d(p, r) = d_2(\bar{p}, \bar{r})$.*

It is known from Euclidean geometry that the comparison triangles defined above always exist and are unique up to isometry so, we will denote any comparison triangle corresponding to the points $p, q, r \in M$ as $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$. For a chosen geodesic triangle $\Delta(p, q, r)$ we will denote the edges joining its vertices by $[p, q], [q, r]$ and $[p, r]$ and the corresponding edges in the comparison triangle $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ by $[\bar{p}, \bar{q}], [\bar{q}, \bar{r}]$ and $[\bar{p}, \bar{r}]$.

Definition 1.6.7. *Let (M, d) be a geodesic space and let $\Delta(p, q, r)$ be a geodesic triangle. Given $z, w \in \{p, q, r\}$ with $z \neq w$, for any $x \in [z, w]$ we say that $\bar{x} \in [\bar{z}, \bar{w}]$ is a **comparison point for** x if $d(z, x) = d_2(\bar{z}, \bar{x})$.*

Now we can finally define CAT(0) spaces.

Definition 1.6.8. *Let (M, d) be a geodesic space. We say that a geodesic triangle satisfies the **CAT(0) inequality** if for all $x, y \in \Delta(p, q, r)$ we have that $d(x, y) \leq d_2(\bar{x}, \bar{y})$. A **CAT(0) space** is a geodesic space for which every geodesic triangle satisfies the CAT(0) inequality.*

It can be shown that CAT(0) spaces are uniquely geodesic.

Example 1.6.4. *It is easy to see that that inner product spaces are CAT(0) spaces.*

The following proposition will allow us to easily show that bounded and complete CAT(0) spaces have the FPP for nonexpansive mappings.

Proposition 1.6.3. *Every complete and bounded CAT(0) space has uniform normal structure.*

Theorem 1.6.1. *Every complete and bounded CAT(0) space has the FPP for nonexpansive mappings.*

Proof: It follows straight from Proposition 1.6.3 and Theorem 1.4.2. ■

In Section 3.2 we will extend the previous Theorem.

1.7 Uniform relative normal structure

The concept of uniform relative normal structure was introduced by P. Soardi in [44] as a geometric property in Banach spaces related to the normal structure and led to the existence of fixed points for nonexpansive mappings. This property was useful to cover the case of L_∞-spaces and, more generally, abstract M-spaces (see [43, Chapter 2, Section 7] and [44]) where the standard normal structure or uniform normal structure do not generally work, in particular when complex Banach lattices are considered. The concept was later used by A. To-Ming Lau [45, Theorem 1] to obtain a common fixed point for (onto) isometries defined on a closed convex bounded subset of a Banach space. In [29, Chapter 5], the concept was defined in the general environment of metric spaces as we present in this section.

Definition 1.7.1. *A metric space* (M, d) *is said to have* ***uniform relative normal structure*** *(URNS for short) if there exists some* $c \in (0, 1)$ *such that, for every nonempty* $A \in \mathcal{A}(M)$ *with* $0 < \operatorname{diam}(A)$*, the following conditions are satisfied:*

i) There exists $z_A \in M$ *with*

$$D(z_A, A) \leq c \cdot \operatorname{diam}(A),$$

ii) For z_A *as above and* $x \in M$ *with* $D(x, A) \leq c \cdot \operatorname{diam}(A)$*, we have that*

$$d(x, z_A) \leq c \cdot \operatorname{diam}(A).$$

When we need to emphasize the constant c*, we say that* (M, d) *has* c*-URNS.*

There is a subtle but important difference between Definition 1.4.1 and Definition 1.7.1 since the point z_A given in Definition 1.7.1 does not necessarily belong to the set A. Because of that, the extra condition ii) is needed to assure the existence of a fixed point for a nonexpansive mapping. The following result was also proven by Soardi in [44] in the context of Banach spaces.

Theorem 1.7.1. *[29, Theorem 5.6] Let* (M, d) *be a bounded metric which has uniform relative normal structure and such that* $\mathcal{A}(M)$ *is compact. Then,* (M, d) *has the FPP for nonexpansive mappings.*

We will not present the proof of Theorem 1.7.1 since in the next chapter we will extend the concept of URNS and we will also state and prove an extension of this theorem.

We finish this chapter by presenting the following characterization of URNS.

Proposition 1.7.1. *A metric space* (M, d) *has* c*-URNS if and only if for each* $A \in \mathcal{A}(M)$ *with* $0 < \operatorname{diam}(A)$ *we have that*

$$B[A, c \cdot \operatorname{diam}(A)] \cap B[B[A, c \cdot \operatorname{diam}(A)], c \cdot \operatorname{diam}(A)] \neq \emptyset.$$

Proof: Let (M, d) be a metric space.
($\Longrightarrow$) Suppose (M, d) has c-URNS. Consider $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ and let z_A be an element in M as in conditions i) and ii).

Since $D(z_A, A) = \sup\{d(z_A, a) : a \in A\} \leq c \cdot \operatorname{diam}(A)$, it follows that $d(z_A, a) \leq c \cdot \operatorname{diam}(A)$ for all $a \in A$ which implies that $z_A \in B[A, c \cdot \operatorname{diam}(A)]$.

Given $x \in B[A, c \cdot \operatorname{diam}(A)]$ we have that $d(x, a) \leq c \cdot \operatorname{diam}(A)$ for all $a \in A$ which implies that $D(x, A) = \sup\{d(x, a) : a \in A\} \leq c \cdot \operatorname{diam}(A)$. Then, condition ii) implies that $d(x, z_A) \leq c \cdot \operatorname{diam}(A)$ and therefore,

$$z_A \in B[B[A, c \cdot \operatorname{diam}(A)], c \cdot \operatorname{diam}(A)].$$

Hence,

$$B\left[A, c \cdot \operatorname{diam}(A)\right] \cap B\left[B\left[A, c \cdot \operatorname{diam}(A)\right], c \cdot \operatorname{diam}(A)\right] \neq \emptyset$$

and then, since A was an arbitrary element of $\mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ it follows that

$$B\left[A, c \cdot \operatorname{diam}(A)\right] \cap B\left[B\left[A, c \cdot \operatorname{diam}(A)\right], c \cdot \operatorname{diam}(A)\right] \neq \emptyset$$

for any such set.

($\Longleftarrow$) Suppose that for each $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ we have that

$$B\left[A, c \cdot \operatorname{diam}(A)\right] \cap B\left[B\left[A, c \cdot \operatorname{diam}(A)\right], c \cdot \operatorname{diam}(A)\right] \neq \emptyset.$$

Let $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ and take

$$z_A \in B\left[A, c \cdot \operatorname{diam}(A)\right] \cap B\left[B\left[A, c \cdot \operatorname{diam}(A)\right], c \cdot \operatorname{diam}(A)\right].$$

Since $z_A \in B\left[A, c \cdot \operatorname{diam}(A)\right]$ we have that $d(z_A, x) \leq c \cdot \operatorname{diam}(A)$ which implies that

$$D(z_A, A) = \sup\left\{d(z_A, x) : x \in A\right\} \leq c \cdot \operatorname{diam}(A).$$

Given $x \in M$ such that $D(x, A) \leq c \cdot \operatorname{diam}(A)$ we have that $d(x, a) \leq c \cdot \operatorname{diam}(A)$ for all $a \in A$ which implies that $x \in B\left[A, c \cdot \operatorname{diam}(A)\right]$. Since $z_A \in B\left[B\left[A, c \cdot \operatorname{diam}(A)\right], c \cdot \operatorname{diam}(A)\right]$ we have that $d(z_A, x) \leq c \cdot \operatorname{diam}(A)$.

Hence, conditions i) and ii) are satisfied and then, since A was an arbitrary element of $\mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$ it follows that both conditions are satisfied for any such set and therefore, (M, d) has c-URNS.

$\blacksquare$

Chapter 2

Some results regarding the FPP for nonexpansive mappings

In this chapter we will present some recent results revolving around the FPP for nonnexpansive mappings, most of which can be found in the article [19] by Rafael Espínola García, María Japón and myself.

In the first section we make a brief presentation of some sequences spaces which will be used in the other sections.

In the second section we present some subsets of $(c, \|\cdot\|_\infty)$ with the FPP for nonexpansive mappings.

In the third section we introduce the concept of (p, q)-uniform relative normal structure ((p, q)-URNS for short) and show how it is related to the FPP for nonexpansive mappings. We also relate hyperconvexity and (p, q)-URNS.

In the fourth section we apply what we did in the previous section to obtain some examples of sets with the FPP for nonexpansive mappings.

2.1 Some important sequence spaces

In this section we will make a quick review of some important sequence spaces which will show up in the following sections. Although in this work we mainly deal with real sequence spaces, for the sake of generality, the definitions in this section will be presented by letting the scalar field $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. *Given a real number* $1 \leq p < +\infty$ *we define*

$$\ell_p := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < +\infty \right\}.$$

We also define

$$\ell_\infty := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < +\infty \right\},$$

$$c := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \lim_{n\to\infty} x_n \textit{ exists and is finite} \right\}$$

and

$$c_0 := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0 \right\}.$$

For $1 \leq p < +\infty$, ℓ_p is called the set of **absolutely p-summable sequences** and ℓ_∞ is simply the set of bounded sequences in $\mathbb{K}$. It is easy to see that for all $p, q \in [1, +\infty)$ with $p < q$ it is true that $\ell_p \subsetneq \ell_q \subsetneq \ell_\infty$. It is also easy to see that $c_0 \subsetneq c \subsetneq \ell_\infty$ and $\ell_p \subsetneq c_0$ for all $p \in [1, +\infty)$. Any set on the previous definition can be made into a $\mathbb{K}$-vector space by considering coordinatewise sum and scalar multiplication.

Moreover, if for each bounded sequence $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ we define

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|,$$

it follows that $\|\cdot\|_\infty$ is a norm on ℓ_∞ and $(c_0, \|\cdot\|_\infty), (c, \|\cdot\|_\infty)$ and $(\ell_\infty, \|\cdot\|_\infty)$ are all Banach spaces.

It can be shown that for any $p \in [1, +\infty)$, $(\ell_p, \|\cdot\|_\infty)$ is **not** a Banach space. In the next proposition we will present a norm on ℓ_p which makes it into a Banach space.

Proposition 2.1.1. *Let $p \in [1, +\infty)$ and consider the function $\|\cdot\|_p : \ell_p \to [0, +\infty)$ given by*

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

for all $x = (x_n) \in \ell_p$.

We have that $\|\cdot\|_p$ is a norm on ℓ_p and $\left(\ell_p, \|\cdot\|_p\right)$ is a Banach space.

Proofs of the results above can be found in the first chapter of [21].

Our next goal is to present the Orlicz sequence spaces. Before that, we will talk about more general concepts and then, these spaces will show up as a particular case of a modular space.

Definition 2.1.2. *Let X be a $\mathbb{K}$-vector space, whose zero vector is $\mathbf{0}$. A mapping $\rho : X \to [0, +\infty]$ is said to be a **convex modular** if it has the following properties:*

(i) $\rho(x) = 0$ *if and only if* $x = \mathbf{0}$*;*

(ii) $\rho(\alpha x) = \rho(x)$ *for all* $x \in X$ *and for every* $\alpha \in \mathbb{K}$ *such that* $|\alpha| = 1$*;*

(iii) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for all* $x, y \in X$ *whenever* $\alpha \geq 0, \beta \geq 0$ *and* $\alpha + \beta = 1$.

The following proposition shows us hot to use convex modulares to build Banach spaces from a given vector space.

Proposition 2.1.2. *Let* X *be a* $\mathbb{K}$*-vector space with a convex modular* ρ*. Then, the set*

$$X_\rho = \left\{ x \in X : \rho\left(\frac{x}{\lambda}\right) < +\infty \text{ for some } \lambda > 0 \right\}$$

is a vector subspace of X*. A vector space as* X_ρ *is called a* ***modular space****.*

Moreover, the function

$$\|\cdot\|_\rho : X_\rho \to [0, +\infty)$$

defined by

$$\|x\|_\rho = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

is a norm on X_ρ *and* $\left(X_\rho, \|\cdot\|_\rho\right)$ *is a Banach space. Such a norm is usually called a* ***Luxemburg norm****.*

Example 2.1.1. *Consider the vector space* $\mathbb{K}^{\mathbb{N}}$*. Given a sequence* $(p_n)_{n\in\mathbb{N}}$ *in* $[1, +\infty)$ *the mapping* $\rho : \mathbb{K}^{\mathbb{N}} \to [0, +\infty]$ *given by*

$$\rho(x) = \sum_{n=1}^{\infty} |x_n|^{p_n}$$

for all $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ *is a convex modular.*

Now, we are ready to introduce the Orlicz sequence spaces.

Definition 2.1.3. *Let* $(p_n)_{n\in\mathbb{N}}$ *be a sequence in* $[1, +\infty)$ *and let* ρ *be the modular defined in the previous example. Then, we define the Orlicz sequence space* ℓ_{p_n} *as the modular space associated to* ρ*, that is*

$$\ell_{p_n} = \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \rho\left(\frac{x}{\lambda}\right) < +\infty \text{ for some } \lambda > 0 \right\}.$$

If we call $\|\cdot\|_{p_n}$ the Luxemburg norm associated to ℓ_{p_n}, it follows from Proposition 2.1.2 that $\left(\ell_{p_n}, \|\cdot\|_{p_n}\right)$ is a Banach space.

For more about Orlicz sequence spaces, modular spaces and related concepts (including the proofs of the previous results) see for example [39],[40],[41] and [27, Chapter 4].

2.2 On subsets of c with the FPP for nonexpansive mappings

It is well-known that for the Banach spaces $(c_0, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$, any of its convex weakly compact subsets $(K, \|\cdot\|_\infty)$ has the FPP for nonexpansive mappings (see [7]). Moreover, it was shown in [12, 14, 15] that convex weakly compact sets are the only convex closed subsets of $(c_0, \|\cdot\|_\infty)$ with the FPP for nonexpansive mappings, that is, if a closed and convex subset of $(c_0, \|\cdot\|_\infty)$ has the FPP for nonexpansive mappings then, it has to be weakly compact. This raised the question on whether this characterization of weak compactness was true for $(c, \|\cdot\|_\infty)$. This question was answered negatively in [22], although weak compactness in $(c, \|\cdot\|_\infty)$ has actually been characterized in terms of fixed point properties for some larger family of mappings. In fact, it was shown in [26] that a closed and convex subset of $(c, \|\cdot\|_\infty)$ is weakly compact if, and only if, every (so-called) cascading nonexpansive self-mapping defined on it has a fixed point.

In [22] the authors introduced the following subset of c:

$$W = \left\{x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty : 1 \geq x_1 \geq x_2 \geq \ldots \geq 0\right\}.$$

In the article it was shown that $(W, \|\cdot\|_\infty)$ is a closed, bounded and convex subset of $(c, \|\cdot\|_\infty)$ which is non-weakly compact and has the FPP for nonexpansive mappings. The set W was the first known example in the literature of a subset of $(c, \|\cdot\|_\infty)$ with the mentioned properties and its existence closed the door to a possible characterization result for weakly compactness in $(c, \|\cdot\|_\infty)$ as the one we have previously described, although it opened the scenario to considering fixed point theorems for a larger family of mappings as it can be seen in [26].

After defining W, for each sequence $q = (q_n)_{n\in\mathbb{N}} \in (0, +\infty)$ for which there exists positive real numbers A and B such that $A \leq q_n \leq B$ for all $n \in \mathbb{N}$, the authors of [22] defined the set

$$W_q = \left\{(q_n y_n)_{n\in\mathbb{N}} \in \ell_\infty : (y_n)_{n\in\mathbb{N}} \in W\right\}$$

and showed that each W_q has the same properties as those described for W.

The authors of [22] showed that the key fact why the set W and all the sets W_q have the FPP for nonexpansive mappings is that they are all hyperconvex. In the article several proofs of the hyperconvexity of (W, d_∞) were presented using different techniques. None of the proofs was an elementary based purely on the definition of hyperconvexity. So, we will start this chapter by providing such a proof.

Proposition 2.2.1. *The metric space (W, d_∞) is a hyperconvex metric subspace of $(\ell_\infty(\mathbb{N}), d_\infty)$, where*

$$W = \left\{x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty : 1 \geq x_1 \geq x_2 \geq \ldots \geq 0\right\}.$$

Proof: Let $\left\{B_W\left((x_\alpha(n))_{n\in\mathbb{N}}, r_\alpha\right)\right\}_{\alpha\in\Gamma}$ be a family of closed balls in W such that $d_\infty\left((x_\alpha(n))_{n\in\mathbb{N}}, (x_\beta(n))_{n\in\mathbb{N}}\right) \leq r_\alpha + r_\beta$ for all $\alpha, \beta \in \Gamma$.

Observe that $W \subset [0,1]^{\mathbb{N}}$ which gives us that for each $\alpha \in \Gamma$,

$$B_W\left((x_\alpha(n))_{n\in\mathbb{N}}, r_\alpha\right) = W \cap B_{\ell_\infty(\mathbb{N})}\left((x_\alpha(n))_{n\in\mathbb{N}}, r_\alpha\right) =$$
$$W \cap [0,1]^{\mathbb{N}} \cap \prod_{n\in\mathbb{N}} [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha] =$$
$$W \cap \left([0,1]^{\mathbb{N}} \cap \prod_{n\in\mathbb{N}} [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right) =$$
$$W \cap \left(\prod_{n\in\mathbb{N}} \left([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right)\right).$$

Thus,

$$\bigcap_{\alpha\in\Gamma} B_W\left((x_\alpha(n))_{n\in\mathbb{N}}, r_\alpha\right) =$$
$$\bigcap_{\alpha\in\Gamma}\left(W \cap \left(\prod_{n\in\mathbb{N}} \left([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right)\right)\right) =$$
$$W \cap \bigcap_{\alpha\in\Gamma}\left(\prod_{n\in\mathbb{N}} \left([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right)\right) =$$
$$W \cap \prod_{n\in\mathbb{N}}\left(\bigcap_{\alpha\in\Gamma} \left([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right)\right).$$

Let $\gamma \notin \Gamma$ and for each $n \in \mathbb{N}$ let $x_\gamma(n) = \dfrac{1}{2}$ and $r_\gamma = \dfrac{1}{2}$. This gives us that $[0,1] = [x_\gamma(n) - r_\gamma, x_\gamma(n) + r_\gamma]$ for all $n \in \mathbb{N}$. Now if we let $\Gamma' = \Gamma \cup \{\gamma\}$, for each $n \in \mathbb{N}$ we have that

$$\bigcap_{\alpha\in\Gamma} \left([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right) = [0,1] \cap \bigcap_{\alpha\in\Gamma} [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha] =$$
$$\bigcap_{\alpha\in\Gamma'} [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha].$$

It follows from what we have seen in Example 1.5.1 that for each $n \in \mathbb{N}$,

$$\bigcap_{\alpha\in\Gamma'} [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha] = [a_n, b_n]$$

where $a_n = \sup\{\alpha \in \Gamma' : x_\alpha(n) - r_\alpha\}$ and $b_n = \inf\{\alpha \in \Gamma' : x_\alpha(n) + r_\alpha\}$.

Thus,

$$\prod_{n\in\mathbb{N}}\left(\bigcap_{\alpha\in\Gamma'}[x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right) = \prod_{n\in\mathbb{N}}[a_n, b_n]$$

which tells us in particular that

$$(b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}\left(\bigcap_{\alpha\in\Gamma'}[x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha]\right).$$

Now, given $n \in \mathbb{N}$, since $x_\alpha(n+1) + r_\alpha \leq x_\alpha(n) + r_\alpha$ for all $\alpha \in \Gamma'$, it follows that

$$b_{n+1} = \inf\left\{\alpha \in \Gamma' : x_\alpha(n+1) + r_\alpha\right\} \leq \inf\left\{\alpha \in \Gamma' : x_\alpha(n) + r_\alpha\right\} = b_n$$

which gives us that $0 \leq b_n \leq b_{n+1} \leq 1$ for all $n \in \mathbb{N}$ and therefore, $(b_n)_{n\in\mathbb{N}} \in W$.

Hence, $W \cap \prod_{n\in\mathbb{N}}\left(\bigcap_{\alpha\in\Gamma}([0,1] \cap [x_\alpha(n) - r_\alpha, x_\alpha(n) + r_\alpha])\right) \neq \emptyset$ and therefore, (W, d_∞) is a hyperconvex metric subspace of $(\ell_\infty(\mathbb{N}), d_\infty)$.

■

Since the sets presented in [22] were all hyperconvex, at the end of the article, its authors say that they do not know of an example of a non-weakly compact, closed, bounded and convex subset of $(c, \|\cdot\|_\infty)$ which has the FPP for nonexpansive mappings and is not hyperconvex. Our next example shows that such a set indeed exists.

Example 2.2.1. *Let $Q = \left\{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 + x_3 = 1\right\}$, let W be as previously described. If we set*

$$M = \left(Q, \|\cdot\|_{\infty,3}\right) \oplus (W, \|\cdot\|_\infty),\ \textit{where } \|\cdot\|_{\infty,3} \textit{ is the sup norm in } \mathbb{R}^3,$$

then $(M, \|\cdot\|_\infty)$ is a non-weakly compact, closed, bounded and convex subset of c which is not hyperconvex but still has the FPP for nonexpansive mappings.

Proof: Since $W \subset c$, it is obvious that $M \subset c$. It is also obvious that M is closed, bounded and convex. We claim that W has uniform normal structure. Indeed, observe that $\left(Q, \|\cdot\|_{\infty,3}\right)$ is isometric to $\left(P, \|\cdot\|_{\infty,3}\right)$ where

$$P = \left\{(x_1, x_2, x_3) \in [0,1]^2 \times [-1,0] : x_1 + x_2 + x_3 = 0\right\}.$$

Also, P is contained in $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\right\}$ which is a 2-dimensional subspace of $\mathbb{R}^3$. Thus, it follows from Proposition 1.4.1 that P with the $\|\cdot\|_{\infty,3}$ norm has $\dfrac{2}{3}$-uniform normal structure and therefore, by isometry, $\left(Q, \|\cdot\|_{\infty,3}\right)$ has $\dfrac{2}{3}$-uniform normal structure too.

Since $(W, \|\cdot\|_{\infty})$ is hyperconvex, it follows from Proposition 1.5.6 that $(W, \|\cdot\|_{\infty})$ has $\dfrac{1}{2}$-uniform normal structure and therefore, Proposition 1.4.3 tells us that $(M, \|\cdot\|_{\infty})$ has $\dfrac{2}{3}$-uniform normal structure.

Thus, it follows from Theorem 1.4.2 that $(M, \|\cdot\|_{\infty})$ has the FPP for nonexpansive mappings.

To show that $(M, \|\cdot\|_{\infty})$ is not hyperconvex we consider the family of closed balls in $(M, \|\cdot\|_{\infty})$ given by:

$$\left\{B\left(e_i, \frac{1}{2}\right) : 1 \leq i \leq 3\right\}$$

where e_i stands for the i-th standard basis element in c, that is,

$$e_i = \left(0, 0, \ldots, 0, \underbrace{1}_{i\text{-th coordinate}}, 0, \ldots, 0, 0, \ldots\right).$$

If $(M, \|\cdot\|_{\infty})$ were hyperconvex, since $\|e_i - e_j\|_{\infty} = 1 = \dfrac{1}{2} + \dfrac{1}{2}$ whenever $i \neq j$, this collection of balls should have nonempty intersection within M. However, if there existed, $x = (x_n)_{n\in\mathbb{N}}$ in these three balls, it would be the case that

$$|x_i - 1| \leq \frac{1}{2} \text{ and } |x_i| \leq \frac{1}{2} \text{ for all } 1 \leq i \leq 3.$$

Thus, we would have that $x_1 = x_2 = x_3 = \dfrac{1}{2}$ which contradicts the fact that $x \notin M$. Hence $(M, \|\cdot\|_{\infty})$ is not hyperconvex.

Finally, since $(M, \|\cdot\|_{\infty})$ is a direct sum with a non-weakly compact subset of c, it follows that M is not weakly compact either.

■

The previous example can be modified by replacing the first term of the direct sum by any non hyperconvex, bounded, closed and convex subset of any finite dimensional normed space since such sets, as stated in Proposition 1.4.1, have uniform normal structure.

A direct sum, as the previous example, may be seen as a too straightforward construction and one might wonder if there is any such example which cannot be obtained in that way. The next example we will present shows that it is indeed possible to do so. Before we proceed to the example, we will make a brief detour to introduce a few definitions and results which will be used in it but deserve a more general presentation.

Definition 2.2.1. *Let I be a nonempty set of indices and let A be a nonempty and bounded subset of $\left(\ell_\infty(I), \|\cdot\|_\infty\right)$. For each $\alpha \in I$ we define*

$$i_A(\alpha) = \inf_{x \in A}\{x(\alpha)\}, s_A(\alpha) = \sup_{x \in A}\{x(\alpha)\} \text{ and } z_A(\alpha) = \frac{i_A(\alpha) + s_A(\alpha)}{2}.$$

We also define

$$i(A) = (i_A(\alpha))_{\alpha \in I}, s(A) = (s_A(\alpha))_{\alpha \in I} \text{ and } z(A) = (z_A(\alpha))_{\alpha \in I}.$$

Lemma 2.2.1. *Let I be a set of indices and let A be a nonempty and bounded subset of $\left(\ell_\infty(I), \|\cdot\|_\infty\right)$. Then, $|s_A(\alpha) - i_A(\alpha)| \leq \operatorname{diam}(A)$ for all $\alpha \in I$ and $A \subset B\left(z(A), \frac{1}{2}\operatorname{diam}(A)\right)$.*

Proof: Let A be as above and let $\delta = \operatorname{diam}(A)$.

For any given $x, y \in A$ we have that $\|x - y\|_\infty \leq \delta$ which implies that $|x(\alpha) - y(\alpha)| \leq \delta$ for all $\alpha \in I$ and therefore,

$$|x(\alpha) - y(\alpha)| \leq \delta \text{ for all } x, y \in A \text{ and for all } \alpha \in I.$$

Now, if we fix $\alpha \in I$, since for each fixed $y \in A$ it is true that $-\delta \leq x(\alpha) - y(\alpha) \leq \delta$ for all $x \in A$, we obtain that $-\delta \leq s_A(\alpha) - y(\alpha) \leq \delta$ for all $y \in A$ and therefore, $-\delta \leq s_A(\alpha) - i_A(\alpha) \leq \delta$.

Hence, $|s_A(\alpha) - i_A(\alpha)| \leq \delta$ for all $\alpha \in I$.

Now, if we fix $\alpha \in I$, for every $x \in A$ we have that $i_A(\alpha) \leq x(\alpha) \leq s_A(\alpha)$ which gives us that

$$i_A(\alpha) - \frac{i_A(\alpha) + s_A(\alpha)}{2} \leq x(\alpha) - \frac{i_A(\alpha) + s_A(\alpha)}{2} \leq s_A(\alpha) - \frac{i_A(\alpha) + s_A(\alpha)}{2}$$

and therefore, $-\dfrac{s_A(\alpha) - i_A(\alpha)}{2} \leq x(\alpha) - z_A(\alpha) \leq \dfrac{s_A(\alpha) - i_A(\alpha)}{2}$.

Thus, $|x(\alpha) - z_A(\alpha)| \leq \dfrac{s_A(\alpha) - i_A(\alpha)}{2} \leq \dfrac{\delta}{2}$ for all $\alpha \in I$ and for all $x \in A$ which implies that $\|x - z(A)\|_\infty \leq \dfrac{\delta}{2}$ for all $x \in A$.

Hence, $A \subset B\left(z(A), \dfrac{\delta}{2}\right)$.

■

Example 2.2.2. *Let*

$$D = \left\{x = (x_n)_{n \in \mathbb{N}} \in [0,1]^{\mathbb{N}} : x_1 + x_2 + x_3 = 1, x_3 \geq x_4 \geq \ldots \geq 0\right\}.$$

Then $(D, \|\cdot\|_\infty)$ is a bounded, closed, convex and non-weakly compact subset of $(c, \|\cdot\|_\infty)$ which is not hyperconvex and has the FPP for nonexpansive mappings.

Proof: It is obvious that $D \subset c$ and D is closed, bounded and convex. Similar arguments to those used in the Example 2.2.1 show that D is neither weakly compact nor hyperconvex.

We will apply Theorem 1.4.2 to prove that D has the FPP for nonexpansive mappings.

It suffices to show that D has $\frac{2}{3}$-uniform normal structure. Let $A \in \mathcal{A}(D)$ such that $\delta = \operatorname{diam}(A) > 0$. For each $n \in \mathbb{N}$, let

$$a_n = i_A(n),\ b_n = s_A(n) \text{ and } z_n = z_A(n).$$

Given $n \geq 3$ since $x_n \geq x_{n+1}$ for all $x \in A$, it follows that $a_n \geq a_{n+1}$ and $b_n \geq b_{n+1}$ which implies that $z_n \geq z_{n+1}$. However, we cannot assure that $z_1 + z_2 + z_1 = 1$ and therefore, we do not know if $z(A) = (z_n)_{n\in\mathbb{N}} \in D$.

Let τ be the coordinatewise convergence topology on ℓ_∞ and consider the projection mapping:

$$\begin{array}{rcl} \pi : (\ell_\infty, \tau) & \to & \left(\mathbb{R}^3, \|\cdot\|_{\infty,3}\right) \\ x = (x_n)_{n\in\mathbb{N}} & \mapsto & (x_1, x_2, x_3) \end{array}$$

Observe that D is τ-compact which implies that any element of $\mathcal{A}(D)$ is τ-compact. Since A is also convex, it follows that $\pi(A)$ is a compact and convex subset of $\left(\mathbb{R}^3, \|\cdot\|_{\infty,3}\right)$. From Proposition 1.4.1 together with the translation argument we used in Example 2.2.1, we know that there exists $w = (w_n)_{n\in\mathbb{N}} \in A$ such that

$$\|\pi(w) - \pi(x)\|_\infty \leq \frac{2}{3}\operatorname{diam}(\pi(A)) \leq \frac{2}{3}\delta \text{ for all } x \in A.$$

This implies that, for every $x \in A$, $|w_n - x_n| \leq \frac{2}{3}\delta$ for $1 \leq n \leq 3$. For $n \geq 4$ we already know that $w_n \in [a_n, b_n]$.

Now, we are going to show that $A \cap B\left[A, \frac{2}{3}\delta\right] \neq \emptyset$. To this end, let us define

$$\tilde{w}_n = \begin{cases} w_n & \text{if } \ 1 \leq n \leq 3, \\ \min\{w_3, z_n\} & \text{if } \ n \geq 4. \end{cases}$$

Observe that $\tilde{w}_1 + \tilde{w}_2 + \tilde{w}_3 = 1$ and $|\tilde{w}_n - x_n| \leq \frac{2}{3}\delta$ for $1 \leq n \leq 3$ and for all $x \in A$. Also, since $0 \leq a_n \leq w_n \leq w_3$ and $a_n \leq z_n \leq b_n$ for all $n \geq 4$, it follows that $\tilde{w}_n \in [a_n, b_n]$ for $n \in \mathbb{N}$.

Since $A \in \mathcal{A}(D)$ we have that

$$A = \bigcap_{i\in I} B\left(p^i, r_i\right) \cap D$$

for a certain collection of centers $p^i = \left(p^i_n\right)_{n\in\mathbb{N}} \in D$ for $i \in I$. Therefore, given $x = (x_n)_{n\in\mathbb{N}} \in A$, we have that $x \in B\left(p^i, r_i\right)$ for all $i \in I$. In particular, we have that $\left|x_n - p^i_n\right| \leq r_i$ for all $n \in \mathbb{N}$ and $i \in I$. Thus, given $n \in \mathbb{N}$,

$$p^i_n - r_i \leq x_n \leq p^i_n + r_i$$

for all $x \in A$ and $i \in I$. Therefore, $p^i_n - r_i \leq a_n \leq b_n \leq p^i_n + r_i$ and we can conclude that

$$[a_n, b_n] \subset \bigcap_{i\in I} \left[p^i_n - r_i, p^i_n + r_i\right].$$

Hence,

$$\tilde{w} \in \bigcap_{i\in I} B\left(p^i, r_i\right)$$

and therefore, since $\tilde{w} \in D$, $\tilde{w} \in A$.

Let $x = (x_n)_{n\in\mathbb{N}} \in A$. We already know that $|\tilde{w}_n - x_n| \leq \frac{2}{3}\delta$ for $1 \leq n \leq 3$. Let $n \geq 4$. We have the following three cases to study:

- Case $z_n \leq w_3$. In this case we have that $\tilde{w}_n = z_n$ and then it follows from Lemma 2.2.1 that
$$|\tilde{w}_n - x_n| = |z_n - x_n| \leq \frac{1}{2}\delta \leq \frac{2}{3}\delta.$$
- Case $w_3 < z_n$ and $x_n \leq z_n$. In this case, since $\tilde{w}_n$, $x_n \in [a_n, b_n]$ and z_n is the middle point of $[a_n, b_n]$, we have that
$$|x_n - \tilde{w}_n| \leq \frac{1}{2}\delta \leq \frac{2}{3}\delta.$$
- Case $w_3 < z_n$ and $z_n < x_n$. In this case we have that
$$|x_n - \tilde{w}_n| = |x_n - w_3| = x_n - w_3 \leq x_3 - w_3 \leq \frac{2}{3}\delta.$$

Since these are the only possible cases, we can conclude that

$$|x_n - \tilde{w}_n| \leq \frac{2}{3}\delta \text{ for all } n \geq 4.$$

Hence, $|\tilde{w}_n - x_n| \leq \frac{2}{3}\delta$ for all $n \in \mathbb{N}$ and $x \in A$. Therefore, $\tilde{w} \in \bigcap_{y\in A} B\left(y, \frac{2}{3}\delta\right) = B\left[A, \frac{2}{3}\delta\right]$ and, finally,

$$A \cap B\left[A, \frac{2}{3}\delta\right] \neq \emptyset.$$

Now, since A was an arbitrary admissible subset of D with positive diameter, we have that D has $\frac{2}{3}$-uniform normal structure and, by Theorem 1.4.2, D has the FPP for nonexpansive mappings.

■

2.3 The (p, q)-uniform relative normal structure

In this section we introduce a concept that is a formal extension of the uniform relative normal structure that and which will be applied to obtain some fixed point results as well as stability of the FPP for nonexpansive mappings.

Definition 2.3.1. *A metric space (M, d) is said to have (p, q)-URNS for some $p > 0$ and $q \in (0, 1)$ if*

$$B\left[A, p \cdot \operatorname{diam}(A)\right] \cap B\left[B\left[A, p \cdot \operatorname{diam}(A)\right], q \cdot \operatorname{diam}(A)\right] \neq \emptyset$$

for every $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A)$.

Observe that if we take $p = q = c \in (0, 1)$ in the previous definition we obtain c-URNS and so (p, q)-URNS provides a formal extension of both normal structure and uniform normal structure.

In what follows we will show that, under standard assumptions, bounded metric spaces with (p, q)-URNS have the FPP for nonexpansive mappings. Before we can show this result, we will present some technical lemmas. In all lemmas, (M, d) is a bounded metric space and $T : M \to M$ is a nonexpansive mapping. In the following proofs we will make extensive use of the sets presented in Definition 1.1.4.

Lemma 2.3.1. *Let $A \in \mathcal{A}_T(M)$ and let $s > 0$ such that $A \cap B[A, s] \neq \emptyset$. If we set*

$$\tilde{A} := \operatorname{cov}_T(A \cap B[A, s]),$$

it holds that $\operatorname{diam}(\tilde{A}) \leq s$.

Proof: If we manage to show that $\tilde{A} \subset \tilde{A} \cap B[\tilde{A}, s]$, we will automatically obtain that $\operatorname{diam}(\tilde{A}) \leq s$. So, we will show that

i) $A \cap B[A, s] \subset \tilde{A} \cap B[\tilde{A}, s]$ and

ii) $\tilde{A} \cap B[\tilde{A}, s]$ is T-invariant.

Since $\tilde{A} \cap B[\tilde{A}, s] \in \mathcal{A}(M)$, the conclusion follows.

i) From the definition, $A \cap B[A, s] \subset \tilde{A}$. Since A is T-invariant and admissible and also $A \cap B[A, s] \subset A$, we have $\tilde{A} \subset A$ and then it follows from item iii) of Lemma 1.1.1 that $B[A, s] \subset B[\tilde{A}, s]$.

   Thus, $A \cap B[A, s] \subset B[\tilde{A}, s]$ and so

$$A \cap B[A, s] \subset \tilde{A} \cap B[\tilde{A}, s].$$

ii) Let $z \in \tilde{A} \cap B[\tilde{A}, s]$. Since $\tilde{A}$ is T-invariant, we have that $Tz \in \tilde{A}$. It remains to be shown that $Tz \in B[\tilde{A}, s]$ which is equivalent to the fact that $\tilde{A} \subset B(Tz, s)$. Thus, it suffices to show that

$$\tilde{A} \cap B(Tz, s) \in \{L \in \mathcal{A}_T(M) : L \supset A \cap B[A, s]\}.$$

It is obvious that $\tilde{A} \cap B(Tz, s) \in \mathcal{A}(M)$. Since $B[A, s] \subset B[\tilde{A}, s]$ and $Tz \in \tilde{A}$, given $x \in A \cap B[A, s]$, it follows that $d(x, Tz) \leq s$, which implies that $A \cap B[A, s] \subset B(Tz, s)$ and therefore,

$$A \cap B[A, s] \subset \tilde{A} \cap B(Tz, s).$$

Now, take $y \in \tilde{A} \cap B(Tz, s)$. Since $\tilde{A}$ is T-invariant, it follows that $Ty \in \tilde{A}$. Since T is nonexpansive, $y \in \tilde{A}$ and $z \in B[\tilde{A}, s]$ we have

$$d(Ty, Tz) \leq d(y, z) \leq s,$$

which implies that $Ty \in B(Tz, s)$. Thus, $\tilde{A} \cap B(Tz, s)$ is T-invariant.

From the above, we conclude that $\tilde{A} \cap B[\tilde{A}, s] \in \mathcal{A}_T(M)$ and contains $A \cap B[A, s]$. Thus, $\tilde{A} \subset \tilde{A} \cap B[\tilde{A}, s]$ and $\operatorname{diam}(\tilde{A}) \leq s$.

■

Lemma 2.3.2. *If $\mathcal{A}(M)$ is compact and $A \in \mathcal{A}_T(M)$. Then, there exists $A_0 \subset A$ such that:*

i) $A_0 \in \mathcal{A}_T(M)$ and

ii) $B[A_0, r] \in \mathcal{A}_T(M)$ whenever $B[A_0, r]$ is nonempty.

Proof: Let $\mathcal{L}_A := \{L \in \mathcal{A}_T(M) : \ L \subset A\}$. Since $A \in \mathcal{A}_T(M)$ we have that $\mathcal{L}_A \neq \emptyset$. Since, $\mathcal{A}(M)$ is compact, by proceeding as in the proof of Lemma 1.1.3 we can find a minimal element A_0 of $\mathcal{L}_A$ and therefore, i) is proven.

Since $A_0 \in \mathcal{A}_T(M)$ we have that $\operatorname{cov}(T(A_0)) \subset A_0$, and so

$$T(\operatorname{cov}(T(A_0))) \subset T(A_0) \subset \operatorname{cov}(T(A_0)).$$

Therefore, $\operatorname{cov}(T(A_0)) \in \mathcal{L}_A$. The minimality of A_0 implies that

$$A_0 = \operatorname{cov}(T(A_0)).$$

Now, let $r > 0$ be such that $B[A_0, r] \neq \emptyset$. Given $x \in B[A_0, r]$ we have that $d(x, y) \leq r$ for all $y \in A_0$. Then, since T is nonexpansive, it follows that $d(Tx, Ty) \leq d(x, y) \leq r$ for all $y \in A_0$. Thus, $Tx \in B(Ty, r)$ for all $y \in A_0$ which implies that $Tx \in B[T(A_0), r]$ and therefore,

$$T(B[A_0, r]) \subset B[T(A_0), r].$$

Item iii) of Proposition 1.1.1 tell us that $B\left[T\left(A_0\right),r\right]=B\left[\operatorname{cov}\left(T\left(A_0\right)\right),r\right]$ and then since, $A_0=\operatorname{cov}\left(T\left(A_0\right)\right)$ we have that $B\left[T\left(A_0\right),r\right]=B\left[A_0,r\right]$.

Hence, $T\left(B\left[A_0,r\right]\right)\subset B\left[A_0,r\right]$ which implies that $B\left[A_0,r\right]\in\mathcal{A}_T\left(M\right)$ and therefore, ii) is proven.

■

Lemma 2.3.3. *If $\mathcal{A}\left(M\right)$ is compact and $A\in\mathcal{A}_T\left(M\right)$. Given $A_0\subset A$ as in Lemma 2.3.2, if $B\left[A_0,r\right]\cap B\left[B\left[A_0,r\right],s\right]\neq\emptyset$, then the set*

$$\tilde{A}_0:=\operatorname{cov}_T\left(B\left[A_0,r\right]\cap B\left[B\left[A_0,r\right],s\right]\right)$$

satisfies:

i) $\tilde{A}_0$ *is T-invariant.*

ii) $\tilde{A}_0\subset B\left[A_0,r\right]$.

iii) $\operatorname{diam}(\tilde{A}_0)\leq s$.

Proof: Assertion i) follows from item ii) of Proposition 1.1.1. Assertion ii) follows from the definition of $\tilde{A}_0$ and from item ii) of Lemma 2.3.2. Assertion iii) follows from the definition of $\tilde{A}_0$ and Lemma 2.3.1.

■

With the previous lemmas in hand we are now ready to show the analogous of Theorem 1.7.1 for (p,q)-URNS.

Theorem 2.3.1. *Let (M,d) be a bounded metric space such that:*

i) $\mathcal{A}\left(M\right)$ *is compact.*

ii) (M,d) *has (p,q)-URNS for some $p>0$ and $q\in(0,1)$.*

Then (M,d) has the FPP for nonexpansive mappings.

Proof: Let (M,d) be as above and let $T:M\to M$ be a nonexpansive mapping.

In order to make the proof clearer, we will denote $\operatorname{diam}\left(A_n\right)$ and $\operatorname{diam}(\tilde{A}_n)$ by δ_n and $\tilde{\delta}_n$ respectively.

Now, take $A_0\in\mathcal{A}_T\left(M\right)$ resulting from letting $A=M$ in Lemma 2.3.2. If A_0 is a singleton then, we already have a fixed point of T.

If A_0 is not a singleton, since (M,d) has (p,q)-URNS we have that

$$B\left[A_0,p\delta_0\right]\cap B\left[B\left[A_0,p\delta_0\right],q\delta_0\right]\neq\emptyset.$$

Thus, defining $\tilde{A}_0:=\operatorname{cov}_T\left(B\left[A_0,p\delta_0\right]\cap B\left[B\left[A_0,p\delta_0\right],q\delta_0\right]\right)$ as in Lemma 2.3.3, we have that

$$\tilde{A}_0\in\mathcal{A}_T\left(M\right),\ \tilde{A}_0\subset B\left[A_0,p\delta_0\right]\text{ and }\tilde{\delta}_0\leq q\delta_0.$$

Now, taking $A=\tilde{A}_0$ and applying Lemma 2.3.2, we find $A_1\in\mathcal{A}_T\left(M\right)$ such that $A_1\subset\tilde{A}_0$ which implies that

$$A_1 \subset B\left[A_0, p\delta_0\right] \text{ and } \delta_1 \leq q\delta_0 .$$

If A_1 is a singleton then, we already have a fixed point of T.
If A_1 is not a singleton, since (M,d) has (p,q)-URNS we have that

$$B\left[A_1, p\delta_1\right] \cap B\left[B\left[A_1, p\delta_1\right], q\delta_0\right] \neq \emptyset.$$

Thus, defining $\tilde{A}_1 := \operatorname{cov}_T\left(B\left[A_1, p\delta_1\right] \cap B\left[B\left[A_1, p\delta_1\right], q\delta_1\right]\right)$ as in Lemma 2.3.3, we have that

$$\tilde{A}_1 \in \mathcal{A}_T(M),\ \tilde{A}_1 \subset B\left[A_1, p\delta_1\right] \text{ and } \tilde{\delta}_1 \leq q\delta_1.$$

Now, taking $A = \tilde{A}_1$ and applying Lemma 2.3.2, we find $A_2 \in \mathcal{A}_T(M)$ such that $A_2 \subset \tilde{A}_1$ which implies that

$$A_2 \subset B\left[A_1, p\delta_1\right] \text{ and } \delta_2 \leq q\delta_1.$$

Proceeding inductively, if there exists n_0 such that A_{n_0} is a singleton then, we already have a fixed of T. Otherwise, we can construct a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_T(M)$ having the following properties for every $n \in \mathbb{N}$:

(1) $A_n \subset B\left[A_{n-1}, p\delta_{n-1}\right]$;
(2) $\delta_n \leq q\delta_{n-1}$.

Assuming we have a sequence as above, for each $n \in \mathbb{N}$ we can choose $x_n \in A_n$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$. Then, given $n \in \mathbb{N}$, it follows from (2) that $\delta_n \leq q^n\delta_0$. Since (1) tells us that $A_{n+1} \subset B\left[A_n, p\delta_n\right]$, we have that $d(x_n, x_{n+1}) \leq p\delta_n \leq pq^n\delta_0$.

Thus, since $q \in (0,1)$, it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d). Moreover, since $\mathcal{A}(M)$ is compact it follows from Lemma 1.1.3 that (M,d) is complete and therefore, $(x_n)_{n\in\mathbb{N}}$ is convergent. Let $x = \lim_{n\to\infty} x_n$.

The triangle inequality and the nonexpansivity of T imply that for each $n \in \mathbb{N}$ we have that

$$\begin{aligned} d(Tx, x) \leq d(Tx, Tx_n) + d(Tx_n, x_n) + d(x_n, x) \leq \\ d(x, x_n) + d(Tx_n, x_n) + d(x_n, x) \leq \\ d(Tx_n, x_n) + 2d(x_n, x) . \end{aligned}$$

Since $x = \lim_{n\to\infty} x_n$, we have that $\lim_{n\to\infty} d(x_n, x) = 0$.
Since $x_n, Tx_n \in A_n$ for all $n \in \mathbb{N}$, it follows that

$$d(Tx_n, x_n) \leq \delta_n \leq q^n\delta_0$$

and therefore since $q \in (0,1)$, we have that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

Thus, $d(Tx, x) = 0$ which implies that x is a fixed point of T.
Hence, (M, d) has the FPP for nonexpansive mappings.

■

In the next theorem we will see that, given a hyperconvex metric space (M, d) and a metric d_1 equivalent to d which is "close enough" to it then, (M, d_1) has (p, q)-URNS.

Theorem 2.3.2. *Let (M, d) be a hyperconvex metric space and let d_1 be a metric equivalent to d such that*

$$ad(x, y) \leq d_1(x, y) \leq bd(x, y) \text{ for all } x, y \in M.$$

Then, for any $L \subset M$ with $\operatorname{diam}_{d_1}(L) > 0$ we have

$$B_{d_1}\left[L, \frac{b}{2a}\operatorname{diam}_{d_1}(L)\right] \cap B_{d_1}\left[B_{d_1}\left[L, \frac{b}{2a}\operatorname{diam}_{d_1}(L)\right], \frac{b^2}{2a^2}\operatorname{diam}_{d_1}(L)\right] \neq \emptyset.$$

In particular, if $\frac{b}{a} < \sqrt{2}$ then, (M, d_1) has $\left(\frac{b}{2a}, \frac{b^2}{2a^2}\right)$-URNS.

Proof: Let (M, d) and d_1 be as above.

Now, let $L \subset M$ be such that $\delta_1 = \operatorname{diam}_{d_1}(L) > 0$. We will begin by showing that

$$A = B_d\left[L, \frac{1}{2a}\delta_1\right] \cap B_d\left[B_{d_1}\left[L, \frac{b}{2a}\delta_1\right], \frac{b}{2a^2}\delta_1\right] \neq \emptyset.$$

Since A is an intersection of closed balls on the hyperconvex metric space (M, d), if we manage to check that for any given two balls $B(x, r_1)$ and $B(y, r_2)$ in the family of balls whose intersection equals to A, it is true that $d(x, y) \leq r_1 + r_2$ then, the hyperconvexity of (M, d) will imply that $A \neq \emptyset$.

It suffices to consider the following three cases:

- Case 1: $x, y \in L$. In this case we have that

$$d(x, y) \leq \frac{1}{a}d_1(x, y) \leq \frac{1}{a}\delta_1 = \frac{1}{2a}\delta_1 + \frac{1}{2a}\delta_1.$$

- Case 2: $x \in L$ and $y \in B_{d_1}\left[L, \frac{b}{2a}\delta_1\right]$. In this case we have that

$$d(x, y) \leq \frac{1}{a}d_1(x, y) \leq \frac{b}{2a^2}\delta_1 \leq \frac{b}{2a^2}\delta_1 + \frac{1}{2a}\delta_1.$$

- Case 3: $x, y \in B_{d_1}\left[L, \frac{b}{2a}\delta_1\right]$. In this case we have that for any $z \in L$,

$$d(x,y) \le \frac{1}{a} d_1(x,y) \le \frac{1}{a} d_1(x,z) + \frac{1}{a} d_1(z,y) \le \frac{b}{2a^2}\delta_1 + \frac{b}{2a^2}\delta_1.$$

Thus, as we mentioned before it follows from the hyperconvexity of (M,d) that $A \ne \emptyset$.

Now, from the relation between d and d_1 that $B_{d_1}(x, ar) \subset B_d(x,r) \subset B_{d_1}(x, br)$ for all $x \in M$ and $r > 0$ which implies that $B_{d_1}[P, ar] \subset B_d[P,r] \subset B_{d_1}[P, br]$ for all $P \subset M$ and $r > 0$. Thus, we have that

$$B_d\left[L, \frac{1}{2a}\delta_1\right] \subset B_{d_1}\left[L, \frac{b}{2a}\delta_1\right]$$

and

$$B_d\left[B_{d_1}\left[L, \frac{b}{2a}\delta_1\right], \frac{b}{2a^2}\delta_1\right] \subset B_{d_1}\left[B_{d_1}\left[L, \frac{b^2}{2a}\delta_1\right], \frac{b^2}{2a^2}\delta_1\right]$$

which gives us that

$$A \subset B_{d_1}\left[L, \frac{b}{2a}\delta_1\right] \cap B_{d_1}\left[B_{d_1}\left[L, \frac{b^2}{2a}\delta_1\right], \frac{b^2}{2a^2}\delta_1\right]$$

and therefore,

$$B_{d_1}\left[L, \frac{b}{2a}\delta_1\right] \cap B_{d_1}\left[B_{d_1}\left[L, \frac{b^2}{2a}\delta_1\right], \frac{b^2}{2a^2}\delta_1\right] \ne \emptyset.$$

Finally, if $\frac{b}{a} < \sqrt{2}$ we have that $\frac{b^2}{2a^2} < 1$ and then, it follows from what we have shown above that for each $L \in A_{d_1}(M)$ with $\operatorname{diam}_{d_1}(L) > 0$ we have that

$$B_{d_1}\left[L, \frac{b}{2a}\operatorname{diam}_{d_1}(L)\right] \cap B_{d_1}\left[B_{d_1}\left[L, \frac{b}{2a}\operatorname{diam}_{d_1}(L)\right], \frac{b^2}{2a^2}\operatorname{diam}_{d_1}(L)\right] \ne \emptyset.$$

Hence, (M, d_1) has $\left(\frac{b}{2a}, \frac{b^2}{2a^2}\right)$-URNS.

■

Now, we can show the following stability result regarding the FPP for nonexpansive mappings.

Corollary 2.3.1. *Let (M,d) be a bounded hyperconvex metric space and let d_1 be a metric equivalent to d such that $ad(x,y) \le d_1(x,y) \le bd(x,y)$ for all $x, y \in M$ and $\frac{b}{a} < \sqrt{2}$. If the family $\mathcal{A}_{d_1}(M)$ is compact then the metric space (M, d_1) has the FPP for nonexpasive mappings.*

Proof: It follows straight from Theorem 2.3.1 and Theorem 2.3.2.

■

Even though the compactness of $\mathcal{A}_{d_1}(M)$ must be required in Corollary 2.3.1, in the next section we will present several examples where this condition comes for free.

2.4 More examples of sets with the FPP for nonexpansive mappings

Example 2.4.1. *Let I be a nonempty index set and consider the hyperconvex metric space $(\ell_\infty(I), d_\infty)$. Let τ be the product topology on $\ell_\infty(I)$, which is the topology of the coordinatewise convergence.*

Observe that given a τ-closed and bounded set A, there exists a cartesian product of closed intervals B such that $A \subset B$. Moreover, it follows from Tychonoff's theorem that B is τ-compact and then, since A is τ-closed, we have that A is τ-compact. In resume, every τ-closed and bounded set is τ-compact.

Thus, if we let M be a τ-closed and bounded subset of $\ell_\infty(I)$ and consider the metric space (M, d_∞) then, by considering a metric d_1 on M which is equivalent to d_∞ and such that the closed balls of (M, d_1) are τ-closed we have that $\mathcal{A}_{d_1}(M)$ is compact.

Now, taking $(a_i)_{i\in I}, (b_i)_{i\in I} \in \ell_\infty(I)$ with $a_i \le b_i$ for all $i \in I$ and letting $M = \prod_{i\in I}[a_i, b_i]$, since M is an admissible subset of (ℓ_∞, d_∞), it follows from Proposition 1.5.3 that (M, d_∞) is a hyperconvex metric space.

Thus, for any metric d_1 on M for which it is true that

$$ad_\infty(x,y) \le d_1(x,y) \le bd_\infty(x,y)$$

for all $x, y \in M$ and such $\frac{b}{a} < \sqrt{2}$ and whose closed balls are coordinatewise closed, it follows from Corollary 2.3.1 that (M, d_1) has the FPP for nonexpansive mappings.

If we particularize to the case $I = \mathbb{N}$, then the coordinatewise topology on ℓ_∞ coincides with the $\sigma(\ell_\infty, \ell_1)$-topology for bounded sets of ℓ_∞. In case that a metric on ℓ_∞ comes from an equivalent norm $\|\cdot\|$, it is also worth noting that the the closed balls for the $\|\cdot\|$ norm are $\sigma(\ell_\infty, \ell_1)$-closed if and only if $\|\cdot\|$ is a dual norm, that is, there exists an equivalent norm $\|\cdot\|'$ on ℓ_1 for which the dual $(\ell_1, \|\cdot\|')^*$ is isometric to $(\ell_\infty, \|\cdot\|)$ (see for instance [21, Lemma 8.8]). This leads us to the following corollary.

Corollary 2.4.1. *Let M be a bounded subset of ℓ_∞. Assume that M is coordinatewise closed and $(M, \|\cdot\|_\infty)$ is hyperconvex. Let $\|\cdot\|$ be a dual norm on ℓ_∞ such that*

$$\|x\|_\infty \le \|x\| \le b\,\|x\|_\infty \text{ for all } x \in M.$$

Then $(M, \|\cdot\|)$ *has the FPP for nonexpansive mappings if* $b < \sqrt{2}$.

We next apply the above stability result to deduce the fulfillment of the FPP for some closed convex bounded subsets of ℓ_∞ endowed with a Luxemburg norm.

In case the sequence $(p_n)_{n\in\mathbb{N}}$ is unbounded, $\left(\ell_{p_n}, \|\cdot\|_{p_n}\right)$ contains an isometric copy of ℓ_∞ so, once more, every failure of the FPP has its own isometric reflection in $\left(\ell_{p_n}, \|\cdot\|_{p_n}\right)$ (see for instance [13, Theorem 2.3]). We will next prove that, under some additional restriction over the divergence of the sequence $(p_n)_{n\in\mathbb{N}}$, we can still prove that there are some closed convex bounded sets with the FPP for the Luxemburg norm.

Example 2.4.2. *Let* $(p_n)_{n\in\mathbb{N}}$ *be an unbounded sequence in* $[1, +\infty)$ *such that there exists* $b \in \left(1, \sqrt{2}\right)$ *for which* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{b}\right)^{p_n} \le 1$. *Let* M *be a bounded subset of* ℓ_∞ *with* $(M, \|\cdot\|_\infty)$ *hyperconvex. Then* $\left(M, \|\cdot\|_{p_n}\right)$ *has the FPP.*

Proof: Let $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$. If x is the sequence whose all entries are 0 we have that $\|x\|_{p_n} = 0 = \|x\|_\infty$. If x is not that sequence then, under the assumptions on b, we have that

$$\sum_{n=1}^{\infty}\left|\frac{x_n}{b\,\|x\|_\infty}\right|^{p_n} \le \sum_{n=1}^{\infty}\left(\frac{1}{b}\right)^{p_n} \le 1.$$

Thus, it follows from the definition of $\|\cdot\|_{p_n}$ that $\|x\|_{p_n} \le b\,\|x\|_\infty$.

Moreover, since $(p_n)_{n\in\mathbb{N}}$ is unbounded, it follows that the identity mapping is an isomorphism from ℓ_∞ into ℓ_{p_n}, which implies that $\|x\|_\infty \le \|x\|_{p_n}$ for all $x \in \ell_\infty$. Therefore,

$$\|x\|_\infty \le \|x\|_{p_n} \le b\,\|x\|_\infty$$

for all $x \in \ell_\infty$.

Furthermore, the closed unit balls for the norm $\|\cdot\|_{p_n}$ are $\sigma(\ell_\infty, \ell_1)$-closed, since $\|\cdot\|_{p_n}$ is the dual norm of the corresponding Orlicz space ℓ_{q_n} with $\dfrac{1}{p_n} + \dfrac{1}{q_n} = 1$ endowed with the Luxemburg norm. Applying Corollary 2.4.1 we obtain the result.

■

From what we have previously seen, we can go further in the study of the FPP for bounded non-weakly compact subsets of c as we can see below in the final example of this chapter.

Example 2.4.3. *Consider again the set below which was introduced in [22]*

$$W = \{x = (x_n) \in \ell_\infty : 1 \geq x_1 \geq x_2 \geq \ldots \geq 0\}.$$

From Corollary 2.3.1 and Example 2.4.1, we know that given a metric d for which there exists $1 < a < \sqrt{2}$ such that

$$d_\infty(x,y) \leq d(x,y) \leq a d_\infty(x,y)$$

for all $x, y \in W$ and for which the closed d-balls are coordinatewise closed in ℓ_∞, then (W, d) has the FPP. In particular, $\left(W, \|\cdot\|_{p_n}\right)$ has the FPP where $(p_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Example 2.4.2.

Chapter 3

Fixed points and common fixed for orbit-nonexpansive mappings in metric spaces

In this chapter we will present some recent results revolving around the FPP for orbit-nonexpansive mappings and common fixed points for families of mappings which satisfy some kind of orbit-nonexpansivity condition. Most of the results presented in this chapter can be found in the article [20] by Rafael Espínola García, María Japón and myself.

In the first section we will present the concept of orbit of a self-mapping and the class of orbit-nonexpansive mappings over a metric space which will be shown to contain the class of nonexpansive mappings. Some properties of orbit-nonexpansive mappings and examples of such mappings will be presented. From the examples we will see that, unlike nonexpansive mappings, orbit-nonexpansive mappings need not be continuous.

In the second section we introduce the notion of a family of interlaced orbit-nonexpansive mappings and study how the notion of normal structure leads us to the existence of common fixed points for such families. We also define what it means for a group of self-mappings to act on a metric space. Moreover, we will show that our result applies to the class of orbit-nonexpansive mappings and to a group of orbit-nonexpansive mappings acting on a metric space therefore, extending previous results found in the literature.

In the third section we will study how the concept of (p, q)-URNS introduced in Section 2.3 implies the existence of fixed points for families of interlaced orbit-nonexpansive mappings and therefore, for orbit-nonexpansive mappings. Once more we will show that our result extends previous results found in the literature.

3.1 Orbit-nonexpasive mappings

Definition 3.1.1. *Let T be a self-mapping on a metric space (M,d). Given $x \in M$ we define the orbit of x with respect to T as the set*

$$O_T(x) := \{x\} \cup \{T^n x : n \geq 1\} = \{T^n x : n \geq 0\}$$

where we consider T^0 to be the identity mapping on M.

The following definition was first introduced by A. Nicolae in [42] under the name of **nonexpansive with respect to orbits**, (see [1] for example).

Definition 3.1.2. *A self-mapping $T : M \to M$ defined on a metric space (M,d) is said to be **orbit-nonexpansive** if*

$$d(Tx, Ty) \leq D(x, O_T(y)) \text{ for all } x, y \in M.$$

We chose the name orbit-nonexpansive because it is shorter which implies in shorter statements. It is clear that every nonexpansive mapping defined over a metric space (M,d) is orbit-nonexpasive since $y \in O_T(y)$ for all $y \in M$. The following proposition shows us an interesting property of orbit-nonexpansive mappings.

Proposition 3.1.1. *Let $T : M \to M$ be an orbit-nonexpansive mapping over a metric space (M,d). Then, for every $n \in \mathbb{N}$ we have that*

$$d(T^n x, T^n y) \leq D(x, O_T(y)) \text{ for all } x, y \in M. \tag{$*$}$$

Proof: Let T be as above. Since T is orbit-nonexpansive we have that

$$d(Tx, Ty) \leq D(x, O_T(y)) \text{ for all } x, y \in M$$

that is, ($*$) holds for $n = 1$.

Now, suppose that for some integer $k \geq 1$ we have that

$$d\left(T^k x, T^k y\right) \leq D(x, O_T(y)) \text{ for all } x, y \in M.$$

Observe first that

$$d\left(T^{k+1}x, T^{k+1}y\right) = d\left(TT^k x, TT^k y\right) \leq D\left(T^k x, O_T\left(T^k y\right)\right) =$$

$$\sup_{m \in \mathbb{N} \cup \{0\}} \left\{d\left(T^k x, T^m T^k y\right)\right\} = \sup_{m \in \mathbb{N} \cup \{0\}} \left\{d\left(T^k x, T^k T^m y\right)\right\}$$

for all $x, y \in M$.

Now, given $x, y \in M$ and $m \in \mathbb{N} \cup \{0\}$ it follows from our hypothesis that

$$d\left(T^k x, T^k T^m y\right) \leq D(x, O_T(T^m y)).$$

Moreoever, we have that

$$D\left(x,O_T\left(T^m y\right)\right)=\sup_{n\in\mathbb{N}\cup\{0\}}\left\{d\left(x,T^nT^m y\right)\right\}=\sup_{n\geq m}\left\{d\left(x,T^n y\right)\right\}\leq D\left(x,O_T\left(y\right)\right).$$

Thus, $\sup_{m\in\mathbb{N}\cup\{0\}}\left\{d\left(T^k x,T^kT^m y\right)\right\}\leq D\left(x,O_T\left(y\right)\right)$ for all $x,y\in M$ and for all $m\in\mathbb{N}\cup\{0\}$ and therefore, $d\left(T^{k+1}x,T^{k+1}y\right)\leq D\left(x,O_T\left(y\right)\right)$ for all $x,y\in M$.

Hence, $(*)$ holds for all $n\in\mathbb{N}$.

■

Given a bounded metric space (M,d) and T a nonexpansive mapping with respect to orbits over M, it is obvious that the orbits of all points under T are bounded. The following proposition tells us that on a generic metric space (possibly unbounded), given a mapping T which is orbit-nonexpansive, either the orbits of all points under T are bounded or the orbits of all points under T are unbounded.

Proposition 3.1.2. *Let T be an orbit-nonexpansive mapping over a metric space (M,d). Then, $O_T\left(x\right)$ is bounded for some $x\in M$ if and only if $O_T\left(y\right)$ is bounded for all $y\in M$.*

Proof: Let T be as above.

$(\Longrightarrow)$ Suppose there exists $x\in M$ such that $O_T\left(x\right)$ is bounded and let $\delta=\operatorname{diam}\left(O_T\left(x\right)\right)$. Given $y\in M$, we have that

$$d\left(x,T^n y\right)\leq d\left(x,T^n x\right)+d\left(T^n x,T^n y\right)\ \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

Since $T^n x\in O_T\left(x\right)$ for all $n\in\mathbb{N}\cup\{0\}$ we have that

$$d\left(x,T^n x\right)\leq\delta \text{ and } d\left(y,T^n x\right)\leq d\left(y,x\right)+d\left(x,T^n x\right)\leq d\left(y,x\right)+\delta$$

for all $n\in\mathbb{N}\cup\{0\}$. In particular, we have that $D\left(y,O_T\left(x\right)\right)\leq d\left(y,x\right)+\delta$. Also, we know that $d\left(x,y\right)\leq D\left(y,O_T\left(x\right)\right)$ and it follows from Proposition 3.1 that $d\left(T^n x,T^n y\right)=d\left(T^n y,T^n x\right)\leq D\left(y,O_T\left(x\right)\right)$ for all $n\in\mathbb{N}$. So, it follows from what we have seen above, that $d\left(T^n x,T^n y\right)\leq d\left(y,x\right)+\delta$.

Therefore, $d\left(x,T^n y\right)\leq\delta+D\left(y,O_T\left(x\right)\right)\leq d\left(y,x\right)+\delta$ for all $n\in\mathbb{N}\cup\{0\}$ which implies that $O_T\left(y\right)\subset B\left(x,d\left(y,x\right)+\delta\right)$.

Hence, $O_T\left(y\right)$ is bounded for all $y\in M$.

$(\Longleftarrow)$ If $O_T\left(y\right)$ is bounded for all $y\in M$ then, in particular $O_T\left(x\right)$ is bounded for some $x\in M$.

■

It is worth noticing that if (M,d) is an unbounded metric space and T is a self-mapping over M such that $O_T\left(x\right)$ is unbounded for all $x\in M$

then, given any $x, y \in M$ we have that $D(x, O_T(y)) = +\infty$ which implies that $d(Tx, Ty) \leq D(x, O_T(y))$ and therefore, T is orbit-nonexpansive. In this case, T might not have any fixed point as we can see looking at the translations in $\mathbb{R}^n$ presented in Example 1.2.3.

The next example, due to S.Prus, shows that even if T is an orbit-nonexpansive mapping whose all orbits are bounded, the existence of fixed points is not guaranteed.

Example 3.1.1. *Consider the mapping $T : (\ell_\infty, d_\infty) \to (\ell_\infty, d_\infty)$ given by*

$$Tx = (1 + \limsup x_n, x_1, x_2, \ldots, x_n, \ldots) \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty.$$

First, recall that for any $z = (z_n)_{n\in\mathbb{N}}$ and $w = (w_n)_{n\in\mathbb{N}}$ we have that

$$\limsup z_n + \liminf w_n \leq \limsup (z_n + w_n)$$

which tells us that for any $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ in ℓ_∞ we have that

$$\limsup x_n - \limsup y_n = \limsup x_n + \liminf(-y_n) \leq \limsup(x_n - y_n) \leq$$

$$\sup_{n\in\mathbb{N}} \{|x_n - y_n|\} = d_\infty(x, y) = \|x - y\|_\infty$$

and

$$\limsup y_n - \limsup x_n = \limsup y_n + \liminf(-x_n) \leq \limsup(y_n - x_n) \leq$$

$$\sup_{n\in\mathbb{N}} \{|x_n - y_n|\} = d_\infty(x, y) = \|x - y\|_\infty$$

which implies that

$$|1 + \limsup x_n - (1 + \limsup y_n)| = |\limsup x_n - \limsup y_n| \leq d_\infty(x, y) =$$

$$\|x - y\|_\infty$$

Hence,

$$d_\infty(Tx, Ty) = \|Tx - Ty\|_\infty =$$

$$\sup(\{|\limsup x_n - \limsup y_n|\} \cup \{|x_m - y_m| : m \in \mathbb{N}\}) =$$

$$\sup_{m\in\mathbb{N}} \{|x_m - y_m|\} = d_\infty(x, y) = \|x - y\|_\infty$$

for all $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ in ℓ_∞ and therefore, T is an isometry which tells us in particular that T is orbit-nonexpansive.

Since T is an isometry, given $x, y \in \ell_\infty$ we have that $\|T^m x - T^m y\|_\infty = \|x - y\|_\infty$ for all $m \in \mathbb{N}$. In particular, if we let $\boldsymbol{0}$ denote the zero vector of ℓ_∞ we have that $\|T^m x - T^m(\boldsymbol{0})\|_\infty = \|x - \boldsymbol{0}\|_\infty = \|x\|_\infty$ for all $x \in \ell_\infty$ and for all $m \in \mathbb{N}$ which implies that

$$\|T^m x - T(\boldsymbol{0})\|_\infty \leq \|T^m x - T^m(\boldsymbol{0})\|_\infty + \|T^m(\boldsymbol{0}) - T(\boldsymbol{0})\|_\infty =$$

$$\|x\|_\infty + \|T^m(\boldsymbol{0}) - T(\boldsymbol{0})\|_\infty$$

for all $x \in \ell_\infty$ *and for all* $m \in \mathbb{N}$.

Now, observe that $T(\boldsymbol{0}) = (1, 0, 0, 0, \ldots)$ *which implies that*

$$T^2(\boldsymbol{0}) = TT(\boldsymbol{0}) = T(1, 0, 0, \ldots) = (1, 1, 0, 0 \ldots)$$

which implies that

$$T^3(\boldsymbol{0}) = TT^2(\boldsymbol{0}) = T(1, 1, 0, 0 \ldots) = (1, 1, 1, 0 \ldots).$$

Proceeding inductively, we can show that $T^m(\boldsymbol{0}) = \left(\underbrace{1, 1, 1, 1, \ldots, 1}_{\text{first } m \text{ coordinates}}, 0, 0, \ldots,\right)$ *for all* $m \in \mathbb{N}$ *and therefore,*

$$\|T^m(\boldsymbol{0}) - T(\boldsymbol{0})\|_\infty = \left\| \left(\underbrace{0, 1, 1, 1, \ldots, 1}_{\text{first } m \text{ coordinates}}, 0, 0, \ldots,\right) \right\|_\infty = 1$$

for all $m \in \mathbb{N}$.

Thus,

$$\|T^m x - T(\boldsymbol{0})\|_\infty \leq \|x\|_\infty + 1$$

for all $x \in \ell_\infty$ *and for all* $m \in \mathbb{N}$ *and then, since*

$$\left\|T^0 x\right\|_\infty = \|x\|_\infty \leq \|x\|_\infty + 1,$$

it follows that $O_T(x) \subset B(T(\boldsymbol{0}), \|x\|_\infty + 1)$ *for all* $x \in \ell_\infty$.

Hence, $O_T(x)$ *is bounded for all* $x \in \ell_\infty$.

Now, observe that if there were $x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty$ *such that*

$$(1 + \limsup x_n, x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_2, x_3, \ldots, x_n, \ldots)$$

then, on one hand we would have $x_1 = 1 + \limsup x_n$ *and on the other hand we would have* $x_1 = x_2 = x_3 = \ldots = x_n = \ldots$ *The previous sequence of equalities implies that* $\limsup x_n = x_1$. *Thus, since the equation* $x_1 = 1 + x_1$ *has no real solution, we obtain a contradiction. Therefore,* T *has no fixed points.*

Since having all orbits bounded is not enough to ensure the existence of fixed points for orbit-nonexpansive mappings, we will need to look for other properties which we can impose on the metric space in order to guarantee the existence of such points. This will be done in the next section.

It is easy to see that nonexpansivity implies (uniform) continuity. The following two examples show that being orbit-nonexpansive does not require continuity. The first example can be found in [42] (Example 5.2) and the second one can be found [20] (Example 2.2). Both examples will be defined over a closed interval where the metric considered is the one induced by the standard metric of $\mathbb{R}$.

Example 3.1.2. *Let $T:[0,1]\to[0,1]$ be the mapping defined by*

$$Tx=\begin{cases} \dfrac{x}{4} & if \quad x\in\left[0,\dfrac{1}{2}\right), \\ \dfrac{x}{2} & if \quad x\in\left[\dfrac{1}{2},1\right]. \end{cases}$$

The proof that the mapping in the previous example is orbit-nonexpansive can be found in [42]. In the next example we present a mapping that is orbit-nonexpansive and for which it is not possible to find a nondegenerate closed T-invariant interval upon which the mapping T is continuous. The proof below follows the same ideas the one we have just mentioned.

Example 3.1.3. *Let $T:[-1,1]\to[-1,1]$ be the mapping defined by*

$$Tx=\begin{cases} \dfrac{x}{3} & if \quad x\in[-1,1]\setminus\mathbb{Q}, \\ -\dfrac{x}{3} & if \quad x\in\mathbb{Q}\cap[-1,1]. \end{cases}$$

There are several cases to study.

Case 1: *$x,y\in[-1,1]\setminus\mathbb{Q}$. In this case we have that*

$$d(Tx,Ty)=\left|\frac{x}{3}-\frac{y}{3}\right|=\frac{1}{3}|x-y|\le|x-y|\le D(x,O_T(y)).$$

Case 2: *$x,y\in\mathbb{Q}\cap[-1,1]$. In this case we have that*

$$d(Tx,Ty)=\left|-\frac{x}{3}-\left(-\frac{y}{3}\right)\right|=\frac{1}{3}|x-y|\le|x-y|\le D(x,O_T(y)).$$

Case 3: *$x\in[-1,1]\setminus\mathbb{Q}$ and $y\in\mathbb{Q}\cap[-1,1]$. In this case we have that*

$$d(Tx,Ty)=\left|\frac{x}{3}-\left(-\frac{y}{3}\right)\right|=\left|\frac{x}{3}+\frac{y}{3}\right| \quad and$$

$$D(x,O_T(y))=\sup\left\{|x-y|,\left|x+\frac{y}{3}\right|,\left|x-\frac{y}{9}\right|,\left|x+\frac{y}{27}\right|,\dots\right\}.$$

If $x > 0$ and $y \geq 0$ or $x < 0$ and $y \leq 0$ then,

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq \left|x + \frac{y}{3}\right| \leq D\left(x, O_T\left(y\right)\right).$$

If $y < 0 < x$ or $x < 0 \leq y$ then,

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq |x - y| \leq D\left(x, O_T\left(y\right)\right).$$

Case 4: *$x \in \mathbb{Q} \cap [-1, 1]$ and $y \in [-1, 1] \setminus \mathbb{Q}$. In this case we have that*

$$d\left(Tx, Ty\right) = \left|-\frac{x}{3} - \frac{y}{3}\right| = \left|\frac{x}{3} + \frac{y}{3}\right| \quad \textit{and}$$

$$D\left(x, O_T\left(y\right)\right) = \sup\left\{|x - y|, \left|x - \frac{y}{3}\right|, \left|x - \frac{y}{9}\right|, \left|x - \frac{y}{27}\right|, \ldots\right\}.$$

If $x = 0$ then, for all $y \in [-1, 1]$ we have that

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq |y| = D\left(x, O_T\left(y\right)\right).$$

If $0 < y \leq 2x$ or $2x \leq y < 0$ then,

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq |x| = D\left(x, O_T\left(y\right)\right).$$

If $0 < 2x \leq y$ or $y \leq 2x < 0$ then,

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq |x - y| = D\left(x, O_T\left(y\right)\right).$$

If $y < 0 < x$ or $x < 0 < y$ then,

$$\left|\frac{x}{3} + \frac{y}{3}\right| \leq \left|x - \frac{y}{3}\right| \leq D\left(x, O_T\left(y\right)\right).$$

Hence, $d\left(Tx, Ty\right) \leq D\left(x, O_T\left(y\right)\right)$ for all $x, y \in [-1, 1]$ and therefore, T is orbit-nonexpansive. Moreover, T fails to be nonexpansive on any T-invariant nondegenerate closed interval of $[-1, 1]$, what easily follows from the fact that T is discontinuous everywhere except for the origin, which is fixed for T.

Even though orbit-nonexpansive mappings need not be continuous, we still have the following proposition.

Proposition 3.1.3. *Let T be an orbit-nonexpansive mapping over a metric space (M, d) such that $Fix(T) \neq \emptyset$. Then, T is continuous at every $x \in Fix(T)$.*

Proof: Let T be as above and take $x_0 \in \operatorname{Fix}(T)$.

Since $Tx_0 = x_0$ it follows that $T^n x_0 = x_0$ for all $n \in \mathbb{N}$ which implies that $O_T(x_0) = \{x_0\}$ and therefore, $D(x, O_T(x_0)) = d(x, x_0)$ for all $x \in M$. In particular, since T is orbit-nonexpansive, we have that

$$d(Tx, Tx_0) \leq D(x, O_T(x_0)) = d(x, x_0)$$

for all $x \in M$.

Thus, for any given $\epsilon > 0$ if $x \in M$ is such that $d(x, x_0) < \epsilon$ then, it follows that $d(Tx, Tx_0) < \epsilon$ and therefore, T is continuous at x_0.

Hence, T is continuous at every $x \in \operatorname{Fix}(T)$.

■

Given a metric space (M, d), it is easy to see that besides containing the class of nonexpansive mappings, the class of orbit-nonexpansive also contains the class of self-mappings which satisfy $d(Tx, Ty) \leq \max\{d(x, y), d(x, Ty)\}$ for all $x, y \in M$. We finish this section by presenting the widely studied class of mean nonexpansive mappings (introduced in [47]) and showing that this class is also contained in the class of orbit-nonexpasive mappings. For more on this class of mappings see also [46] and [48].

Definition 3.1.3. *A self-mapping $T : M \to M$ defined on a metric space (M, d) is said to be* ***mean nonexpansive*** *if there exist $a \geq 0$ and $b \geq 0$ with $a + b \leq 1$ such that*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Ty) \ \textit{ for all } x, y \in M.$$

Proposition 3.1.4. *If T is a mean nonexpansive mapping over a metric space (M, d) then T is orbit-nonexpansive.*

Proof: Let T be as above. Then, there exist $a \geq 0$ and $b \geq 0$ with $a + b \leq 1$ such that

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Ty) \ \text{ for all } x, y \in M$$

which implies that

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Ty) \leq aD(x, O_T(y)) + bD(x, O_T(y)) =$$

$$(a + b) D(x, O_T(y)) \leq D(x, O_T(y))$$

for all $x, y \in M$ and therefore, T is orbit-nonexpansive.

■

3.2 Common fixed points of families of interlaced orbit-nonexpansive mappings

In this section we introduce the notion of a family of interlaced orbit-nonexpansive mappings and study how the notion of normal structure introduced in Definition 1.3.1 leads us to achieve positive results concerning the existence of common fixed points for such families. We will then show that our fixed point result not only applies to the class of orbit-nonexpansive mappings but also to a group of orbit-nonexpansive mappings acting on a metric space. Common fixed point results for a commutative family of orbit-nonexpansive mappings will also be obtained.

Definition 3.2.1. *Let (M, d) be a metric space and let $\mathcal{F}$ be a family of self-mappings on M. We say that $\mathcal{F}$ is* ***a family of interlaced orbit-nonexpansive mappings*** *if*

$$d\,(Tx, Sy) \leq \sup\left\{D\,(x, O_R\,(y)) : R \in \mathcal{F}\right\}$$

for all $S, T \in \mathcal{F}$ and for all $x, y \in M$.

The next proposition shows us that orbit-nonexpansive mappings are closely related to the previous definition.

Proposition 3.2.1. *Let T be an orbit-nonexpansive mapping over a metric space (M, d). Then, $\mathcal{F} = \{T\}$ is a family of interlaced orbit-nonexpansive mappings.*

Proof: Let T be as above and let $\mathcal{F} = \{T\}$. Since T is orbit-nonexpansive, we have that

$$d\,(Tx, Ty) \leq D\,(x, O_T\,(y))$$

for all $x, y \in M$.

Now, since $\mathcal{F}$ has T as its single element, given $x, y \in M$ it follows that

$$\sup\left\{D\,(x, O_R\,(y)) : R \in \mathcal{F}\right\} = D\,(x, O_T\,(y))$$

Hence, $\mathcal{F}$ is a family of interlaced orbit-nonexpansive mappings.

■

Before we present our next proposition we will need a few definitions.

Definition 3.2.2. *A set $\mathcal{G}$ together with a binary operation $*$ is said to be* ***group*** *if the following requirements are satisfied:*

i) For all $f, g, h \in \mathcal{G}$, we have that $(f * g) * h = f * (g * h)$;

ii) There exists a unique $e \in \mathcal{G}$ such that $e * g = g = g * e$ for all $g \in \mathcal{G}$ and this unique e is called ***the identity element*** *of the group.*

iii) *For every $g \in \mathcal{G}$ there exists a unique $h \in \mathcal{G}$ such that $g * h = e = h * g$ and for each g this unique h is denoted by g^{-1} and it is called **the inverse** of g.*

Example 3.2.1. *Let M be a set and let $\mathcal{G}$ be the set of all bijections defined on M. Then, the set $(\mathcal{G}, \circ)$ is a group where $\circ$ is the composition of mappings. The identity element of this group is the identity mapping on M and for each $g \in \mathcal{G}$ the inverse of g is the inverse mapping of g.*

Next we define the action of a group over a set.

Definition 3.2.3. *Let M be a set and $(\mathcal{G}, *)$ a group. A mapping $\varphi : \mathcal{G} \times M \to M$ is said to be a (left) **group action** on M if it satisfies the following requirements:*

i) *$\varphi(e, x) = x$ for all $x \in M$, where e is the identity element of $(\mathcal{G}, *)$;*

ii) *$\varphi(f, \varphi(g, x)) = \varphi(f * g, x)$ for all $f, g \in \mathcal{G}$ and for all $x \in M$.*

When we have M, $(\mathcal{G}, *)$ and φ as above we say simply say that $(\mathcal{G}, *)$ acts on M.

Whenever there is no risk of confusion, we will drop the symbol of the binary operation and write ab instead of $a * b$. We will also write $f(x)$ or fx to denote $\varphi(f, x)$. This is precisely what we have been doing when working with iterates of a self-mapping T.

Observe that if a group $(\mathcal{G}, *)$ acts on a set M then, for each fixed $g \in \mathcal{G}$ the mapping

$$\begin{aligned} \varphi_g : M &\to M \\ x &\mapsto gx \end{aligned}$$

is a bijection. Also, $\varphi_{fg} = \varphi_f \circ \varphi_g$ for all $f, g \in \mathcal{G}$ and if e is the identity element of $(\mathcal{G}, *)$ then, φ_e is the identity mapping on M.

Example 3.2.2. *Let M be a set and let $(\mathcal{G}, \circ)$ be a group of mappings defined on M, where again $\circ$ is the composition of mappings. Then,*

$$\begin{aligned} \varphi : \mathcal{G} \times M &\to M \\ (g, x) &\mapsto g(x) \end{aligned}$$

is an action of $(\mathcal{G}, \circ)$ on M. Moreover, for each $g \in \mathcal{G}$ we have that $\varphi_g = g$.

Whenever we say that a group $(\mathcal{G}, \circ)$ of mappings defined on a set M acts on M, we mean the action presented in the previous example.

Now, we can finally present our next proposition.

Proposition 3.2.2. *Let (M, d) be a bounded metric space and let $(\mathcal{G}, \circ)$ be a group of self-mappings acting on M, each of which is orbit-nonexpansive. Then, the family G is interlaced orbit-nonexpansive.*

Proof: Let (M, d) and $(\mathcal{G}, \circ)$ be as above.

Given $f, g \in \mathcal{G}$, since $(\mathcal{G}, \circ)$ is a group we have that

$$g = eg = \left(ff^{-1}\right) g = f\left(f^{-1}g\right)$$

and then since f is orbit-nonexpansive it follows that

$$d\left(fx, gy\right) = d\left(fx, f\left(f^{-1}g\right) y\right) \leq D\left(x, O_f\left(f^{-1}gy\right)\right)$$

for all $x, y \in M$.

Once more, since $(\mathcal{G}, \circ)$ is a group it follows that $f^{-1}g \in \mathcal{G}$ which implies that for any $x, y \in M$ we have that

$$D\left(x, O_f\left(f^{-1}gy\right)\right) \leq \sup\left\{D\left(x, O_h\left(y\right)\right) : h \in \mathcal{G}\right\}.$$

Hence, $\mathcal{G}$ is a family of interlaced orbit-nonexpansive mappings.

■

Before we present the main result of this section we must recall the definition of a one-local retract of a metric space.

Definition 3.2.4. *A subset D of a metric space (M, d) is a* ***one-local retract*** *of M if, for any family of closed balls $\mathcal{B} = \{B\left(x_i, r_i\right)\}_{i \in I}$ whose centers are in D and such that $\bigcap_{i \in I} B\left(x_i, r_i\right) \neq \emptyset$ it is the case that*

$$D \cap \bigcap_{i \in I} B\left(x_i, r_i\right) \neq \emptyset.$$

More details about one-local retracts can be found in [31], where it is shown that they can be considered as a generalization of nonexpansive retracts, enjoying some structural properties of great interest.

Theorem 3.2.1. *Let (M, d) be a bounded metric space which has normal structure and such that $\mathcal{A}(M)$ is compact. Let $\mathcal{F}$ be a family of interlaced orbit-nonexpansive self-mappings on M. Then, there exists a common fixed point to all mappings in $\mathcal{F}$. Moreover, the common fixed point set of $\mathcal{F}$, $Fix(\mathcal{F})$, is a one-local retract of M.*

Proof: Let (M, d) and $\mathcal{F}$ be as above.

Since $\mathcal{A}(M)$ is compact, it follows from Lemma 1.1.3 that we can take a minimal element A_0 of $\mathcal{A}_{\mathcal{F}}(M)$ with respect to set inclusion.

Since $T(A_0) \subset A_0$ for all $T \in \mathcal{F}$ and $A_0 \in \mathcal{A}(M)$ we have that

$$\operatorname{cov}\left(\bigcup_{T \in \mathcal{F}} T\left(A_0\right)\right) \subset A_0$$

which gives us that $S\left(\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\right)\right)\subset S\left(A_0\right)\subset\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\right)$ for all $S\in\mathcal{F}$ and therefore, $\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\right)\in\mathcal{A}_{\mathcal{F}}\left(M\right)$. Thus, it follows from the minimality of A_0 that

$$A_0=\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\right).$$

Since $A_0\neq\emptyset$, there exists $x_0\in A_0$. We affirm that $A_0=\{x_0\}$. Suppose that A_0 has more than one element. Then, it follows that $0<\operatorname{diam}\left(A_0\right)<+\infty$ and therefore, since (M,d) has normal structure, it follows from Proposition 1.3.2 that there exists $0<r<\operatorname{diam}\left(A_0\right)$ such that $A_0\cap B\left[A_0,r\right]\neq\emptyset$. Now, take $a_0\in A_0\cap B\left[A_0,r\right]$ and fix $S\in\mathcal{F}$. Since $T\left(A_0\right)\subset A_0$ for all $T\in\mathcal{F}$ and $a_0\in B\left[A_0,r\right]$, we have that $d\left(a_0,a\right)\leq r$ and $d\left(a_0,T^na\right)\leq r$ for all $a\in A_0$, for all $T\in\mathcal{F}$ and for all $n\in\mathbb{N}$ which gives us that

$$D\left(a_0,O_T\left(a\right)\right)=\sup_{n\in\mathbb{N}\cup\{0\}}\left\{d\left(a_0,T^na\right)\right\}\leq r$$

for all $a\in A_0$ and for all $T\in\mathcal{F}$.

Hence, since $\mathcal{F}$ is a family of interlaced orbit-nonexpansive self-mappings on M, it follows that for all $a\in A_0$ and for all $T\in\mathcal{F}$ we have that

$$d\left(Sa_0,Ta\right)\leq\sup\left\{D\left(a_0,O_T\left(a\right)\right):T\in\mathcal{F}\right\}\leq r.$$

Thus, for every fixed $T\in\mathcal{F}$ we have that $Ta\in B\left(Sa_0,r\right)$ for all $a\in A_0$ which implies that $T\left(A_0\right)\subset B\left(Sa_0,r\right)$ for all $T\in\mathcal{F}$ and therefore, $\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\subset B\left(Sa_0,r\right)$. Since $B\left(Sa_0,r\right)\in\mathcal{A}\left(M\right)$, what we have just shown implies that

$$A_0=\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}}T\left(A_0\right)\right)\subset B\left(Sa_0,r\right).$$

Since S was an arbitrary element of $\mathcal{F}$ we have that $A_0\subset B\left(Sa_0,r\right)$ for all $S\in\mathcal{F}$ which implies that $Sa_0\in B\left[A_0,r\right]$ for all $S\in\mathcal{F}$. Also, since $S\left(A_0\right)\subset A_0$ for all $S\in\mathcal{F}$ and $a_0\in A_0$, we have that $Sa_0\in A_0\cap B\left[A_0,r\right]$ for all $S\in\mathcal{F}$. Thus, since a_0 was an arbitrary element in $A_0\cap B\left[A_0,r\right]$, we have that

$$S\left(A_0\cap B\left[A_0,r\right]\right)\subset A_0\cap B\left[A_0,r\right]$$

for all $S\in\mathcal{F}$ and therefore, $A_0\cap B\left[A_0,r\right]\in\mathcal{A}_{\mathcal{F}}\left(M\right)$.

Finally, observe that given any two elements $x,y\in A_0\cap B\left[A_0,r\right]$, since $x\in A_0$ and $y\in B\left[A_0,r\right]$ we have that $d\left(x,y\right)\leq r<\operatorname{diam}\left(A_0\right)$ which implies that $\operatorname{diam}(A_0 \cap B[A_0, r]) < \operatorname{diam}(A_0)$ and therefore, $A_0 \cap B[A_0, r]$ is a proper subset of A_0 which contradicts the minimality of A_0.

Thus, $A_0 = \{x_0\}$ and therefore, since $T(A_0) \subset A_0$ for all $T \in \mathcal{F}$, if follows that x_0 is a common fixed point for all $T \in \mathcal{F}$.

In order to complete the proof, we next show that $\operatorname{Fix}(\mathcal{F})$ is a one-local retract of M. Indeed, let $\{B(x_i, r_i)\}_{i \in I}$ be a family of closed balls where $x_i \in \operatorname{Fix}(\mathcal{F})$ for all $i \in I$ and such that $B = \bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$.

Since B is admissible, Proposition 1.1.4 tells us that $\mathcal{A}(B)$ is compact and Proposition 1.3.1 tell us that $(B, d_{|B})$ has normal structure.

Now, observe that given $y \in B$, for each $i \in I$ and for each $T \in \mathcal{F}$ we have that

$$d(Ty, x_i) = d(Ty, Tx_i) \leq \sup\{D(y, O_S(x_i)) : S \in \mathcal{F}\} = d(y, x_i) \leq r_i.$$

which implies that $T(B) \subset B$ for all $T \in \mathcal{F}$.

Applying the first part of this proof to the metric space $(B, d_{|B})$ we have that there exists $x_0 \in B$ such that $T(x_0) = x_0$ for all $T \in \mathcal{F}$, which shows that $\operatorname{Fix}(\mathcal{F}) \cap B \neq \emptyset$.

Hence, $\mathcal{F} \cap B \neq \emptyset$ and therefore, $\mathcal{F}$ is a one-local retract of M.

■

If we put Theorem 3.2.1 together with Proposition 3.2.2, we obtain the next common fixed point result for the action of a group. The result seems to be unknown even for the case of nonexpansivity.

Corollary 3.2.1. *Let (M, d) be a bounded metric space with normal structure such that $\mathcal{A}(M)$ is compact. Let $\mathcal{G}$ be a group of orbit-nonexpansive mappings acting over M. Then, there exists $x \in M$ such that $g(x) = x$ for all $g \in \mathcal{G}$.*

Observe that Corollary 3.2.1 clearly extends the common fixed point results given for onto isometries defined on a convex weakly compact subset of a Banach space with normal structure proved by Brodskiĭ and Mil'man in [9] (see also [25]), and on a bounded hyperconvex metric space stated in [37, Proposition 1.2].

When only a single mapping is considered, we obtain the following result.

Corollary 3.2.2. *Let (M, d) be a bounded metric space with normal structure and such that $\mathcal{A}(M)$ is compact. Let $T : M \to M$ be an orbit-nonexpansive mapping. Then, T has a fixed point. Moreover, the fixed point set of T, Fix(T), is a one-local retract of M.*

In the proof of Theorem 3.2.1, the boundedness of the metric space was used to assure that M belongs to the family of admissible sets. We could ask whether the boundedness of the metric space could be replaced by the boundedness of the orbits of the mapping T. Example 3.1.1 in the previous section tells us that this is not the case since the mapping T presented there is a fixed-point free orbit-nonexpansive mapping, defined over (ℓ_∞, d_∞), whose all orbits are bounded. But, we know that (ℓ_∞, d_∞) is a hyperconvex metric space and therefore, it follows from Proposition 1.5.6 and Proposition 1.5.5 it has (uniform) normal structure and $\mathcal{A}(\ell_\infty)$ is compact.

The theorem below was proved by Khamsi in [31].

Theorem 3.2.2. *[31, Theorem 8] Let (M, d) be a bounded metric space with normal structure and such that $\mathcal{A}(M)$ is compact. Then any commuting family of nonexpansive self-mappings on M has a common fixed point. Moreover, the set of common fixed points is a one-local retract of M.*

Following the same arguments as in [31] and by using the fact that the set of all fixed points is a one-local retract of M, we can derive the next common fixed point theorem which is a strict generalization of the previous theorem (see also [16, Theorem 6.2] for the particular case of hyperconvex metric spaces).

Corollary 3.2.3. *Let (M, d) be a bounded metric space with normal structure such that $\mathcal{A}(M)$ is compact. Then any commutative family $\mathcal{F}$ of orbit-nonexpansive self-mappings on M has a common fixed point. Moreover, the set $Fix(\mathcal{F})$ of the common fixed points of $\mathcal{F}$ is a one-local retract of M.*

Proof: The proof of this corollary follows in the same way as those of Theorems 7 and 8 in [31].

■

Corollary 3.2.3 also extends the theorem below.

Theorem 3.2.3. *[1, Theorem 2.2] Let M be a convex weakly compact subset of a Banach space X with weak normal structure and $T : M \to M$ an orbit-nonexpansive mapping. Then T has a fixed point.*

Also, since Lemma 1.4.1 and Theorem 1.6.3 imply that for every bounded complete CAT(0) space the family of admissible sets is compact, we obtain that Corollary 3.2.3 also extends the following theorem.

Theorem 3.2.4. *[42, Theorem 5.1] Let (M, d) be a bounded complete CAT(0) metric space and $T : M \to M$ an orbit-nonexpansive mapping. Then T has a fixed point.*

It is worth mentioning that the more general class of uniformly convex metric spaces studied in [18] satisfies the conditions of Corollary 3.2.3.

We finish this section with some applications to the Banach space setting:

Corollary 3.2.4. *Let M be a convex closed bounded set of a Banach space X satisfying one of the following conditions:*

i) X has weak normal structure and M is a weakly compact set, *ii) X is a dual space with weak* normal structure and M is a weak* compact set,*

iii) $X = L_1[0,1]$ and M is a compact in measure set.

Then every commuting family $\mathcal{F}$ of orbit-nonexpansive self-mappings on M has a common fixed point. Additionally, the set $Fix(\mathcal{F})$ of common fixed points of this family of mappings is a one-local retract of M. Furthermore, if $\mathcal{G}$ is a group of orbit-nonexpansive mappings defined on M, there exists $x \in M$ such that $g(x) = x$ for all $g \in \mathcal{G}$.

3.3 Uniform relative normal structure and fixed points for orbit-nonexpansive mappings

In this section we prove that, under standard assumptions, the (p,q)-uniform relative normal structure, defined in Section 1.7 implies the existence of fixed points for orbit-nonexpansive mappings. In fact, we will, in the orbit-nonexpansive setting, extend for a single mapping and for the action of groups, Soardi's Theorem in [44], its metric version given in [29, Theorem 5.6] and its corresponding generalization obtained in [19, Section 4]. Since we have already observed in Section 3.1, orbit-nonexpansivity does not imply continuity and therefore, some arguments need to be modified in order to show the existence of fixed points.

The main result of the section is an extension of Theorem 2.3.1 (which is the main theorem of [19, Section 4]). The result will be proved for the case of a general interlaced orbit-nonexpansive family and we will obtain, as particular cases, a fixed point result for a single orbit-nonexpansive mapping and a common fixed point result for the action of a group of such mappings. Before we can proceed, we will extend lemmas 2.3.1, 2.3.2 and 2.3.3 to the context of families of interlaced orbit-nonexpansive mappings.

In the following lemmas, (M,d) is a bounded metric space and $\mathcal{F}$ is a familiy of interlaced orbit-nonexpansive mappings.

Lemma 3.3.1. *Let $A \in \mathcal{A}_{\mathcal{F}}(M)$ and let $s > 0$ such that $A \cap B[A,s] \neq \emptyset$. If we set*

$$\tilde{A} := \operatorname{cov}_{\mathcal{F}}(A \cap B[A,s]),$$

it holds that $\operatorname{diam}(\tilde{A}) \leq s$.

Proof: If we manage to show that $\tilde{A} \subset \tilde{A} \cap B[\tilde{A}, s]$, we will automatically obtain that $\operatorname{diam}(\tilde{A}) \leq s$. So, we will show that

i) $A \cap B[A,s] \subset \tilde{A} \cap B[\tilde{A}, s]$ and

ii) $\tilde{A} \cap B[\tilde{A}, s]$ is T-invariant for all $T \in \mathcal{F}$.

Since $\tilde{A} \cap B[\tilde{A}, s] \in \mathcal{A}(M)$, the conclusion follows.

i) From the definition, $A \cap B[A, s] \subset \tilde{A}$. Since A is T-invariant for all $T \in \mathcal{F}$ and admissible and also $A \cap B[A, s] \subset A$, we have $\tilde{A} \subset A$ and then it follows from item iii) of Lemma 1.1.1 that $B[A, s] \subset B[\tilde{A}, s]$.

Thus, $A \cap B[A, s] \subset B[\tilde{A}, s]$ and so

$$A \cap B[A, s] \subset \tilde{A} \cap B[\tilde{A}, s].$$

ii) Let $z \in \tilde{A} \cap B[\tilde{A}, s]$. Since $\tilde{A}$ is T-invariant for all $T \in \mathcal{F}$, we have that $Tz \in \tilde{A}$ for all $T \in \mathcal{F}$. It remains to be shown that $Tz \in B[\tilde{A}, s]$ for all $T \in \mathcal{F}$ which is equivalent to the fact that $\tilde{A} \subset B(Tz, s)$ for all $T \in \mathcal{F}$. Thus, it suffices to show that for each $T \in \mathcal{F}$,

$$\tilde{A} \cap B(Tz, s) \in \{L \in \mathcal{A}_{\mathcal{F}}(M) : L \supset A \cap B[A, s]\}.$$

It is obvious that for each $T \in \mathcal{F}$, $\tilde{A} \cap B(Tz, s) \in \mathcal{A}(M)$. Since $B[A, s] \subset B[\tilde{A}, s]$ and $Tz \in \tilde{A}$ for all $T \in \mathcal{F}$, given $x \in A \cap B[A, s]$, it follows that $d(x, Tz) \leq s$, which implies that $A \cap B[A, s] \subset B(Tz, s)$ for all $T \in \mathcal{F}$ and therefore,

$$A \cap B[A, s] \subset \tilde{A} \cap B(Tz, s)$$

for all $T \in \mathcal{F}$.

Now, fix $T \in \mathcal{F}$. For any given $y \in \tilde{A} \cap B(Tz, s)$, since $\tilde{A}$ is R-invariant for all $R \in \mathcal{F}$, it follows that $R^n y \in \tilde{A}$ for all $R \in \mathcal{F}$ and for all $n \in \mathbb{N}$. Then, since $z \in B[\tilde{A}, s]$ we have that

$$d(z, R^n y) \leq s \text{ for all } R \in \mathcal{F} \text{ and for all } n \in \mathbb{N} \cup \{0\}$$

and therefore, $D(z, O_R(y)) = \sup\limits_{n \in \mathbb{N} \cup \{0\}} \{d(z, R^n y)\} \leq s$ for all $R \in \mathcal{F}$.

Since $\mathcal{F}$ is a family of interlaced orbit-nonexpansive mappings we have that

$$d(Sy, Tz) \leq \sup\{D(z, O_R(y)) : R \in \mathcal{F}\} \leq s \text{ for all } S \in \mathcal{F}$$

which implies that $Sy \in B(Tz, s)$ for all $S \in \mathcal{F}$. Thus, for each $T \in \mathcal{F}$, we have that $S\left(\tilde{A} \cap B(Tz, s)\right) \subset \tilde{A} \cap B(Tz, s)$ for all $S \in \mathcal{F}$ and therefore, $\tilde{A} \cap B(Tz, s) \in \mathcal{A}_{\mathcal{F}}(M)$.

From the above, we conclude that $\tilde{A} \cap B[\tilde{A}, s] \in \mathcal{A}_{\mathcal{F}}(M)$ and contains $A \cap B[A, s]$. Thus, $\tilde{A} \subset \tilde{A} \cap B[\tilde{A}, s]$ and $\text{diam}(\tilde{A}) \leq s$.

■

Lemma 3.3.2. *If $\mathcal{A}(M)$ is compact and $A \in \mathcal{A}_{\mathcal{F}}(M)$. Then, there exists $A_0 \subset A$ such that:*

i) $A_0 \in \mathcal{A}_{\mathcal{F}}(M)$ *and*

ii) $B[A_0, r] \in \mathcal{A}_{\mathcal{F}}(M)$ *whenever* $B[A_0, r]$ *is nonempty.*

Proof: Let $\mathcal{L}_A := \{L \in \mathcal{A}_{\mathcal{F}}(M) : \ L \subset A\}$. Since $A \in \mathcal{A}_{\mathcal{F}}(M)$ we have that $\mathcal{L}_A \neq \emptyset$. Since, $\mathcal{A}(M)$ is compact, by proceeding as in the proof of Lemma 1.1.3 we can find a minimal element A_0 of $\mathcal{L}_A$ and therefore, i) is proven.

Since $A_0 \in \mathcal{A}_{\mathcal{F}}(M)$ we have that $T(A_0) \subset A_0$ for all $T \in \mathcal{F}$ and so,

$$\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}} T(A_0)\right) \subset A_0$$

which gives us that $S\left(\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}} T(A_0)\right)\right) \subset S(A_0) \subset \operatorname{cov}\left(\bigcup_{T\in\mathcal{F}} T(A_0)\right)$ for all $S \in \mathcal{F}$ and therefore, $\operatorname{cov}\left(\bigcup_{T\in\mathcal{F}} T(A_0)\right) \in \mathcal{A}_{\mathcal{F}}(M)$. Thus, it follows from the minimality of A_0 that

$$A_0 = \operatorname{cov}\left(\bigcup_{T\in\mathcal{F}} T(A_0)\right).$$

Now, let $r > 0$ be such that $B[A_0, r] \neq \emptyset$. Fix $T \in \mathcal{F}$. Given $x \in B[A_0, r]$, since $R(A_0) \subset A_0$ for all $R \in \mathcal{F}$ it follows that for any $y \in A_0$ it is true that $R^n y \in A_0$ for all $R \in \mathcal{F}$ and for all $n \in \mathbb{N}$ which implies that for any $y \in A_0$ we have that

$$d(x, R^n y) \leq r \text{ for all } R \in \mathcal{F} \text{ and for all } n \in \mathbb{N} \cup \{0\}$$

and therefore, $D(x, O_R(y)) = \sup_{n\in\mathbb{N}\cup\{0\}} \{d(x, R^n y)\} \leq r$ for all $R \in \mathcal{F}$.

Then, since $\mathcal{F}$ is a family of interlaced orbit-nonexpansive mappings, it follows that

$$d(Tx, Sy) \leq \sup\{D(x, O_R(y)) : R \in \mathcal{F}\} \leq r \text{ for all } y \in A_0$$

Thus, $Tx \in B(Sy, r)$ for all $y \in A_0$ and for all $S \in \mathcal{F}$ which implies that $Tx \in B\left[\bigcup_{S\in\mathcal{F}} S(A_0), r\right]$ and therefore,

$$T(B[A_0, r]) \subset B\left[\bigcup_{S\in\mathcal{F}} S(A_0), r\right] \text{ for all } T \in \mathcal{F}.$$

Item iii) of Proposition 1.1.1 tell us that

$$B\left[\bigcup_{S\in\mathcal{F}} S\left(A_0\right), r\right] = B\left[\operatorname{cov}\left(\bigcup_{S\in\mathcal{F}} S\left(A_0\right)\right), r\right]$$

and then since, $A_0 = \operatorname{cov}\left(\bigcup_{S\in\mathcal{F}} S\left(A_0\right)\right)$ we have that

$$B\left[\bigcup_{S\in\mathcal{F}} S\left(A_0\right), r\right] = B\left[A_0, r\right].$$

Hence, $T\left(B\left[A_0, r\right]\right) \subset B\left[A_0, r\right]$ for all $T \in \mathcal{F}$ which implies that $B\left[A_0, r\right] \in \mathcal{A}_{\mathcal{F}}\left(M\right)$ and therefore, ii) is proven.

■

Lemma 3.3.3. *If $\mathcal{A}\left(M\right)$ is compact and $A \in \mathcal{A}_T\left(M\right)$. Given $A_0 \subset A$ as in Lemma 3.3.2, if $B\left[A_0, r\right] \cap B\left[B\left[A_0, r\right], s\right] \neq \emptyset$, then the set*

$$\tilde{A}_0 := \operatorname{cov}_{\mathcal{F}}\left(B\left[A_0, r\right] \cap B\left[B\left[A_0, r\right], s\right]\right)$$

satisfies:

i) $\tilde{A}_0$ *is T-invariant.*

ii) $\tilde{A}_0 \subset B\left[A_0, r\right]$.

iii) $\operatorname{diam}(\tilde{A}_0) \leq s$.

Proof: Assertion i) follows from item ii) of Proposition 1.1.1. Assertion ii) follows from the definition of $\tilde{A}_0$ and from item ii) of 3.3.2. Assertion iii) follows from the definition of $\tilde{A}_0$ and Lemma 3.3.1.

■

With the previous lemmas in hand we are now ready to show the main result of this section. Although the proof below has the same structure as the one of Theorem 2.3.1, it is important to highlight that since the continuity of the mappings involved is no longer assured, we need to use a more laborious argument in order to show that the point we construct is indeed a commom fixed point for the family of mappings.

Theorem 3.3.1. *Let (M, d) be a bounded metric space with (p, q)-URNS for some $0 < q < 1$ and such that $\mathcal{A}\left(M\right)$ is compact. Let $\mathcal{F}$ be a family of interlaced orbit-nonexpansive self-mappings on M. Then, there exists a common fixed point for all mappings in $\mathcal{F}$.*

Proof: In order to make the proof clearer, we will denote $\operatorname{diam}\left(A_n\right)$ and $\operatorname{diam}\left(\tilde{A}_n\right)$ by δ_n and $\tilde{\delta}_n$ respectively.

Now, take $A_0 \in \mathcal{A}_{\mathcal{F}}\left(M\right)$ resulting from letting $A = M$ in Lemma 3.3.2. If A_0 is a singleton then, we already have a common fixed point for all mappings in $\mathcal{F}$.

If A_0 is not a singleton, since (M,d) has (p,q)-URNS we have that

$$B[A_0, p\delta_0] \cap B[B[A_0, p\delta_0], q\delta_0] \neq \emptyset.$$

Thus, defining $\tilde{A}_0 := \operatorname{cov}_{\mathcal{F}}(B[A_0, p\delta_0] \cap B[B[A_0, p\delta_0], q\delta_0])$ as in Lemma 3.3.3, we have that

$$\tilde{A}_0 \in \mathcal{A}_{\mathcal{F}}(M),\ \tilde{A}_0 \subset B[A_0, p\delta_0] \text{ and } \tilde{\delta}_0 \leq q\delta_0.$$

Now, taking $A = \tilde{A}_0$ and applying Lemma 3.3.2, we find $A_1 \in \mathcal{A}_{\mathcal{F}}(M)$ such that $A_1 \subset \tilde{A}_0$ which implies that

$$A_1 \subset B[A_0, p\delta_0] \text{ and } \delta_1 \leq q\delta_0.$$

If A_1 is a singleton then, we already have a common fixed point for all mappings in $\mathcal{F}$.

If A_1 is not a singleton, since (M,d) has (p,q)-URNS we have that

$$B[A_1, p\delta_1] \cap B[B[A_1, p\delta_1], q\delta_0] \neq \emptyset.$$

Thus, defining $\tilde{A}_1 := \operatorname{cov}_{\mathcal{F}}(B[A_1, p\delta_1] \cap B[B[A_1, p\delta_1], q\delta_1])$ as in Lemma 3.3.3, we have that

$$\tilde{A}_1 \in \mathcal{A}_{\mathcal{F}}(M),\ \tilde{A}_1 \subset B[A_1, p\delta_1] \text{ and } \tilde{\delta}_1 \leq q\delta_1.$$

Now, taking $A = \tilde{A}_1$ and applying Lemma 3.3.2, we find $A_2 \in \mathcal{A}_{\mathcal{F}}(M)$ such that $A_2 \subset \tilde{A}_1$ which implies that

$$A_2 \subset B[A_1, p\delta_1] \text{ and } \delta_2 \leq q\delta_1.$$

Proceeding inductively, if there exists n_0 such that A_{n_0} is a singleton then, we already have a common fixed point for all mappings in $\mathcal{F}$. Otherwise, we can construct a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\mathcal{F}}(M)$ having the following properties for every $n \in \mathbb{N}$:

(1) $A_n \subset B[A_{n-1}, p\delta_{n-1}]$;
(2) $\delta_n \leq q\delta_{n-1}$.

Assuming we have a sequence as above, for each $n \in \mathbb{N}$ we can choose $x_n \in A_n$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$. Then, given $n \in \mathbb{N}$, it follows from (2) that $\delta_n \leq q^n\delta_0$. Since (1) tells us that $A_{n+1} \subset B[A_n, p\delta_n]$, we have that $d(x_n, x_{n+1}) \leq p\delta_n \leq pq^n\delta_0$.

Thus, since $q \in (0,1)$, it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d). Moreover, since $\mathcal{A}(M)$ is compact it follows from Lemma 1.1.3 that (M,d) is complete and therefore, $(x_n)_{n\in\mathbb{N}}$ is convergent. Let $x = \lim_{n\to\infty} x_n$.

Now, for each $T \in \mathcal{F}$, since $T(A_n) \subset A_n$ for all $n \in \mathbb{N}$, we have

$$D\left(x_n, O_T\left(x_n\right)\right) = \sup_{m \in \mathbb{N}\cup\{0\}} \left\{d\left(x_n, T^m x_n\right)\right\} \leq \delta_n$$

and

$$D\left(x, O_T\left(x_n\right)\right) = \sup_{m \in \mathbb{N}\cup\{0\}} \left\{d\left(x, T^m x_n\right)\right\} \leq$$

$$\sup_{m \in \mathbb{N}\cup\{0\}} \left\{d\left(x, x_n\right) + d\left(x_n, T^m x_n\right)\right\} =$$

$$d\left(x, x_n\right) + \sup_{m \in \mathbb{N}\cup\{0\}} \left\{d\left(x_n, T^m x_n\right)\right\} =$$

$$d\left(x, x_n\right) + D\left(x_n, O_T\left(x_n\right)\right) \leq d\left(x, x_n\right) + \delta_n$$

for all $n \in \mathbb{N}$ which implies

$$\sup\left\{D\left(x, O_T\left(x_n\right)\right) : T \in \mathcal{F}\right\} \leq d\left(x, x_n\right) + \delta_n \text{ for all } n \in \mathbb{N}.$$

Since $\mathcal{F}$ is a family of interlaced orbit-nonexpansive mappings, for every $S \in \mathcal{F}$ we have that

$$d\left(Sx, Sx_n\right) \leq \sup\left\{D\left(x, O_T\left(x_n\right)\right) : T \in \mathcal{F}\right\} \leq d\left(x, x_n\right) + \delta_n$$

for all $n \in \mathbb{N}$
which implies that

$$d\left(x, Sx\right) \leq d\left(x, x_n\right) + d\left(x_n, Sx_n\right) + d\left(Sx_n, Sx\right) \leq 2d\left(x, x_n\right) + 2\delta_n$$

for all $n \in \mathbb{N}$.

Thus, since $\lim_{n\to} \left(2d\left(x, x_n\right) + 2\delta_n\right) = 0$ we obtain that $Sx = x$ for all $S \in \mathcal{F}$, that is, x is a common fixed point for all mappings in $\mathcal{F}$.

■

Corollary 3.3.1. *Let (M, d) be a bounded metric space with uniform relative normal structure and such that $\mathcal{A}(M)$ is compact. Let $T : M \to M$ be an orbit-nonexpansive mapping. Then, T has a fixed point.*

Corollary 3.3.2. *Let (M, d) be a bounded metric space with uniform relative normal structure and such that $\mathcal{A}(M)$ is compact. Let $\mathcal{G}$ be a group of orbit-nonexpansive on M. Then, there exists some $x \in M$ such that $g(x) = x$ for all $g \in \mathcal{G}$.*

We end this section by showing the following stability result regarding the FPP for a family of interlaced orbit-nonexpansive self-mappings on M.

Corollary 3.3.3. *Let (M, d) be a bounded hyperconvex metric space and let d_1 be a metric equivalent to d such that $ad(x, y) \le d_1(x, y) \le bd(x, y)$ for all $x, y \in M$ and $\frac{b}{a} < \sqrt{2}$. If the family $\mathcal{A}_{d_1}(M)$ is compact then, the metric space (M, d_1) any family of interlaced orbit-nonexpansive self-mappings on M has a common fixed point.*

Proof: It follows straight from Theorem 3.3.1 and Theorem 2.3.2.

■

3.4 Some open questions

In this final section we leave some open questions which arose from our research.

Throughout this work, the compactness of the family of admissible sets has been assumed in many of our results in order to guarantee the existence of minimal invariant sets. As we had already mentioned, in many occasions this assumption arises from intrinsic conditions of the metric space, as we can see in the examples presented in Section 2.4. In particular, this holds for weak compact or weak*-compact domains in Banach spaces. Additionally, as it was mentioned before, the uniform normal structure implies compactness of the family of admissible sets (see [30] or [29, Theorem 5.4]) for complete metric spaces. Whether the compactness of the family of admissible sets can be deduced from the URNS in complete metric spaces is still an open problem. We also don't know either whether the hypothesis of the compactness of $\mathcal{A}_{d_1}(M)$ in Corollaries 2.3.1 and 3.3.3 can be dropped.

The examples of non-weakly compact closed convex subsets of c with the FPP given in [22] and in Section 2.2 of our work are all coordinatewise closed. We wonder whether there is a convex closed subset C of c with the FPP for nonexpansive mappings, which additionally fails to be closed for the coordinatewise topology. In the particular case that C is a subset of c_0 with the FPP, we know that C is weakly compact (see [12, 14, 15]) and therefore, closed coordinatewise. If C is a convex weakly compact subset of c, it has the FPP but it is also compact for the coordinatewise topology.

Finally, we don't know whether or not the thesis of Corollary 2.4.1 would hold without the assumption of $\|\cdot\|$ being an equivalent dual norm.

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