SIAM J. CONTROL OPTIM.
Vol. 58, No. 4, pp. 2188–2205

OBSERVABILITY INEQUALITIES ON MEASURABLE SETS FOR THE STOKES SYSTEM AND APPLICATIONS*

FELIPE W. CHAVES-SILVA†, DIEGO A. SOUZA‡, AND CAN ZHANG§

Abstract. In this paper, we establish spectral inequalities on measurable sets of positive Lebesgue measure for the Stokes operator, as well as observability inequalities on space-time measurable sets of positive measure for nonstationary Stokes system. The latter extends the result established recently by Wang and Zhang [*SIAM J. Control Optim.,* 55 (2017), pp. 1862–1886] to the case of observations from subsets of positive measure in both time and space variables. Furthermore, we present their applications in the shape optimization problem, as well as the time optimal control problem for the Stokes system. In particular, we give a positive answer to an open question raised by Privat, Trélat, and Zuazua [*Arch. Rational Mech. Anal.,* 216 (2015), pp. 921–981].

Key words. spectral inequality, observability inequality, Stokes system, shape optimization, time optimal control

AMS subject classifications. 49Q10, 76D55, 93B05

DOI. 10.1137/18M117652X

1. Introduction and main results.

Let $T > 0$, and let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded connected open set with a smooth boundary $\partial\Omega$. We will use the notation $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$, and we will denote by $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x})$ the outward unit normal vector to Ω at $\mathbf{x} \in \partial\Omega$. Throughout the paper spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldface letters.

The present paper deals with an observability inequality on measurable sets of positive measure for the Stokes system

$$
(1.1)\qquad \left|\begin{array}{lll}
\mathbf{z}_t - \Delta\mathbf{z} + \nabla q = \mathbf{0} & \text{in} & Q,\\
\operatorname{div}\mathbf{z} = 0 & \text{in} & Q,\\
\mathbf{z} = \mathbf{0} & \text{on} & \Sigma,\\
\mathbf{z}(\cdot,0) = \mathbf{z}_0 & \text{in} & \Omega.
\end{array}\right.
$$

System (1.1) is a linearization of the Navier–Stokes system for a homogeneous viscous incompressible fluid (with unit density and unit kinematic viscosity) subject to

*Received by the editors March 21, 2018; accepted for publication (in revised form) May 26, 2020; published electronically July 29, 2020.
https://doi.org/10.1137/18M117652X
Funding: The first author was partially supported by CNPq - Brazil by grant 2019/0014 of Paraíba State Research Foundation (FAPESQ-PB) and by ERC project 32084, Semiclassical Analysis of Partial Differential Equations, ERC-2012-ADG. The second author was partially supported by CNPq - Brazil, by Propesq - UFPE edital Qualis A, by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research program and by CAPES-PRINT, 88881.311964/2018-01. The third author was supported by the National Natural Science Foundation of China under grants 11971363 and 11771344 and was also partially supported by The Fundamental Research Funds for Central Universities under grant 2042019kf0023 and by the Academic Team Building Plan for Young Scholars from Wuhan University under grant 413100085.
†Department of Mathematics, Federal University of Paraíba, UFPB, CEP 58051-900, João Pessoa, PB, Brazil (fchaves@mat.ufpb.br).
‡Department of Mathematics, Federal University of Pernambuco, CEP 50740-545, Recife, PE, Brazil (diego.souza@dmat.ufpe.br).
§Corresponding author. School of Mathematics and Statistics, Wuhan University; Computational Science Hubei Key Laboratory, Wuhan University, Wuhan 430072, China (zhangcansx@163.com).

homogeneous Dirichlet boundary conditions. Here, $\mathbf{z}$ is the $\mathbb{R}^N$-valued velocity field and q stands for the scalar pressure.

Our motivation to obtain an observability inequality on measurable sets for the Stokes system (1.1) comes from the well-known fact that observability inequalities are equivalent to controllability properties. In the case we are dealing with, this will be equivalent to the null controllability of system (1.1) with bounded controls acting on measurable sets with positive measure and will have important applications in shape optimization problems, in the study of the bang-bang property for time optimal control problems, and also distributed control problems for system (1.1) (see section 3).

Observability inequalities for (1.1) from a cylinder $\omega \times (0,T)$, with $\omega \subset \Omega$ being a nonempty open set, have been proved in different ways by several authors in the past few years. For instance, in [11], the observability inequality for the Stokes system is obtained by means of global Carleman inequalities for parabolic equations with zero Dirichlet boundary conditions (see also [6] and [10]). Another proof is given in [12] by means of Carleman inequalities for parabolic equations with nonhomogeneous Dirichlet boundary conditions applied to the system satisfied by the vorticity $curl\, z$. More recently, in [5], a new proof was established based on a spectral inequality in terms of finite sums of eigenfunctions of the Stokes operator.

Concerning observability inequalities over general Lebesgue measurable sets in space and time variables, as far as we know, the first result was obtained in [2] for the heat equation in a bounded and locally star-shaped domain, and later extended in [7] and [8] to the case of parabolic systems with time-independent analytic coefficients associated to possibly non–self-adjoint elliptic differential operators and higher order parabolic evolutions with the analytic coefficients depending on space and time variables, respectively, when the boundary of the bounded domain in which the equation evolves is analytic. We also refer the interested reader to [1, 18, 21] for some earlier and closely related results on this subject.

For the Stokes system, the only result we know is the one in [22], which shows an observability inequality from a measurable subset with positive measure in the time variable. The argument in [22] is mainly based on the theory of analytic semigroups. In this paper, however, we can extend the result in [22] to the case of observations from sets of positive measure in both time and space variables, by using also the spatial analyticity of solutions to the Stokes operators.

Before presenting our main results, we first introduce the usual spaces in the context of fluid mechanics:

$$\mathbf{V} = \{\mathbf{y} \in \mathbf{H}_0^1(\Omega)^N;\ \operatorname{div} \mathbf{y} = 0\},$$

$$\mathbf{H} = \{\mathbf{y} \in \mathbf{L}^2(\Omega)^N;\ \operatorname{div} \mathbf{y} = 0,\ \mathbf{y} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}.$$

In what follows, the following notation will be used frequently: $B_R(x_0)$ denotes a ball in $\mathbb{R}^N$ of radius $R > 0$ and with center $x_0 \in \Omega$; $|\omega|$ is the Lebesgue measure of a subset $\omega \subset \Omega$, and $C(\dots)$ stands for a positive constant depending only on the parameters within the brackets, and it may vary from line to line in the context.

The first main result is a L^1-observability inequality from measurable sets with positive measure for system (1.1).

THEOREM 1.1. *Let $B_{4R}(\mathbf{x}_0) \subset \Omega$. For any measurable subset $\mathcal{M} \subset B_R(\mathbf{x}_0) \times (0,T)$ with positive measure, there exists a positive constant $C_{obs} = C(N, R, \Omega, \mathcal{M}, T)$ such that the observability inequality*

$$\|\mathbf{z}(T,\cdot)\|_{\mathbf{H}} \le C_{obs} \int_{\mathcal{M}} |\mathbf{z}(\mathbf{x},t)|\, d\mathbf{x}dt \tag{1.2}$$

holds for all $\mathbf{z}_0 \in \mathbf{H}$.

Remark 1.2. When the observation set is $\mathcal{M} = B_R(\mathbf{x}_0) \times (0,T)$, one can see that the observability constant C_{obs} has the form $Ce^{C/T}$ with $C = C(N,\Omega,R) > 0$. This is in accordance with the very recent result [5, Theorem 1.1].

Remark 1.3. The above technical assumption imposed on the measurable set $\mathcal{M}$ is just to simplify the statement of the main result. Without loss of generality, for any measurable set $\mathcal{M} \subset \Omega \times (0,T)$ with positive measure, one can always assume that

$$\mathcal{M} \subset B_R(\mathbf{x}_0) \times (0,T) \quad \text{with} \quad B_{4R}(\mathbf{x}_0) \subset \Omega \tag{1.3}$$

for some $R > 0$ and $\mathbf{x}_0 \in \mathbb{R}^N$. Indeed, by Lebesgue's density theorem, one may choose a new measurable set $\widetilde{\mathcal{M}} \subset \mathcal{M}$ such that (1.3) holds and $|\widetilde{\mathcal{M}}| \ge c|\mathcal{M}|$, for some constant $0 < c < 1$.

The argument we shall use to prove Theorem 1.1 relies mainly on the telescoping series method, the propagation of smallness for real-analytic functions on measurable sets, as well as a spectral inequality for Stokes system.

We next start to introduce the spectral inequality: Let $\{\mathbf{e}_j\}_{j\ge1}$ be the sequence of eigenfunctions of the Stokes system

$$\left|\begin{array}{lll} -\Delta \mathbf{e}_j + \nabla p_j = \lambda_j \mathbf{e}_j & \text{in} & \Omega, \\ \operatorname{div} \mathbf{e}_j = 0 & \text{in} & \Omega, \\ \mathbf{e}_j = \mathbf{0} & \text{on} & \partial\Omega, \end{array}\right. \tag{1.4}$$

with the sequence of eigenvalues $\{\lambda_j\}_{j\ge1}$ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \cdots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = +\infty.$$

The following was proved in [5].

THEOREM 1.4 (see [5, Theorem 3.1]). *For any nonempty open subset $\mathcal{O} \subset \Omega$, there exists a constant $C = C(N,\Omega,\mathcal{O}) > 0$ such that*

$$\sum_{\lambda_j \le \Lambda} a_j^2 = \int_\Omega \left| \sum_{\lambda_j\le\Lambda} a_j \mathbf{e}_j(\mathbf{x}) \right|^2 d\mathbf{x} \le Ce^{C\sqrt{\Lambda}} \int_{\mathcal{O}} \left| \sum_{\lambda_j\le\Lambda} a_j \mathbf{e}_j(\mathbf{x}) \right|^2 d\mathbf{x} \tag{1.5}$$

for any sequence of real numbers $\{a_j\}_{j\ge1} \in \ell^2$ and any positive number Λ.[1]

We mention that the spectral inequality (1.5) allows one to control the low frequencies of the Stokes system with a precise estimate on the cost of controllability with respect to the frequency length which, combined with the exponential decay of solutions of (1.1), thus implies the null controllability of Stokes system with $\mathbf{L}^2$-controls applied to arbitrarily small open sets.

The second main result is an extension of the spectral inequality (1.5) from open sets to measurable sets of positive measure.

[1] Recall that $\ell^2 \triangleq \{\{a_j\}_{j\ge1} : \sum_{j=1}^{+\infty} a_j^2 < +\infty\}$.

THEOREM 1.5. *Let $B_{4R}(\mathbf{x}_0) \subset \Omega$, and let $\omega \subset B_R(\mathbf{x}_0)$ be a measurable set with positive measure. Then, there exists a constant $C = C(N, R, \Omega, |\omega|) > 0$ such that*

$$\left(\sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \leq Ce^{C\sqrt{\Lambda}} \int_\omega \left| \sum_{\lambda_j \leq \Lambda} a_j \mathbf{e}_j(\mathbf{x}) \right| d\mathbf{x} \tag{1.6}$$

for all $\Lambda > 0$ and any sequence of real numbers $\{a_j\}_{j\geq 1} \in \ell^2$.

Remark 1.6. It is worth mentioning that the inequality (1.6) leads to a null controllability result for the Stokes system with $\mathbf{L}^\infty$-controls (see Theorem 3.6 below), which is a refined controllability result.

The proof of Theorem 1.5 strongly depends on quantitative estimates of the interior spatial analyticity for finite sums of eigenfunctions of the Stokes system (1.4). As far as we know, for the Navier–Stokes equations, the qualitative analyticity has been first analyzed in [13] and [14], where the authors consider a nonlinear elliptic system satisfied by the velocity z and the vorticity *curl* z and show the interior analyticity for the velocity z. However, since the boundary condition for the *curl* z is not prescribed, the analyticity up to the boundary cannot be achieved by this method.

In this paper, in order to establish the spectral inequality (1.6), we adapt and combine the arguments in [13] and [14] and [2, Theorem 5] to the low frequencies of the Stokes system.

The rest of the paper is organized as follows. In section 2, we shall present the proofs of Theorems 1.1 and 1.5, respectively. Section 3 deals with several applications of main theorems for shape optimization and time optimal control problems of Stokes system. Finally, in Appendix A, we prove real-analytic estimates for solutions of the Poisson equation.

2. Proofs of main results.

2.1. Spectral inequality on measurable sets. This subsection is devoted to the proof of Theorem 1.5. Compared with the proof of [2, Theorem 5] for the Laplace operator, we here encounter the difficulty due to the pressure in the Stokes system. To circumvent that, we instead consider the equation satisfied by the *curl* of the low frequencies, which is an equation without pressure but with no boundary conditions. This allows us recover and quantify the interior real-analytic estimates based on the *curl* operator.

We begin with an estimate of the propagation of smallness for real-analytic functions on measurable sets with positive measure, which plays a core ingredient in the proof of Theorem 1.5.

LEMMA 2.1. *Assume that $\mathbf{f} : B_{2R}(\mathbf{x}_0) \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is real-analytic and verifies*

$$|\partial_x^\alpha \mathbf{f}(\mathbf{x})| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}} \quad \textit{for } \mathbf{x} \in B_{2R}(\mathbf{x}_0),\ \alpha \in \mathbb{N}^N,$$

with some $M > 0$ and $0 < \rho \leq 1$. For any measurable set $\omega \subset B_R(\mathbf{x}_0)$ with positive measure, there are positive constants $C = C(R, N, \rho, |\omega|)$ and $\theta = \theta(R, N, \rho, |\omega|)$, with $\theta \in (0, 1)$, such that

$$\|\mathbf{f}\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \leq C \left(\int_\omega |\mathbf{f}(\mathbf{x})|\, d\mathbf{x} \right)^\theta M^{1-\theta}.$$

The above-mentioned local observability inequality for real-analytic functions was first established in [20]. The interested reader can also find a simpler proof of Lemma 2.1 in [1, section 3] and a more general extension in [7, Lemma 2].

Such a kind of stability estimate is called as the Hadamand three spheres theorem when ω is a nonempty open ball. The fact that the smallest set ω is allowed to be Lebesgue measurable while the constants depend only on its measure are particularly useful in our proof below.

Proof of Theorem 1.5. For each real number $\Lambda > 0$ and each sequence $\{a_j\}_{j\geq 1} \in \ell^2$, we define

$$\mathbf{u}_\Lambda(\mathbf{x}) = \sum_{\lambda_j \leq \Lambda} a_j \mathbf{e}_j(\mathbf{x}), \ \ \mathbf{x} \in \Omega,$$

and

$$\mathbf{v}_\Lambda(\mathbf{x}, s) = \sum_{\lambda_j \leq \Lambda} a_j e^{s\sqrt{\lambda_j}} d\mathbf{e}_j(\mathbf{x}), \quad (\mathbf{x}, s) \in \Omega \times (-1, 1),$$

where d denotes the curl operator.[2]

Because $\mathbf{v}_\Lambda(\cdot, 0) = d\mathbf{u}_\Lambda$ and $\operatorname{div}_\mathbf{x} \mathbf{u}_\Lambda = 0$, we have

$$\Delta_\mathbf{x} \mathbf{u}_\Lambda(\mathbf{x}) = d^* \mathbf{v}_\Lambda(\mathbf{x}, 0), \ \ \mathbf{x} \in \Omega, \tag{2.1}$$

where d^* is the adjoint of d.

Let us now obtain an estimate of the propagation of smallness for $\mathbf{u}_\Lambda$ on measurable sets with positive measure. According to Lemma 2.1, it is sufficient to quantify the analytic estimates of higher-order derivatives of $\mathbf{u}_\Lambda$.

Since $\mathbf{v}_\Lambda(\cdot, \cdot)$ satisfies

$$-\partial_{ss}^2 \mathbf{v}_\Lambda(\mathbf{x}, s) - \Delta_\mathbf{x} \mathbf{v}_\Lambda(\mathbf{x}, s) = 0, \quad (\mathbf{x}, s) \in \Omega \times (-1, 1),$$

we have that $d^* \mathbf{v}_\Lambda$ verifies

$$-\partial_{ss}^2 d^* \mathbf{v}_\Lambda(\mathbf{x}, s) - \Delta_\mathbf{x} d^* \mathbf{v}_\Lambda(\mathbf{x}, s) = 0, \quad (\mathbf{x}, s) \in \Omega \times (-1, 1),$$

and, using Lemma A.1 in Appendix with $f \equiv 0$, $d^* \mathbf{v}_\Lambda$ is real-analytic in $B_{4R}(\mathbf{x}_0, 0) \subset \mathbb{R}^{N+1}$, and the following estimate holds

$$\|\partial_\mathbf{x}^\alpha \partial_s^\beta d^* \mathbf{v}_\Lambda\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0, 0))} \leq C \frac{(|\alpha| + \beta)!}{(\rho R)^{|\alpha| + \beta}} \left(\fint_{B_{4R}(\mathbf{x}_0, 0)} |d^* \mathbf{v}_\Lambda(\mathbf{x}, s)|^2 d\mathbf{x} ds \right)^{1/2}$$
$$\forall \alpha \in \mathbb{N}^N, \ \beta \geq 0,$$

where the positive constants ρ and C only depend on the dimension N. Note that here we are assuming that $B_{4R}(\mathbf{x}_0, 0) \subset \Omega \times (-1, 1)$, which is always possible because $B_{4R}(\mathbf{x}_0) \subset \Omega$ and Remark 1.3.

[2]In fact, d is the differential which maps 1-forms into 2-forms. When a vector field w is identified with a 1-form, then dw can be identified with a $\frac{1}{2}N(N-1)$-dimensional vector. We can also see d as an operator $d : \mathcal{D}'(\Omega)^N \to \mathcal{D}'(\Omega)^{N^2}$ whose entries are given by

$$(d\mathbf{u})_{i,j} = \partial_{x_j} u_i - \partial_{x_i} u_j \quad (1 \leq i, j \leq N),$$

and d^* is its adjoint operator.

Taking $\beta = 0$ in the previous estimate, we readily obtain

$$\|\partial_x^\alpha d^*\mathbf{v}_\Lambda(\cdot,0)\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0))} \le C\frac{|\alpha|!}{(\rho R)^{|\alpha|}}\left(\fint_{B_{4R}(\mathbf{x}_0,0)}|d^*\mathbf{v}_\Lambda(\mathbf{x},s)|^2 d\mathbf{x}ds\right)^{1/2} \quad \forall\alpha\in\mathbb{N}^N. \tag{2.2}$$

To bound the right-hand side in (2.2), we set

$$\mathbf{w}_\Lambda(\mathbf{x},s) = \sum_{\lambda_j\le\Lambda} a_j e^{s\sqrt{\lambda_j}}\mathbf{e}_j(\mathbf{x}),\quad (\mathbf{x},s)\in\Omega\times(-1,1),$$

and then the following estimate holds

$$\begin{aligned}\|d^*\mathbf{v}_\Lambda\|^2_{\mathbf{L}^2(B_{4R}(\mathbf{x}_0,0))} &\le C\|\mathbf{w}_\Lambda\|^2_{L^2((-1,1);\mathbf{H}^2(\Omega))}\\ &\le C\int_{-1}^1\|\mathbf{A}\mathbf{w}_\Lambda(\cdot,s)\|^2_{\mathbf{H}}\,ds,\end{aligned}$$

where we have used the fact that there exists $C = C(N,\Omega) > 0$ such that

$$\frac{1}{C}\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)} \le \|\mathbf{A}\mathbf{y}\|_{\mathbf{H}} \le C\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)}\quad \forall\mathbf{y}\in D(\mathbf{A}),$$

with $\mathbf{A}$ being the Stokes operator.[3]

Since $\{\mathbf{e}_j\}_{j\ge1}$ is an orthonormal basis of $\mathbf{H}$, the last estimate yields

$$\|d^*\mathbf{v}_\Lambda\|^2_{\mathbf{L}^2(B_{4R}(\mathbf{x}_0,0))} \le Ce^{C\sqrt{\Lambda}}\sum_{\lambda_j\le\Lambda}a_j^2 \tag{2.3}$$

for some $C > 0$.

Therefore, combining (2.2) and (2.3), we have

$$\|\partial_x^\alpha d^*\mathbf{v}_\Lambda(\cdot,0)\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0))} \le C\frac{|\alpha|!}{(\rho R)^{|\alpha|}}e^{C\sqrt{\Lambda}}\left(\sum_{\lambda_j\le\Lambda}a_j^2\right)^{1/2}\quad\forall\alpha\in\mathbb{N}^N, \tag{2.4}$$

where $C = C(N,\Omega)$.

Since $\mathbf{u}_\Lambda$ solves the Poisson equation (2.1), we have that $\mathbf{u}_\Lambda$ is real-analytic whenever the exterior force $d^*\mathbf{v}_\Lambda(\cdot,0)$ is real analytic. Now, thanks to (2.4), we can apply again Lemma A.1 to obtain that

$$\|\partial_x^\alpha\mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \le (R\tilde\rho)^{-|\alpha|-1}|\alpha|!\left(\|\mathbf{u}_\Lambda\|_{\mathbf{L}^2(B_{2R}(\mathbf{x}_0))} + Ce^{C\sqrt{\Lambda}}\left(\sum_{\lambda_j\le\Lambda}a_j^2\right)^{1/2}\right)\quad\forall\alpha\in\mathbb{N}^N$$

for some constant $\tilde\rho > 0$.

Noticing that

$$\|\mathbf{u}_\Lambda\|^2_{\mathbf{L}^2(B_{2R}(\mathbf{x}_0))} \le \|\mathbf{u}_\Lambda\|^2_{\mathbf{H}} = \sum_{\lambda_j\le\Lambda}a_j^2,$$

[3] The Stokes operator $\mathbf{A} : D(\mathbf{A}) \longrightarrow \mathbf{H}$ is defined by $\mathbf{A} = -P\Delta$, with $D(\mathbf{A}) = \{\mathbf{y}\in\mathbf{V} :\ \mathbf{A}\mathbf{y}\in\mathbf{H}\}$ and $P : \mathbf{L}^2(\Omega) = \mathbf{H}\oplus\mathbf{H}^\perp \longrightarrow \mathbf{H}$ is the Leray projection.

one can see that

$$\|\partial_x^\alpha \mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \le \frac{|\alpha|!}{(\rho R)^{|\alpha|}} e^{K\sqrt{\Lambda}} \left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2} \quad \forall \alpha \in \mathbb{N}^N, \tag{2.5}$$

where ρ and K are positive constants independent of Λ.

Applying (2.5) and Lemma 2.1 to the real-analytic function $\mathbf{u}_\Lambda$, we obtain the estimate

$$\|\mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \le C \left(\int_\omega |\mathbf{u}_\Lambda(\mathbf{x})|\, d\mathbf{x}\right)^\theta \left(e^{K\sqrt{\Lambda}} \left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2}\right)^{1-\theta} \tag{2.6}$$

for some constants $C = C(N, R, \Omega, |\omega|) > 0$ and $\theta = \theta(N, R, \Omega, |\omega|) \in (0,1)$.

On the other hand, by the spectral inequality given in Theorem 1.4, there exists $C = C(\Omega, R, N)$ such that

$$\left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2} \le Ce^{C\sqrt{\Lambda}} \|\mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))}.$$

The above inequality and (2.6) then lead to

$$\left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2} \le Ce^{C\sqrt{\Lambda}} \left(\int_\omega |\mathbf{u}_\Lambda(\mathbf{x})|\, d\mathbf{x}\right)^\theta \left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{(1-\theta)/2},$$

which gives us the desired observability inequality

$$\left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2} \le Ce^{C\sqrt{\Lambda}} \int_\omega |\mathbf{u}_\Lambda(\mathbf{x})|\, d\mathbf{x}.$$

This finishes the proof. ∎

2.2. Observability inequality on measurable sets in space-time variables. This subsection is devoted to the proof of Theorem 1.1. We begin with an interpolation estimate for the solutions of the Stokes system, which is a consequence of the spectral inequality given in Theorem 1.5 and the exponential decay of solutions of the Stokes system. This can also be seen as a quantitative estimate of strong unique continuation of solutions to the Stokes system. We refer the reader to [2, 7, 22] for closely related results concerning the strong unique continuation property for general parabolic equations.

Proposition 2.2. *Let $B_{4R}(\mathbf{x}_0) \subset \Omega$, and let $\omega \subset B_R(\mathbf{x}_0)$ be a measurable set with positive measure. Then, there exists $C = C(\Omega, |\omega|) > 0$ such that*

$$\|\mathbf{z}(\cdot,t)\|_{\mathbf{H}} \le \left(Ce^{\frac{C}{t-s}} \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\omega)}\right)^{1/2} \|\mathbf{z}(\cdot,s)\|_{\mathbf{H}}^{1/2} \quad \forall \mathbf{z}_0 \in \mathbf{H},$$

where $0 \le s < t \le T$ and $\mathbf{z}$ is the solution of (1.1) associated to $\mathbf{z}_0$.

Proof. It suffices to prove the estimate in the case $s = 0$.

For any $\Lambda > 0$, we set

$$\mathbf{H}_\Lambda \triangleq \operatorname{span}\{\mathbf{e}_j; \lambda_j \leq \Lambda\}.$$

Given $\mathbf{z}_0 \in \mathbf{H}$, the solution $\mathbf{z}$ of (1.1) can be split into $\mathbf{z} = \mathbf{z}_\Lambda + \mathbf{z}_\Lambda^\perp$, where $\mathbf{z}_\Lambda$ and $\mathbf{z}_\Lambda^\perp$ are the solutions of (1.1) (together with some pressures) associated to $\mathbf{z}_{0,\Lambda} \in \mathbf{H}_\Lambda$ and $\mathbf{z}_{0,\Lambda}^\perp \in \mathbf{H}_\Lambda^\perp$,[4] $\mathbf{z}_0 = \mathbf{z}_{0,\Lambda} + \mathbf{z}_{0,\Lambda}^\perp$, respectively. Moreover, one has

$$\mathbf{z}_\Lambda(\cdot, t) \in \mathbf{H}_\Lambda \quad \text{and} \quad \|\mathbf{z}_\Lambda^\perp(\cdot, t)\|_{\mathbf{H}} \leq e^{-\Lambda t}\|\mathbf{z}_0\|_{\mathbf{H}} \tag{2.7}$$

for every $t > 0$.

From (1.6) and (2.7), for each $t > 0$ we have

$$\begin{aligned}
\|\mathbf{z}(\cdot, t)\|_{\mathbf{H}} &\leq \|\mathbf{z}_\Lambda(\cdot, t)\|_{\mathbf{H}} + \|\mathbf{z}_\Lambda^\perp(\cdot, t)\|_{\mathbf{H}} \\
&\leq Ce^{C\sqrt{\Lambda}}\|\mathbf{z}_\Lambda(\cdot, t)\|_{\mathbf{L}^1(\omega)} + e^{-\Lambda t}\|\mathbf{z}_0\|_{\mathbf{H}} \\
&\leq Ce^{C\sqrt{\Lambda}}\left(\|\mathbf{z}(\cdot, t)\|_{\mathbf{L}^1(\omega)} + \|\mathbf{z}_\Lambda^\perp(\cdot, t)\|_{\mathbf{H}}\right) + e^{-\Lambda t}\|\mathbf{z}_0\|_{\mathbf{H}} \\
&\leq Ce^{C\sqrt{\Lambda}}\left(\|\mathbf{z}(\cdot, t)\|_{\mathbf{L}^1(\omega)} + e^{-\Lambda t}\|\mathbf{z}_0\|_{\mathbf{H}}\right) + e^{-\Lambda t}\|\mathbf{z}_0\|_{\mathbf{H}} \\
&\leq \widehat{C}e^{\widehat{C}\sqrt{\Lambda} - \frac{\Lambda}{2}t}\left(e^{\frac{\Lambda}{2}t}\|\mathbf{z}(\cdot, t)\|_{\mathbf{L}^1(\omega)} + e^{-\frac{\Lambda}{2}t}\|\mathbf{z}_0\|_{\mathbf{H}}\right) \\
&\leq \widetilde{C}e^{\frac{\widetilde{C}}{t}}\|\mathbf{z}(\cdot, t)\|_{\mathbf{L}^1(\omega)}^{1/2}\|\mathbf{z}_0\|_{\mathbf{H}}^{1/2},
\end{aligned}$$

where in the last inequality we used the fact that

$$C_1\sqrt{\Lambda} - \frac{t\Lambda}{2} \leq \frac{C_1^2}{2t} \quad \text{for any } \Lambda > 0$$

as well as the following lemma.

Lemma 2.3 ([19]). *Let C_1, C_2 be positive and M_0, M_1, and M_2 be nonnegative. Assume there exist $C_3 > 0$ and $\delta_0 > 0$ such that $M_0 \leq C_3 M_1$ and*

$$M_0 \leq e^{-C_1\delta}M_1 + e^{C_2\delta}M_2$$

for every $\delta \geq \delta_0$. Then, there exits $C_0 > 0$ such that

$$M_0 \leq C_0 M_1^{C_2/(C_1+C_2)} M_2^{C_1/(C_1+C_2)}.$$

For the proof of Theorem 1.1, we will use the following result concerning the property of Lebesgue density point for a measurable set in $\mathbb{R}$.

Lemma 2.4 ([18], Proposition 2.1). *Let E be a measurable set in $(0, T)$ with positive measure, and let ℓ be a density point of E[5]. Then, for each $\mu > 1$, there is $\ell_1 = \ell_1(\mu, E)$ in (ℓ, T) such that the sequence $\{\ell_m\}_{m\geq 1}$ defined as*

$$\ell_{m+1} = \ell + \mu^{-m}(\ell_1 - \ell), \ m = 1, 2, \ldots$$

satisfies

$$|E \cap (\ell_{m+1}, \ell_m)| \geq \frac{1}{3}(\ell_m - \ell_{m+1}) \ \ \forall m \geq 1. \tag{2.8}$$

[4] $\mathbf{H}_\Lambda^\perp = \operatorname{span}\{\mathbf{e}_j; \lambda_j > \Lambda\}$.

[5] Let E a measurable set of $\mathbb{R}$. A point $x \in E$ is a density point if

$$\lim_{h\to 0} \frac{|E \cap (x - h, x + h)|}{2h} = 1.$$

Proof of Theorem 1.1. For each $t \in (0,T)$, let us define the slice

$$\mathcal{M}_t = \{x \in \Omega : (x,t) \in \mathcal{M}\}$$

and

$$E = \left\{ t \in (0,T); |\mathcal{M}_t| \geq \frac{|\mathcal{M}|}{2T} \right\}.$$

From Fubini's theorem, it follows that $\mathcal{M}_t \subset \Omega$ is measurable for almost everywhere (a.e.) $t \in (0,T)$, E is measurable in $(0,T)$ and

$$|E| \geq \frac{|\mathcal{M}|}{2|B_R(\mathbf{x}_0)|} \quad \text{and} \quad \chi_E(t)\chi_{\mathcal{M}_t}(\mathbf{x}) \leq \chi_{\mathcal{M}}(\mathbf{x},t), \text{ in } \Omega \times (0,T).$$

For a.e. $t \in E$, we apply Proposition 2.2 to $\mathcal{M}_t$ to find a constant $C = C(\Omega, R, |\mathcal{M}|/(T|B_R(\mathbf{x}_0)|))$ such that

$$\|\mathbf{z}(\cdot,t)\|_{\mathbf{H}} \leq \left(Ce^{\frac{C}{t-s}} \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)} \right)^{1/2} \|\mathbf{z}(\cdot,s)\|_{\mathbf{H}}^{1/2} \tag{2.9}$$

for $0 \leq s < t$.

Let ℓ be any density point in E. For $\mu > 1$ (to be chosen later), we denote by $\{\ell_m\}_{m\geq 1}$ the strictly monotone decreasing sequence associated to ℓ and μ as in Lemma 2.4. For each $m \geq 1$, we set

$$\tau_m = \ell_{m+1} + \frac{(\ell_m - \ell_{m+1})}{6};$$

hence,

$$|E \cap (\tau_m, \ell_m)| = |E \cap (\ell_{m+1}, \ell_m)| - |E \cap (\ell_{m+1}, \tau_m)| \geq \frac{(\ell_m - \ell_{m+1})}{6}. \tag{2.10}$$

Taking $s = \ell_{m+1}$ in (2.9), we get

$$\|\mathbf{z}(\cdot,t)\|_{\mathbf{H}} \leq \left(Ce^{\frac{C}{\ell_m - \ell_{m+1}}} \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)} \right)^{1/2} \|\mathbf{z}(\cdot,\ell_{m+1})\|_{\mathbf{H}}^{1/2} \text{ for a.e. } t \in E \cap (\tau_m, \ell_m). \tag{2.11}$$

Then, using the L^2 energy estimate for the Stokes system, integrating (2.11) with respect to t over $E \cap (\tau_m, \ell_m)$ and using (2.10), we obtain

$$\|\mathbf{z}(\cdot,\ell_m)\|_{\mathbf{H}} \leq \left(Ce^{\frac{C}{\ell_m - \ell_{m+1}}} \int_{\ell_{m+1}}^{\ell_m} \chi_E(t) \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)}\, dt \right)^{1/2} \|\mathbf{z}(\cdot,\ell_{m+1})\|_{\mathbf{H}}^{1/2},$$

which implies that

$$\|\mathbf{z}(\cdot,\ell_m)\|_{\mathbf{H}} \leq \epsilon \|\mathbf{z}(\cdot,\ell_{m+1})\|_{\mathbf{H}} + \epsilon^{-1} Ce^{\frac{C}{\ell_m - \ell_{m+1}}} \int_{\ell_{m+1}}^{\ell_m} \chi_E(t) \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)}\, dt$$

for any $\epsilon > 0$.

Taking $\epsilon = e^{-\frac{1}{2(\ell_m - \ell_{m+1})}}$ in the above inequality, we have

$$e^{-\frac{C+\frac{1}{2}}{\ell_m - \ell_{m+1}}} \|\mathbf{z}(\cdot,\ell_m)\|_{\mathbf{H}} - e^{-\frac{C+1}{\ell_m - \ell_{m+1}}} \|\mathbf{z}(\cdot,\ell_{m+1})\|_{\mathbf{H}} \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E(t) \|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)}\, dt. \tag{2.12}$$

Finally, choosing $\mu = \frac{2(C+1)}{2C+1}$, where C is any constant for which inequality (2.12) holds, we readly obtain

$$
\begin{aligned}
& e^{-\frac{C+\frac{1}{2}}{\ell_m-\ell_{m+1}}}\|\mathbf{z}(\cdot,\ell_m)\|_{\mathbf{H}} - e^{-\frac{C+\frac{1}{2}}{\ell_{m+1}-\ell_{m+2}}}\|\mathbf{z}(\cdot,\ell_{m+1})\|_{\mathbf{H}} \\
& \qquad \le C\int_{\ell_{m+1}}^{\ell_m}\chi_E(t)\|\mathbf{z}(\cdot,t)\|_{\mathbf{L}^1(\mathcal{M}_t)}\,dt \quad \forall m\ge 1,
\end{aligned} \tag{2.13}
$$

because $\mu(\ell_{m+1}-\ell_{m+2}) = \ell_m - \ell_{m+1}$ for all $m \ge 1$.

This way, adding the telescoping series in (2.13) from $m=1$ to $+\infty$ and using the fact that $\lim\limits_{m\to\infty}\ell_m = \ell$, we have that

$$
\begin{aligned}
e^{-\frac{C+\frac{1}{2}}{\ell_1-\ell_2}}\|z(\cdot,\ell_1)\|_{\mathbf{H}} &= \sum_{m=1}^{\infty}\left[e^{-\frac{C+\frac{1}{2}}{\ell_m-\ell_{m+1}}}\|z(\cdot,\ell_m)\|_{\mathbf{H}} - e^{-\frac{C+\frac{1}{2}}{\ell_{m+1}-\ell_{m+2}}}\|z(\cdot,\ell_{m+1})\|_{\mathbf{H}}\right] \\
&\le C\int_{\mathcal{M}\cap(\Omega\times[\ell,\ell_1])}|\mathbf{z}(\mathbf{x},t)|\,d\mathbf{x}dt.
\end{aligned}
$$

The L^2 energy estimate for the Stokes system and the fact that $\ell < \ell_m < T$ for every $m \in \mathbb{N}$ lead to the following observability inequality

$$
\|\mathbf{z}(\cdot,T)\|_{\mathbf{H}} \le C\int_{\mathcal{M}\cap(\Omega\times[l,l_1])}|\mathbf{z}(\mathbf{x},t)|\,d\mathbf{x}dt,
$$

with some constant $C = C(N,R,\Omega,\mathcal{M},T) > 0$. This completes the proof. ∎

3. Applications.

3.1. Shape optimization problems.

As an interesting application of Theorem 1.5, we analyze the following shape optimization problem formulated in [17].

Let $\{\beta_j^\nu\}_{j\in\mathbb{N}}$ be a sequence of independent real random variables on a probability space $(\mathrm{X},\mathcal{F},\mathbb{P})$ having mean equal to 0, variance equal to 1, and a super exponential decay (for instance, independent Gaussian or Bernoulli random variables; see [4, Assumption (3.1)] for more details). For every $\nu\in\mathrm{X}$, the solution of (1.1) corresponding to the initial datum

$$
\mathbf{z}_0^\nu = \sum_{j\ge 1}\beta_j^\nu a_j\mathbf{e}_j, \quad \text{with } \{a_j\}_{j\ge 1}\in\ell^2, \tag{3.1}
$$

is given by

$$
\mathbf{z}^\nu(\cdot,t) = \sum_{j\ge 1}\beta_j^\nu a_j e^{-t\lambda_j}\mathbf{e}_j. \tag{3.2}
$$

Given $L\in(0,1)$, we define the set of admissible designs

$$
\mathcal{U}_L = \Big\{\chi_\omega \in L^\infty(\Omega;\{0,1\}) : \omega\subset\Omega \text{ is a measurable subset of measure } |\omega| = L|\Omega|\Big\}.
$$

For each $\chi_\omega\in\mathcal{U}_L$, we then define the randomized observability constant by

$$
C_{T,rand}(\chi_\omega) = \inf_{\|\mathbf{z}^\nu(T)\|=1}\mathbb{E}\int_0^T\int_\omega|\mathbf{z}^\nu(x,t)|^2 d\mathbf{x}dt.
$$

Using (3.2), the properties of random variables β_j^ν, and the change of variable $b_j = a_j e^{-T\lambda_j}$, we deduce that

$$C_{T,rand}(\chi_\omega) = \inf_{\sum_{j=1}^{\infty} |b_j|^2=1} \mathbb{E} \int_0^T \int_\omega \left| \sum_{j\geq 1} \beta_j^\nu b_j e^{t\lambda_j} \mathbf{e}_j(\mathbf{x}) \right|^2 d\mathbf{x}dt,$$

where $\mathbb{E}$ is the expectation over the space $\mathbb{X}$ with respect to the probability measure $\mathbb{P}$.

From Fubini's theorem and the independence of the random variables $\{\beta_j^\nu\}_{j\in\mathbb{N}}$, a simple computation gives

$$C_{T,rand}(\chi_\omega) = \inf_{j\geq 1} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\omega |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x}.$$

We now consider the optimal design problem of maximizing the randomized observability constant $C_{T,rand}(\chi_\omega)$ over the set of admissible designs $\mathcal{U}_L$. In other words, we study the problem

$$(P^T): \quad \sup_{\chi_\omega \in \mathcal{U}_L} C_{T,rand}(\chi_\omega) = \sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j\geq 1} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\omega |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x}. \tag{3.3}$$

The optimal shape design problem (3.3) models the best sensor shape and location problem for the control of the Stokes system (1.1).

We have the following result.

THEOREM 3.1. *The problem (P^T) has a unique solution.*

Proof. We only have to check the following two conditions:

(i) If there exists $E \subset \Omega$ of positive Lebesgue measure, an integer $m \in \mathbb{N}^*$, $\beta_1, \dots, \beta_m \in \mathbb{R}^+$, and $C \geq 0$ such that $\sum_{j=1}^m \beta_j |\mathbf{e}_j(\mathbf{x})|^2 = C$ almost everywhere on E, then there must hold $C = 0$ and $\beta_1 = \beta_2 = \cdots = \beta_m = 0$.

(ii) For every $a \in L^\infty(\Omega; [0,1])$ such that $\int_\Omega a(\mathbf{x}) \, d\mathbf{x} = L|\Omega|$, one has

$$\liminf_{j\to+\infty} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\Omega a(x) |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x} > \frac{e^{2T\lambda_1} - 1}{2\lambda_1}.$$

By the analyticity of the eigenfunctions of Stokes system with homogeneous Dirichlet boundary conditions, it is not difficult to show that the first condition holds.

For the second condition, notice that there exists $\epsilon > 0$ and $E \subset \Omega$ of positive measure such that $a \geq \epsilon \chi_E$ and

$$\int_\Omega a(x) |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x} \geq \epsilon \int_E |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x}.$$

From Theorem 1.5, we easily see that

$$\liminf_{j\to+\infty} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\Omega a(x) |\mathbf{e}_j(\mathbf{x})|^2 \, d\mathbf{x} = +\infty.$$

From [17, Theorem 1], it follows that problem (P^T) has a unique solution. ∎

Remark 3.2. The optimal set given by Theorem 3.1 is open and semianalytic.[6] This follows from the fact that the eigenfunctions of the Stokes system with homogeneous Dirichlet boundary conditions are analytic.

Remark 3.3. A proof of Theorem 3.1 when Ω is the unit disk of $\mathbb{R}^2$ can be found in [17]. However, the proof there relies on an explicit knowledge of the eigenfunctions of the Stokes operator, which of course cannot be extended to the case of general domains or higher dimensions. For the general case, the key point is the obtainment of a uniform observability inequality with observations on measurable sets of positive measure. Thus, Theorem 1.5 gives a positive answer to the shape optimization problem for Stokes system raised in [17].

3.2. Null controllability for Stokes system with bounded controls. Let ω be a nonempty open subset of Ω, and consider the following controlled Stokes system

$$(3.4)\qquad \left|\begin{array}{lll} \mathbf{u}_t - \Delta\mathbf{u} + \nabla p = \mathbf{v}\chi_\omega & \text{in} & Q, \\ \operatorname{div}\mathbf{u} = 0 & \text{in} & Q, \\ \mathbf{u} = \mathbf{0} & \text{on} & \Sigma, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0 & \text{in} & \Omega. \end{array}\right.$$

It is well known that for any $T>0$, $\mathbf{u}_0 \in \mathbf{H}$, and $\mathbf{v} \in \mathbf{L}^2(\omega\times(0,T))$, there exists exactly one solution $(\mathbf{u},p)$ to the Stokes equations (3.4) with

$$\mathbf{u} \in C^0([0,T];\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \ p \in L^2(0,T;U),$$

where

$$U := \left\{\psi \in H^1(\Omega); \int_\Omega \psi(\mathbf{x})\,d\mathbf{x} = 0\right\}.$$

In the context of the Stokes system (3.4), for $1 \le p \le \infty$, the $\mathbf{L}^p$- *null controllability* problem at time T reads as follows:

For any $\mathbf{u}_0 \in \mathbf{H}$*, can one find a control* $\mathbf{v} \in \mathbf{L}^p(\omega\times(0,T))$ *such that the associated solution to* (3.4) *satisfies*

$$(3.5)\qquad \mathbf{u}(\cdot,T) = 0 \quad \textit{in} \quad \Omega?$$

The following result is well known.

THEOREM 3.4. *For any nonempty open subset* ω *of* Ω *and any* $T>0$*, the Stokes system* (3.4) *is* $\mathbf{L}^2$*-null controllable.*

For the proof, we refer the reader to [5, 10, 11].

In practice it would be interesting to take the control steering the solution of the Stokes system to rest to be in $\mathbf{L}^\infty(\omega\times(0,T))$. Nevertheless, to the best of our knowledge, it is not clear how to construct $\mathbf{L}^\infty(\omega\times(0,T))$ controls from $\mathbf{L}^2(\omega\times(0,T))$ controls. Notice that for the case of the heat equation this is always possible since one can use local regularity results (for more details, see [3]), which is no longer the case for the Stokes system.

The observability inequality established in Theorem 1.1 allows us to conclude stronger controllability properties for the Stokes system (3.4). In fact it is possible to control the Stokes system with $\mathbf{L}^\infty$-controls supported in any measurable set of positive measure:

[6]Here, it is understood that the optimal set is unique up to the set of zero measure. A subset of a real analytic finite-dimensional manifold is said to be semianalytic if it can be written in terms of equalities and inequalities of real analytic functions.

THEOREM 3.5. *For any $T > 0$ and any measurable set of positive measure $\gamma \subset \Omega \times [0,T]$, the Stokes system* (3.4) *is $\mathbf{L}^\infty$-null controllable with a control $\mathbf{v}$ supported in γ and having the following estimate*

$$\|\mathbf{v}\|_{\mathbf{L}^\infty(\gamma)} \le C_{obs}(T,\gamma)\|\mathbf{u}_0\|_{\mathbf{H}}, \tag{3.6}$$

where $C_{obs}(T,\gamma)$ is the observability constant given by Theorem 1.1 *for the control domain γ at time T.*

In particular, in the case of a nonempty open subset, we have the following.

COROLLARY 3.6. *For any nonempty open subset ω of Ω and any $T > 0$, the Stokes system* (3.4) *is $\mathbf{L}^\infty$-null-controllable.*

3.3. Time optimal control problem for Stokes system. Let $|\cdot|_r : \mathbb{R}^N \to [0,\infty)$ be the r-Euclidean norm in $\mathbb{R}^N$, i.e.,

$$|\mathbf{x}|_r = \begin{cases} (|x_1|^r + \cdots + |x_N|^r)^{\frac{1}{r}} & \text{if} \quad r \in [1,\infty), \\ \max\{|x_1|,\dots,|x_N|\} & \text{if} \quad r = \infty \end{cases}$$

for every $\mathbf{x} \in \mathbb{R}^N$.

For $r \in [1,\infty]$ fixed and any $M > 0$, we consider the *set of admissible controls*

$$\mathcal{U}_{ad}^{M,r} = \big\{\mathbf{v} \in \mathbf{L}^\infty(\omega \times [0,\infty))\,;\, |\mathbf{v}(\mathbf{x},t)|_r \le M \ \text{ a.e. in } \ \omega \times [0,\infty)\big\},$$

and for $\mathbf{u}_0 \in \mathbf{H}$ given, we define the *set of reachable states starting from* $\mathbf{u}_0$:

$$\mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r}) = \Big\{\mathbf{u}(\cdot,\tau)\,;\, \tau > 0 \text{ and } \mathbf{u} \text{ is the solution of (3.4) with } \mathbf{v} \in \mathcal{U}_{ad}^{M,r}\Big\}.$$

Thanks to Theorem 3.6, it follows that $\mathbf{0} \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$ for any $\mathbf{u}_0 \in \mathbf{H}$.

In this section, we study the following time optimal control problem:

Given $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$, find $\mathbf{v}_r^\star \in \mathcal{U}_{ad}^{M,r}$ such that the corresponding solution $\mathbf{u}^\star$ of (3.4) *satisfies*

$$\mathbf{u}^\star(\tau_r^\star(\mathbf{u}_0,\mathbf{u}_f)) = \mathbf{u}_f, \tag{3.7}$$

where $\tau_r^\star(\mathbf{u}_0,\mathbf{u}_f)$ is the minimal time needed to steer the initial datum $\mathbf{u}_0$ to the target $\mathbf{u}_f$ with controls in $\mathcal{U}_{ad}^{M,r}$, i.e.,

$$\tau_r^\star(\mathbf{u}_0,\mathbf{u}_f) = \min_{\mathbf{v}\in\mathcal{U}_{ad}^{M,r}} \{\tau\,;\, \mathbf{u}(\cdot,\tau) = \mathbf{u}_f\}. \tag{3.8}$$

Time optimal control problems are well known for the heat equation; see, for instance, [15, 18]. However, we are not aware any result for the Stokes system.

We have the following result.

THEOREM 3.7. *Let $M > 0$ and $r \in [1,\infty]$ be given. For every $\mathbf{u}_0 \in \mathbf{H}$ and any $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$, the time optimal problem* (3.8) *has at least one solution. Moreover, any optimal control $\mathbf{v}_r^\star$ satisfies the bang-bang property: $|\mathbf{v}_r^\star(\mathbf{x},t)|_r = M$ for a.e. $(\mathbf{x},t) \in \omega \times [0,\tau_r^\star(\mathbf{u}_0,\mathbf{u}_f)]$.*

Proof. Since $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$, there exists a minimizing sequence $(\tau_n, \mathbf{v}_n)_{n\ge1}$ such that $\tau_n \xrightarrow[n\to\infty]{} \tau_r^\star(\mathbf{u}_0,\mathbf{u}_f)$ and $(\mathbf{v}_n)_{n\ge1} \subset \mathcal{U}_{ad}^{M,r}$ has the property that the associated solution $\mathbf{u}_n$ to (3.4) satisfies $\mathbf{u}_n(\cdot,\tau_n) = \mathbf{u}_f$ for all $n \ge 1$. Also, because

$(\mathbf{v}_n)_{n\geq 1} \subset \mathcal{U}_{ad}^{M,r}$, it follows that $(\mathbf{v}_n)_{n\geq 1}$ converges weak-$\star$ to some vector-function $\mathbf{v}^\star \in \mathcal{U}_{ad}^{M,r}$ in $\mathbf{L}^\infty(\omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)))$.

Claim: $\mathbf{v}^\star$ is a solution of the time optimal problem (3.7).

Proof of the Claim. We only have to show that $\mathbf{u}^\star(\cdot, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)) = \mathbf{u}_f$, where $\mathbf{u}^\star$ is the solution of (3.4) associated to $\mathbf{v}^\star$.

To show this, let $\bar{\mathbf{u}}$ be the solution of (3.4) with $\mathbf{v} \equiv \mathbf{0}$ and $\mathbf{w} = \mathbf{u}^\star - \bar{\mathbf{u}}$, $\mathbf{w}_n = \mathbf{u}_n - \bar{\mathbf{u}}$ solutions of

$$\left|\begin{array}{lll} \mathbf{w}_t - \Delta \mathbf{w} + \nabla \pi = \mathbf{v}^\star 1_\omega & \text{in} & Q, \\ \operatorname{div} \mathbf{w} = 0 & \text{in} & Q, \\ \mathbf{w} = \mathbf{0} & \text{on} & \Sigma, \\ \mathbf{w}_n(0) = \mathbf{0} & \text{in} & \Omega, \end{array}\right.$$

and

$$\left|\begin{array}{lll} \mathbf{w}_{n,t} - \Delta \mathbf{w}_n + \nabla \pi_n = \mathbf{v}_n 1_\omega & \text{in} & Q, \\ \operatorname{div} \mathbf{w}_n = 0 & \text{in} & Q, \\ \mathbf{w}_n = \mathbf{0} & \text{on} & \Sigma, \\ \mathbf{w}_n(0) = \mathbf{0} & \text{in} & \Omega, \end{array}\right.$$

respectively.

Now, thanks to the continuity in time of $\bar{\mathbf{u}}$ and that $\tau_n \xrightarrow[n\to\infty]{} \tau^\star(\mathbf{u}_0, \mathbf{u}_f)$, it follows that $\bar{\mathbf{u}}(\cdot, \tau_n) \xrightarrow[n\to\infty]{} \bar{\mathbf{u}}(\cdot, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f))$ in $\mathbf{H}$. Moreover, it is not difficult to see that

$$\langle \mathbf{w}_n(\tau_n) - \mathbf{w}_n(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle \to 0 \quad \forall \varphi \in \mathbf{H},$$

$$\langle \mathbf{w}_n(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle \to \langle \mathbf{w}(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle \quad \forall \varphi \in \mathbf{H},$$

and

$$\langle \mathbf{w}_n(\tau_n), \varphi \rangle \to \langle \mathbf{w}(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle \quad \forall \varphi \in \mathbf{H}.$$

Since $\mathbf{u}_f = \bar{\mathbf{u}}(\cdot, \tau_n) + \mathbf{w}_n(\cdot, \tau_n)$, we have that $\langle \mathbf{u}_f, \varphi \rangle = \langle \bar{\mathbf{u}}(\cdot, \tau_n) + \mathbf{w}_n(\cdot, \tau_n), \varphi \rangle$ for all $\varphi \in \mathbf{H}$ and $\langle \mathbf{u}_f, \varphi \rangle = \langle \bar{\mathbf{u}}(\cdot, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)) + \mathbf{w}(\cdot, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle = \langle \mathbf{u}^\star(\cdot, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)), \varphi \rangle$ for all $\varphi \in \mathbf{H}$. ☐

Now, let us show that any optimal control $\mathbf{v}^\star \in \mathcal{U}_{ad}^{M,r}$ satisfies the bang-bang property. To do this, we argue by contradiction.

We consider $\mathbf{u}^\star$ the corresponding state (with some pressure) to (3.4) and suppose that there exist $\epsilon > 0$ and a measurable set of positive measure $\gamma \subset \omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f))$ such that

$$|\mathbf{v}^\star(\mathbf{x}, t)|_r < M - \epsilon \quad ((\mathbf{x}, t) \in \gamma). \tag{3.9}$$

Choosing $\delta_0 > 0$ small enough such that

$$\left\{\begin{array}{l} \tau_0 = \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) - \delta_0 > 0, \\ \text{the set } \Gamma = \{(\mathbf{x}, t) \in \omega \times (0, \tau_0) \,:\, (\mathbf{x}, t) \in \gamma\} \text{ has positive measure,} \end{array}\right.$$

and using the time continuity of $\mathbf{u}^\star$, there exists $\delta \in (0, \delta_0)$ such that

$$\|\mathbf{u}_0 - \mathbf{u}^\star(\cdot, \delta)\|_{\mathbf{H}} < \frac{\epsilon}{C_{obs}(\tau_0, \Gamma)}, \tag{3.10}$$

where $C_{obs}(\tau_0, \Gamma)$ is the observability constant given by Theorem 1.1 for the control domain Γ at time τ_0.

From Theorem 3.5, there exists a control $\mathbf{v} \in \mathbf{L}^\infty(\omega \times (0, \tau_0))$ with

$$\begin{cases} \operatorname{supp} \mathbf{v} \subset \Gamma, \\ \text{the associated solution } \mathbf{u} \text{ satisfies } \mathbf{u}(\cdot, 0) = \mathbf{u}_0 - \mathbf{u}^\star(\cdot, \delta) \text{ and } \mathbf{u}(\cdot, \tau_0) = \mathbf{0}, \\ \|\mathbf{v}\|_{\mathbf{L}^\infty(\Gamma)} \le C_{obs}(\tau_0, \Gamma)\|\mathbf{u}_0 - \mathbf{u}^\star(\delta)\|_{\mathbf{H}}. \end{cases}$$

Thus, from (3.10) we have that

$$\|\mathbf{v}\|_{\mathbf{L}^\infty(\omega \times (0,\tau_0))} \le \epsilon.$$

Now, let $\widehat{\mathbf{v}} \in \mathbf{L}^\infty(\omega \times (0, \tau_0))$ be defined by

$$\widehat{\mathbf{v}}(x, t) = \mathbf{v}^\star(\mathbf{x}, t + \delta) + \mathbf{v}(\mathbf{x}, t) \quad (t \in [0, \tau_0]).$$

Noticing that $\tau_0 + \delta \le \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)$, using the fact that $\operatorname{supp} \mathbf{v} \subset \Gamma$ and estimate (3.9), it follows that $\widehat{\mathbf{v}} \in \mathcal{U}_{ad}^{M,r}$.

Finally, setting $\widehat{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}^\star(\mathbf{x}, t+\delta) + \mathbf{u}(\mathbf{x}, t)$ and $\widehat{p}(\mathbf{x}, t) = p^\star(\mathbf{x}, t+\delta) + p(\mathbf{x}, t)$, we have that $\widehat{\mathbf{u}}(\cdot, 0) = \mathbf{u}_0$, $\widehat{\mathbf{u}}(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) - \delta) = \mathbf{u}_f$ and that

$$\widehat{\mathbf{u}}_t - \Delta\widehat{\mathbf{u}} + \nabla\widehat{p} = \widehat{\mathbf{v}}1_\omega.$$

Hence, $\widehat{\mathbf{v}} \in \mathcal{U}_{ad}^{M,r}$ is a control which steers $\mathbf{u}_0$ to $\mathbf{u}_f$ at time $\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) - \delta$. This contradicts with the definition of $\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)$ and thus the desired bang-bang property holds. ∎

About the uniqueness of the optimal control for the problem (3.8), using some ideas from [9], we have the following result.

PROPOSITION 3.8. *Let $M > 0$ and $r \in (1, \infty)$. For any $\mathbf{u}_0 \in \mathbf{H}$ and every $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$, the time optimal control problem* (3.7)–(3.8) *has a unique solution $\mathbf{v}_r^\star$ which satisfies a bang-bang property: $|\mathbf{v}_r^\star(\mathbf{x}, t)|_r = M$ for a.e. $(\mathbf{x}, t) \in \omega \times [0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)]$.*

Proof. The existence of solution and the bang-bang property is a consequence of Theorem 3.7. We only have to prove the uniqueness of solution. Thus, let $\mathbf{v}$ and $\mathbf{h}$ be two time optimal controls in $\mathcal{U}_{ad}^{M,r}$. Thanks to the linearity, $\mathbf{w} = \frac{1}{2}(\mathbf{v} + \mathbf{h})$ is also a time optimal control. From Theorem 3.7, $\mathbf{w}$ also satisfies the bang-bang property. Therefore, we have that $|\mathbf{v}(\mathbf{x}, t)|_r = |\mathbf{h}(\mathbf{x}, t)|_r = |\mathbf{w}(\mathbf{x}, t)|_r = M$, a.e. in $\omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_1))$. Now, if $\mathbf{v}(\mathbf{x}, t) \neq \mathbf{h}(\mathbf{x}, t)$ in a measurable set of positive measure $\mathcal{D} \subset \omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_1))$, then, thanks to the fact that any norm $|\cdot|_r$ for $r \in (1, \infty)$ is *uniformly convex* in $\mathbb{R}^N$, we have that $|\mathbf{w}(\mathbf{x}, t)|_r < M$ a.e. in $\mathcal{D} \subset \omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_1))$. This contradicts with the bang-bang property for $\mathbf{w}$. ∎

Remark 3.9. Related to the results of this section, one could ask for a stronger bang-bang property for time optimal controls: given an optimal control $\mathbf{v}^\star$, is it true that each component of $\mathbf{v}^\star$ satisfies $|\mathbf{v}_i^\star(\mathbf{x}, t)| = M$ for a.e. $(\mathbf{x}, t) \in \omega \times [0, \tau^\star(\mathbf{u}_0, \mathbf{u}_f)]$ and all $i = 1, \dots, N$?

Arguing as in Theorem 3.7, one can see that this problem is related to the null controllability property on measurable sets for the Stokes system with only one control (see, e.g., [6]), which we believe it is true at least for $N = 2$.

Appendix A. Real-analytic estimates for solutions to the Poisson equation.

In this appendix we prove the following lemma which was used in the proof of Theorem 1.5.

LEMMA A.1. *Assume that f is a real-analytic function in $B_R(\mathbf{x}_0)$ verifying*

$$|\partial_{\mathbf{x}}^\alpha f(\mathbf{x})| \leq \frac{M|\alpha|!}{(R\rho_0)^{|\alpha|}} \quad \forall\ \mathbf{x} \in B_R(\mathbf{x}_0) \text{ and } \alpha \in \mathbb{N}^N, \tag{A.1}$$

with some positive constants M and ρ_0. Let $u \in H^2(B_R(\mathbf{x}_0))$ satisfy the Poisson equation

$$-\Delta u = f \text{ in } B_R(\mathbf{x}_0). \tag{A.2}$$

Then, u is real-analytic in $B_{R/2}(\mathbf{x}_0)$. Further, it verifies the estimate

$$\|\partial_{\mathbf{x}}^\alpha u\|_{L^\infty(B_{R/2}(\mathbf{x}_0))} \leq \frac{|\alpha|!}{(R\tilde{\rho})^{|\alpha|+1}} \left(\|u\|_{L^2(B_R(\mathbf{x}_0))} + M\right) \text{ for all } \alpha \in \mathbb{N}^N, \tag{A.3}$$

where $\tilde{\rho}$ is a constant depending only on the dimension N and ρ_0.

A proof of the lemma A.1 for $f \equiv 0$ can be found in [16]. For the sake of completeness, we give a sketch proof for the nonhomogeneous case.

Proof. By a rescaling argument, it suffices to prove the estimate (A.3) when $R = 1$ and $\mathbf{x}_0 = \mathbf{0}$.

Since f is real-analytic in $B_1(\mathbf{0})$, by the interior regularity for solutions of elliptic equations, we have that u is smooth in $B_1(\mathbf{0})$. Hence, we have that

$$-\Delta\partial_{\mathbf{x}}^\alpha u(\mathbf{x}) = \partial_{\mathbf{x}}^\alpha f(\mathbf{x}) \quad \forall\ \mathbf{x} \in B_1(\mathbf{0})$$

for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$.

Multiplying the above equation by $(1-|\mathbf{x}|^2)^{2(|\alpha|+1)}\partial_{\mathbf{x}}^\alpha u$ gives

$$-(1-|\mathbf{x}|^2)^{2(|\alpha|+1)}\partial_{\mathbf{x}}^\alpha u(\mathbf{x})\Delta\partial_{\mathbf{x}}^\alpha u(\mathbf{x}) = (1-|\mathbf{x}|^2)^{2(|\alpha|+1)}\partial_{\mathbf{x}}^\alpha u(\mathbf{x})\partial_{\mathbf{x}}^\alpha f(\mathbf{x}) \quad \forall\, \mathbf{x} \in B_1(\mathbf{0}), \tag{A.4}$$

and integration by parts gives

$$\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2(|\alpha|+1)}|\nabla\partial_{\mathbf{x}}^\alpha u|^2\, d\mathbf{x} = 4(|\alpha|+1)\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2|\alpha|+1}(\nabla\partial_{\mathbf{x}}^\alpha u \cdot \mathbf{x})\partial_{\mathbf{x}}^\alpha u\, d\mathbf{x}$$
$$+ \iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2(|\alpha|+1)}\partial_{\mathbf{x}}^\alpha u\partial_{\mathbf{x}}^\alpha f\, d\mathbf{x}.$$

Now, thanks to the Young inequality, we have the following estimate:

$$\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2(|\alpha|+1)}|\nabla\partial_{\mathbf{x}}^\alpha u|^2 d\mathbf{x} \leq [16(|\alpha|+1)^2+1]\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2|\alpha|}|\partial_{\mathbf{x}}^\alpha u|^2\, d\mathbf{x}$$
$$+ \iint_{B_1(\mathbf{0})} |\partial_{\mathbf{x}}^\alpha f|^2\, d\mathbf{x}.$$

Since f satisfies (A.1), we get

$$\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2(|\alpha|+1)}|\nabla\partial_{\mathbf{x}}^\alpha u|^2\, d\mathbf{x} \leq 17(|\alpha|+1)^2\iint_{B_1(\mathbf{0})} (1-|\mathbf{x}|^2)^{2|\alpha|}|\partial_{\mathbf{x}}^\alpha u|^2\, d\mathbf{x}$$
$$+ |B_1(0)|\left|\frac{M|\alpha|!}{\rho_0^{|\alpha|}}\right|^2.$$

Therefore, we obtain

$$\left\|(1-|\mathbf{x}|^2)^{|\alpha|+1}\nabla\partial_{\mathbf{x}}^{\alpha}u\right\|_{L^2(B_1(\mathbf{0}))} \leq 5\left[(|\alpha|+1)\left\|(1-|\mathbf{x}|^2)^{|\alpha|}\partial_{\mathbf{x}}^{\alpha}u\right\|_{L^2(B_1(\mathbf{0}))} + \frac{M|\alpha|!}{\rho_0^{|\alpha|}}\right] \tag{A.5}$$

for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$. In particular, taking $\alpha = (0, \ldots, 0)$, we deduce the estimate

$$\left\|(1-|\mathbf{x}|^2)\nabla u\right\|_{L^2(B_1(\mathbf{0}))} \leq 5\left(\|u\|_{L^2(B_1(\mathbf{0}))} + M\right).$$

By induction, we have the inequality

$$\left\|(1-|\mathbf{x}|^2)^{|\alpha|}\partial_{\mathbf{x}}^{\alpha}u\right\|_{L^2(B_1(\mathbf{0}))} \leq \rho^{-|\alpha|-1}|\alpha|!\left(\|u\|_{L^2(B_1(\mathbf{0}))} + M\right) \tag{A.6}$$

for some constant $0 < \rho < \min\{\rho_0, 1/6\}$ and every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$. Finally, it is not difficult to see that the estimate (A.6) leads to (A.3). ∎

Acknowledgments. The authors would like to appreciate Prof. E. Trélat and Prof. G. Lebeau for the stimulating conversations during this work.

REFERENCES

[1] J. Apraiz and L. Escauriaza, *Null-control and measurable sets*. ESAIM: COCV, 19 (2013), 239–254.

[2] J. Apraiz, L. Escauriaza, G. Wang, and C. Zhang, *Observability inequalities and measurable sets*, J. Eur. Math. Soc., 16 (2014), pp. 2433–2475.

[3] O. Bodart, M. González-Burgos, and R. Pérez-García, *Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity*, Comm. Partial Differential Equations, 29 (2004), pp. 1017–1050.

[4] N. Burq and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations.* I. *Local theory*, Invent. Math., 173 (2008), pp. 449–475.

[5] F. W. Chaves-Silva and G. Lebeau, *Spectral inequality and optimal cost of controllability for the Stokes system*, ESAIM: COCV, 22 (2016), pp. 1137–1162.

[6] J. M. Coron and S. Guerrero, *Null controllability of the N-dimensional Stokes system with N-1 scalar controls*, J. Differential Equations, 246 (2009), pp. 2908–2921.

[7] L. Escauriaza, S. Montaner, and C. Zhang, *Observation from measurable sets for parabolic analytic evolutions and applications*, J. Math. Pures Appl., 104 (2015), pp. 837–867.

[8] L. Escauriaza, S. Montaner, and C. Zhang, *Analyticity of solutions to parabolic evolutions and applications*, SIAM J. Math. Anal., 49 (2017), pp. 4064–4092.

[9] H. Fattorini, *Infinite Dimensional Linear Control Systems; the Time Optimal and Norm Optimal Problems*, North-Holland Math. Stud. 201, Elsevier, Amsterdam, 2005.

[10] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J. P. Puel, *Local exact controllability of the Navier-Stokes system*, J. Math. Pures Appl., 83 (2004), pp. 1501–1542.

[11] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series 34, Research Institute of Mathematics, Seoul National University, Seoul, 1996.

[12] O. Yu. Imanuvilov, J. P. Puel, and M. Yamamoto, *Carleman estimates for parabolic equations with nonhomogeneous boundary conditions*, Chin. Ann. Math. Ser. B, 30 (2009), pp. 333–378.

[13] C. Kahane, *On the spatial analyticity of solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., 33 (1969), pp. 386–405.

[14] K. Masuda, *On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation*, Proc. Japan Acad., 43 (1967), pp. 827–832.

[15] S. Micu, I. Roventa, and M. Tucsnak, *Time optimal boundary controls for the heat equation*, J. Funct. Anal., 263 (2012), pp. 25–49.

[16] C. B. Morrey and L. Nirenberg, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Commun. Pur. Appl. Math., X (1957), pp. 271–290.

[17] Y. Privat, E. Trélat, and E. Zuazua, *Optimal shape and location of sensors for parabolic equations with random initial data*, Arch. Ration. Mech. Anal., 216 (2015), pp. 921–981.

[18] K. D. Phung and G. Wang, *An observability estimate for parabolic equations from a general measurable set in time and its applications*, J. Eur. Math. Soc., 15 (2013), pp. 681–703.

[19] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*, Asympt. Analysis, 10 (1995), pp. 95–115.

[20] S. Vessella, *A continuous dependence result in the analytic continuation problem*, Forum Math., 11 (1999), pp. 695–703.

[21] G. Wang, *L^∞-Null controllability for the heat equation and its consequences for the time optimal control problem*, SIAM J. Control Optim., 47 (2008), pp. 1701–1720.

[22] G. Wang and C. Zhang, *Observability estimate from measurable sets in time for some evolution equations*, SIAM J. Control Optim., 55 (2017), pp. 1862–1886.