Automatica 153 (2023) 111053

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Robust coalitional model predictive control with plug-and-play capabilities ☆

Eva Masero [a,*], Pablo R. Baldivieso-Monasterios [b], José M. Maestre [a,c], Paul A. Trodden [b]

[a] Departament of Ingeniería de Sistemas y Automática, Universidad de Sevilla, Av. Camino de los Descubrimientos s/n, 41092 Sevilla, Spain
[b] Department of Automatic Control and Systems Engineering, University of Sheffield, Mappin Street, S1 3JD Sheffield, UK
[c] Department of Systems Science, Graduate School of Informatics, Kyoto University, Japan

ARTICLE INFO

Article history:
Received 1 October 2022
Received in revised form 23 February 2023
Accepted 13 March 2023
Available online xxxx

Keywords:
Distributed networked systems
Model predictive control
Control by clustering
Plug-and-play events
Tubes

ABSTRACT

This article presents a distributed implementation of a model predictive controller with information exchange to manage a distributed networked system of coupled dynamic subsystems. We propose a coalitional control method, where local controllers coalesce into clusters to improve performance, as a tool to solve plug-and-play problems. Our main contribution is a tube-based coalitional approach that employs online optimized invariant sets. These sets are instrumental in guaranteeing recursive feasibility and stability when faced with plug-and-play operations, *i.e.,* subsystems joining or leaving the network. We also explore the inherent robustness properties to absorb disturbances not covered by the tubes without the need to group local controllers. Finally, the simulation results show the benefits of our proposed control method.

1. Introduction

Networked systems are composed of numerous physically coupled distributed subsystems. Controlling the overall system while satisfying constraints and guaranteeing stability is not straightforward. For that reason, many studies have focused on distributed model predictive control (MPC) schemes (Christofides et al., 2013; Maestre & Negenborn, 2014), in which these challenging issues can be successfully handled.

The idea behind distributed control schemes is that each subsystem is managed by a local controller that forms the so-called *agent* (see Fig. 1), which can exchange information to improve global performance. Typically, the control network follows a fixed topology with prearranged enabled communication links between agents, where the cooperation effort sets a trade-off between conservatism and performance. Exploiting this trade-off by changing the control topology is the essence of coalitional MPC strategies (Chanfreut et al., 2021; Fele et al., 2017). According to how clusters of agents —the so-called *coalitions*— are selected, coalitional control schemes can be sorted into: (i) top-down architectures, if a supervisory controller decides the control topology (Barreiro-Gomez & Zhu, 2022; Jain et al., 2018; Núñez et al., 2015), and (ii) bottom-up structures, if the formation of coalitions is chosen at the agent level without global knowledge (Baldivieso-Monasterios & Trodden, 2021; Maxim & Caruntu, 2021; Mi et al., 2019). In the current work, we follow a top-down control architecture.

One of the key challenges in coalitional control, and indeed distributed control, is handling the disturbances that each agent experiences owing to the dynamic coupling between subsystems. Robust control techniques have been used to address this challenge. A first approach when designing local controllers is to consider couplings as bounded additive uncertainties to ensure stability and a suitable global performance (Richards & How, 2007). The most conservative way to model the presence of uncertainties is the Min–Max MPC (Scokaert & Mayne, 1998), which optimizes the control input for worst-case disturbances. The idea of *rigid* tubes proposed by Langson et al. (2004) has also become popular to guarantee robust stability for constrained linear systems (Mayne et al., 2005; Trodden & Richards, 2010). However, a significant drawback is induced by the tightening of local constraint sets by margins that may conservatively outer-bound the disturbances a local subsystem will experience. Further methods have been developed to minimize the conservatism of tube-based methods; *e.g.,* Riverso and Ferrari-Trecate (2012) propose applying tube-based control twice to exploit the region of attraction of the subsystem for the planned state trajectories of neighbors. Lucia et al. (2015) present a contract-based

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Marcello Farina under the direction of Editor Ian R. Petersen.

* Corresponding author.
E-mail addresses: evamasero@us.es (E. Masero), p.baldivieso@sheffield.ac.uk (P.R. Baldivieso-Monasterios), pepemaestre@us.es, pepemaestre@sys.i.kyoto-u.ac.jp (J.M. Maestre), p.trodden@sheffield.ac.uk (P.A. Trodden).

https://doi.org/10.1016/j.automatica.2023.111053
0005-1098/

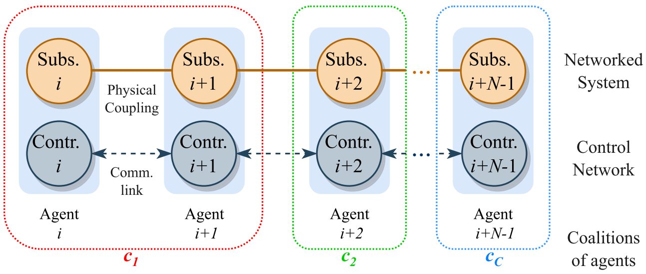


Fig. 1. Coalitional control scheme of a networked system composed by N agents, which can cluster into coalitions: $\{c_1, c_2, \ldots, c_C\}$.

distributed MPC strategy with recursive feasibility and input-to-state stability, where subsystems communicate to their neighbors the sequence of future values of coupling variables. Raković et al. (2012) introduce homothetic tubes in which the diameters of the state and control tubes are optimized online. In Trodden and Maestre (2017), coupling disturbances are rejected via optimized tubes, which are reconfigured online in order to more accurately outer-bound the disturbance set a subsystem experiences. A similar idea to the latter is the one pursued in the current work but expanded to a coalitional scheme.

Another major challenge of controlling distributed networked systems is that they are subject to changes in instrumentation (*e.g.*, sensors, and actuators) and subsystems that can be added or removed. Most control approaches lack flexibility to handle these changes and may require a redesign of the entire control system, which might not be feasible due to the costs of shutdown and start-up processes. In this context, the term Plug-and-Play (PnP) control is defined as a way to automatically reconfigure the controller after *plug-in* or *plug-out* of the system components. Several interpretations of PnP control have emerged in the literature over the last decades. For example, both Bendtsen et al. (2011) and Stoustrup (2009) explore a gradual reconfiguration of the control system after identifying the new hardware. Regarding fault-tolerant control, Bodenburg et al. (2014), Patton et al. (2007) and Riverso et al. (2016) employ PnP operations to automatically recover the control objective after process failures. In the field of microgrid applications (Dörfler et al., 2014; Lou et al., 2016), the PnP capability of the controller allows one to handle unknown and variable network conditions. Another interpretation made by Bodenburg et al. (2016), Lucia et al. (2015), and Riverso et al. (2014) proposes the design of distributed control schemes capable of dealing with plug-in and plug-out subsystems, guaranteeing global performance and stability. These schemes involve information transmission for the adaptation of local controllers affected by the PnP operation. Additionally, any PnP operations that negatively impact the feasibility and stability of the entire system are rejected.

Unlike the previously mentioned approaches, this article addresses the formation of clusters to avoid rejection of PnP operations. We also investigate the inherent subsystems' robustness not to redesign the controllers affected by PnP events. In particular, we cover this gap by proposing a tube-based coalitional MPC method with plug-and-play features for distributed linear networked systems. The subsystems, which are physically coupled, present constraints sets that can be scaled down by each agent, similar to what is proposed by Trodden and Maestre (2017) but with the difference that the agents here employ two scaling factors to build an inherent robust margin in order to absorb additional disturbances arisen from the PnP operations. In contrast to earlier studies where trajectories are exchanged among agents, our methodology allows sharing scaling factors among neighbors to reconfigure the disturbance sets. Stability guarantees for closed-loop control of the system are also provided by a terminal constraint formulation with positively invariant sets. Whereas previous studies proposed offline PnP operations (Riverso et al., 2014), we consider that they are performed in real time, and switching dynamics can be introduced. Therefore, the control topology can be reconfigured online in response to physical changes in the system. The main contributions of our work are:

- A tube-based coalitional MPC scheme in which agents can group in *coalitions* to find a trade-off between performance and communication costs. Moreover, coalitions are formed if agents cannot tolerate their local disturbances and disbanded when the feasibility is not spoiled and cost benefits are obtained.
- The introduction of *plug-and-play* operations by adding and removing subsystems in real time, while the controllers are automatically reconfigured to adapt to new characteristics of the network.
- The use of *public* and *private* scaling factors for constraint sets. Public information is broadcast in the system, while private information is individual and confidential for each subsystem. The rationale for these separate factors is to explore the inherent robustness properties to accommodate disturbances not covered by the tubes (*e.g.*, plug-and-play events) without the need for grouping local controllers.

Index of contents: Section 2 defines the problem settings. Section 3 formulates the tube-based MPC approach for the system in a distributed coalition setting. Section 4 details the coalitional control algorithm. Section 5 presents plug-and-play operations. Section 6 analyzes the recursive feasibility and stability. Section 7 illustrates the simulation results. Section 8 summarizes the main findings.

Notation. $\mathbb{N}_{0+}$ and $\mathbb{N}_{+}$ are the sets of non-negative and positive integers. $\mathbb{R}^n$ refers to an n-dimension set of real numbers. I denotes the identity matrix. For sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, the Minkowski sum is $\mathcal{X} \oplus \mathcal{Y} \triangleq \{x+y : x \in \mathcal{X},\ y \in \mathcal{Y}\}$; the Pontryagin difference is $\mathcal{X} \ominus \mathcal{Y} \triangleq \{z \in \mathbb{R}^n : \mathcal{Y} \oplus \{z\} \subseteq \mathcal{X}\}$ for $\mathcal{Y} \subseteq \mathcal{X}$; the subtraction operation is $\mathcal{X} \setminus \mathcal{Y} = \{x \in \mathcal{X} : x \notin \mathcal{Y}\}$; and the Cartesian product is $\mathcal{X} \times \mathcal{Y} \triangleq \{(x, y) : x \in \mathcal{X},\ y \in \mathcal{Y}\}$. If $\{\mathcal{X}_i\}_{i\in\mathcal{N}}$ is a finite family of sets indexed by $\mathcal{N} = \{1, \ldots, N\}$, then the Cartesian product $\prod_{i\in\mathcal{N}} \mathcal{X}_i$ is defined as $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N = \{(x_1, \ldots, x_N) : x_1 \in \mathcal{X}_1, \ldots, x_N \in \mathcal{X}_N\}$. The image of a set $\mathcal{X} \subseteq \mathbb{R}^n$ under a linear mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is given by $A\mathcal{X} \triangleq \{Ax : x \in \mathcal{X}\}$, and the diameter of the set is denoted as $\operatorname{diam}(\mathcal{X}) = \sup\{|x - y| : x, y \in \mathcal{X}\}$. The l_a-norm of the vector $x \in \mathbb{R}^n$ with $a \in \mathbb{N}_+$ is $\|x\|_a$, and $\|x\|_Q^2 = x^\top Q x$ with Q being a weighting matrix. The cardinality of $\mathcal{A}$ is denoted by $|\mathcal{A}|$; $\emptyset$ denotes the empty set. A polytope P is a bounded intersection of a finite set of half-spaces defined as $P = \{x \in \mathbb{R}^n : Gx \leqslant g\}$ with $G \in \mathbb{R}^{m\times n}$ and $g \in \mathbb{R}^m$. A set $\Omega \in \mathbb{R}^n$ is a robust positively invariant (RPI) set for system $x^+ = f(x, w)$ with constraints $\mathcal{X}$ and $\mathcal{W}$ if $\forall x \in \Omega \subset \mathcal{X}$ and $\forall w \in \mathcal{W}$, the system evolution fulfills $x^+ \in \Omega$. A set Ω is robust control invariant (RCI) for dynamics $x^+ = Ax + Bu + w$ with constraint sets $(\mathcal{X}, \mathcal{U}, \mathcal{W})$ if for any $x \in \Omega \subseteq \mathcal{X}$, there exists a control law $u = \mu(x) \in \mathcal{U}$ such that $x^+ \in \Omega$, for all $w \in \mathcal{W}$; the control law $\mu(\cdot)$ is said to induce invariance over the set Ω. A function $\alpha : \mathbb{R} \to \mathbb{R}$ is $\mathcal{K}$ continuous, if it is non-decreasing, and $\alpha(0) = 0$; it is $\mathcal{K}_\infty$ if it is also a radially unbounded function.

2. Problem formulation

This section describes the dynamics and constraints of the system, subsystems, and coalitions. We also describe how the information is exchanged amongst members of the network and the control objective.

2.1. System dynamics and constraints

Let us define a linear time-invariant (LTI) networked system:

$$x_{\mathcal{N}}^{+} = A_{\mathcal{N}} x_{\mathcal{N}} + B_{\mathcal{N}} u_{\mathcal{N}} + w_{\mathcal{N}}^{\mathrm{e}}, \tag{1}$$

where $x_{\mathcal{N}} \in \mathbb{R}^q$ is the current state and $x_{\mathcal{N}}^{+}$ its successor, $u_{\mathcal{N}} \in \mathbb{R}^r$ is the control input, and $w_{\mathcal{N}}^{\mathrm{e}}$ is the external disturbance. The system can be decomposed into a set of dynamically coupled subsystems $\mathcal{N} = \{1, 2, \ldots, N\}$, whose dynamics are

$$\begin{aligned} x_i^{+} &= A_{ii} x_i + B_{ii} u_i + w_i, \\ w_i &= \sum_{j \in \mathcal{M}_i} (A_{ij} x_j + B_{ij} u_j) + w_i^{\mathrm{e}}, \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^{q_i}$ and $u_i \in \mathbb{R}^{r_i}$ are the state and control input of subsystem $i \in \mathcal{N}$, and $w_i \in \mathbb{R}^{q_i}$ is the sum of the coupling through states and inputs with its neighbors $j \in \mathcal{M}_i \triangleq \{j \in \mathcal{N} \setminus \{i\} : [A_{ij} \;\; B_{ij}] \neq 0\}$ plus the external uncertainty w_i^{e}, which is assumed to be bounded by $\mathcal{W}_i^{\mathrm{e}}$.

Assumption 1 (*Controllability*)**.** The pair (A_{ii}, B_{ii}) is controllable for each $i \in \mathcal{N}$.

Each subsystem $i \in \mathcal{N}$ has restricted its states $x_i \in \mathcal{X}_i$ and inputs $u_i \in \mathcal{U}_i$.

Assumption 2 (*Constraints Sets*)**.** The sets $\mathcal{X}_i \subset \mathbb{R}^{q_i}$, $\mathcal{U}_i \subset \mathbb{R}^{r_i}$, and $\mathcal{W}_i^{\mathrm{e}} \subset \mathbb{R}^{q_i}$ are compact convex sets that contain the origin in their non-empty interiors.

2.2. Coalition dynamics and constraints

The approach of this work is to let subsystems cluster in the so-called *coalitions* to improve performance and deal with unexpected disturbances, such as plug-and-play operations.

Definition 1 (*Cooperation Topology*)**.** A cooperation topology Λ organizes the set of subsystems $\mathcal{N} = \{1, \ldots, N\}$ into a set of coalitions $\mathcal{C} = \{c_1, \ldots, c_C\}$ with $C \leqslant N$, satisfying:

- A coalition $c \in \mathcal{C}$ is a non-empty cluster of subsystems with $c \subseteq \mathcal{N}$, *i.e.*, it can range from a subsystem $c = \{i\}$ to the grand coalition $c = \mathcal{N}$.
- Coalitions are non-overlapping: $c \cap d = \varnothing$ for all $c \neq d$ and $c, d \in \mathcal{C}$.
- $\mathcal{C}$ defines a covering of $\mathcal{N}$, *i.e.*, $\bigcup_{c \in \mathcal{C}} c = \mathcal{N}$.

The discrete-time dynamics of coalition $c \in \mathcal{C}$ is

$$\begin{aligned} x_c^{+} &= A_{cc} x_c + B_{cc} u_c + w_c, \\ w_c &= \textstyle\sum_{d \in \mathcal{M}_c} (A_{cd} x_d + B_{cd} u_d) + w_c^{\mathrm{e}}, \end{aligned} \tag{3}$$

where $x_c = (x_i)_{i \in c}$ and $u_c = (u_i)_{i \in c}$ are the aggregate state and control input of the subsystems within the coalition c, which are, respectively, constrained by the sets $\mathcal{X}_c$ and $\mathcal{U}_c$. The state and input matrices are $A_{cc} = [A_{ij}]_{i,j \in c}$ and $B_{cc} = [B_{ij}]_{i,j \in c}$, and $w_c \in \mathcal{W}_c$ is the disturbance term due to the coupling with other coalitions plus external noise. The set of neighbors of coalition c is $\mathcal{M}_c \triangleq \{d \in \mathcal{C} \setminus \{c\} : [A_{cd} \;\; B_{cd}] \neq 0\}$.

Assumption 3. The constraints sets of coalition c are $\mathcal{X}_c = \prod_{i \in c} \mathcal{X}_i$ and $\mathcal{U}_c = \prod_{i \in c} \mathcal{U}_i$.

2.3. Control network

Let us define a cooperation control network described by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of *agents* and $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of links. Each enabled link $l_{ij} = l_{ji} \in \mathcal{L}$ connecting agents i and j is assumed to provide a bidirectional information flow that involves a fixed cooperation cost $\mathrm{c}_{\mathrm{link}} \in \mathbb{R}_{0+}$. The set of active links in the control network defines the controller cooperation topology Λ. Thus, the cardinality of the topology, *i.e.*, $|\Lambda|$, denotes the number of active links. Note that if $\mathrm{c}_{\mathrm{link}} = 0$, there will be no incentive to adopt a different topology from the centralized one because it provides the best performance from a control point of view.

Table 1
Relationship between the cooperation topologies and their sets of coalitions for a networked system $\mathcal{N} = \{1, 2, 3, 4\}$.

$\mathcal{T}$	l_{12}	l_{23}	l_{34}	$\mathcal{C}$
Λ_1	0	0	0	$\{\{1\}, \{2\}, \{3\}, \{4\}\}$
Λ_2	0	0	1	$\{\{1\}, \{2\}, \{3, 4\}\}$
Λ_3	0	1	0	$\{\{1\}, \{2, 3\}, \{4\}\}$
Λ_4	0	1	1	$\{\{1\}, \{2, 3, 4\}\}$
Λ_5	1	0	0	$\{\{1, 2\}, \{3\}, \{4\}\}$
Λ_6	1	0	1	$\{\{1, 2\}, \{3, 4\}\}$
Λ_7	1	1	0	$\{\{1, 2, 3\}, \{4\}\}$
Λ_8	1	1	1	$\{\{1, 2, 3, 4\}\}$

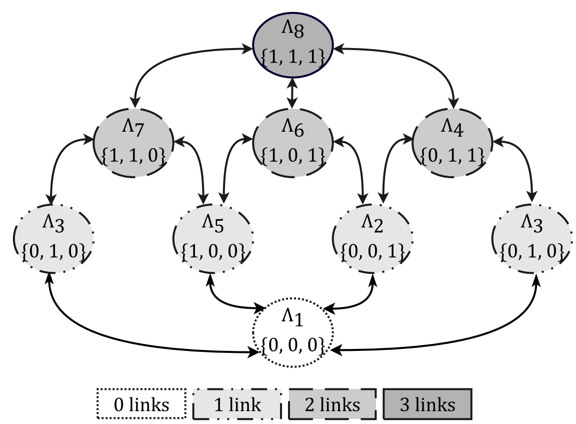


Fig. 2. The partially ordered set of cooperation topologies for a networked system $\mathcal{N} = \{1, 2, 3, 4\}$.

Given the total number of links $|\mathcal{L}|$, there are $2^{|\mathcal{L}|}$ different cooperation topologies, which are grouped into a set $\mathcal{T} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2^{|\mathcal{L}|}}\}$. For convenience, $\Lambda_1 = \Lambda_{\mathrm{dec}}$ represents the decentralized topology (all links are disabled), and $\Lambda_{2^{|\mathcal{L}|}} = \Lambda_{\mathrm{cen}}$ denotes the centralized topology (full cooperation). As an example, Table 1 shows the relationship between the topologies and their coalitional structures for a networked system $\mathcal{N} = \{1, 2, 3, 4\}$. The set $\mathcal{T}$ is a partially ordered set of cooperation topologies regarding active cooperation links, as shown in Fig. 2.

Assumption 4 (*Controllability of Coalitions*)**.** For any $\Lambda \in \mathcal{T}$, each pair (A_{cc}, B_{cc}) is controllable for any $c \in \mathcal{C}$.

Note that there could exist systems that satisfy Assumption 1 but not Assumption 4. If there are topologies that do not meet Assumption 4, they can be removed from the set of topologies we consider.

2.4. Optimal control problem

The control objective of MPC is to regulate the state of the networked system to its origin while satisfying all constraints and minimizing the following system-wide cost function in a finite prediction horizon N_{p}:

$$J_{\mathcal{N}}(x_{\mathcal{N}}, u_{\mathcal{N}}, \Lambda) = \sum_{k=0}^{N_{\mathrm{p}}} \ell\big(x_{\mathcal{N}}(k), u_{\mathcal{N}}(k)\big) + g(\Lambda),$$

where $\ell(\cdot)$ measures the distance to the origin and $g(\Lambda) = c_{\text{link}}|\Lambda|$ penalizes the amount of communication needed in the cooperation topology $\Lambda \in \mathcal{T}$.

3. Tube-based coalitional MPC

In this section, we formulate the tube-based MPC approach for the system in a distributed coalition setting. First, we define the notion of time-varying constraint and disturbance sets based on the scaling factors that reduce the conservatism of the tube approaches. Second, we explain the coalitional MPC problem. Finally, we detail the ingredients of the tube approach.

3.1. Time-varying sets

For a topology Λ with coalitional structure $\mathcal{C}$, by Definition 1, the constraint sets for each $c \in \mathcal{C}$ can be scaled by $\alpha_c, \beta_c \in \mathbb{R}_{0+}$:

$$\mathcal{X}_c(\alpha_c) = \alpha_c \mathcal{X}_c, \quad \mathcal{U}_c(\beta_c) = \beta_c \mathcal{U}_c, \tag{4}$$

where the sets of hard constraints are given by $\mathcal{X}_c = \mathcal{X}_c(1)$ and $\mathcal{U}_c = \mathcal{U}_c(1)$. Clearly, any state and control pair satisfying $(x_c, u_c) \in \mathcal{X}_c(\alpha_c) \times \mathcal{U}_c(\beta_c)$ satisfies the hard constraints if $\alpha_c \in [0, 1]$ and $\beta_c \in [0, 1]$. Taking into account the dynamics (3) and the constraints (4) of coalition c, the disturbance w_c is bounded by the set:

$$\mathcal{W}_c(\alpha, \beta) = \Big(\bigoplus_{d \in \mathcal{M}_c} A_{cd}\mathcal{X}_d(\alpha_d) \oplus B_{cd}\mathcal{U}_d(\beta_d) \Big) \oplus \mathcal{W}_c^{\text{e}}, \tag{5}$$

where the external disturbances are also assumed to be bounded by $\mathcal{W}_c^{\text{e}}$. The set $\mathcal{W}_c(\alpha, \beta)$ depends on all of c's neighbors, *i.e.,* (α_d, β_d) for all $d \in \mathcal{M}_c$; the notation $\mathcal{W}_c(\alpha, \beta)$ makes this dependency explicit and aims to simplify the notational burden. Given any compact set $\mathcal{W}_c \subset \mathbb{R}^{q_c}$, the triplet $(\mathcal{X}_c(\alpha_c), \mathcal{U}_c(\beta_c), \mathcal{W}_c)$ defines an RCI set $\Omega_c(\mathcal{W}_c)$, if it exists, such that $\Omega_c(\mathcal{W}_c) \subseteq \mathcal{X}_c(\alpha_c)$ and $\bigcup_{x_c \in \Omega_c(\mathcal{W}_c)} \mu(x_c) \subseteq \mathcal{U}_c(\beta_c)$. The set $\Omega_c(\cdot)$ can be parameterized by the disturbance set that affects the dynamics.

In our context, the existence of invariant sets determines a measure of robustness against the disturbances arising from the coupling; this is evident in a tube MPC setting, where the constraints are tightened according to their invariant sets. In our setting, we aim to use scaling factors to reduce conservative behaviors arising from aggressive constraint tightening to improve performance. However, this improvement of performance requires a starting point, and for this reason, we invoke the following assumption for each subsystem (current and future plug-in subsystems) to guarantee the existence of a family of RCI sets corresponding to the original constraints and the decentralized topology.

Assumption 5. There exists an RCI set $\Omega_c(\mathcal{W}_c(\mathbf{1}, \mathbf{1})) \subset \mathcal{X}_c(1)$ with $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{|\mathcal{M}_c|}$ for all $c \in \mathcal{C}$ for the decentralized cooperation topology Λ_{dec}.

3.2. Coalitional MPC problem

A tube-based approach has two main components: a nominal, *i.e.,* disturbance-free dynamics $z_c^+ = A_{cc}z_c + B_{cc}v_c$ regulated by an MPC controller with tightened constraints; and the error dynamics $e_c^+ = A_{cc}e_c + B_{cc}\mu(e_c) + w_c$ where $e_c = x_c - z_c$ is confined to an invariant set that, in our case, is the RCI set $\Omega_c(\mathcal{W}_c(\alpha, \beta))$. The nominal constraint sets are functions of this invariant set such that

$$\begin{aligned} \mathcal{Z}_c(\alpha_c) &\triangleq \mathcal{X}_c(\alpha_c) \ominus \Omega_c(\mathcal{W}_c(\alpha, \beta)), \\ \mathcal{V}_c(\beta_c) &\triangleq \mathcal{U}_c(\beta_c) \ominus \mu\big(\Omega_c(\mathcal{W}_c(\alpha, \beta))\big). \end{aligned} \tag{6}$$

Let us define $V_c^{\text{o}} = \{v_c^{\text{o}}(0), \dots, v_c^{\text{o}}(N_{\text{p}} - 1)\}$ and $Z_c^{\text{o}} = \{z_c^{\text{o}}(0), \dots, z_c^{\text{o}}(N_{\text{p}})\}$, respectively, as the optimal control and state sequences on a prediction horizon N_{p}. The control objective of coalition c is to regulate the nominal state z_c to the origin whilst minimizing the N_{p}-horizon cost $J_c(Z_c, V_c)$, defined in the next section, subject to constraints:[1]

$$\begin{aligned} V_c^{\text{o}} = \arg\min_{V_c} \quad & J_c\big(Z_c, V_c\big), \\ \text{s.t.} \quad & z_c(0) = \tilde{z}_c, \\ & z_c(t+1) = A_{cc}z_c(t) + B_{cc}v_c(t), \\ & z_c(t) \in \mathcal{Z}_c(\alpha_c), \ \ v_c(t) \in \mathcal{V}_c(\beta_c), \\ & z_c(N_{\text{p}}) \in \Omega_c^{\text{f}}, \end{aligned} \tag{7}$$

where $t = 0, \dots, N_{\text{p}} - 1$, $\tilde{z}_c$ is the current value of the nominal state, and α_c, β_c are the coalition scaling factors for state and input constraint sets, respectively. From Assumption 3 and the properties of the Cartesian product, the coalition scaling factors are the corresponding values to be fulfilled: $\alpha_c\mathcal{X}_c = \prod_{i \in c} \alpha_i \mathcal{X}_i$, and $\beta_c\mathcal{U}_c = \prod_{i \in c} \beta_i \mathcal{U}_i$. When a coalition is disbanded, $\alpha_c\mathcal{X}_c = \prod_{i \in c} \alpha_c \mathcal{X}_i$, which implies that individual scaling factors $\alpha_i = \alpha_c$ for all $i \in c$. A similar observation holds for the input scaling factor β_c. Moreover, the set Ω_c^{f} is a terminal set that depends on the scaling factors and is assumed to satisfy the following.

Assumption 6. The terminal set Ω_c^{f} is positively invariant (PI) for the nominal dynamics $z_c^+ = A_{cc}z_c + B_{cc}v_c$, that is, $(A_{cc} + B_{cc}K_c^{\text{f}})\Omega_c^{\text{f}} \subseteq \Omega_c^{\text{f}}$ with $\Omega_c^{\text{f}} \subseteq \mathcal{Z}_c(\alpha_c)$ and $K_c^{\text{f}}\Omega_c^{\text{f}} \subseteq \mathcal{V}_c(\beta_c)$ under a control law $v_c = K_c^{\text{f}} z_c$.

Note that although the terminal set is considered to be merely PI and not RPI, we can select $\Omega_c^{\text{f}} = 0$ if needs be in order to reduce computation efforts in high-order dynamics.

3.3. Coalition cost function

The arguments of the finite-horizon cost $J_c\big(\cdot, \cdot\big)$ for each $c \in \mathcal{C}$ are the N_{p}–sequence of control actions V_c, and the $N_{\text{p}} + 1$–sequence of states Z_c. This cost function is defined as:

$$J_c\big(Z_c, V_c\big) = \sum_{t=0}^{N_{\text{p}}-1} \Big(\ell_c\big(z_c(t), v_c(t)\big) \Big) + f_c\big(z_c(N_{\text{p}})\big), \tag{8}$$

where $\ell_c(z_c, v_c) \triangleq \|z_c\|_{Q_c}^2 + \|v_c\|_{R_c}^2$ is the stage cost, which penalizes *nominal* state z_c and input v_c weighted by matrices $Q_c \succ 0$ and $R_c \succ 0$. The function $f_c(z_c) = \|z_c\|_{P_c}^2$ with $P_c \succ 0$ is the terminal cost designed such that $z_c^\top P_c z_c - z_c^{+\top} P_c z_c^+ \geqslant \ell_c(z_c, K_c^{\text{f}} z_c)$. Therefore, $z_c^\top P_c z_c$ is a control Lyapunov function, and the local stability of coalition c is guaranteed around the origin.

3.4. Outer bounding of RCI sets

The explicit computation of RCI sets is computationally costly and increases as the size of the coalition grows. Since the RCI sets tighten the constraint sets as (6), we can employ an outer bound of the RCI set with the idea of reducing the computational burden and making the approach suitable for high-order dynamics (Baldivieso-Monasterios & Trodden, 2021). One can rely on the implicit existence of an RCI set to guarantee closed-loop feasibility and stability.

[1] The control problem of each coalition can be solved by a local controller that works as a leader or distributed among the agents in the coalition (Franzè et al., 2018).

As proposed by Raković et al. (2007), we can formulate a linear programming (LP) problem to find an RCI set Ω_c as:

$$\min\{\epsilon : \phi \in \Phi\}, \tag{9}$$

where $\phi = (\mathbf{M}_{H_c}, a_c, b_c, \epsilon)$ and $\Phi = \{\phi : \mathbf{M}_{H_c} \in \mathcal{M}_{H_c},\ \Omega_c \subseteq a_c\mathcal{X}_c,\ \mu_c(\Omega_c) \subseteq b_c\mathcal{U}_c,\ (a_c, b_c) \in [0,1] \times [0,1],\ q_x a_c + q_u b_c \leqslant \epsilon\}$ with weights q_x and q_u to provide the constants (a_c, b_c) that scale the state and input constraint sets, and matrices $\mathbf{M}_{H_c} = (M_0, \ldots, M_{H_c-1})$, with $M_l \in \mathbb{R}^{q_c \times r_c}$ and $l = 1, \ldots, H_c - 1$, characterizing the optimized RCI set for a system $x_c^+ = A_{cc}x_c + B_{cc}u_c + w_c$ constrained in $(\mathcal{X}_c, \mathcal{U}_c, \mathcal{W}_c)$ as

$$\Omega_c = \bigoplus_{h=0}^{H_c-1} D_h(\mathbf{M}_{H_c})\mathcal{W}_c, \text{ and } \mu_c(\Omega_c) = \bigoplus_{h=0}^{H_c-1} M_h\mathcal{W}_c.$$

The set of control inputs that induce invariance $\mu_c(\Omega_c)$ is defined as $\mu_c(\Omega_c) \triangleq \{u_c \in \mathcal{U}_c : x_c^+ \in \Omega_c, \forall w_c \in \mathcal{W}_c\}$. For $h = 0, \ldots, H_c$, matrices $D_h(\mathbf{M}_{H_c})$ are defined as

$$D_h(\mathbf{M}_{H_c}) = \begin{cases} I & \text{if } h = 0, \\ A_{cc}^h + \sum_{l=0}^{h-1} A_{cc}^{h-1-l}B_{cc}M_l & \text{if } h \geqslant 1, \end{cases}$$

such that $D_H(\mathbf{M}_{H_c}) = 0$ provided that H_c is greater than or equal to the controllability index of (A_{cc}, B_{cc}). The set of matrices that meet these criteria is $\mathcal{M}_{H_c} \triangleq \{\mathbf{M}_{H_c} : D_{H_c}(\mathbf{M}_{H_c}) = 0\}$. Constraint satisfaction is guaranteed if $\Omega_c \subseteq a_c\mathcal{X}_c$ and $\mu_c(\Omega_c) \subseteq b_c\mathcal{U}_c$. Consequently, the constraint sets (6) can be replaced with

$$\begin{aligned} \mathcal{Z}_c(\alpha_c, a_c) &= \mathcal{X}_c(\alpha_c) \ominus a_c\mathcal{X}_c(\alpha_c) = (1 - a_c)\mathcal{X}_c(\alpha_c), \\ \mathcal{V}_c(\beta_c, b_c) &= \mathcal{U}_c(\beta_c) \ominus b_c\mathcal{U}_c(\beta_c) = (1 - b_c)\mathcal{U}_c(\beta_c), \end{aligned} \tag{10}$$

which implies that, in (7), terminal set Ω_c^{f} also depends on $\alpha_c, \beta_c, a_c, b_c$, that is, $\Omega_c^{\mathrm{f}}(\alpha_c, \beta_c, a_c, b_c)$.

3.5. Tube-based approach

We propose an MPC strategy based on optimized tubes to control the coalition dynamics (3) through the control law:

$$u_c = v_c^{\mathrm{o}}(z_c) + \mu_c(x_c - z_c), \tag{11}$$

where v_c^{o} is the first element in the optimized input sequence (*i.e.*, $V_c^{\mathrm{o}}(1)$), and $\mu_c(x_c - z_c)$ is the RCI control law. Moreover, each coalition c has constraint sets that can be scaled down by its controller. Our approach goes a step further than that of Trodden and Maestre (2017) by separating the scaling factors for constraint sets into two types: *public* and *private*. The core idea behind this segregation of factors is to create an extra robustness margin to handle uncertainties not covered by the tubes, such as PnP events. This approach allows for the accommodation of uncertainty locally, without requiring any significant reconfiguration of the control system. Conversely, the method that uses a single scaling factor (Trodden & Maestre, 2017) may require full cooperation to address disturbances caused by PnP events, or these events may even cause infeasibility of the controllers/optimal control problems.

The rationale of two scaling factors is that agents maintain and optimize *private* scaling factors that tightly bound their predicted trajectories but communicate larger *public* scaling factors to neighbors; thus, the gap between private and public values allows agents to absorb locally disturbances. In particular, sets $(\mathcal{X}_c, \mathcal{Z}_c, \mathcal{U}_c, \mathcal{V}_c, \mathcal{W}_c, \Omega_c, \Omega_c^{\mathrm{f}})$ are parameterized by time-varying public scaling factors $(\alpha_c^{\mathrm{pub}}, \beta_c^{\mathrm{pub}})$, and private scaling factors $(\alpha_c^{\mathrm{priv}}, \beta_c^{\mathrm{priv}})$ are added as optimization variables to the following nominal problem, which replaces (7):

$$\begin{aligned} J_{N,c}^{\mathrm{o}}(z_c) = \min_{V_c,\ \alpha_c^{\mathrm{priv}},\ \beta_c^{\mathrm{priv}}} \ & J_c(Z_c, V_c) + \tau_\alpha\ \alpha_c^{\mathrm{priv}} + \tau_\beta\ \beta_c^{\mathrm{priv}}, \\ \text{s.t. } & (\alpha_c^{\mathrm{priv}}, \beta_c^{\mathrm{priv}}) \in [0,1]^2, \quad z_c(0) = \tilde{z}_c, \\ & z_c(t+1) = A_{cc}z_c(t) + B_{cc}v_c(t), \quad t = 0, \ldots, N_{\mathrm{p}} - 1, \\ & z_c(t) \in \alpha_c^{\mathrm{priv}}\mathcal{Z}_c(\alpha_c^{\mathrm{pub}}, a_c), \quad t = 1, \ldots, N_{\mathrm{p}} - 1, \\ & v_c(t) \in \beta_c^{\mathrm{priv}}\mathcal{V}_c(\beta_c^{\mathrm{pub}}, b_c), \quad t = 0, \ldots, N_{\mathrm{p}} - 1, \\ & z_c(N_{\mathrm{p}}) \in \Omega_c^{\mathrm{f}}(\alpha_c^{\mathrm{pub}}, \beta_c^{\mathrm{pub}}, a_c, b_c), \end{aligned} \tag{12}$$

where $J_{N,c}^{\mathrm{o}}(z_c)$ is the value function and weights $\tau_\alpha, \tau_\beta \in \mathbb{R}_+$.

As a consequence of allowing the constraints sets to shrink, the dynamics of public scaling factors arise naturally, as detailed in the next lemma.

Lemma 1. *Given topology Λ, each coalition $c \in \mathcal{C}$ has public scaling factors $(\alpha_c^{\mathrm{pub}}, \beta_c^{\mathrm{pub}})$ at time instant k, which parameterize constraint sets $(\mathcal{X}_c(\alpha_c^{\mathrm{pub}}), \mathcal{U}_c(\beta_c^{\mathrm{pub}}))$ and evolve as:*

$$\begin{aligned} \alpha_c^{\mathrm{pub}+} &= \alpha_c^{\mathrm{pub}}\big(a_c + \alpha_c^{\mathrm{priv}}(1 - a_c)\big), \\ \beta_c^{\mathrm{pub}+} &= \beta_c^{\mathrm{pub}}\big(b_c + \beta_c^{\mathrm{priv}}(1 - b_c)\big). \end{aligned} \tag{13}$$

Proof. By solving problem (12), the successor state $x_c^+ = z_c^+ + e_c^+$ depends on the nominal state $z_c^+ \in \alpha_c^{\mathrm{priv}}\mathcal{Z}_c(\alpha_c^{\mathrm{pub}}, a_c)$ and the state mismatch $e_c^+ \in a_c\mathcal{X}_c(\alpha_c^{\mathrm{pub}})$. Taking into account (10):

$$\begin{aligned} x_c^+ &\in \big(\alpha_c^{\mathrm{priv}}(1 - a_c)\mathcal{X}_c(\alpha_c^{\mathrm{pub}})\big) \oplus a_c\mathcal{X}_c(\alpha_c^{\mathrm{pub}}) \\ &\in \big(\alpha_c^{\mathrm{priv}}(1 - a_c) + a_c\big)\alpha_c^{\mathrm{pub}}\mathcal{X}_c \\ &\in \alpha_c^{\mathrm{pub}+}\mathcal{X}_c. \end{aligned}$$

Therefore, the state constraint set $\mathcal{X}_c$ at instant $k+1$ is scaled by a parameter $\alpha_c^{\mathrm{pub}+} = \alpha_c^{\mathrm{pub}}\big(a_c + \alpha_c^{\mathrm{priv}}(1 - a_c)\big)$. In a similar way, the successor control input: $u_c^+ = v_c^{\mathrm{o}}(0) + \mu_c(e_c^+)$, where $v_c^{\mathrm{o}} \in \beta_c^{\mathrm{priv}}\mathcal{V}_c(\beta_c^{\mathrm{pub}}, b_c)$ and $\mu_c(e_c^+) \in b_c\mathcal{U}_c(\beta_c^{\mathrm{pub}})$, satisfies $u_c^+ \in \beta_c^{\mathrm{pub}+}\mathcal{U}_c$. □

The significance of Lemma 1 is that since $0 < a_c < 1$, $(a_c + \alpha_c^{\mathrm{priv}}(1 - a_c))$ is a number less than one whenever $\alpha_c^{\mathrm{priv}} < 1$, thus public scaling factors are reduced at a rate given by α_c^{priv}. If $\alpha_c^{\mathrm{priv}} = 1$, then the public scaling factor will remain constant.

4. Top-down control algorithm

We implement a top-down coalitional MPC algorithm, which is divided into an upper and a lower control layer.

4.1. Upper control layer

Every $T_{\mathrm{up}} \in \mathbb{N}_+$ time steps, a central supervisor executes Alg. 1 to select the best cooperation topology that ensures recursive feasibility. Since the number of topologies increases combinatorially with the number of subsystems, we consider a suitable subset of $\mathcal{T}$ to reduce this potential bottleneck.

Definition 2 (*Set of Potential Successor Topologies*)**.** Let Λ_{cur} be the current topology, we define the set of the potential successor topologies $\mathcal{T}_{\mathrm{new}} \subseteq \mathcal{T}$ based on the Hamming distance between two topologies:

$$\mathcal{T}_{\mathrm{new}} \triangleq \{\Lambda \in \mathcal{T} : \mathrm{dist}(\Lambda_{\mathrm{cur}}, \Lambda) \leqslant 1\}. \tag{14}$$

For example, if $\Lambda_{\mathrm{cur}} = \Lambda_5$, the set of potential successor topologies whose distance is less than or equal to 1 is $\mathcal{T}_{\mathrm{new}} = \{\Lambda_1, \Lambda_5, \Lambda_6, \Lambda_7\}$, as shown in Fig. 2.

Alg. 1: Upper control layer

Initial data: $\mathcal{X}_i, \mathcal{U}_i, H_i, K_i^{\mathrm{f}}, \ \forall i \in \mathcal{N}, T_{\mathrm{up}}, N_{\mathrm{p}}, \tau_\alpha, \tau_\beta, \mathrm{c}_{\mathrm{link}}$.
Start: $z_i(0) = x_i(0)$; $\Lambda_{\mathrm{cur}} = \Lambda_{\mathrm{cen}}$; $\alpha_i^{\mathrm{pub}}, \alpha_i^{\mathrm{priv}}, \beta_i^{\mathrm{pub}}, \beta_i^{\mathrm{priv}} = 1, \ \forall i \in \mathcal{N}$.
Inputs: $\Lambda_{\mathrm{cur}}, \alpha_i^{\mathrm{pub}}, \beta_i^{\mathrm{pub}}, \forall i \in \mathcal{N}$. **Output:** Λ_{new}

1: Given Λ_{cur}, measure the current values of states $\tilde{x}_c$ and $\tilde{z}_c, \forall c \in \mathcal{C}$.
2: Calculate $\mathcal{T}_{\mathrm{new}}$ as (14).
3: **for** each $\Lambda_{\mathrm{new}} \in \mathcal{T}_{\mathrm{new}}$ **do**:
4: Compute $\mathcal{W}_c$ as (5), Ω_c by (9), and $\Omega_c^{\mathrm{f}}, \forall c \in \mathcal{C}$.
5: **if** $\nexists \Omega_c$ for any c **then**:
6: Mark this Λ_{new} as infeasible, and go to Step 3.
7: **end if**
8: **for** each $c \in \mathcal{C}$ **do**:
9: Solve (12) setting $\tau_\alpha, \tau_\beta = 0$ and $\alpha_c^{\mathrm{priv}}, \beta_c^{\mathrm{priv}} = 1$ to obtain the control sequence U_c via (11).
10: Gauge $\Gamma_c \triangleq \sum_{t=1}^{N_{\mathrm{p}}} \big(\ell_c(x_c(t), u_c(t)) + \mathrm{c}_{\mathrm{link}} |\Lambda_c|\big)$.
11: **end for**
12: Compute the cost $\Gamma_\Lambda = \sum_{c \in \mathcal{C}} \Gamma_c$ for Λ_{new}.
13: **end for**
14: **if** all $\Lambda_{\mathrm{new}} \in \mathcal{T}_{\mathrm{new}}$ are marked as infeasible **then**:
15: Any c with $\nexists \Omega_c$ clusters with the neighbor $d \in \mathcal{M}_c$ with the largest diameter of $\mathcal{W}_d$, and update Λ_{cur}.
16: Go to Step 2.
17: **else**
18: Select topology $\Lambda_{\mathrm{new}} \in \mathcal{T}_{\mathrm{new}}$ with the lowest cost Γ_Λ.
19: Send Λ_{new} to the lower layer (Alg. 2).
20: **end if**

4.2. Lower control layer

Each time instant k, each coalition $c \in \mathcal{C}$ execute Algorithm 2 according to the current public scaling factors and topology Λ_{cur}:

Alg. 2: Lower control layer

Initial data: $\mathcal{X}_c, \mathcal{U}_c, K_c^{\mathrm{f}}, N_{\mathrm{p}}, H_c, \tau_\alpha, \tau_\beta, \sigma_c = 0$.
Inputs: $\Lambda_{\mathrm{cur}}, x_c, \alpha_c^{\mathrm{pub}}, \beta_c^{\mathrm{pub}}, \ \forall c \in \mathcal{C}$
Outputs: $\alpha_c^{\mathrm{pub}+}, \beta_c^{\mathrm{pub}+}, x_c^+$

1: Calculate $\mathcal{W}_c$ as (5), Ω_c by (9), and Ω_c^{f}.
2: Solve (12) to obtain $V_c^{\mathrm{o}}, \alpha_c^{\mathrm{priv}}$, and β_c^{priv}.
3: Apply $v_c^{\mathrm{o}} = V_c^{\mathrm{o}}(1)$ to attain z_c^+, and u_c to obtain x_c^+.
4: Get $\alpha_c^{\mathrm{pub}+}$ and $\beta_c^{\mathrm{pub}+}$ as (13), and share them with $\mathcal{M}_c$.
5: Compute $\mathcal{W}_c^+$ and then Ω_c^+.
6: **if** Ω_c^+ do not exist or $x_c^+ - z_c^+ \notin \Omega_c^+$, for any $c \in \mathcal{C}$ **then**:
7: Active a flag $\sigma_c = 1$ and share it to the network.
8: **end if**
9: **if** any σ_c is active in the network **then**:
10: Set $\alpha_c^{\mathrm{pub}+}$ and $\beta_c^{\mathrm{pub}+}$ with the current scaling values.
11: **end if**

5. Plug-and-play operations

Adding (removing) subsystems to (from) the system changes the physical configuration of the network. Consequently, it may force the redesign of the cooperation control topology for stability and performance reasons. We consider the following:

(a) Instants k_{plug}^- and k_{plug}^+ are, respectively, infinitesimal instants before and after a plug-and-play operation.
(b) *Plug-in* and *plug-out* are allowed: $\mathcal{N}(k)$[2] can grow or shrink and, correspondingly, each $\mathcal{M}_i(k)$ can grow or shrink owing to the addition or removal of subsystems.
(c) Plug-and-play operations in which the subsystems are partially disconnected (connected) from (to) the network, *e.g.*, $\mathcal{N}(k_{\mathrm{plug}}^-) = \mathcal{N}(k_{\mathrm{plug}}^+)$ but $\mathcal{M}_i(k_{\mathrm{plug}}^-) \neq \mathcal{M}_i(k_{\mathrm{plug}}^+)$ for some $i \in \mathcal{N}(k_{\mathrm{plug}}^-)$, are not permitted.
(d) The PnP operations are executed sequentially and requested to the supervisor, which triggers the execution of the upper layer and may adapt the cooperation topology to the new system scenario for stability and performance reasons. If there were several PnP operation requests at k_{plug}^-, these may be queued and executed in a FIFO fashion. Furthermore, one could let agents form a coalition to, for example, connect to the system altogether at once.

5.1. Adding subsystems

The current cooperation topology Λ for system $\mathcal{N}$ has a coalitional structure $\mathcal{C}_\Lambda$. Consider a new subsystem $N+1$ dynamically defined by its corresponding state and input matrices, constraint sets $(\mathcal{X}_{N+1}, \mathcal{U}_{N+1})$ and K_{N+1}^{f} that is plugged into the system, which yields the following set of dynamically coupled subsystems:

$$\mathcal{N}(k) = \begin{cases} \{1, 2, \ldots, N\} & \text{if } k < k_{\mathrm{plug}}, \\ \{1, 2, \ldots, N, N+1\} & \text{if } k \geqslant k_{\mathrm{plug}}. \end{cases}$$

Therefore, the new topology $\tilde{\Lambda}$ for $\mathcal{N} \cup \{N+1\}$ has a coalitional structure $\mathcal{C}_{\tilde{\Lambda}} = \mathcal{C}_\Lambda \cup \{N+1\}$. The PnP operation changes the dynamics of the overall system, and also the set of possible cooperation topologies from $\mathcal{T}$ to $\tilde{\mathcal{T}}$. Due to the couplings, the disturbances of its neighbors $\mathcal{M}_{N+1}$ grow and the recursive feasibility may be lost. To prevent that from happening, at k_{plug}^+, we allow the cooperation topology to change according to Alg. 1.

Assumption 7. The new subsystem $N+1$, with neighbors $\mathcal{M}_{N+1} \subset \mathcal{N}$ and constraint sets $(\alpha_{N+1}^{\mathrm{pub}} \mathcal{X}_{N+1}, \beta_{N+1}^{\mathrm{pub}} \mathcal{U}_{N+1})$ with $(\alpha_{N+1}^{\mathrm{pub}}, \beta_{N+1}^{\mathrm{pub}}) \in (0, 1)^2$, joins the system at time $k_{\mathrm{plug}} > 0$, *i.e.*, $\mathcal{N}(k) = \mathcal{N} \cup \{N+1\}$. Moreover, there exists an RCI set $\Omega_{N+1}\big(\mathcal{W}_{N+1}(\alpha^{\mathrm{pub}}, \beta^{\mathrm{pub}})\big)$ for the new element, and its initial state x_{N+1} is feasible.

5.2. Removing subsystems

Let us assume that a subsystem i is unplugged from the system at the instant k_{unplug}:

$$\mathcal{N}(k) = \begin{cases} \{1, 2, \ldots, i, \ldots, N\} & \text{if } k < k_{\mathrm{unplug}}, \\ \{1, 2, \ldots, i-1, i+1, \ldots, N\} & \text{if } k \geqslant k_{\mathrm{unplug}}. \end{cases}$$

This PnP operation changes the dynamics of the overall system, the graph from $\mathcal{G}$ to $\tilde{\mathcal{G}}$, and the set of potential successor topologies from $\mathcal{T}$ to $\tilde{\mathcal{T}}$. Since the disturbances of their neighbors $j \in \mathcal{M}_i(k)$ decrease, the recursive feasibility and stability are not endangered. We could then execute Alg. 1 to select another topology that improves performance or maintains the current cooperation setting, which is computationally less expensive.

6. Feasibility and stability

In this section, we describe the properties of recursive feasibility and stability of the proposed algorithm. First, we focus on the recursive feasibility side of the problem. We highlight the potential issues that may arise and how the operations of

[2] Henceforth, explicit time-dependent notation will be employed in the case of ambiguity.

adding and removing elements of the network affect the overall feasibility. Then, leveraging the feasibility results, we study the stability of the closed-loop system with respect to a compact neighborhood of the equilibrium.

6.1. Feasibility sets

Our feasibility analysis of the closed-loop system begins with a characterization of the feasible sets of optimization problem (12) for each coalition c associated with a topology Λ. The feasible input set of (12) contains all control sequences $\{v_c(0), \ldots, v_c(N_{\text{p}}-1)\}$ and scaling factors $(\alpha_c^{\text{priv}}, \beta_c^{\text{priv}})$ that can be parameterized by the initial nominal state, *i.e.*, $\mathbb{V}_c^N(z_c) \subset \mathcal{V}_c^N \times [0,1]^2$. The feasible state set for coalition c is therefore $\mathbb{Z}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}}) = \{z_c \in \mathcal{Z}_c(\alpha_c^{\text{pub}}, a_c) : \mathbb{V}_c^N(z_c) \neq \emptyset\}$; since the public scaling factors are not necessarily constant, the feasible set varies over time. The feasible state set for the *true* dynamics of coalition $c \in \mathcal{C}$ is simply $\mathbb{X}_c^N = \mathbb{Z}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}}) \oplus \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}}, \beta^{\text{pub}}))$. A state $x_c = z_c + e_c \in \mathbb{X}_c^N$ is said to be recursively feasible if $\mathbb{V}_c^N(z_c^+) \neq \emptyset$ and $e_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}+}, \beta^{\text{pub}+}))$ where z_c^+, e_c^+, $\alpha_c^{\text{pub}+}$, and $\beta_c^{\text{pub}+}$ are, respectively, the successor nominal state, the error, and the state and input scaling factors. In the case of constant scaling factors, we recover the traditional definition of recursive feasibility. In the following, we study the nuances of the recursive feasibility of our proposed scheme.

6.2. Recursive feasibility for unchanging coalitions

Parameterization of constraints by scaling factors results in changes in the RCI sets used by the tube-based controller. In the next lemma, we explore the relationship between these scaling factors and the RCI sets.

Lemma 2 (*Smaller W_c Implies Smaller $\Omega_c(W_c)$*). *Suppose that $\Omega_c(\gamma_c W_c)$ is RCI w.r.t. the constraint sets $(\alpha_c \mathcal{X}_c, \beta_c \mathcal{U}_c)$ and disturbance $\gamma_c W_c$ where W_c is a compact and convex set and γ_c is the smallest positive number such that $\gamma_c W_c \supseteq \mathcal{W}_c(\alpha, \beta) = \left(\bigoplus_{d \in \mathcal{M}_c} A_{cd}\alpha_d \mathcal{X}_d \oplus B_{cd}\beta_d \mathcal{U}_d\right) \oplus \mathcal{W}_c^{\text{e}}$. Suppose, in addition, that $\alpha_c^+ \leqslant \alpha_c$ and $\beta_c^+ \leqslant \beta_c$ for all $c \in \mathcal{C}$, then $\gamma_c^+ \leqslant \gamma_c$ and $\mathcal{W}_c(\alpha^+, \beta^+) \subseteq \mathcal{W}_c(\alpha, \beta)$ with γ_c^+ a scaling factor associated with α^+ and β^+. Furthermore, $\Omega_c(\gamma_c^+ W_c) \subseteq \Omega_c(\gamma_c W_c)$ is the RCI with disturbance set $\gamma_c^+ W_c$ and constraint sets $(\alpha_c^+ \mathcal{X}_c, \beta_c^+ \mathcal{U}_c)$.*

Proof. The smallest outer scaling γ_c is the tightest scaling factor such that $\gamma_c = \arg\inf\{\lambda(\xi W_c \setminus \mathcal{W}_c(\alpha, \beta))$ where $\lambda(\cdot)$ is the Lebesgue measure in $\mathbb{R}^{q_c}$. Since $\alpha_c^+ \leqslant \alpha_c$ and $\beta_c^+ \leqslant \beta_c$ for all coalitions, then it is straightforward to show that $\mathcal{W}_c(\alpha^+, \beta^+) \subseteq \mathcal{W}_c(\alpha, \beta)$. The optimality of γ_c^+ implies that $\lambda(\gamma_c^+ W_c \setminus \mathcal{W}_c(\alpha^+, \beta^+)) \leqslant \lambda(\xi W_c \setminus \mathcal{W}_c(\alpha^+, \beta^+))$ for all $\xi > 0$; in particular, this relation holds for $\xi = \gamma_c$. The standard properties of the Lebesgue measure imply $\lambda(\xi W_c \setminus \mathcal{W}_c(\alpha^+, \beta^+)) = \lambda(\xi W_c) - \lambda(\mathcal{W}_c(\alpha^+, \beta^+))$ and $\lambda(\xi W_c) = \xi^{q_c}\lambda(W_c)$. Using these properties with $\lambda(W_c) > 0$ implies $\gamma_c^+ \leqslant \xi$; in particular, $\xi = \gamma_c$. The RCI condition of $\Omega_c(S_c)$ holds for all $S_c \subseteq \gamma_c W_c$, especially for $S_c = \gamma_c^+ W_c$. □

Lemma 2 guarantees that the RCI set $\Omega_c(\mathcal{W}_c(\alpha^{\text{pub}}, \beta^{\text{pub}}))$ remains an RCI set when the disturbances seen by coalition $c \in \mathcal{C}$ shrink as a result of optimization (12). The following ancillary result is a key element of the feasibility properties of our proposed controller.

Lemma 3. *Suppose Assumption 2 holds. Consider $a_c, b_c, \alpha_c^{\text{pub}}, \beta_c^{\text{pub}} \in [0,1]$ for all $c \in \mathcal{C}$ with dynamics (13). The following holds:*

(i) $a_c^+ \leqslant a_c \iff \alpha_c^{\text{priv}} \mathcal{Z}(\alpha_c^{\text{pub}}, a_c) \subseteq \mathcal{Z}(\alpha_c^{\text{pub}+}, a_c^+)$,
(ii) $b_c^+ \leqslant b_c \iff \beta_c^{\text{priv}} \mathcal{V}(\beta_c^{\text{pub}}, b_c) \subseteq \mathcal{V}(\beta_c^{\text{pub}+}, b_c^+)$.

Proof. if: Taking into account definition (10), the inclusion $\alpha_c^{\text{priv}} \alpha_c^{\text{pub}}(1-a_c)\mathcal{X}_c \subseteq \alpha_c^{\text{pub}+}(1-a_c^+)\mathcal{X}_c$ holds if and only if $\alpha_c^{\text{pub}+}(1-a_c^+) - \alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c) \geqslant 0$. We now prove the above inequality. From (13),

$$\begin{gathered}\alpha_c^{\text{pub}+}(1-a_c^+) - \alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c) = \\ \alpha_c^{\text{pub}}\big(1-(1-a_c)(1-\alpha_c^{\text{priv}})\big)(1-a_c^+) - \alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c),\end{gathered}$$

further manipulation and our hypothesis, $a_c^+ \leqslant a_c$, yield

$$\begin{aligned}\alpha_c^{\text{pub}+}(1-a_c^+) - \alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c) &\geqslant \\ \alpha_c^{\text{pub}}(1-a_c)\big(1-(1-a_c)(1-\alpha_c^{\text{priv}}) - \alpha_c^{\text{priv}}\big) &\geqslant \\ \alpha_c^{\text{pub}} a_c (1-a_c)(1-\alpha_c^{\text{priv}}) &\geqslant 0.\end{aligned}$$

only if: We prove it using properties of the Lebesgue measure

$$\begin{gathered}\lambda\big(\mathcal{Z}_c(\alpha_c^{\text{pub}+}, a_c{}^+) \setminus \mathcal{Z}_c(\alpha_c^{\text{pub}}, a_c)\big) = \\ \lambda\big(\mathcal{Z}_c(\alpha_c^{\text{pub}+}, a_c{}^+)\big) - \lambda\big(\mathcal{Z}_c(\alpha_c^{\text{pub}}, a_c)\big) = \\ (\alpha_c^{\text{pub}+}(1-a_c^+))^{q_c}\lambda(\mathcal{X}_c) - (\alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c))^{q_c}\lambda(\mathcal{X}_c) \geqslant 0,\end{gathered}$$

that is,

$$\alpha_c^{\text{pub}+}(1-a_c^+) \geqslant \alpha_c^{\text{priv}}\alpha_c^{\text{pub}}(1-a_c).$$

Since $\alpha_c^{\text{pub}} \geqslant \alpha_c^{\text{pub}+}$, we have: $1 - a_c^+ - \alpha_c^{\text{priv}} + \alpha_c^{\text{priv}} a_c \geqslant 0$ for all $\alpha_c^{\text{priv}} \leqslant 1$. Taking the limit as $\alpha_c^{\text{priv}} \to 1$ yields $a_c^+ \leqslant a_c$. The proof of the input set follows *mutatis mutandis*. □

The above lemma has profound implications; it gives us a way to assess how the nominal sets change when the RCI sets are updated or when the disturbances created by coupling diminish as the state evolves. We are now in a position to establish our first result concerning recursive feasibility under unchanging coalitions.

Proposition 1 (*Feasibility of the Tail*). *Suppose $(\alpha_c^{\text{pub}}, \beta_c^{\text{pub}}) \in [0,1]^2$, and $V_c^{\text{o}} = \{v_c^{\text{o}}(0), \ldots, v_c^{\text{o}}(N_{\text{p}}-1)\}$ is feasible for $x_c = z_c + e_c \in \mathbb{X}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$. Consider $\alpha_c^{\text{pub}+} \leqslant \alpha_c^{\text{pub}}$ and $\beta_c^{\text{pub}+} \leqslant \beta_c^{\text{pub}}$ by Lemma 1. If $a_c^+ \leqslant a_c$ and $e_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}+}, \beta^{\text{pub}+}))$, then $\tilde{V}_c^+ = \{v_c^{\text{o}}(1), \ldots, v_c^{\text{o}}(N_{\text{p}}-1), K_c^{\text{f}} z_c(N_{\text{p}})\}$ is feasible for x_c^+.*

Proof. Fix $(\alpha_c^{\text{pub}}, \beta_c^{\text{pub}}) \in [0,1]^2$ for all $c \in \mathcal{C}$. Given a feasible state x_c, the successor nominal state and error satisfy $z_c^+ \in \mathcal{Z}_c(\alpha_c^{\text{pub}}, a_c)$ and $e_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}}, \beta^{\text{pub}}))$, respectively. The feasibility of the tail relies on the fact that $z_c^+ \in \mathcal{Z}_c(\alpha_c^{\text{pub}+}, a_c^+)$ by Lemma 3, and the hypothesis $e_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}+}, \beta^{\text{pub}+}))$. We note that the set $\Omega_c^{\text{f}}(\alpha_c^{\text{pub}}, \beta_c^{\text{pub}}, a_c, b_c)$ remains an invariant set that satisfies Assumption 6 for the successor constraint pairs $(\mathcal{Z}_c(\alpha_c^{\text{pub}+}, a_c^+), \mathcal{V}_c(\beta_c^{\text{pub}+}, b_c^+))$. This leads to the tail being a feasible solution for z_c^+. □

Following Proposition 1, we always meet the assumption $e_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}+}, \beta^{\text{pub}+}))$ because of Step 6 of Alg. 2 which checks $x_c^+ - z_c^+ \in \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}+}, \beta^{\text{pub}+}))$ before updating $\alpha_c^{\text{pub}+}$, $\beta_c^{\text{pub}+}$, $\forall c \in \mathcal{C}$. In this way, we obtain the recursive feasibility for the case of unchanging coalitions. These results are the cornerstone of the results concerning a change of coalitions or when plug-and-play operations occur. Another consequence of Proposition 1 is that for each topology Λ that admits a family of RCI sets $\{\Omega_c\}_{c \in \mathcal{C}}$, the value function of (12), *i.e.*, $J_{N,c}^{\text{o}}(\cdot)$ for all $c \in \mathcal{C}$, is a Lyapunov function for the nominal dynamics as summarized in the next corollary.

Corollary 1. *Suppose Assumptions 1, 2, and 6 hold, for all $c \in \mathcal{C}$ for a fixed topology Λ with $J_c(\cdot,\cdot)$ continuously differentiable, positive* *definite, and strictly convex in its arguments. Then, for all* $z_c \in \mathbb{Z}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$,

$$J_{N,c}^{o}(z_c) \geqslant \eta_{1,c}(|z_c|), \tag{15a}$$

$$J_{N,c}^{o}(z_c) \leqslant \eta_{2,c}(|z_c|), \tag{15b}$$

$$J_{N,c}^{o}(z_c^{+}) \leqslant J_{N,c}^{o}(z_c) - \eta_{1,c}(|z_c|), \tag{15c}$$

where $\eta_{1,c}(\cdot)$ *and* $\eta_{2,c}(\cdot)$ *are* $\mathcal{K}_\infty$.

Proposition 1 and Corollary 1 for constant scaling factors recover the traditional notions of recursive feasibility and stability of MPC controllers. In fact, given an initial state $x_\Lambda(0) \in \mathbb{X}_\Lambda^N(\mathbf{1}, \mathbf{1})$, any point in its time evolution $x_\Lambda(k) \in \mathbb{X}_\Lambda^N(\alpha^{\text{pub}}(k), \beta^{\text{pub}}(k))$ is also feasible for the initial time. This observation serves as the cornerstone of future analysis.

6.3. Recursive feasibility for changing coalitions

Given a topology Λ with coalitions $\mathcal{C} = \{c_1, \dots, c_C\}$, the overall N_p–step feasible set is given by the product of individual feasible sets $\mathbb{X}_\Lambda^N = \prod_{c\in\mathcal{C}} \mathbb{X}_c^N$. To ensure the recursive feasibility of our proposed strategy, it is enough to guarantee that for a given topology Λ, and feasible state $x_\Lambda = (x_{c_1}, \dots, x_{c_C}) \in \mathbb{X}_\Lambda^N$, the successor state satisfies $x_\Lambda^+ \in \mathbb{X}_{\Lambda^+}^N$ where Λ^+ is the successor topology. However, as mentioned in Baldivieso-Monasterios et al. (2019), the feasible sets corresponding to two different topologies Λ_1 and Λ_2 do not have a direct relationship between them. In fact, there exist feasible points for a topology that are infeasible for a different one. To characterize recursive feasibility for the case of changing coalitions, we rely on the concepts of feasibility and the strong feasibility of a state x_Λ, introduced in Baldivieso-Monasterios and Trodden (2021). A state x_Λ is said to be recursively feasible if $x_\Lambda \in \mathbb{X}_\Lambda^N$, and it is *strongly recursively feasible* if $x_\Lambda \in \prod_{c\in\mathcal{C}} \mathbb{Z}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$. Proposition 5.5 in Baldivieso-Monasterios and Trodden (2021) states that feasibility becomes a strong feasibility if the coupling between coalitions is sufficiently weak. The concept of strong feasibility coupled with the monotonicity of the scaling factor dynamics (13) will help us characterize feasible topology switches.

On the other hand, following Alg. 1, the topologies contained in $\mathcal{T}_\text{new}$ differ by a maximum of one communication link from the current topology, *i.e.*, $\text{dist}(\Lambda, \Lambda') \leqslant 1$ for $\Lambda' \in \mathcal{T}_\text{new}$. This implies that for Λ with $\mathcal{C}_\Lambda = \{c_1, \dots, c_k, c_{k+1}, \dots, c_C\}$, the new topologies Λ' can have either $\{c_1, \dots, \{c_k, c_{k+1}\}, \dots, c_C\}$ or $\{c_1, \dots, c_{k'}, c_{k''}, c_{k+1}, \dots, c_C\}$ as coalition structures. This observation motivates the following definition:

Definition 3. Given two topologies $\Lambda_1, \Lambda_2 \in \mathcal{T}$, Λ_1 refines Λ_2 (or Λ_2 coarsens Λ_1) if every member of Λ_1 is contained in some member of Λ_2.

For refinement, coalition c_k of Λ_1 has been split into $c_{k'}$ and $c_{k''}$ in Λ_2. Without loss of generality, we can assume that only one subsystem separates from the coalition *i.e.*, $c_k = \{i_k\} \cup c_{k'}$. We note that the state and control input of coalition c_k are, respectively, $x_{c_k} \in \mathbb{R}^{q_{c_k}}$ and $u_{c_k} \in \mathbb{R}^{r_{c_k}}$, where $q_{c_k} = q_{i_k} + q_{c_{k'}}$ and $r_{c_k} = r_{i_k} + r_{c_{k'}}$. The disturbance sets satisfy:[3] $\mathcal{W}_{i_k} = (\bigoplus_{e\in\mathcal{M}_{c_k}\cup\{c_{k'}\}} A_{i_k e}\mathcal{X}_e \oplus B_{i_k e}\mathcal{U}_e) \oplus \mathcal{W}_{i_k}^{\text{e}}$ and $\mathcal{W}_{c_{k'}} = (\bigoplus_{e\in\mathcal{M}_{c_k}\cup\{i_k\}} A_{c_{k'} e}\mathcal{X}_e \oplus B_{c_{k'} e}\mathcal{U}_e) \oplus \mathcal{W}_{c_{k'}}^{\text{e}}$. Following the properties of the Minkowski sum and the Cartesian product, the disturbance set for finer coalitions $\tilde{\mathcal{W}}_{c_k}$ is:

$$\begin{aligned}\tilde{\mathcal{W}}_{c_k} &= \mathcal{W}_{i_k} \times \mathcal{W}_{c_{k'}}\\ &= \Big(\bigoplus_{e\in\mathcal{M}_{i_k}} A_{i_k e}\mathcal{X}_e \oplus B_{i_k e}\mathcal{U}_e\Big) \times \Big(\bigoplus_{e\in\mathcal{M}_{c_{k'}}} A_{c_{k'} e}\mathcal{X}_e \oplus B_{c_{k'} e}\mathcal{U}_e\Big)\\ &\quad \oplus (A_{i_k c_{k'}}\mathcal{X}_{c_{k'}} \oplus B_{i_k c_{k'}}\mathcal{U}_{c_{k'}}) \times (A_{c_{k'} i_k}\mathcal{X}_{i_k} \oplus B_{c_{k'} i_k}\mathcal{U}_{i_k})\\ &\quad \oplus \mathcal{W}_{c_k}^{\text{e}} = \mathcal{W}_{c_k} \oplus \mathcal{W}_{c_{k'} i_k} \supseteq \mathcal{W}_{c_k}.\end{aligned}$$

Therefore, $\tilde{\mathcal{W}}_{c_k}$ has extra terms of the form $\mathcal{W}_{c_{k'} i_k}$ for each of the missing interconnections, and the prediction model changes from (A_{cc}, B_{cc}) to $(A_{cc}^D, B_{cc}^D) = (\text{diag}(A_{i_k i_k}, A_{c_{k'} c_{k'}}), \text{diag}(B_{i_k i_k}, B_{c_{k'} c_{k'}}))$. The overall disturbance sets for the system for both topologies satisfy $\mathcal{W}_{\Lambda_1} = \prod_{c\in\mathcal{C}_{\Lambda_1}} \mathcal{W}_c \subseteq \prod_{d\in\mathcal{C}_{\Lambda_2}} \mathcal{W}_d = \mathcal{W}_{\Lambda_2}$. As a result, topology refinement introduces a counter-nesting of disturbance sets and RCI sets. The latter follows from the observation: $\mathcal{W}_{c_k} \subseteq \mathcal{W}_{i_k} \times \mathcal{W}_{c_{k'}}$ implies $\Omega_{c_k}(\mathcal{W}_{c_k}) \subseteq \Omega_{c_k}(\mathcal{W}_{c_{k'}} \times \mathcal{W}_{i_k})$. Using the definition of an RCI set, it is straightforward to see that $\Omega_{c_k}(\mathcal{W}_{c_{k'}} \times \mathcal{W}_{i_k})$ is also an RCI for coalition $c_{k'} \cup i_k$.

Our study of the recursive feasibility of the system in closed loop with Algorithms 1 and 2 hinges on the idea that state constraint sets shrink as public scaling factors decrease. The following theorem tackles the problem of recursive feasibility when the topology of the system is allowed to change according to Alg. 1. In this theorem, we focus mainly on the case where the public scaling factors strictly decrease *i.e.*, $\alpha_c^{\text{pub}+} < \alpha_c^{\text{pub}}$; the case of equality has been addressed in Baldivieso-Monasterios and Trodden (2021, Proposition 5.5, 5.6).

Theorem 1 (*Recursive Feasibility*)**.** *Suppose Assumptions 1, 2, 5, and 6 hold. In addition, suppose that for all* $c \in \mathcal{C}$, $J_c(\cdot,\cdot)$ *is continuously differentiable, positive definite and strictly convex in its arguments. Given a topology* Λ, *if the state pair* (z_Λ, e_Λ) *satisfies* $(z_\Lambda, e_\Lambda) \in \prod_{c\in\mathcal{C}_\Lambda} \mathbb{Z}_c^N(\alpha^{\text{pub}}, \beta^{\text{pub}}) \times \Omega_c(\mathcal{W}_c(\alpha^{\text{pub}}, \beta^{\text{pub}}))$ *for some* $\alpha_\Lambda^{\text{pub}} = (\alpha_{c_1}^{\text{pub}}, \dots, \alpha_{c_C}^{\text{pub}})$ *and* $\beta_\Lambda^{\text{pub}} = (\beta_{c_1}^{\text{pub}}, \dots, \beta_{c_C}^{\text{pub}})$. *Considering* $\alpha_c^{\text{pub}+} < \alpha_c^{\text{pub}}$ *by Lemma 1, there exists a time* $k > 0$ *for which the state pair satisfies* $\big(z_\Lambda(k), e_\Lambda(k)\big) \in \prod_{c\in\mathcal{C}_{\Lambda(k)}} \mathbb{Z}_c^N\big(\alpha^{\text{pub}}(k), \beta^{\text{pub}}(k)\big) \times \Omega_c\big(\mathcal{W}_c\big(\alpha^{\text{pub}}(k), \beta^{\text{pub}}(k)\big)\big)$ *with the successor topology* $\Lambda(k)$ *selected according Algorithm* 1.

Proof. Given an initial feasible state $x_\Lambda = z_\Lambda + e_\Lambda \in \mathbb{X}_\Lambda^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$ with $z_\Lambda \in \mathbb{Z}_\Lambda^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$ and $e_\Lambda \in \Omega_\Lambda(\mathcal{W}_\Lambda(\alpha^{\text{pub}}, \beta^{\text{pub}}))$, the successor state satisfies $x_\Lambda^+ \in \mathbb{X}_\Lambda^N(\alpha^{\text{pub}+}, \beta^{\text{pub}+}) \subset \alpha_\Lambda^{\text{pub}+}\mathcal{X}$. Furthermore, by hypothesis, the state enters the interior of the feasible set at a rate $\gamma = \min\{\alpha_\Lambda^{\text{pub}} - \alpha_\Lambda^{\text{pub}+}\}$; in fact, the solution of (13) yields $\alpha_c^{\text{pub}}(h) = \alpha_c^{\text{pub}}(1 - (1 - \alpha_c^{\text{priv}})(1 - a_c))^h$ where α_c^{pub} is the initial value of the public scaling factor. On the other hand, the feasible state $x_\Lambda \in \mathbb{X}_\Lambda^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$ becomes strongly feasible after a time $h^* > 0$ such that $x_\Lambda \in \mathbb{Z}_\Lambda^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$ with $h^* = \inf\{h > 0 \colon a_c \leqslant \max_{c\in\mathcal{C}_\Lambda} \alpha_c^{\text{pub}}(h) - \alpha_c^{\text{pub}}\}$. Following Baldivieso-Monasterios and Trodden (2021, Proposition 5.2), strong feasibility implies feasibility under topology coarsening.

Under refinement of the topologies, we have that $\Omega_\Lambda(\mathcal{W}_\Lambda(\alpha^{\text{pub}}, \beta^{\text{pub}})) \subseteq \Omega_{\Lambda'}(\mathcal{W}_{\Lambda'}(\alpha^{\text{pub}}, \beta^{\text{pub}}))$ for $\Lambda' \in \mathcal{T}_\text{new}$. Since feasible regions for each topology are compact sets, we have that there exists $\delta \in (0, 1)$ such that $\delta\mathbb{Z}_\Lambda^N(\alpha^{\text{pub}}(g), \beta^{\text{pub}}(g)) \subset \mathbb{Z}_{\Lambda'}^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$. Now, there exists, similarly to the previous case, $g^* = \inf\{g > 0 \colon \delta \leqslant \max_{c\in\mathcal{C}_\Lambda} \alpha_c^{\text{pub}}(g) - \alpha_c^{\text{pub}}\}$. After g^* samples, the state satisfies $x_\Lambda(g^*) \in \delta\mathcal{X}$ and, by the recursive feasibility of topology Λ, $x_\Lambda(g^*) \in \mathbb{X}_\Lambda^N(\alpha^{\text{pub}}, \beta^{\text{pub}})$. The choice of g^* implies $x_\Lambda(g^*) \in \mathbb{X}_{\Lambda'}^N(\alpha^{\text{pub}}(g^*), \beta^{\text{pub}}(g^*))$. Therefore, the state x_Λ is recursively feasible after $k = \min\{h^*, g^*\}$. □

Remark 1. Despite changes in cooperation topology and plug-and-play operations, which alter the size of disturbance sets

[3] In the following, we drop the dependency on the scaling factors to ease the notational burden, *e.g.*, $\mathcal{W}_c = \mathcal{W}_c(\alpha^{\text{pub}}, \beta^{\text{pub}})$.

$\mathcal{W}_c$, the recursive feasibility is maintained taking into account Theorem 1. In general, unplugged operations compromise neither the feasibility and stability of the system nor the satisfaction of the constraints.

Corollary 2 (*Recursive Feasibility with Plug-and-Play Operations*). *Suppose the assumptions of Theorem 1 and Assumption 7 hold. When a new subsystem $N+1$ is added to the system, for all its neighbors $l \in \mathcal{M}_{N+1}$, there exists $\gamma_l \in (0,1)$ such that:*

$$\mathcal{W}_l(\alpha^{\text{pub}}, \beta^{\text{pub}}) \oplus A_{l(N+1)}\alpha^{\text{pub}}_{N+1}\mathcal{X}_{N+1} \oplus B_{l(N+1)}\beta^{\text{pub}}_{N+1}\mathcal{U}_{N+1} \subset \gamma_l \mathcal{W}_l(\alpha_h, \beta_h),$$

for some $(\alpha_h, \beta_h) \in (0,1)^{|\mathcal{N}|}$ and $\gamma_l < \text{diam}(\mathcal{W}_l(\mathbf{1},\mathbf{1}))$, and thus the new system $\mathcal{N}(k)$ is recursively feasible.

Proof. The dynamics of $l \in \mathcal{M}_{N+1}$ and $N+1$ can be described, respectively, as

$$x_l^+ = A_{ll}x_l + B_{ll}u_l + w_l + w_{l(N+1)},$$
$$x_{N+1}^+ = A_{(N+1)(N+1)}x_{N+1} + B_{(N+1)(N+1)}u_{N+1} + w_{N+1},$$

where $w_{l(N+1)} = A_{l(N+1)}x_{N+1} + B_{l(N+1)}u_{N+1}$ and $w_{N+1} = \sum_{j\in\mathcal{M}_{N+1}} A_{(N+1)j}x_j + B_{(N+1)j}u_j$. We note that the disturbance seen by l satisfies $w_l + w_{l(N+1)} \in \mathcal{W}_l(\alpha^{\text{pub}}, \beta^{\text{pub}}) \oplus A_{l(N+1)}\alpha^{\text{pub}}_{N+1}\mathcal{X}_{N+1} \oplus B_{l(N+1)}\beta^{\text{pub}}_{N+1}\mathcal{U}_{N+1}$. By assumption, for $l \in \mathcal{M}_{N+1}$, the set $\Omega_l(\mathcal{W}_l(\alpha^{\text{pub}}, \beta^{\text{pub}}))$ is non-empty and we note, following Theorem 1, that for a time $h < k$ there exist $\alpha_l^{\text{pub}} < \alpha_{l,h} < 1$ and $\beta_l^{\text{pub}} < \beta_{l,h} < 1$ with $\mathcal{W}_l(\alpha^{\text{pub}}, \beta^{\text{pub}}) \subset \mathcal{W}_l(\alpha_h, \beta_h)$ and $\Omega_l(\mathcal{W}_l(\alpha_h, \beta_h))$ invariant. The existence of $\gamma_l \in (0,1)$ that upper bounds $\mathcal{W}_l(\alpha^{\text{pub}}, \beta^{\text{pub}}) \oplus \mathcal{W}_{l(N+1)}$ implies the invariance of $\Omega_l(\gamma_l\mathcal{W}_l(\alpha_h, \beta_h))$ by Lemma 2.

By assumption, the set $\Omega_{N+1}(\mathcal{W}_{N+1}(\alpha^{\text{pub}}, \beta^{\text{pub}}))$ is invariant. Then the set $\Omega_{N+1}(\mathcal{W}_{N+1}) \times \prod_{c\in\mathcal{C}_\Lambda} \Omega_c(\gamma_c\mathcal{W}_c(\alpha_h, \beta_h))$ is the RCI for the system with topology $\tilde{\Lambda}$. These invariance conditions, coupled with the feasibility of x_{N+1}, allow us to conclude that the state $(x_{\mathcal{N}}, x_{N+1})$ is feasible at the time of connection of $N+1$. The recursive feasibility follows from Theorem 1. □

In the above corollary, the recursive feasibility of our approach hinges on the idea of robustness. The initial tube is given by the diameter of $\Omega_c(\mathcal{W}_c(\mathbf{1},\mathbf{1}))$. The public scaling factors decrease as time evolves if there are no plug-in operations and enough time has elapsed, as seen in the proof of Theorem 1. Corollary 2 establishes that a plug-in operation can only occur when two events happen: assuming that the plug-in takes place at a given instant $k > 0$, the states of the system lie within the interior of the feasible set; and the size of the perturbation generated by the new subsystem is bounded and can be contained in the initial tubes of its neighbors. With these two conditions, we allow Alg. 1 to find the most suitable topology $\tilde{\Lambda} \in \tilde{\mathcal{T}}$ for the system.

6.4. Stability analysis

In this section, we study the stability properties of the system in closed-loop with Alg. 1 and Alg. 2. Traditionally, tube-based MPC methods attack the stability analysis by establishing stability conditions on the nominal system by interpreting the value cost function as a Lyapunov function, *i.e.,* similar to Corollary 1. Moreover, the error between the nominal and *true* systems remains bounded within an invariant set, resulting in a notion of stability with respect to a neighborhood of the origin. Our case, however, is different in two aspects. First, the structure of the nominal system is allowed to change every time the controller selects a new operating topology. Second, the scaling factors used in our framework lead to shrinking state constraints. Our analysis relies on this last fact to establish a stronger notion of convergence to the equilibrium point. A preliminary result towards our goal is the following lemma, which allows us to quantify, in a functional way, the effect of a changing RCI set.

Lemma 4. *Suppose Assumptions 1 and 2 hold, the disturbance set satisfies $\mathcal{W}_c = G_c\mathbb{B}_{d_c,\infty} + p_c$ for some matrix $G_c \in \mathbb{R}^{q_c\times d_c}$ and vector $p_c \in \mathbb{R}^{q_c}$ where $\mathbb{B}_{d_c,\infty}$ is the infinite ball in $\mathbb{R}^{d_c}$. For a topology Λ admitting a family of RCI sets $\{\Omega_c\}_{c\in\mathcal{C}}$, for all $c \in \mathcal{C}$ the function $\Psi_c\colon \Omega_c \to \mathbb{R}^+$, where $\Psi_c(x) = \inf\{\lambda > 0\colon x \in \lambda\Omega_c^*\}$ and $\Omega_c^* = \{\xi : \xi^\top x \leqslant 1 \text{ for all } x \in \Omega_c\}$, satisfies, along the trajectories of error dynamics $e_c^+ = A_{cc}e_c + B_{cc}\mu_c(e_c) + w_c$,*

$$\Psi_c(e_c^+) - \Psi_c(e_c) \leqslant -\psi_c(e_c) + \varphi_c(\alpha_c^{\text{pub}})$$

with $\psi_c(\cdot)$ is $\mathcal{K}_\infty$ and $\varphi_c(\cdot)$ is positive definite and continuous.

Proof. Using the standard properties of the polar set, we obtain that $\Psi_c(e_c) = \sup\{e_c^\top r : r \in \Omega_c\}$ for some $e_c \in \mathbb{R}^{q_c}$ is the support function of the RCI set. Since, by construction, this set can be expressed as a Minkowski sum of linear transformations of the set $\mathcal{W}_c$, then

$$\Psi_c(e_c) = \sum_{h=0}^{H_c-1} \sup\{e_c^\top D_h(\mathbf{M}_{H_c})w_c : w_c \in \mathcal{W}_c\}.$$

Given that $\mathcal{W}_c = G_c\mathbb{B}_{d_c,\infty} + p_c$, where we can assume without loss of generality $p_c = \mathbf{0}$, the support function of the zonotope $\mathcal{W}_c$ is $h_{\mathcal{W}_c}(e_c) = \|e_c^\top G_c\|_1$. Using this fact, together with norm equivalence[4] yields

$$\Psi_c(e_c) = \sum_{h=0}^{H_c-1} \|e_c^\top D_h(\mathbf{M}_{H_c})G_c\|_1 \geqslant \sum_{h=0}^{H_c-1} \|G_c^\top D_h(\mathbf{M}_{H_c})^\top e_c\|_2.$$

On the other hand, the RCI control law $\mu_c(e_c)$ ensures $e_c^+ \in \Omega_c \subset a_c\mathcal{X}_c(\alpha_c^{\text{pub}})$. However, by (4), the constraint set can be written as $\mathcal{X}_c = \alpha_c^{\text{pub}}\mathcal{X}_c$ giving $\|e_c^+\|_2 \leqslant \alpha_c^{\text{pub}} a_c \frac{\text{diam}(\mathcal{X}_c)}{2}$. The desired result follows from the bound with $\psi_c(e_c) = \sum_{h=0}^{H_c-1} \|G_c^\top D_h(\mathbf{M}_{H_c})^\top e_c\|_2$ and $\varphi_c(\alpha_c^{\text{pub}}) = \alpha_c^{\text{pub}} a_c \frac{\text{diam}(\mathcal{X}_c)}{2}$, *i.e.,*

$$\Psi_c(e_c^+) - \Psi_c(e_c) \leqslant -\sum_{h=0}^{H_c-1} \|G_c^\top D_h(\mathbf{M}_{H_c})^\top e_c\|_2 + \alpha_c^{\text{pub}} a_c \frac{\text{diam}(\mathcal{X}_c)}{2}. \quad \square$$

Theorem 2 (*Monotonic Shrinking of Constraint Sets*). *Suppose Assumptions 1, 2, 5 and 6 hold. For all $c \in \mathcal{C}$, $J_c(\cdot,\cdot)$ is continuously differentiable, positive definite and strictly convex in its arguments. For all $i \in \mathcal{N}$, the sequence of public scaling factors $\{(\alpha_i^{\text{pub}}, \beta_i^{\text{pub}})(k)\}_{k\in\mathbb{N}}$ with $(\alpha_i^{\text{pub}}, \beta_i^{\text{pub}})(0) = (1,1)$ is monotonic, but in a finite number of points. Furthermore, the sequence of sets $\{\mathcal{X}_i(\alpha_i^{\text{pub}}(k))\}_{k\in\mathbb{N}}$ converges to a compact and convex set $\bar{\Omega}_{\mathcal{N}} \subseteq \prod_{i\in\mathcal{N}} \mathcal{X}_i$.*

Proof. By construction and fixing a topology $\Lambda \in \mathcal{T}$, each constraint set is scaled at time step k as $\mathcal{X}_c(k) = \alpha_c^{\text{pub}}(k)\mathcal{X}_c$ and $\mathcal{U}_c(k) = \beta_c^{\text{pub}}(k)\mathcal{U}_c$. These constraints induce disturbance and RCI sets according to Lemma 2 for each coalition in Λ, *i.e.,*

$$\mathcal{W}_c(k) = \Big(\bigoplus_{d\in\mathcal{M}_c} A_{cd}\mathcal{X}_d(k) \oplus B_{cd}\mathcal{U}_d(k)\Big) \oplus \mathcal{W}_c^{\text{e}} \subseteq \gamma_c\mathcal{W}_c(\mathbf{1},\mathbf{1})$$

for some $\gamma_c \in (0,1)$ and $\Omega_c(\mathcal{W}_c(k)) \subset a_c\mathcal{X}_c(k)$. These sets lead to the definitions of the nominal tightened constraints of (10); the solution of the optimal control problem (12) for a nominal state $z_c(k)$ yields an optimal sequence $V_c^o(k)$ and two private scaling factors $(\alpha_c^{\text{priv}}, \beta_c^{\text{priv}})(k)$ that force $z_c(k+1) \in \alpha_c^{\text{priv}}(k)\mathcal{Z}_c(\alpha_c^{\text{pub}}(k), a_c(k))$. Following Corollary 1, the value function of the nominal system behaves as a Lyapunov function, which implies that $z_c(\cdot)$ converges towards its equilibrium if the coalition does not change. Furthermore,

$$x_c(k+1) \in \alpha_c^{\text{priv}}(k)(1-a_c(k))\alpha_c^{\text{pub}}(k)\mathcal{X}_c \oplus a_c(k)\alpha_c^{\text{pub}}(k)\mathcal{X}_c$$

[4] Let $x \in \mathbb{R}^n$, $\alpha, \beta > 0$ and $a, b \geqslant 1$, then $\beta\|x\|_a \leqslant \|x\|_b \leqslant \alpha\|x\|_a$.

since $e_c(k+1) \in a_c(k)\alpha_c^{\text{pub}}(k)\mathcal{X}_c$ by invariance of $\Omega_c(\mathcal{W}_c(k))$. The successor state then satisfies:

$$x_c(k+1) \in \alpha_c^{\text{pub}}(k+1)\mathcal{X}_c,$$

where $\alpha_c^{\text{pub}}(k+1) = \alpha_c^{\text{pub}}(k)\big(a_c(k)+\alpha_c^{\text{priv}}(k)(1-a_c(k))\big)$, and forms a decreasing sequence only if $\alpha_c^{\text{priv}} \in (0,1)$ and a_c decreases. We note here that the sequence of $\{a_c(k)\}_{k\in\mathbb{N}}$ does not need to converge toward zero because of exogenous disturbances $\mathcal{W}_c^{\text{e}}$; we claim that there exists a lower bound a_c^{e} on each $a_c(k)$ based on the exogenous disturbance $\mathcal{W}_c^{\text{e}}$. In fact, the RCI set $\Omega_c(\mathcal{W}_c^{\text{e}}) \subseteq a_c^{\text{e}}\mathcal{X}_c$, following Lemma 2, is a minimal element in the RCI sets with respect to the set inclusion since $\mathcal{W}_c^{\text{e}} \subseteq \mathcal{W}_c(k)$ for all $k \geqslant 0$. The implication of this is that $a_c(k) \geqslant a_c^{\text{e}}$ for all $k > 0$ and there exists a converging subsequence $\{a_c(k_j)\}_{k_j}$. This, together with the fact that $\{(\alpha_c^{\text{priv}}, \beta_c^{\text{priv}})(k)\}_{k\in\mathbb{N}}$ lies in the interior of $[0,1]$, implies that $\{(\alpha_c^{\text{pub}}, \beta_c^{\text{pub}})(k)\}_{k\in\mathbb{N}}$ has at least an accumulation point $(\bar{\alpha}_c^{\text{pub}}, \bar{\beta}_c^{\text{pub}}) \in (0,1)^2$ since by definition all elements of the sequence are less than 1. Therefore, the state constraints satisfy $\mathcal{X}_c(k) \to \bar{\alpha}_c^{\text{pub}}\mathcal{X}_c$ following the standard arguments of the set convergence theory (see Rockafellar & Wets, 1998). As a consequence of Theorem 1, a change of topology implies that the region of convergence is given by the union over all possible topologies, *i.e.*, $\bar{\Omega}_{\mathcal{N}} = \bigcup_{\Lambda\in\mathcal{T}}\prod_{c\in\mathcal{C}_\Lambda}\bar{\alpha}_c^{\text{pub}}\mathcal{X}_c$. □

The set where the constraint sets converge to an invariant set is composed of two parts: the first one given by the exogenous disturbance $\prod_{i\in\mathcal{N}}\mathcal{W}_i^{\text{e}}$ and the second one $\bigcup_{\Lambda\in\mathcal{T}}\prod_{c\in\mathcal{C}_\Lambda}\bigoplus_{d\in\mathcal{M}_c}\bar{\alpha}_d^{\text{pub}}A_{cd}\mathcal{X}_d \oplus \bar{\beta}_d^{\text{pub}}B_{cd}\mathcal{U}_d$ that captures the effect of using a distributed controller, *i.e.*, the disturbance arising from exchanging information between a coalition and its neighbors. The private scaling factors do not necessarily need to converge; the only requirement is for them to lie in the interior of the unit interval to allow the convergence of the public factors. The fact that there exists a subsequence of the scaling factors that converges implies that we can allow these sequences to increase or stagnate for a finite number of steps before returning to a monotonic one. This fact follows from our recursive feasibility results, which guarantee that Alg. 1 can find suitable topologies to cope with the addition of new subsystems that may increase the size of the disturbance sets.

Corollary 3 (*Stability of a Neighborhood of the Origin*). *Suppose the assumptions of Theorem 2 hold. The state for the system $x_{\mathcal{N}}(\cdot)$ is asymptotically stable with respect to the set $\bigcup_{\Lambda\in\mathcal{T}}\prod_{c\in\mathcal{C}_\Lambda}\Omega_c(\mathcal{W}_c^{\text{e}})$.*

Proof. Given a feasible initial state $x_{\mathcal{N}}(0)$, Theorem 1 guarantees that the state evolution is contained within the feasible set $\bigcup_{\Lambda\in\mathcal{T}}\prod_{c\in\mathcal{C}_\Lambda}\mathcal{X}_c(1)$ for all forward times. In addition, Theorem 2 ensures that these feasible sets monotonically converge towards a compact set $\bar{\Omega}_{\mathcal{N}}$. These two facts together imply that there exists a time k_0 for which $x_{\mathcal{N}}(k_0) \in \bar{\Omega}_{\mathcal{N}} \oplus \mathbb{B}_{q,2}(\varepsilon)$ where $\mathbb{B}_{q,2}(\varepsilon)$ is the 2−ball with radius $\varepsilon > 0$. For all $k < k_0$, the state satisfies $\|x_c(k)\|_2 \leqslant \text{diam}(\mathcal{X}_c(\alpha_c^{\text{pub}}(k)))$ for all $c \in \mathcal{C}_\Lambda$ for some $\Lambda \in \mathcal{T}$; the right-hand side of the inequality is a decreasing function of the scaling factors, which implies $d(x_{\mathcal{N}}, \bar{\Omega}_{\mathcal{N}}) \to 0$. Fixing a topology Λ, using Corollary 1 and Lemma 4, the candidate for the overall Lyapunov function within $\mathbb{B}_{q,2}(\varepsilon)\oplus\prod_{c\in\mathcal{C}_\Lambda}\bar{\alpha}_c\mathcal{X}_c$ for the composite system $(x_{\mathcal{N}}, z_{\mathcal{N}})$ is $\Upsilon_\Lambda(x_c, z_c) = \sum_{c\in\mathcal{C}_\Lambda}J^{\text{o}}_{N,c}(z_c) + \Psi_c(x_c - z_c)$, which is an Input-to-State stable Lyapunov function. Furthermore, applying LaSalle's invariance principle, we can conclude the asymptotic stability of a neighborhood of equilibrium point. □

7. Illustrative example

We consider the coupled-truck system presented in Trodden and Maestre (2017), where trucks are coupled by dampers and springs with their immediate neighbors, as shown in Fig. 3.

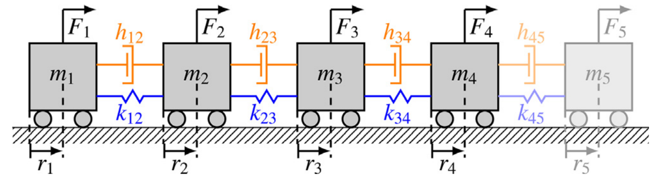


Fig. 3. System compound of an array of four coupled trucks. At time step k_{plug}, a fifth truck is plugged into the system.

Table 2
Damping factors [N · s/m], spring constants [N/m], and masses [kg].

	Damping [N · s/m]	Spring [N/m]	Mass [kg]
	$h_{12} = 0.3$	$k_{12} = 0.5$	$m_1, m_3 = 3$
	$h_{23} = 0.4$	$k_{23} = 0.7$	$m_2, m_4 = 2$
	$h_{34} = 0.3$	$k_{34} = 0.6$	
Case 1:	$h_{45} = 1$	$k_{45} = 1.5$	$m_5 = 6$
Case 2:	$h_{45} = 0.1$	$k_{45} = 0.08$	$m_5 = 2$

Each truck i is modeled by second-order dynamics:

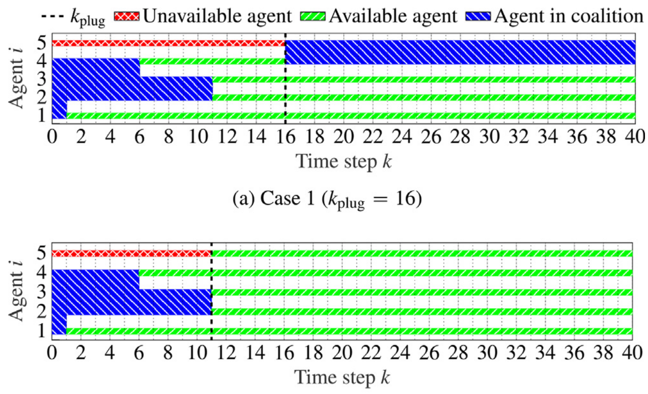
$$\underbrace{\begin{bmatrix}\dot{r}_i\\ \dot{v}_i\end{bmatrix}}_{\dot{x}_i} = \underbrace{\begin{bmatrix}0 & 1\\ -\dfrac{1}{m_i}\sum_{j\in\mathcal{M}_i}k_{ij} & -\dfrac{1}{m_i}\sum_{j\in\mathcal{M}_i}h_{ij}\end{bmatrix}}_{A_{ii}}\underbrace{\begin{bmatrix}r_i\\ v_i\end{bmatrix}}_{x_i} + \underbrace{\begin{bmatrix}0\\ 100\end{bmatrix}}_{B_{ii}}u_i$$
$$+ \sum_{j\in\mathcal{M}_i}\underbrace{\begin{bmatrix}0 & 0\\ \dfrac{1}{m_i}\sum_{j\in\mathcal{M}_i}k_{ij} & \dfrac{1}{m_i}\sum_{j\in\mathcal{M}_i}h_{ij}\end{bmatrix}}_{A_{ij}}\begin{bmatrix}r_j\\ v_j\end{bmatrix} + w_i^{\text{e}},$$

where the state x_i of each truck i is composed of its displacement from the equilibrium position r_i and its velocity v_i. Each agent can apply a horizontal force $F_i = B_{ii}u_i$ with u_i being the control input. Moreover, we consider a bounded exogenous disturbance $|w_i^{\text{e}}| \leqslant [0.0025, 0.0025]^\top$ for all agents. Table 2 displays the model parameters used in the simulations that will be performed for two case studies. A discrete-time model with sample time $T_s = 0.2\,\text{s}$ that approximates the continuous-time model is employed to simulate and control each subsystem.

The control problem is to lead the subsystems from their initial states: $x_1(0) = [1.5, 0]^\top$, $x_2(0) = [-0.5, 0]^\top$, $x_3(0) = [1, 0]^\top$, $x_4(0) = [-1, 0]^\top$, and $x_5(k_{\text{plug}}) = [1, 0]^\top$ to the origin, while satisfying constraints $|r_i| \leqslant 2\,\text{m}$, $|v_i| \leqslant 1\,\text{m/s}$, $|u_i| \leqslant 1\,\text{N/kg}$ and handling a plugged subsystem. Therefore, the system is formed by $N = 4$ trucks for $k < k_{\text{plug}}$ and composed of $N = 5$ for $k \geqslant k_{\text{plug}}$. At first, the maximum number of cooperation links is $|\mathcal{L}| = 3$ and there are eight cooperation topologies $\mathcal{T} = \{\Lambda_1, \ldots, \Lambda_8\}$; after the plug-in: $|\mathcal{L}| = 4$ and $\mathcal{T} = \{\Lambda_1, \ldots, \Lambda_{16}\}$.

The weighting matrices for the state and input for all $i \in \mathcal{N}$ are, respectively, $Q_i = I$ and $R_i = 100$, and aggregated as $Q_c = \text{diag}(Q_i)_{i\in c}$ and $R_c = \text{diag}(R_i)_{i\in c}$. The LQR terminal controller $K_c^{\text{f}} = \text{diag}(K_i^{\text{f}})_{i\in c}$, where $K_1^{\text{f}} = [-0.0365, -0.0460]$, $K_2^{\text{f}} = [-0.0334, -0.0443]$, $K_3^{\text{f}} = [-0.0345, -0.0450]$, $K_4^{\text{f}} = [-0.0341, -0.0446]$, $K_5^{\text{f}} = [-0.0370, -0.0462]$, and the terminal weight matrix $P_c = \text{diag}(P_i)_{i\in c}$, where

$$P_1 = \begin{bmatrix}4.3327 & -2.7765\\ -2.7765 & 3.9817\end{bmatrix}, \quad P_2 = \begin{bmatrix}4.2137 & -2.7240\\ -2.7240 & 3.9148\end{bmatrix},$$
$$P_3 = \begin{bmatrix}4.2571 & -2.7424\\ -2.7424 & 3.9393\end{bmatrix}, \quad P_4 = \begin{bmatrix}4.2411 & -2.7359\\ -2.7359 & 3.9293\end{bmatrix},$$
$$P_5 = \begin{bmatrix}4.3527 & -2.7859\\ -2.7859 & 3.9931\end{bmatrix}.$$



(a) Case 1 ($k_{\text{plug}} = 16$)

(b) Case 2 ($k_{\text{plug}} = 11$)

Fig. 4. Formation of coalitions in the two case studies.

Table 3
Numerical results comparison for the two case studies.

(a) Costs in Case 1					
Tube methods	$\bar{t}_c$ [s]	J_{perf}	J_{coop}	J_{total}	J_{total} [%]
Cen. MPC	1.13	28.49	14.5	43.99	–
Coal. MPC	0.94	28.94	4	32.94	23.39%
Dec. MPC	–	–	–	–	–

(b) Costs in Case 2					
Tube methods	$\bar{t}_c$ [s]	J_{perf}	J_{coop}	J_{total}	J_{total} [%]
Cen. MPC	1.19	28.57	15	43.57	–
Coal. MPC	0.89	28.93	1.5	30.43	30.16%
Dec. MPC	0.51	29.13	0	29.13	33.16%

7.1. Simulation results

Two simulations of length $N_{\text{sim}} = 40$ have been performed using MATLAB® on Windows with a PC Intel® Core™ i7-8700 CPU at 3.20 GHz and 16 GB RAM. We have also used `YALMIP` (Lofberg, 2004) with `quadprog` solver, the `MPT 3.0` (Herceg et al., 2013), and the `PnPMPC toolbox` (Riverso et al., 2013). The MPC methods consider a prediction horizon $N_{\text{p}} = 10$, the upper-layer period $T_{\text{up}} = 5$, the parameter $\tau_\alpha = 5 \cdot 10^{-5}$, and the cost per active cooperation link $c_{\text{link}} = 0.1$. Since we consider external disturbances, our proposed tube-based coalitional MPC algorithm is compared with two other tube-based methods that do not employ scaling factors: centralized MPC (full cooperation between agents) and decentralized MPC (without communication between local agents).

Fig. 4 presents the evolution of the cooperation topology with the coalitional MPC strategy in two case studies starting with the great coalition $c = \{\{1, 2, 3, 4\}\}$. Every $T_{\text{up}} = 5$ time step from $k = 1$, the supervisor decides the cooperation topology. In both cases, for $k < 6$, there are two coalitions $c_1 = \{1\}$ and $c_2 = \{2, 3, 4\}$, and three coalitions $c_1 = \{1\}$, $c_2 = \{2, 3\}$, and $c_3 = \{4\}$ for $6 \leqslant k < 11$. Afterwards, in Case 1 (Fig. 4(a)), all agents work decentralized until a new subsystem $i = 5$ is connected to the system in $k_{\text{plug}} = 16$. Since agent $i = 4$ cannot deal with its new disturbances, it forms a coalition with agent $i = 5$ until the end of the simulation. Conversely, in Case 2 (Fig. 4(b)), the coupling between agents $i = 4$ and $i = 5$ is lower, thus agent $i = 4$ can handle the increase in disturbances caused by the plug-in subsystem without collaborating with its neighbors.

The sequence of the outer bounds of RCI sets, the scaling factors, the volume of sets $\left(\alpha_i^{\text{pub}}\mathcal{X}_i \text{ and } \alpha_i^{\text{priv}}\mathcal{Z}_i(\alpha_i^{\text{pub}}, a_i)\right)$ for Case 1 are depicted in Fig. 5. The outer bounds of the RCI sets are calculated by solving the LP problem (9) with weights $q_x = 10$ and $q_u = 1$. As shown, the volume of set $\alpha_i^{\text{pub}}\mathcal{X}_i$ monotonically decreases for all $i \in \mathcal{N}$ despite changes in the scaling factors. The values of a_i and α_i^{pub} shown in Fig. 5a do not exactly converge to zero due to the exogenous disturbances w_i^{e}, but to very small values; at $k = 40$:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0.061 \\ 0.018 \\ 0.027 \\ 0.092 \\ 0.092 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha_1^{\text{pub}} \\ \alpha_2^{\text{pub}} \\ \alpha_3^{\text{pub}} \\ \alpha_4^{\text{pub}} \\ \alpha_5^{\text{pub}} \end{bmatrix} = \begin{bmatrix} 0.023 \\ 0.033 \\ 0.030 \\ 0.027 \\ 0.184 \end{bmatrix}.$$

These variables directly affect the RCI sets (*i.e.,* the cross sections of tubes), which represent the admissible disturbance set around the nominal trajectories and are defined as $a_i\mathcal{X}_i(\alpha_i^{\text{pub}})$ (recall (10)). As observed in Fig. 6, the tubes *shrink* and *grow* due to the coalition breakups and changes in the scaling factors α_i^{pub}, $\forall i \in \mathcal{N}(k)$.[5] When a coalition is disbanded or a new subsystem is plugged, the agents involved or the neighbors are, respectively, subject to further disturbances and, therefore, at that time instant, their tubes can grow to cope with more uncertainty. For example, in Fig. 6, the tubes of agents $i = 2$ and $i = 3$ grow at $k = 11$ due to its coalition breakdown. At the end of the simulation, the tubes are as small as possible to cover the external disturbances w_i^{e} that affect each subsystem locally.

Fig. 7 depicts the evolution of position and velocity, and the control inputs of the five trucks for coalitional MPC and centralized MPC in Case 1. As shown, the local states $x_i = [r_i, v_i]^\top$ reach their origin despite the disturbances caused by the plug-in, the coupling and the external noise.

Finally, Table 3 shows a comparison of the numerical results obtained with all the MPC methods for the two case studies.

The total cost J_{total} is the sum of the accumulated performance cost during the simulation:

$$J_{\text{perf}} = \sum_{k=1}^{N_{\text{sim}}} \left(\|x_{\mathcal{N}}(k)\|^2_{Q_{\mathcal{N}}} + \|u_{\mathcal{N}}(k)\|^2_{R_{\mathcal{N}}} \right),$$

and the accumulated cooperation cost, which penalizes the number of links of Λ at the instant k:

$$J_{\text{coop}} = \sum_{k=1}^{N_{\text{sim}}} c_{\text{link}} \, |\Lambda(k)|.$$

The average computing time per coalition, $\bar{t}_c$ [s], is calculated as follows:

$$\bar{t}_c = \frac{\sum_{k=1}^{N_{\text{sim}}} \left(\sum_{c \in \mathcal{C}_\Lambda} t_c(k) \,/\, |\mathcal{C}_\Lambda| \right)}{N_{\text{sim}}},$$

where t_c and $|\mathcal{C}_\Lambda|$ denote, respectively, the time per coalition and the total number of coalitions in topology Λ at time step k. The average computation times of the supervisory layer (Alg. 2) in the coalitional method for Case 1 and Case 2 are 43.2 s and 58.9 s, respectively. Note that faster implementations would require more computing power and more efficient programming languages than MATLAB®, such as C and C++ programming. Furthermore, the proposed coalitional method brings several other potential advantages compared to a fully centralized implementation (*e.g.*, the removal of a single point of failure, and enhanced privacy/security) that warrant the additional time spent on a supervisory layer. In any case, a full comparison of the three methods is more complex and nuanced, as computation time

[5] The tube cross section of any agent within a coalition whose cardinality is $|c| \geqslant 2$ will be in $\mathbb{R}^{\geqslant 4}$, so we have projected it in $\mathbb{R}^2$ to be able to represent it.

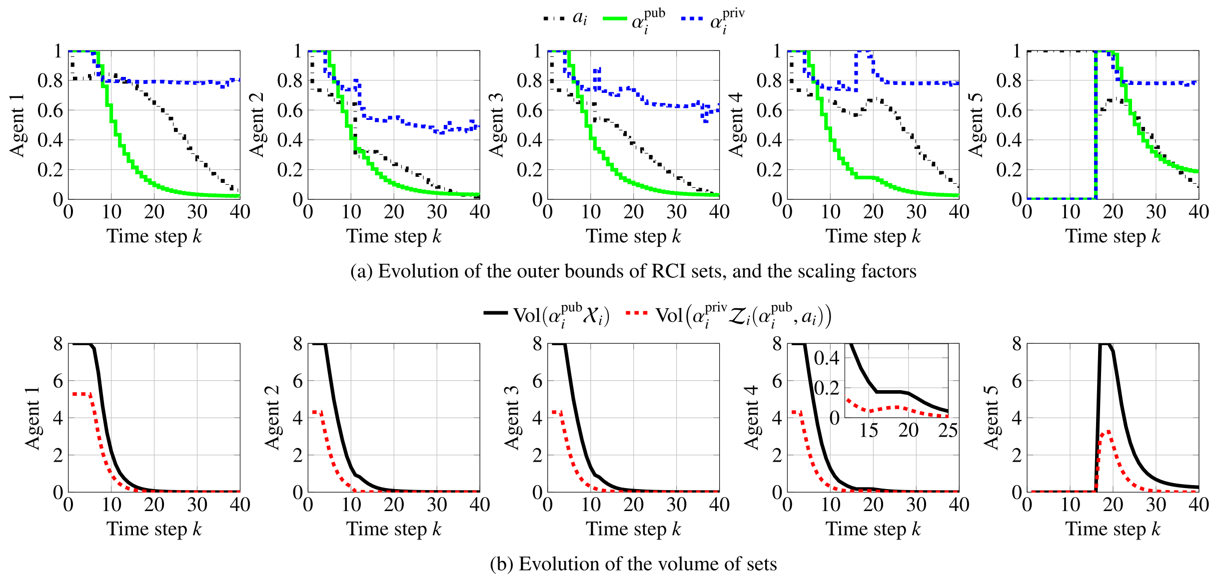


Fig. 5. Results with the coalitional strategy for Case 1 ($k_{\text{plug}} = 16$).

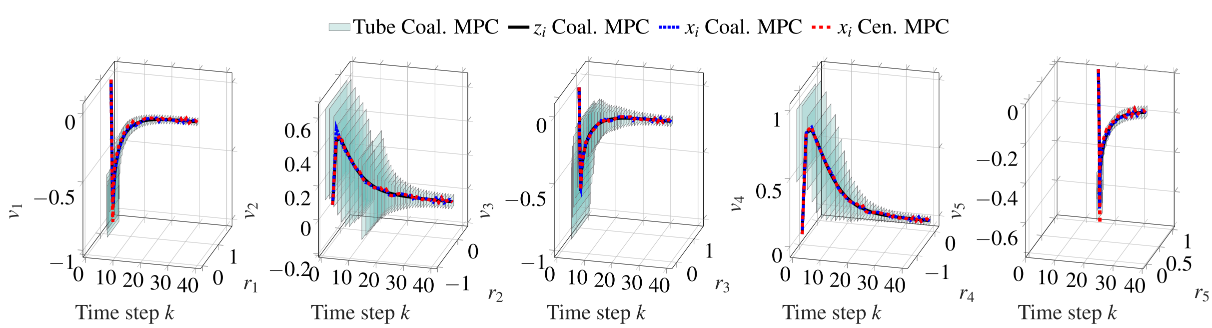


Fig. 6. Tube evolution of each agent for Case 1 ($k_{\text{plug}} = 16$).

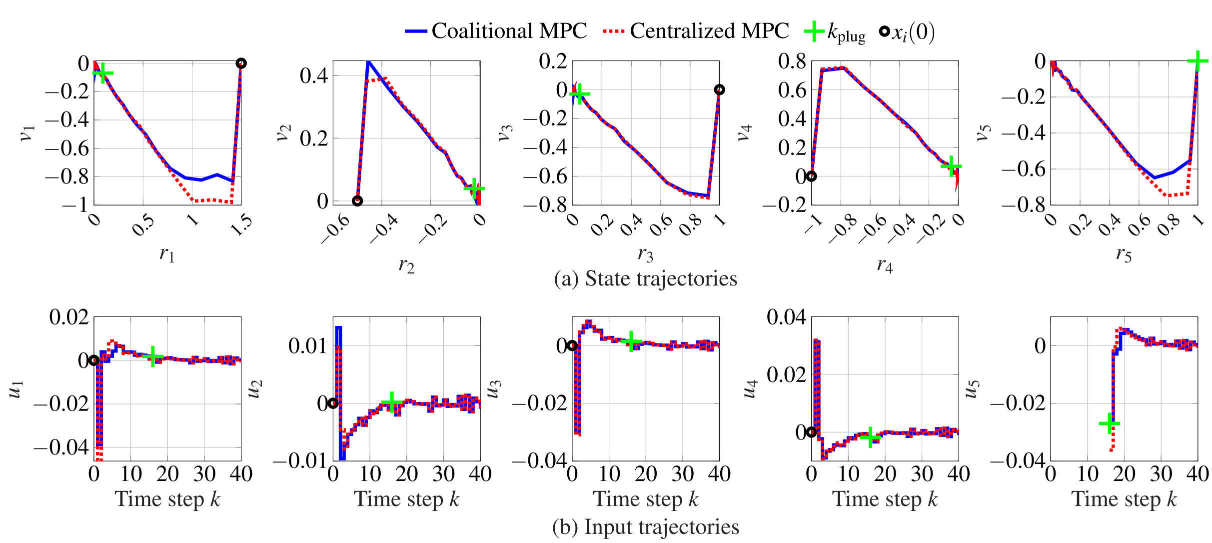


Fig. 7. State and input trajectory of each truck for Case 1 ($k_{\text{plug}} = 16$).

and closed-loop performance are just two of several aspects to consider.

As shown in Table 3, our approach achieves close control performance over centralized MPC, which provides the best. Adding the cooperation costs, the coalitional MPC algorithm outperforms the centralized MPC with a total cost reduction of 23.39% in Case 1 (see Table 3a) and a 30.16% in Case 2 (see Table 3b). Note that the decentralized approach is the most convenient option in terms of cooperation and computation payload, but may result in lower performance or even infeasibility due to the difficulty in managing interactions (*e.g.*, dynamic couplings between subsystems and PnP operations) while ensuring constraint satisfaction. As shown in the numerical results, the decentralized MPC only outperforms the other methods in Case 2, where the dynamic coupling —and especially that between the plugged-in agent and its neighbor— is weak. However, decentralized control cannot be implemented in Case 1 because it becomes infeasible due to the increase in disturbances caused by the plug-in event. This fact reinforces the need for coalitional strategies to control networked systems with subsystems joining and leaving the network.

8. Conclusions

We propose a robust coalitional MPC based on optimized tubes that can handle plug-and-play events. Our approach allows agents to exchange information about their public scaled constraint sets —which shrink as long as the system comes close to the origin— and to cluster into coalitions to reject disturbances and improve performance. Scaling factors for constraint sets are separated into public and private values to create an inherent robustness margin that allows controllers to locally absorb disturbances without a redesign of the control system. Furthermore, plug-and-play operations are successfully performed in real time while maintaining the recursive feasibility and stability of the system. Another finding is the possibility of *plug-in* and *plug-out* coalitions of agents.

Future research lines are the fully distributed implementation of the proposed strategy and its application to potential real systems, such as vehicle platoons and microgrids.

Acknowledgments

The authors thank the support received from the European Research Council for the Advanced Research Grant OCONTSOLAR (ref. 789051), the project C3PO-R2D2 (Grant PID2020-119476RB-I00 funded by MCIN/AEI/ 10.13039/501100011033), and the Spanish Ministry of Science and Innovation under the Predoctoral Training programme for University Staff (Ph.D. grant FPU18/04476).

References

Baldivieso-Monasterios, P. R., & Trodden, P. A. (2021). Coalitional predictive control: Consensus-based coalition forming with robust regulation. *Automatica*, *125*, Article 109380.

Baldivieso-Monasterios, P. R., Trodden, P. A., & Cannon, M. (2019). On feasible sets for coalitional MPC. In *Proceedings of the 58th Conference on Decision and Control* (pp. 4668–4673). IEEE.

Barreiro-Gomez, J., & Zhu, Q. (2022). Coalitional stochastic differential games for networks. *IEEE Control Systems Letters*, *6*, 2707–2712.

Bendtsen, J., Trangbaek, K., & Stoustrup, J. (2011). Plug-and-play control—Modifying control systems online. *IEEE Transactions on Control Systems Technology*, *21*(1), 79–93.

Bodenburg, S., Kraus, V., & Lunze, J. (2016). A design method for plug-and-play control of large-scale systems with a varying number of subsystems. In *Proceedings of the American Control Conference* (pp. 5314–5321). IEEE.

Bodenburg, S., Niemann, S., & Lunze, J. (2014). Experimental evaluation of a fault-tolerant plug-and-play controller. In *Proceedings of the European Control Conference* (pp. 1945–1950). IEEE.

Chanfreut, P., Maestre, J. M., & Camacho, E. F. (2021). A survey on clustering methods for distributed and networked control systems. *Annual Reviews in Control*, *52*, 75–90.

Christofides, P. D., Scattolini, R., Muñoz de la Peña, D., & Liu, J. (2013). Distributed model predictive control: A tutorial review and future research directions. *Computers & Chemical Engineering*, *51*, 21–41.

Dörfler, F., Simpson-Porco, J. W., & Bullo, F. (2014). Plug-and-play control and optimization in microgrids. In *Proceedings of the 53rd Conference on Decision and Control* (pp. 211–216). IEEE.

Fele, F., Maestre, J. M., & Camacho, E. F. (2017). Coalitional control: Cooperative game theory and control. *IEEE Control Systems Magazine*, *37*(1), 53–69.

Franzè, G., Lucia, W., & Tedesco, F. (2018). A distributed model predictive control scheme for leader–follower multi-agent systems. *International Journal of Control*, *91*(2), 369–382.

Herceg, M., Kvasnica, M., Jones, C. N., & Morari, M. (2013). Multi-parametric toolbox 3.0. In *Proceedings of the European Control Conference* (pp. 502–510). IEEE.

Jain, A., Chakrabortty, A., & Biyik, E. (2018). Distributed wide-area control of power system oscillations under communication and actuation constraints. *Control Engineering Practice*, *74*, 132–143.

Langson, W., Chryssochoos, I., Raković, S. V., & Mayne, D. Q. (2004). Robust model predictive control using tubes. *Automatica*, *40*(1), 125–133.

Lofberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the International Conference on Robotics and Automation* (pp. 284–289). IEEE.

Lou, G., Gu, W., Xu, Y., Cheng, M., & Liu, W. (2016). Distributed MPC-based secondary voltage control scheme for autonomous droop-controlled microgrids. *IEEE Transactions on Sustainable Energy*, *8*(2), 792–804.

Lucia, S., Kögel, M., & Findeisen, R. (2015). Contract-based predictive control of distributed systems with plug and play capabilities. *IFAC-PapersOnLine*, *48*(23), 205–211.

Maestre, J. M., & Negenborn, R. R. (Eds.), (2014). *Distributed model predictive control made easy, vol. 69*. Springer.

Maxim, A., & Caruntu, C. -F. (2021). A coalitional distributed model predictive control perspective for a cyber-physical multi-agent application. *Sensors*, *21*(12), 4041.

Mayne, D. Q., Seron, M. M., & Raković, S. V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, *41*(2), 219–224.

Mi, X., Zou, Y., Li, S., & Karimi, H. R. (2019). Self-triggered DMPC design for cooperative multiagent systems. *IEEE Transactions on Industrial Electronics*, *67*(1), 512–520.

Núñez, A., Ocampo-Martinez, C., Maestre, J. M., & De Schutter, B. (2015). Time-varying scheme for noncentralized model predictive control of large-scale systems. *Mathematical Problems in Engineering*, *2015*.

Patton, R. J., Kambhampati, C., Casavola, A., Zhang, P., Ding, S., & Sauter, D. (2007). A generic strategy for fault-tolerance in control systems distributed over a network. *European Journal of Control*, *13*(2–3), 280–296.

Raković, S. V., Kerrigan, E. C., Mayne, D. Q., & Kouramas, K. I. (2007). Optimized robust control invariance for linear discrete-time systems: Theoretical foundations. *Automatica*, *43*(5), 831–841.

Raković, S. V., Kouvaritakis, B., Findeisen, R., & Cannon, M. (2012). Homothetic tube model predictive control. *Automatica*, *48*(8), 1631–1638.

Richards, A., & How, J. P. (2007). Robust distributed model predictive control. *International Journal of Control*, *80*(9), 1517–1531.

Riverso, S., Battocchio, A., & Ferrari-Trecate, G. (2013). PnPMPC toolbox. http://sisdin.unipv.it/pnpmpc/pnpmpc.php.

Riverso, S., Boem, F., Ferrari-Trecate, G., & Parisini, T. (2016). Plug-and-play fault detection and control-reconfiguration for a class of nonlinear large-scale constrained systems. *IEEE Transactions on Automatic Control*, *61*(12), 3963–3978.

Riverso, S., Farina, M., & Ferrari-Trecate, G. (2014). Plug-and-play model predictive control based on robust control invariant sets. *Automatica*, *50*(8), 2179–2186.

Riverso, S., & Ferrari-Trecate, G. (2012). Tube-based distributed control of linear constrained systems. *Automatica*, *48*(11), 2860–2865.

Rockafellar, R. T., & Wets, R. J. B. (1998). *Grundlehren der mathematischen wissenschaften*: *vol. 317*, *Variational analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg.

Scokaert, P. O. M., & Mayne, D. Q. (1998). Min-max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control*, *43*(8), 1136–1142.

Stoustrup, J. (2009). Plug & play control: Control technology towards new challenges. *European Journal of Control*, *15*(3–4), 311–330.

Trodden, P. A., & Maestre, J. M. (2017). Distributed predictive control with minimization of mutual disturbances. *Automatica*, *77*, 31–43.

Trodden, P. A., & Richards, A. (2010). Distributed model predictive control of linear systems with persistent disturbances. *International Journal of Control*, *83*(8), 1653–1663.

Eva Masero received the B.Eng. degree in Industrial Electronics and Automation Engineering from the University of Extremadura, Spain, in 2016, and the M.Eng. degree in Electronics, Robotics and Automation Engineering from the University of Seville, Spain, in 2018. Since 2019, she is a predoctoral fellow in the Department of Systems Engineering and Automation, University of Seville. Her research interests include model predictive control of distributed networks by clustering, robust control, and applications of control in cyber–physical and energy systems.

Pablo R. Baldivieso-Monasterios received the B.Eng. degree in Electronic Engineering from Escuela Militar de Ingeniería, Bolivia, in 2010, the M.Sc. degree in Control Systems from the University of Sheffield, UK, in 2014, and the Ph.D. degree in Robust Distributed Model Predictive Control from the University of Sheffield, UK, in 2018. Since 2018, he is a post-doctoral research associate in the Department of Automatic Control and Systems Engineering, University of Sheffield, UK. His research interests include robust and distributed model predictive and optimization-based control, and game theoretic methods for control and smartgrids.

José M. Maestre holds a Ph.D. degree from the University of Seville, Spain, where he currently works as a full professor. He has held various positions at universities such as TU Delft, University of Pavia and Tokyo Institute of Technology. His research focuses on the control of distributed cyber–physical systems, with a special emphasis on the integration of heterogeneous agents in the control loop. He has published over 200 journal and conference papers, co-edited several books, and led several research projects. Finally, his achievements have been recognized through several awards and honors, including the Spanish Royal Academy of Engineering's medal for his contributions to the predictive control of large-scale systems.

Paul A. Trodden received the M.Eng. degree in Engineering Science from the University of Oxford, UK, in 2003, and the Ph.D. degree in Aerospace Engineering from the University of Bristol, UK, in 2009. He subsequently held a PDRA positions in the Department of Aerospace Engineering, University of Bristol (2009–10), and the School of Mathematics, University of Edinburgh (2010–12). Since 2012, he has been a Lecturer and then Senior Lecturer with the Department of Automatic Control and Systems Engineering, University of Sheffield, UK. His research interests include model predictive and optimization-based control, especially distributed and robust forms, and applications of control and optimization to aerospace and power & energy systems. He is a member of the IEEE CSS Conference Editorial Board and an Associate Editor for IET Smart Grid.