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On the existence of solutions for a parabolic-elliptic chemotaxis model with flux limitation and logistic source

Silvia Sastre-Gomez[1] | **Jose Ignacio Tello**[2]

[1]Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, Seville, Spain

[2]Departamento de Matemáticas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, Madrid, Spain

Correspondence
Silvia Sastre-Gomez, Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, Seville, Spain.
Email: ssastre@us.es

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In this article, we study the existence of solutions of a parabolic-elliptic system of partial differential equations describing the behaviour of a biological species "u" and a chemical stimulus "v" in a bounded and regular domain Ω of $\mathbb{R}^N$. The equation for u is a parabolic equation with a nonlinear second order term of chemotaxis type with flux limitation as

$$-\chi \mathrm{div}(u|\nabla v|^{p-2}\nabla v),$$

for $p > 1$. The chemical substance distribution v satisfies the elliptic equation

$$-\Delta v + v = u.$$

The evolution of u is also determined by a logistic type growth term $\mu u(1-u)$. The system is studied under homogeneous Neumann boundary conditions. The main result of the article is the existence of uniformly bounded solutions for $p < 3/2$ and any $N \geq 2$.

KEYWORDS
bounded solutions, chemotaxis, global existence of solutions

MSC CLASSIFICATION
35A01, 35B09

1 | INTRODUCTION

In the decade of 70s of the last century, Keller and Segel[1,2] described chemotaxis phenomena for a biological species density "u" and a chemical substance concentration "v" using systems of partial differential equations. The Keller–Segel model presents a second order chemotaxis term with a linear dependence of the chemical flux "∇v" in the way

$$-\mathrm{div}(\chi u \nabla v)$$

where χ is a given constant. After the model was proposed, many authors have studied such systems from a mathematical point of view, see for instance Horstmann[3,4] and Hillen and Painter[5] for details concerning mathematical results of chemotaxis models with linear flux.

Rivero et al[6] proposed a chemotaxis term with nonlinear dependence of the chemical flux, given in the form

$$-\frac{\partial}{\partial x}\left(F\left(u,\frac{\partial v}{\partial x}\right)\right), \qquad F\left(u,\frac{\partial v}{\partial x}\right) = arctan\left[\frac{u}{1+u}\frac{\partial v}{\partial x}\right]$$

in a one-dimensional spatial domain.

Saragosti et al[7] considered a flux limitation model of integral type in the form

$$-div(u[v]), \text{ for } u[v] = -\frac{1}{|V|}\int_V f(v\nabla v)dv,$$

where V denotes the set of possible velocities v; see also Perthame et al.[8]

Chertock et al[9] consider the parabolic-parabolic system with a flux limitation defined by

$$F(\nabla v) = \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}.$$

Bianchi et al[10,11] use such a nonlinear dependence to obtain a biomedical model. Bellomo and Winkler[12,13] study the parabolic-elliptic system for a chemotaxis term of the form

$$-\text{div}(uf(|\nabla v|)\nabla v)$$

for

$$f(|\nabla v|) = \frac{\chi}{\sqrt{1+|\nabla v|^2}}$$

and a diffusive term

$$-\text{div}\left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right)$$

to describe the evolution of the biological species u. For more information about this model, see also Chiyoda et al[14] and Mizukami et al.[15]

In Bellomo and Winkler,[12] for $\chi < 1$ and $N \geq 2$, the solutions are global and bounded if the initial mass is smaller than $(\chi^2-1)_+^{-\frac{1}{2}}$. Using an energy method, Winkler[16] has obtained blow up of solutions for

$$f = f(|\nabla v|) = \frac{\chi}{(1+|\nabla v|^2)^{\frac{\alpha}{2}}}$$

and $N > 2$ when

$$\alpha < \frac{N-2}{2(N-1)}.$$

More blow up results can be found in Marras et al.[17]

In this article, we consider a bounded N-dimensional domain, Ω, with regular boundary $\partial\Omega$ and denote by $\vec{\boldsymbol{n}}$ the outward pointing normal vector on the boundary $\partial\Omega$.

The equation for v is restricted to the elliptic case, for simplicity, we assume that v satisfies the Poisson equation and the system studied reads as follows:

$$u_t - \Delta u = -\text{div}(\chi u|\nabla v|^{p-2}\nabla v) + \mu u(1-u) \qquad x\in\Omega,\ t>0, \tag{1.1}$$

$$-\Delta v + v = u \qquad x\in\Omega,\ t>0, \tag{1.2}$$

$$\frac{\partial u}{\partial \vec{\boldsymbol{n}}} = \frac{\partial v}{\partial \vec{\boldsymbol{n}}} = 0 \qquad x\in\partial\Omega,\ t>0, \tag{1.3}$$

$$u(0,x) = u_0(x) \qquad x\in\Omega. \tag{1.4}$$

In Negreanu and Tello,[18] the system is considered for $\mu = 0$, that is, the logistic term does not appear and p satisfies

$$p \in (1, \infty), \text{ if } N = 1 \text{ and } p \in \left(1, \frac{N}{N-1}\right), \text{ if } N \geq 2,$$

to obtain uniform bounds in $L^\infty(\Omega)$.

The complementary case, for $p \in (N/(N-1), 2)$, for $N > 2$ and Ω defined as the unit ball, presents blow up for some initial data when $\mu = 0$; see Tello.[19] The stationary case with flux limitation has been studied in Boccardo and Tello[20]; see also Boccardo and Orsina[21] for the stationary case for the case $p = 2$.

Parabolic-elliptic systems of chemotaxis with logistic terms have been already studied in the literature. In Tello and Winkler,[22] a comparison method is applied to proof the converge of the solution to the homogeneous steady state (see also Galakhov et al[23] and references therein).

In this article, we study the global existence of solutions in $(0, T)$ for any $T < \infty$ under the following assumptions:

$$\Omega \text{ is an open and bounded domain with regular boundary } \partial\Omega, \tag{1.5}$$

$$\begin{cases} p < 2, & N = 2, \\ p \in (1, 3/2), & N \geq 3, \end{cases} \tag{1.6}$$

and the initial data u_0 satisfy

$$u_0 \in C^{2+\alpha}(\Omega), \ \frac{\partial u_0}{\partial \vec{\boldsymbol{n}}} = 0, \ x \in \partial\Omega, \tag{1.7}$$

for some $\alpha \in (0, 1)$.

In this article, we prove the existence of weak solutions to problems (1.1)–(1.4) in the sense of the following definition.

Definition 1.1. We say that $(u, v) \in [L^2(0, T : H^1(\Omega)) \cap H^1(0, T : (H^1(\Omega))')]^2$ is a weak solution to (1.1)-(1.4) if $u\nabla v \in [L^1(\Omega)]^N$ and for any $\varphi, \psi \in L^2(0, T : H^1(\Omega)) \cap H^1(0, T : L^2(\Omega))$, (u, v) satisfies

$$\int_0^T \int_\Omega \varphi u_t dxdt + \int_0^T \int_\Omega \nabla u \nabla \varphi dxdt = \int_0^T \int_\Omega \chi u |\nabla v|^{p-1} \nabla v \nabla \varphi dxdt + \int_0^T \int_\Omega \mu u(1-u)\varphi dxdt, \tag{1.8}$$

$$\int_0^T \int_\Omega \nabla v \nabla \psi dxdt = \int_0^T \int_\Omega (u - v)\, \psi dxdt. \tag{1.9}$$

The existence of weak solutions given in the previous definition is enclosed in the following theorem.

Theorem 1.1. *Under assumptions (1.5)–(1.7), there exists at least a weak solution to (1.1) in $(0, T)$ for any $T < \infty$, in the sense of Definition 1.1. Moreover, there exists a constant c independent of T such that the solution satisfies*

$$\|u(t)\|_{L^\infty(\Omega)} \leq c, \text{ for any } t \in (0, T).$$

The article is organized as follows. In Section 2, we obtain some a priori estimates that are used in Section 3 to proof Theorem 1.1. We use an iterative method based in the Moser–Alikakos iteration that allows us to obtain explicit L^q estimates of the solutions. Then, we pass to the limit to get the boundedness of the solutions in L^∞ norm. Finally, using an approximated problem, we pass to the limit in the weak formulation to obtain the existence of weak solutions.

2 | A PRIORI ESTIMATES FOR THE AUXILIARY PROBLEM

We introduce the following auxiliary problem:

$$u_{nt} - \Delta u_n = -\text{div}\left(\chi u_n \frac{|\nabla v_n|^{p-2}\nabla v_n}{1 + \frac{1}{n}|\nabla v_n|^{p-1}}\right) + \mu u_n(1 - u_n) \qquad x \in \Omega, \ t > 0, \tag{2.1}$$

$$-\Delta v_n + v_n = u_n \qquad x \in \Omega, \ t > 0, \tag{2.2}$$

$$\frac{\partial u_n}{\partial \vec{\boldsymbol{n}}} = \frac{\partial v_n}{\partial \vec{\boldsymbol{n}}} = 0 \qquad x \in \partial\Omega,\ t > 0, \tag{2.3}$$

$$u_n(0,x) = u_0(x) \qquad x \in \Omega. \tag{2.4}$$

In this section, we give some estimates that will be useful to prove the existence of solution of (2.1)–(2.4). In particular, we obtain uniform bounds in $L^\infty(\Omega)$ for u_n and v_n.

In the following result, we obtain an upper bound of the total mass of u_n.

Lemma 2.1. *The total mass of the component u_n of the solution to (2.1) is bounded, $\int_\Omega u_n\,dx \le c_1$, with*

$$c_1 =: \max\left\{\int_\Omega u_0\,dx,\ |\Omega|\right\}.$$

Since the proof of this lemma is the same as in the case $p = 2$, we omit the details.

In the following result, we prove that u_n is nonnegative if the initial data u_0 is nonnegative.

Lemma 2.2. *The solution u_n to (2.1)–(2.4) with $u_0 \ge 0$ satisfies $u_n \ge 0$.*

Proof. We first consider the auxiliary equation

$$u_{nt} - \Delta u_n = -\mathrm{div}\left(\chi u_n \frac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}}\right) + \mu u_n(1-(u_n)_+) \tag{2.5}$$

and define T_h as follows

$$T_h(s) = \begin{cases} -h, & \text{if } s \le -h, \\ s, & \text{if } -h < s < 0, \\ 0, & \text{if } s \ge 0. \end{cases}$$

We denote the primitive of T_h, by Φ_h, which is given by

$$\Phi_h(s) = \begin{cases} -hs - \frac{h^2}{2}, & \text{if } s \le -h, \\ \frac{s^2}{2}, & \text{if } -h < s < 0, \\ 0, & \text{if } s \ge 0, \end{cases}$$

and satisfies $\Phi_h' = T_h$. We multiply (2.5) by $T_h(u_n)$ and integrate by parts to obtain

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_\Omega \Phi_h(u_n)dx + \int_{-h<u_n<0}|\nabla u_n|^2 dx = {} & \chi \textstyle\int_{-h<u_n<0} u_n \frac{|\nabla v_n|^{p-2}\nabla v_n \nabla u_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}}dx \\ & + \mu \textstyle\int_\Omega T_h(u_n)u_n(1-(u_n)_+)dx \\ \le {} & \chi\, n \textstyle\int_{-h<u_n<0} u_n|\nabla u_n|dx + \mu\int_\Omega T_h(u_n)u_n dx \end{aligned}$$

and apply Hölder inequality, and after some computations, we get

$$\frac{d}{dt}\int_\Omega \Phi_h(u_n)dx + \frac{1}{2}\int_{-h<u_n<0}|\nabla u_n|^2 dx \le \frac{\chi^2}{2} n \int_{-h<u_n<0} u_n^2 + 2\mu \int_\Omega \Phi_h(u_n)dx,$$

which implies

$$\frac{d}{dt}\int_\Omega \Phi_h(u_n)dx + \frac{1}{2}\int_{-h<u_n<0}|\nabla u_n|^2 dx \le ch^2 + 2\mu\int_\Omega \Phi_h(u_n)dx.$$

We divide by h and take limits when $h \to 0$ to get

$$\frac{d}{dt}\int_\Omega (-u_n)_+ dx \le 2\mu \int_\Omega (-u_n)_+ dx.$$

We apply Gronwall's lemma, and it results

$$\int_\Omega (-u_n)_+ dx \le e^{2\mu t} \int_\Omega (-u_n(0))_+ dx.$$

Since $(-u_0)_+ = 0$, we get

$$u_n \ge 0.$$

Since $u_n \ge 0$, we have that $(u_n)_+ = u_n$ and u_n satisfies (2.1). □

The following estimate will be useful to obtain L^q estimates for u_n.

Lemma 2.3. *Let $p < \frac{3}{2}$, $N \ge 3$ and q satisfying assumptions*

$$\max\left\{\frac{1}{2}, \frac{N-1}{2} - \frac{N}{4(p-1)}\right\} < q < \frac{N-1}{2},$$

then

$$\int_\Omega u_n^{2q} dx < c_1$$

and

$$\int_0^t \int_\Omega u_n^{2q+1} dx\, ds \le c_2 t + c_3,$$

where $c_1 = c_1(q)$, $c_2 = c_2(q)$ and $c_3 = c_3(q)$ are independent of t.

Proof. We multiply Equation (2.1) by $|u_n|^{2q-2}u_n$, and after integration over Ω, we get

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q} dx + \frac{2q-1}{q^2}\int_\Omega |\nabla u_n^q|^2 dx = \frac{(2q-1)\chi}{q}\int_\Omega u_n^q \frac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}} \nabla u_n^q dx + \mu \int_\Omega u_n^{2q}(1-u_n) dx. \tag{2.6}$$

Notice that

$$\frac{(2q-1)\chi}{q}\int_\Omega u_n^q \frac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}} \nabla u_n^q dx \le \frac{(2q-1)}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx + \frac{(2q-1)\chi^2}{2}\int_\Omega |\nabla v_n|^{2p-2} u_n^{2q} dx$$

and thanks to Hölder inequality

$$\begin{aligned} \int_\Omega u_n^{2q}|\nabla v_n|^{2p-2} dx &\le \left[\int_\Omega |\nabla v_n|^{2(p-1)(2q+1)} dx\right]^{\frac{1}{2q+1}} \left[\int_\Omega u_n^{(2q+1)} dx\right]^{\frac{2q}{2q+1}} \\ &\le \|\nabla v_n\|^{2(p-1)}_{L^{2(p-1)(2q+1)}(\Omega)} \|u_n\|^{2q}_{L^{2q+1}(\Omega)}. \end{aligned}$$

The elliptic regularity of the problem gives the inequality,

$$\|\nabla v_n\|_{L^{2(p-1)(2q+1)}(\Omega)} \le c\|u_n\|_{L^{2q+1}(\Omega)}$$

provided

$$\frac{N(2q+1)}{N-2q-1} > 2(p-1)(2q+1),$$

which is equivalent to

$$\frac{N}{N-2q-1} > 2(p-1). \tag{2.7}$$

We take q such that

$$\frac{N[2p-3]}{2(p-1)} < 2q+1$$

to get

$$\frac{N-1}{2} - \frac{N}{4(p-1)} < q. \tag{2.8}$$

Thanks to (2.7) and (2.8), we have that

$$q \in \left(\max\left\{ \frac{1}{2}, \frac{N-1}{2} - \frac{N}{4(p-1)} \right\}, \frac{N-1}{2} \right).$$

Then, after some computations, we get

$$\frac{(2q-1)\chi}{q} \int_\Omega u_n^q |\nabla v_n|^{p-2} \nabla v_n \nabla u_n^q dx \le \frac{(2q-1)}{2q^2} \int_\Omega |\nabla u_n^q|^2 dx + c\left[\int_\Omega u_n^{2q+1} dx \right]^{1+\frac{2(p-\frac{3}{2})}{2q+1}}$$

we replace the previous inequality in Equation (2.6) to get

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q} dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \le c\left[\int_\Omega u_n^{2q+1} dx \right]^{1+\frac{2(p-\frac{3}{2})}{2q+1}} - \frac{\mu}{2}\int_\Omega u_n^{2q+1} dx + c. \tag{2.9}$$

We apply Poincaré–Wirtinger inequality to the term

$$\int_\Omega |u_n^{2q}| dx$$

in the following way

$$\int_\Omega |u_n^{2q}| dx \le c\int_\Omega |\nabla u_n^q|^2 dx + c\left[\int_\Omega |u_n|^q dx \right]^2$$

then,

$$\frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \ge c\int_\Omega |u_n^{2q}| dx - c\left[\int_\Omega |u_n|^q dx \right]^2.$$

Since

$$\left[\int_\Omega |u_n|^q dx \right]^2 \le \epsilon \int_\Omega |u_n|^{2q+1} dx + c(\epsilon)$$

and

$$\left[\int_\Omega |u_n|^{2q+1} dx \right]^{1+\frac{2(p-\frac{3}{2})}{2q+1}} \le \epsilon \int_\Omega |u_n|^{2q+1} dx + c(\epsilon).$$

Thanks to (2.9), we have

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q} dx + c\int_\Omega u_n^{2q} dx \le -\left(\frac{\mu}{2} - 2\epsilon \right)\int_\Omega |u_n|^{2q+1} dx + c(\epsilon). \tag{2.10}$$

We take $\epsilon < \mu/4$ and apply maximum principle for ODE to obtain

$$\int_\Omega u_n^{2q} dx \le c_1, \qquad \int_0^t \int_\Omega u_n^{2q+1} dx\, ds \le c_2 t + c_3,$$

where $c_1 = c_1(q)$, $c_2 = c_2(q)$ and $c_3 = c_3(q)$ are positive constants and independent of t. □

Remark 2.1. In the previous lemma, the assumption $p < \frac{3}{2}$ is used to obtain the estimate

$$\|\nabla v_n\|_{L^{2(p-1)(2q+1)}(\Omega)} \leq c\|u_n\|_{L^{2q+1}(\Omega)}$$

for any N and any $q < 1$ by using the elliptic regularity of the problem.

In the result below, we give an L^{2q}-estimate of u_n for $N = 2$.

Lemma 2.4. *Let $N = 2$, $p < 2$ and $q \in (1, \infty)$ then,*

$$\int_\Omega u_n^{2q} dx < c_4(q).$$

and $v_n \in W^{1,\infty}(\Omega)$, where $c_4(q)$ is independent of t.

Proof. We proceed as in Lemma 2.3 to obtain

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q} dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \leq c\int_\Omega u_n^{2q}|\nabla v|^{2(p-1)} dx - \frac{\mu}{2}\int_\Omega u_n^{2q+1} dx + c. \tag{2.11}$$

Since

$$\int_\Omega u_n^{2q}|\nabla v_n|^{2(p-1)} dx \leq c\|\nabla v_n\|_{L^\infty(\Omega)}^{2(p-1)}\int_\Omega u_n^{2q} dx,$$

and thanks to the elliptic regularity, we know that

$$\|\nabla v_n\|_{L^\infty(\Omega)} \leq c\|u_n\|_{L^{2+\frac{\epsilon}{q}}(\Omega)},$$

and for $q \geq 1$,

$$\|u_n\|_{L^{2+\frac{\epsilon}{q}}(\Omega)} \leq c\|u_n\|_{L^{2q+\epsilon}(\Omega)}$$

then

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q} dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \leq c\left[\int_\Omega u_n^{2q+\epsilon} dx\right]^{2\frac{p-1+q}{2q+\epsilon}} - \frac{\mu}{2}\int_\Omega u_n^{2q+1} dx + c. \tag{2.12}$$

Thanks to Gagliardo–Nirenberg inequality, we know that

$$\|u_n\|_{L^{2q+\epsilon}(\Omega)} = \|u_n^q\|_{L^{2+\frac{\epsilon}{q}}(\Omega)}^{\frac{1}{q}} \leq c\left[\|\nabla u_n^q\|_{L^2(\Omega)}^{\frac{a}{q}}\|u_n^q\|_{L^1(\Omega)}^{\frac{(1-a)}{q}} + c\|u_n^q\|_{L^1(\Omega)}\right]^{\frac{1}{q}} \tag{2.13}$$

for a satisfying

$$\frac{1}{2+\frac{\epsilon}{q}} = \left(\frac{1}{2} - \frac{1}{N}\right)a + (1-a),$$

which is equivalent to

$$a = \frac{2N(q+\epsilon)}{(2q+\epsilon)(N+2)}.$$

For $N = 2$ and $q = 1$, we have that

$$a = \frac{1+\epsilon}{(2+\epsilon)}.$$

We replace $q = 1$ into (2.13), and thanks to Lemma 2.1, we get

$$\|u_n\|_{L^{2+\epsilon}(\Omega)}^{2+\epsilon} \leq c\|\nabla u_n\|_{L^2(\Omega)}^{1+\epsilon} + c.$$

Then,

$$\left[\int_\Omega u_n^{2+\epsilon}dx\right]^{\frac{2p}{2+\epsilon}} \leq c\|\nabla u_n\|_{L^2(\Omega)}^{\frac{(1+\epsilon)2p}{2+\epsilon}} + c,$$

for ϵ small enough, we get

$$\|\nabla u_n\|_{L^2(\Omega)}^{\frac{(1+\epsilon)2p}{2+\epsilon}} \leq \delta\|\nabla u_n\|_{L^2}^2 + c(\delta)$$

for any $\delta > 0$. We replace into (2.12) to get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega u_n^2dx + \left(\frac{1}{2}-\delta\right)\int_\Omega|\nabla u_n|^2dx \leq c - \frac{\mu}{4}\int_\Omega|u_n|^3dx.$$

Standard computations show the uniform boundedness in time of

$$\int_\Omega u_n^2dx.$$

To end the proof, we take $q \in (1, 2)$ to obtain; thanks to Gagliardo–Nirenberg inequality that

$$\|u_n\|_{L^{2q+\epsilon}(\Omega)} = \|u_n^q\|_{L^{2+\frac{\epsilon}{q}}(\Omega)}^{\frac{1}{q}} \leq c\left[\|\nabla u_n^q\|_{L^2(\Omega)}^a\|u_n^q\|_{L^{\frac{2}{q}}(\Omega)}^{1-a} + c\|u_n^q\|_{L^1(\Omega)}\right]^{\frac{1}{q}} \tag{2.14}$$

for a satisfying

$$\frac{1}{2+\frac{\epsilon}{q}} = \left(\frac{1}{2}-\frac{1}{N}\right)a + \frac{q(1-a)}{2}$$

that is,

$$\frac{q}{2q+\epsilon} - \frac{q}{2} = \left(\frac{1}{2}-\frac{1}{N}\right)a - \frac{qa}{2}$$

since $N = 2$, we have

$$\frac{q}{2q+\epsilon} - \frac{q}{2} = -\frac{qa}{2}$$

and then

$$\left(\frac{2(q-1)+\epsilon}{2q+\epsilon}\right) = a$$

then, for ϵ small enough and q close to 1, and since $p < 2$, we have that

$$a\left(2\frac{p-1+q}{2q+\epsilon}\right) < 2.$$

We replace into (2.12) to obtain

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q}dx + \frac{2q-1}{2q^2}\int_\Omega|\nabla u_n^q|^2dx \leq c\left[\int_\Omega|\nabla u_n^q|^2dx\right]^\beta - \frac{\mu}{2}\int_\Omega u_n^{2q+1}dx + c$$

for $\beta < 2$. We proceed as before to get

$$\int_\Omega u_n^{2q}dx < c$$

for some $q > 1$. We apply elliptic regularity to second equation and get that $v_n \in W^{1,\infty}(\Omega)$. Then, the term

$$\int_\Omega u_n^{2q}|\nabla v_n|^{2(p-1)} \leq c\int_\Omega u_n^{2q} \leq c(\epsilon) + \epsilon\int_\Omega u_n^{2q+1}$$

which implies, for $\epsilon \leq \frac{\mu}{4}$

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q}dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \leq c - \frac{\mu}{4}\int_\Omega u_n^{2q+1}dx.$$

Since

$$\int_\Omega u^{2q}dx \leq \int_\Omega u^{2q+1}dx + |\Omega|,$$

we get

$$\frac{d}{dt}\frac{1}{2q}\int_\Omega u_n^{2q}dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx + \frac{\mu}{4}\int_\Omega u_n^{2q}dx \leq c + |\Omega|.$$

Maximum principle ends the proof. □

In the following lemma, we obtain an estimate of u_n and v_n, for $N \geq 3$.

Lemma 2.5. *Let $N \geq 3$ and $p < \frac{3}{2}$; then, for any $s < \infty$, we have that*

$$\int_\Omega u_n^s \leq c_5$$

and $v_n \in W^{2,N+1}(\Omega) \cap W^{1,\infty}(\Omega)$, where c_5 is independent of s and t.

Proof. Thanks to Lemma 2.3, we have that u_n is uniformly bounded in $L^q(\Omega)$ for $q < N-1$; then we have that, thanks to the elliptic regularity, $v_n \in W^{2,q}(\Omega)$ for $q < N-1$, which implies

$$v_n \in W^{1,\frac{Nq}{N-q}}(\Omega)$$

for any $q \in (1, N-1)$. In particular, we have that, for any $p < 3/2$, $q < N-1$ and close to $N-1$, the inequality

$$2(p-1)(2q+1) < \frac{Nq}{N-q}$$

is satisfied, and then, the term

$$\|\nabla v_n\|_{L^{2(p-1)(2q+1)}(\Omega)}$$

is bounded. Then we proceed as in the case $N = 2$ (see Lemma 2.4) to get that u_n is uniformly bounded in $L^s(\Omega)$ for any $s < \frac{N(N-1)}{2(p-1)} - 1$. Since $p < \frac{3}{2}$, we get that u_n is uniformly bounded in $L^s(\Omega)$ for any $s < N(N-1) - 1$ and we deduce, in view of $N \geq 3$ that $u_n \in L^{N+1}(\Omega)$. Elliptic regularity implies that $v_n \in W^{2,N+1}(\Omega)$, which is included in $W^{1,\infty}(\Omega)$ and the proof ends. □

The following result gives a uniform bound of u_n when $N \geq 3$ and $p < 3/2$.

Lemma 2.6. *Let $N \geq 2$ and $p < \frac{3}{2}$; then, we have that*

$$\|u_n\|_{L^\infty(\Omega)} \leq c_6$$

where c_6 is independent of t.

Proof. Thanks to Lemmas 2.4 and 2.5, we have that $v_n \in W^{1,\infty}(\Omega)$. We take u_n^{2q-1} as test function in the weak formulation of (2.1)–(2.4) to get, after some computations, that

$$\frac{1}{2q}\frac{d}{dt}\int_\Omega u_n^{2q}dx + \frac{2q-1}{2q^2}\int_\Omega |\nabla u_n^q|^2 dx \leq \|\nabla v_n\|_{L^\infty(\Omega)}^{2(p-1)}\frac{\chi^2(2q-1)}{2}\int_\Omega u_n^{2q}dx + \mu\int_\Omega u_n^{2q}dx - \mu\int_\Omega u_n^{2q+1}dx. \tag{2.15}$$

We split the term $\mu\int_\Omega u_n^{2q}dx$ into two parts as follows:

$$\mu\int_\Omega u_n^{2q}dx \le \frac{\mu}{2}\int_\Omega u_n^{2q+1}dx + \mu|\Omega|.$$

In the case $N = 2$, we apply Gagliardo–Nirenberg inequality to the term

$$\int_\Omega u_n^{2q}dx$$

in the following way

$$\|u_n\|_{L^{2q}(\Omega)} = \|u_n^q\|_{L^2(\Omega)}^{\frac{1}{q}} \le c\left[\|\nabla u_n^q\|_{L^2(\Omega)}^{\frac{a}{q}}\|u_n^q\|_{L^1(\Omega)}^{\frac{(1-a)}{q}} + c\|u_n^q\|_{L^1(\Omega)}\right]^{\frac{1}{q}}$$

for a defined by

$$a = \frac{N}{(N+2)}.$$

Then, thanks to Young's inequality

$$\int_\Omega u_n^{2q}dx \le \epsilon\int_\Omega u_n^{2q+1}dx + c\epsilon^{-\frac{N}{2}}$$

for any $\epsilon > 0$. We take

$$\epsilon < \frac{2q-1}{4q^2}\frac{2}{\|\nabla v_n\|_{L^\infty(\Omega)}^{2(p-1)}\chi^2(2q-1)} = \frac{1}{4q^2}\frac{2}{\|\nabla v_n\|_{L^\infty(\Omega)}^{2(p-1)}\chi^2}$$

we have that

$$\|\nabla v_n\|_{L^\infty(\Omega)}^{2(p-1)}\frac{\chi^2(2q-1)}{2}\int_\Omega u_n^{2q}dx \le \frac{2q-1}{4q^2}\int_\Omega|\nabla u_n^q|^2dx + c(1+q^N).$$

We replace the previous inequality into (2.15) to get, in view of

$$u_n^{2q} \le u_n^{2q+1} + 1$$

and

$$c(2q-1)u_n^q \le \frac{\mu}{4}u_n^{2q} + cq^2.$$

For the case $N \ge 3$, we have that

$$\frac{1}{2q}\frac{d}{dt}\int_\Omega u_n^{2q}dx + \frac{2q-1}{4q^2}\int_\Omega|\nabla u_n^q|^2dx + \frac{\mu}{2}\int_\Omega u_n^{2q}dx \le c(q^N+1). \tag{2.16}$$

The previous inequality gives the following one

$$\frac{d}{dt}\int_\Omega u_n^{2q}dx + \frac{\mu}{2}\int_\Omega u_n^{2q}dx \le c(q^{N+1}+1),$$

for any $t > 0$. Maximum principle for O.D.E.s gives that

$$\int_\Omega u_n^{2q}dx \le \max\{\int_\Omega u_n(0)^{2q}dx, \frac{2c}{\mu}(q^{N+1}+1)\},$$

for any $t > 0$. We take square-roots in the previous inequality to obtain that

$$\|u_n\|_{L^q(\Omega)} \le c$$

where c is independent of q. We take limits when $q \to \infty$ to end the proof. □

Lemma 2.7. *Let u_n be the solution to (2.1)–(2.4) satisfying assumptions (1.5)–(1.7); then, for any $T \in (0, \infty)$, we have that u_n satisfies*

$$\int_0^T \int_\Omega |\nabla u_n|^2 dxds \le c_7 T + c_8.$$

Proof. We denote by y the average of u_n, that is,

$$y(t) := \frac{1}{|\Omega|} \int_\Omega u_n dx,$$

which satisfies

$$y' = \mu \int_\Omega u_n(1 - u_n) dx.$$

We multiply Equation (2.1) by $(u_n - y)$ and integrate by parts to get,

$$\frac{d}{dt}\int_\Omega |u_n - y|^2 dx + \int_\Omega |\nabla u_n|^2 dx = \chi \int_\Omega u_n \frac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}} \nabla u_n\, dx + \mu \int_\Omega u_n(1 - u_n)(u_n - y) dx.$$

Thanks to Lemma 2.6

$$\int_\Omega u_n(1 - u_n)(u_n - y) dx \le c$$

and

$$\begin{aligned} \int_\Omega u_n \frac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}} \nabla u_n dx &\le \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + c\int_\Omega u_n^2 |\nabla v_n|^{2p-2} dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + c. \end{aligned}$$

After integration, we get the wished result. □

Lemma 2.8. *Let u_n the solution to (2.1)–(2.4) then, under assumptions (1.5)–(1.7), and for any $T \in (0, \infty)$, we have that*

$$u_{nt} \in L^2(0, T : (H^1(\Omega))').$$

Proof. For simplicity, we denote by X the space $L^2(0, T : H^1(\Omega))$ and by X' its dual, which is equivalent to $L^2(0, T : (H^1(\Omega))')$.

Since

$$u_{nt} = \Delta u_n - \operatorname{div}\left(\chi u_n \frac{|\nabla v_n|^{p-1}\nabla v_n}{1 + \frac{1}{n}|\nabla v_n|} \right) + u_n(1 - u_n),$$

and for any, $w \in L^2(0, T : H^1(\Omega))$, we have that

$$\left|< w, -\Delta u_n >_{X,X'}\right| = \left| \int_0^T \int_\Omega \nabla u_n \nabla w dxdt \right| \le \|u_n\|_{L^2(0,T:H^1(\Omega))} \|w\|_{L^2(0,T:H^1(\Omega))}, \tag{2.17}$$

$$\begin{aligned} \left| < w, \operatorname{div}\left(\chi u_n \frac{|\nabla v_n|^{p-1}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|} \right) >_{X,X'} \right| &= \left| \int_0^T \int_\Omega \chi u_n \frac{|\nabla v_n|^{p-1}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|} \nabla w dxdt \right| \\ &\le c\|u_n\|_{L^\infty(\Omega)} \|v_n\|_{L^2(0,T:W^{1,2p-2}(\Omega))} \|w\|_{L^2(0,T:H^1(\Omega))} \\ &\le c\|w\|_{L^2(0,T:H^1(\Omega))}, \end{aligned}$$

which implies

$$\left| < w, \operatorname{div}\left(\chi u_n \frac{|\nabla v_n|^{p-1} \nabla v_n}{1+\frac{1}{n}|\nabla v_n|} \right) >_{X,X'} \right| \leq c\|w\|_{L^2(0,T:H^1(\Omega))}. \tag{2.18}$$

Finally,

$$\begin{aligned} |< w, u_n(1-u_n)>_{X,X'}| &= \left| \int_0^T \int_\Omega u_n(1-u_n) w dx \right| \\ &\leq \|1-u_n\|_{L^\infty(\Omega)} \|w\|_{L^2(0,T:L^2(\Omega))} \|u_n\|_{L^2(0,T:L^2(\Omega))} \\ &\leq c\|w\|_{L^2(0,T:H^1(\Omega))}, \end{aligned}$$

which gives

$$|< w, u_n(1-u_n)>_{X,X'}| \leq c\|w\|_{L^2(0,T:H^1(\Omega))} \tag{2.19}$$

From (2.17), (2.18) and (2.19), we deduce the result. □

3 | EXISTENCE OF SOLUTIONS

The proof of the existence of solutions is given into several steps. First, we prove the existence of weak solutions of the approximated problems (2.1)–(2.4), and thanks to the parabolic regularity, we obtain the existence of a unique classical solution of the approximated problem. Thanks to the estimates obtained in Section 2, we prove the convergence of the solution of (2.1)–(2.4) to the weak solution of (1.1)–(1.4).

Lemma 3.1. *Let $p < 3/2$, $\alpha \in (0,1)$ and $T < \infty$, and then, under assumptions (1.5)–(1.7), there exists a classical solution*

$$u \in C^{\alpha,2+\alpha}_{t,x}(\Omega_T), \text{ where } \Omega_T = (0,T) \times \Omega,$$

of the approximated problems (2.1)–(2.4) satisfying

$$\|u_n\|_{L^\infty(\Omega)} \leq c$$

where $c < \infty$ is independent of t and T.

Proof. We consider a fixed point argument and define the following set:

$$A := \{w \in L^2(0,T : L^2(\Omega)),\ w \geq 0\ \|w\|_{L^\infty(0,T:L^\infty(\Omega))} \leq c(T)\}$$

where $c(T)$ is the constant obtained in Lemma 2.4 for $N = 2$ and Lemma 2.5 for $N \geq 3$. Now, we consider the function J

$$J : A \subset L^2(0,T : L^2(\Omega)) \to L^2(0,T : L^2(\Omega)).$$

Let $\tilde{u}_n \in L^2(0,T : L^2(\Omega))$ and define $J(\tilde{u}_n) = u_n$ as the solution to the problem

$$u_{nt} - \Delta u_n = -\operatorname{div}\left(\chi u_n | \frac{\nabla v_n|^{p-2} \nabla v_n}{1+\frac{1}{n}|\nabla v_n|} \right) + \mu u_n(1-\tilde{u}_n)$$

where v_n is the solution to

$$-\Delta v_n + v_n = \tilde{u}_n.$$

We first notice that J is a continuous function, and thanks to the estimates obtained in Section 2, we have that

(i) $J(A)$ is a precompact set in A;
(ii) $J(A) \subset A$;
(iii) $J(A)$ is a bounded set in $H^1(0,T : (H^1(\Omega))') \cap L^2(0,T : H^1(\Omega))$.

Thanks to Aubin–Lions lemma, we get the result for any $T < \infty$.
Moreover, we have that

$$v_n \in L^\infty(0,T : W^{2,q}(\Omega)) \cap H^1(0,T : H^1(\Omega)),$$

for any $q < \infty$. Then, we have that $u_{nt} - \Delta u_n \in L^q(0,T : L^q(\Omega))$. Thanks to the parabolic regularity (see, for instance, Quittner and Souplet,[24] Remark 48.3 (ii), p. 439), we obtain

$$u_n \in W^{1,q}(0,T : L^q(\Omega)) \cap L^q(0,T : W^{2,q}(\Omega))$$

for any $q < \infty$. Let Ω_T be defined as follows:

$$\Omega_T = (0,T) \times \Omega,$$

then,

$$W^{1,q}(0,T : L^q(\Omega)) \cap L^q(0,T : W^{2,q}(\Omega)) \subset C^{0,\alpha}(\Omega_T)$$

for any $\alpha < 1$, which implies that $u_n \in C^{0,\alpha}(\Omega_T)$. Thanks to the elliptic regularity, we have that

$$v_n \in C^{0,\alpha}(0,T : W^{2,q}(\Omega)) \cap C^{\alpha,2+\alpha}_{t,x}(\Omega_T)$$

see, for instance, Gilbart and Trudinger,[25] Theorem 6.2, p. 90. We write the equation of u_n as

$$u_{nt} - \Delta u_n + \sum_{i=1}^{N} b_i \frac{\partial u_n}{\partial x_i} = f,$$

where

$$\begin{aligned} b = (b_1 \cdots b_N) &:= \chi u_n \tfrac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}} \\ f(t,x) &:= \mu u_n(1-u_n(t,x)) - \chi u_n div(u_n(t,x)\tfrac{|\nabla v_n|^{p-2}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|^{p-1}}), \end{aligned}$$

and satisfy

$$b \in [C^{0,\alpha}(\Omega_T)]^N, \; f \in C^{0,\alpha}(\Omega_T).$$

We apply Theorem 10, p. 72 in Friedman[26] to obtain that the solution

$$u_n \in C^{\alpha,2+\alpha}_{t,x}(\Omega_T)$$

for any $\alpha \in (0,1)$, which implies that the solution of the approximated problem is a classical solution. Uniqueness of solutions to the approximated problem is obtained by following the standard procedure. □

Lemma 3.2. *Let $p < 3/2$; then, under assumptions (1.5)–(1.7), the weak solution of (2.1)–(2.4) converges to the weak solution of (1.1)–(1.4).*

Proof. We reproduce the steps given in Section 2 to obtain that

$$\begin{aligned} &\textstyle\int_\Omega |\nabla u_n|^2 dx && \le C, \\ &\|u_n\|_{L^\infty(\Omega)} && \le C, \\ &\|v_n\|_{W^{2,q}(\Omega)} && \le C \qquad \text{for any } q < \infty, \\ &\|u_{nt}\|_{L^2(0,T:(H^1(\Omega))')} && \le C(T+1). \end{aligned}$$

Since $u_n \in H^1(0,T : (H^1(\Omega))')$, we have the $v_n \in H^1(0,T : H^1(\Omega))$. We consider the inclusions of spaces

$$W^{2,N+1}(\Omega) \subset W^{1,\infty}(\Omega) \subset H^1(\Omega).$$

Since

$$v_n \in L^\infty(0,T : W^{2,N+1}(\Omega)) \cap H^1(0,T : H^1(\Omega)),$$

we apply Aubin–Lions lemma to deduce that there exists a subsequence v_{n_i} such that

$$v_{n_i} \to v^*$$

strong in

$$C([0,T] : W^{1,\infty}(\Omega)).$$

Moreover,

$$\frac{1}{1+\frac{1}{n_j}|\nabla v_{n_j}|} \to 1, \text{ strong in } L^q(0,T : L^q(\Omega)), \text{ for any } q < \infty,$$

$$|\nabla v_{n_j}|^{p-2}\nabla v_{n_j} \to |\nabla v^*|^{p-2}\nabla v^*, \text{ strong in } C([0,T] : L^\infty(\Omega)).$$

Moreover, there exists u^* and a subsequence u_{n_j} such that

$$\begin{aligned} u_{nj} &\to u^*, \text{ strong in } L^2(0,T : L^2(\Omega)). \\ u_{n_j} &\rightharpoonup u^*, \text{ weak in } L^2(0,T : H^1(\Omega)), \end{aligned}$$

and since $\|u_n\|_{L^\infty(\Omega)} \le C$, we have that

$$u_{n_j} \rightharpoonup u^*, \text{ weak in } L^q(0,T : L^q(\Omega)), \text{ for any } q < \infty,$$

and thanks to Banach–Alaoglu theorem,

$$u_{n_j} \rightharpoonup u^*, \text{weak}^*\text{in} L^\infty(0,T : L^\infty(\Omega)),$$

which implies, in particular, that $u^* \in L^\infty([0,T] : L^\infty(\Omega))$. Thanks to Lemma 2.6, we have that $\|u_n\|_{L^\infty(\Omega)} \le c$, then we obtain that $\|u\|_{L^\infty(\Omega)} \le c_6$. Since (u_n, v_n) satisfies

$$\int_0^T \int_\Omega u_{nt}\varphi dxdt + \int_0^T \int_\Omega \nabla u_n \nabla \varphi dxds = \int_0^T \int_\Omega \chi u_n \frac{|\nabla v_n|^{p-1}\nabla v_n}{1+\frac{1}{n}|\nabla v_n|}\nabla\varphi dxdt + \int_0^T \int_\Omega \mu u_n(1-(u_n)_+)\varphi dxdt, \tag{3.1}$$

$$\int_0^T \int_\Omega \nabla v_n \nabla \psi(s,x) dxdt = \int_0^T \int_\Omega (u_n - v_n)\,\psi(s,x) dxdt, \tag{3.2}$$

for any $\varphi, \psi \in L^2(0,T : H^1(\Omega)) \cap H^1(0,T : L^2(\Omega))$.

We take limits when $n \to \infty$ in (3.1) to obtain that (u^*, v^*) satisfies Definition 1.1 and proves the existence of weak solutions. □

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

ORCID

Silvia Sastre-Gomez https://orcid.org/0000-0002-7082-2726
Jose Ignacio Tello https://orcid.org/0000-0003-2671-7803

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