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The Conjugacy Stability Problem for Parabolic Subgroups in Artin Groups

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Abstract. Given an Artin group A and a parabolic subgroup P, we study if every two elements of P that are conjugate in A, are also conjugate in P. We provide an algorithm to solve this decision problem if A satisfies three properties that are conjectured to be true for every Artin group. This allows to solve the problem for new families of Artin groups. We also partially solve the problem if A has FC-type, and we totally solve it if A is isomorphic to a free product of Artin groups of spherical type. In particular, we show that in this latter case, every element of A is contained in a unique minimal (by inclusion) parabolic subgroup.

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1. Introduction

Artin (or Artin–Tits) groups were defined by Jacques Tits in the 60's. They are groups presented by a finite set of generators S and at most one relation of the form $stst\cdots = tsts\cdots$, for every pair $s,t \in S$, with the same number of letters $m_{s,t}$ at each side of the equality. If there is no relation associated with a pair of generators $s,t \in S$, then we denote $m_{s,t} = \infty$. Then, the presentation of an Artin group is as follows:

$$A_S = \langle S \mid \underbrace{sts\ldots}_{m_{s,t} \text{ elements}} = \underbrace{tst\ldots}_{m_{s,t} \text{ elements}} \quad \forall s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle.$$

These groups are algebraic generalisations of the well-known braid groups on $n+1$ strands [2]:

$$A_n = \left\langle \sigma_1, \ldots, \sigma_n \middle| \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \end{matrix} \right\rangle.$$

A fundamental tool for the study of braid groups is the action by isometries of A_n on the curve complex of the $n+1$-punctured disk $\mathcal{D}_{n+1}$. The curve complex has as vertices (isotopy classes of non-degenerated) simple closed curves in $\mathcal{D}_{n+1}$. For Artin groups, the analogous of simple closed curves are irreducible parabolic subgroups. In fact, there is a bijection between the proper irreducible parabolic subgroups of A_n and the simple closed curves in $\mathcal{D}_{n+1}$ (see an explanation in [9, Section 2].

A *standard parabolic subgroup* A_X is a subgroup generated by a subset of generators $X \subseteq S$. The conjugate of any standard parabolic subgroup by an element of A_S is called a *parabolic subgroup*. The study of parabolic subgroups has been an important source of research in Artin groups over the last forty years. These are natural and easy-to-define subgroups. They are the main ingredient of complexes in which Artin groups act, as the Deligne complex [6, 11] or the complex of irreducible parabolic subgroups [9]. However, as it happens for most questions in Artin groups, basic properties of parabolic subgroups are in general unknown. Some of the facts we know about are the following: In his thesis, Van der Lek [24] proved that a standard parabolic subgroup is again an Artin group, and we also know that they are convex in every case [7]. The structure of centralisers of parabolic subgroups and many of their properties have been well studied only certain cases by Paris [21] and Godelle [12–14], among others; and we only know if the intersection of parabolic subgroups is again a parabolic subgroup for a few families of Artin groups [9, 10, 20].

In this paper, we discuss in which cases embeddings of parabolic subgroups into the Artin group merge conjugacy classes. This is also called the *conjugacy stability* problem for parabolic subgroups.

Definition 1. A parabolic subgroup P of an Artin group A is *conjugacy stable* in A if for every $x, y \in P$ such that $g^{-1}xg = y$, $g \in G$, there is $\hat{g} \in P$ such that $\hat{g}^{-1}x\hat{g} = y$. If P is not conjugacy stable in A we say that the inclusion of P into A *merges conjugacy classes*.

This problem has been solved only for some specific families of Artin groups. [16] proved that parabolic subgroups of braid groups are always conjugacy stable. However, for Artin groups this is not always the case. In Calvez et al. [5], we give an explicit classification for *spherical-type* (or finite type) Artin groups, which are the groups that become finite when adding to their presentation the relations $s^2 = 1$ for every $s \in S$. For large type and FC-type, a simpler question was addressed by Godelle [15]: He studied what happens if in the definition of conjugacy stable we impose g to be an element of S. At the end of Cumplido et al. [10], we completely classify the parabolic subgroups of large Artin groups up to conjugacy stability, using the aforementioned results of Paris and Godelle. The aim of this article is to use these results to prove that conjugacy stability problem can be solved for every Artin group satisfying three properties that are conjectured to always hold in Artin groups.

If for an element α in an Artin group there is a unique minimal (with respect to the inclusion) parabolic subgroup P_α containing α, we say that P_α is the *parabolic closure* of α. We will show:

Theorem A. *Let A be a standardisable* (Definition 14) *Artin group satisfying the ribbon property* (Definition 13) *and such that every element in A has a* *parabolic closure. Then, there is an algorithm that decides whether a parabolic subgroup P of A is conjugacy stable in A or not.*

The existence of parabolic closures—which is a consequence of the intersection of two parabolic subgroups being a parabolic subgroup—and the other two hypotheses of the theorem are conjectured to be true for all Artin groups. In particular, they are known to be true for spherical-type Artin groups [9,12]. The standardisation and ribbon properties are true for FC-type and two-dimensional Artin groups [13,14]. For FC-type, the problem of the intersection of parabolic subgroups is solved for spherical-type parabolic subgroups—the conjugates of some spherical-type standard parabolic subgroup—by Morris-Wright [20]. Using these results and the fact that FC-type Artin groups can be seen as amalgamated free-products of spherical-type Artin groups, we will partially solve the conjugacy stability problem for parabolic subgroups of a FC-type Artin group A. We will totally solve the problem if A is isomorphic to a free-product of spherical-type Artin groups, by proving the existence of parabolic closures in this case (Proposition 28). This is summarized in Theorem B.

Definition 2. Given an Artin group A and a parabolic subgroup P of A, we say that P is *conjugacy quasi-stable* if for every two elements $x, y \in P$ contained in (possibly different) spherical-type parabolic subgroups of A such that $g^{-1}xg = y$ with $g \in A$, there is $z \in P$ such that $z^{-1}xz = y$.

Remark 3. Notice that for spherical-type Artin groups being conjugacy quasi-stable is equivalent to be conjugacy stable.

Theorem B. *Let A be an FC-type Artin group. There is an algorithm that decides whether a given parabolic subgroup P of A is conjugacy quasi-stable in A. In particular, this algorithm can tell whether a spherical-type parabolic subgroup is conjugacy stable or not.*

Moreover, if A is isomorphic to a free product of spherical type Artin groups, then every element of A has a parabolic closure and there is an algorithm that solves the conjugacy stability problem for every parabolic subgroup of A.

This article is structured in the following way: In Sect. 2 we will describe a result of Paris [21] that gives an algorithm to decide when two standard parabolic subgroups are conjugate in any Artin group, and we will give an explicit form for this algorithm; in Sect. 3 we will explain how to modify this algorithm to solve the conjugacy stability problem for parabolic subgroups of Artin groups that satisfy the three hypothesis of Theorem A; in Sect. 4 we will discuss the case of FC-type Artin groups.

Remark 4. After the first preprint of this paper, [3] generalised the results in Cumplido et al. [10] and showed that the intersection of parabolic subgroups is a parabolic subgroup for two-dimensional Artin groups with a Coxeter graph—see next section—in which every vertex is disconnected from at most one other vertex. This completed the set of three hypotheses needed in Theorem A and allowed him two apply Algorithm 4 of Sect. 3 to solve the conjugacy problem in this case.

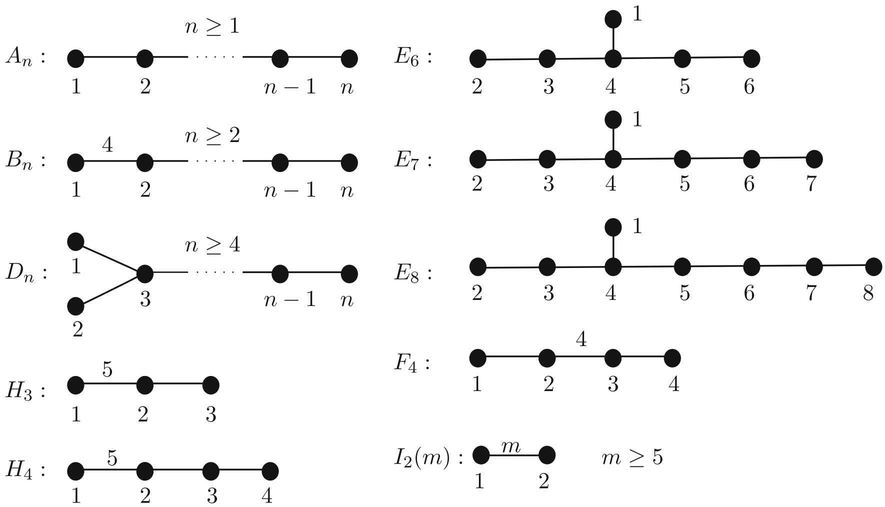

Figure 1. Classification of irreducible Coxeter graphs of finite type

Haettel [17] has also proved the three conjectures for Euclidean Artin groups of type $\tilde{A}$ and $\tilde{C}$, so we know that the main theorem works for these groups.

2. Conjugate Standard Parabolic Subgroups

In this section, we explain in detail the results in Paris, [21] to decide when two standard parabolic subgroups are conjugate in an Artin group A_S. This work is based on the part of Daan Krammer's thesis that solves the conjugacy problem in Coxeter groups, which is published in Krammer, [18]. To begin, we first need to know how to define the Coxeter graph of an Artin group and the classification of Artin groups of spherical type.

Definition 5. The *Coxeter graph* Γ_S of the Artin group A_S is the graph defined by the following data:

- The set of vertices of Γ_S is S.
- There is an edge connecting s and t if and only if $m_{s,t} > 2$. This edge if labeled with $m_{s,t}$ if $m_{s,t} > 3$.

If Γ_S is connected, we say that A_S is *irreducible*.

In Fig. 1, the reader can find the classification [8] of the ten types of irreducible Artin groups of spherical type. All the other Artin groups of spherical type are direct products of irreducible ones. When useful, we will refer to A_S as A_n, B_n, $D_n \dots$, but normally we will say that the Artin group and Coxeter graph are of type A_n, B_n, $D_n \dots$ We denote the generators of A_S by $s_1, s_2, s_3, \dots$, accordingly with the numbering of Fig. 1.

Given an Artin group A_S, the submonoid A_S^+ of A_S generated by S has the exactly same presentation as A_S (seen as monoid) [22]. If A_S is an

Artin group of spherical type, it has a Garside structure. This implies that if A_S has spherical type there is a lattice order $\preccurlyeq$ defined by "$a \preccurlyeq b$ iff $\exists c \in A_S^+$, $ac = b$". The least common multiple of all generators of S is called the Garside element of A_S and is denoted by Δ. By Brieskorn and Saito, [4] we know that the centre $Z(A_S)$ of A_S is generated either by Δ or by Δ^2.

For Artin groups of type $A_n\,(n \geq 2), D_n\,(n \geq 5), E_6$ and $I_2(m)\,(m \geq 5$ and odd), Δ^2 generates the centre of the group. Otherwise, Δ generates the centre of A_S. In the first case, the conjugation by Δ can be seen as a reflection automorphism of Γ_S. These conjugations, that are well-known by experts, are detailed in what follows:

- For A_n, $n \geq 2$, one has that $\Delta^{-1}s_i\Delta = s_{n-i+1}$.
- For D_n, with $n \geq 5$ and n odd, the conjugation by Δ permutes s_1 and s_2 and fixes the other generator.
- For E_6, the conjugation by Δ fixes s_1 and $\Delta^{-1}s_i\Delta = s_{8-i}$ for $i \neq 1$.
- For $I_2(m)$, with $m \geq 5$ and m odd, the conjugation by Δ permutes s_1 and s_2.

We will be specially interested in the Artin groups such that the conjugation by Δ can be seen as a reflection automorphism of Γ_S:

Definition 6. We say that A_S is a *twistable* Artin group if it is one of the following Artin groups of spherical type:

$$A_n,\, n \geq 2; \quad D_n,\, n \geq 5 \text{ and } n \text{ odd}; \quad E_6; \quad I_2(m),\, m \geq 5 \text{ and } m \text{ odd}.$$

Thanks to [24], we also know that a standard parabolic subgroup A_Y is an Artin group having as Coxeter graph $\Gamma_Y \subset \Gamma_S$. If A_Y has spherical type, we denote its Garside element by Δ_Y.

Suppose that A_X is a maximal proper standard parabolic subgroup of a twistable standard parabolic subgroup A_Y of A_S, in other words, $X = Y \setminus \{t\}$, $t \in X$. If A_Y is of type A_n, n odd, suppose that t is not the central generator of A_Y. If A_Y is of type E_6 suppose that t does not correspond to s_1 or s_4 and if A_Y is of type D_n suppose that t corresponds to either s_1 or s_2. Then $\Delta_Y^{-1}A_X\Delta_Y$ is a standard parabolic subgroup of A_Y different from A_X. (If t is one of the forbidden generators, then $\Delta_Y^{-1}A_X\Delta_Y = A_X$). This is the main ingredient of Paris' result, which states that two standard parabolic subgroups A_X and $A_{X'}$ are conjugate if and only if it is possible to go from one to the other by performing those types of conjugations or "twists".

Let $X \subset S$ and define $\mathrm{Adj}(X)$ as the set of vertices in Γ_S that are adjacent to Γ_X. We will consider lists of couples (Y, c), where $Y \subset S$ is a subset of generators and $c \in A_S$ is an element that conjugates the set X to the set Y. For a given $X \subset S$, we will recursively construct the list $\mathcal{V}_X$ as follows. Start the list with the couple $(X, 1)$. For every (Y, c) in the list and for every $t \in \mathrm{Adj}(Y)$, take the connected component $\Gamma_{Y'}$ of $\Gamma_{Y\cup\{t\}}$ containing t. If this component is twistable, conjugate Y by the Garside element $\Delta_{Y'}$ of the component. If the result Z is a subset of generators that is not contained in some couple of the list, add the couple $(Z, c\Delta_{Y'})$. Repeat this process. To prove that the process stops at some point, just observe that the set of standard parabolic subgroups of an Artin group is finite.

Theorem 7. *Given an Artin group A_S and two standard parabolic subgroups A_X and $A_{X'}$, A_X is conjugate to $A_{X'}$ if and only if there is a couple (X', c) in $\mathcal{V}_X$, in which case c is a conjugacy element.*

Proof. This theorem is a reformulation of [21, Theorem 4.1]. We can see that c is a conjugacy element by its own construction. □

In Algorithm 1, we give to Paris' result an explicit algorithmic form. The algorithm tells us when two standard parabolic subgroups A_X and $A_{X'}$ are conjugate. If they are not, it constructs the whole list $\mathcal{V}_X$.

Algorithm 1: Algorithm that finds a conjugating element between two standard parabolic subgroups or tells that it does not exist.

Input : The Coxeter graph Γ_S of an Artin group A_S and two subsets $X, X' \subset S$.
Output: A conjugating element between the parabolic subgroups A_X and $A_{X'}$ or "There is no conjugating element".

```
if |X| ≠ |X'| then
    return "There is no conjugating element";
V = {(X, 1)};
for (Y, c) ∈ V do
    for t ∈ Adj(Y) do
        if the connected component Γ_{Y'} of Γ_{Y∪{t}} containing t is twistable then
            Z = Δ_{Y'}^{-1} Y Δ_{Y'};
            if Z is not the first element of any couple in V then
                V = V ∪ {(Z, cΔ_{Y'})};
            if Z = X' then
                return cΔ_{Y'} ;
return "There is no conjugating element";
```

Example. Consider the spherical-type Artin group E_7, as depicted in Fig. 1. We are going to see that the parabolic subgroup A_X with $X = \{s_1, s_2, s_3, s_4, s_6\}$ is conjugate to $A_{X'}$, where $X' = \{s_2, s_4, s_5, s_6, s_7\}$. First, we take $s_5 \in \mathrm{Adj}(X)$. The set of generators $X \cup \{s_5\} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ defines a connected spherical-type parabolic subgroup isomorphic to E_6, which is twistable. If we conjugate X by the Garside element of $A_{X\cup\{s_5\}}$, we obtain the set of generators $Y = \{s_1, s_2, s_4, s_5, s_6\}$. Now take $s_7 \in \mathrm{Adj}(Y)$. The group defined by $Y \cup \{s_7\}$ has the connected component A_Z, $Z = \{s_1, s_4, s_5, s_6, s_7\}$, which is a (twistable) braid group. Conjugating by the corresponding Garside element, we finally obtain $\Delta_Z^{-1} Y \Delta_Z = X'$.

3. Solution to the Conjugacy Stability Problem

In this section, we will explain two of the three hypotheses of Theorem A, namely the ribbon property and the property of being standardisable. After that, we will construct the main algorithm of this paper to know when the embedding of a standard parabolic subgroup merges conjugacy classes.

We first describe the results of Godelle [12–14] concerning the set of elements conjugating two standard parabolic subgroup of an Artin group A_S. Suppose that A_X, $X \subset S \setminus T$, is a standard parabolic subgroup of spherical type and let $X' = X \setminus \{t\}$, for some $t \in X$. Since X has a spherical type, X' also has a spherical type and we can consider Δ_X and $\Delta_{X'}$. We have that

$$\Delta_X^{-1}\Delta_{X'}A_{X'}\Delta_{X'}^{-1}\Delta_X = \Delta_X^{-1}A_{X'}\Delta_X = A_Y,$$

for some subset $Y \subset X$. The conjugating element $\Delta_{X'}^{-1}\Delta_X$ and its inverse is what Godelle respectively calls an elementary (X', Y)–ribbon and an elementary (Y, X')–ribbon.

In general, for any (not necessarily of spherical type) parabolic subgroup A_T of A_S, if there is $s \in S$ such that the component Γ_U of $\Gamma_{T\cup\{s\}}$ that contains s is of spherical type, we call $r_{T,s} := \Delta_{U\setminus\{s\}}^{-1}\Delta_U$ and its inverse *elementary ribbons*—notice that Γ_U does not need to be twistable—. We say that an element $r = r_1r_2\cdots r_q$ is a (T, T')—*ribbon* if and only if there is a sequence of sets of generators $T = T_1, T_2, \ldots, T_{q+1} = T'$ such that each r_i is an elementary (T_i, T_{i+1})—ribbon. The set of all (T, T')—ribbons is denoted by $\mathrm{Ribb}(T, T')$. When referring to a (T, T')—ribbon without caring about the specific T', we will use the term $(T, -)$—ribbon.

Now we will see some properties about ribbons. The following lemma will allow us to work on some of the proofs using positive elementary ribbons and treat the negative ones as an analogous case:

Lemma 8. *Let A_S be an Artin group, $X \subset S$ and $a \in S \setminus X$. Suppose that Γ_Y is the component of $\Gamma_{X\cup\{a\}}$ that contains a. If there is an elementary ribbon $r_{X,a} = \Delta_{Y\setminus\{a\}}^{-1}\Delta_Y$, then there are $T \subset S$, $s \in S \setminus T$ such that $r_{T,s} = \Delta_{Y\setminus\{s\}}^{-1}\Delta_Y$ and $r_{X,a}^{-1} = \Delta_{Y\setminus\{s\}}\Delta_Y^{-1}$.*

Proof. We have that $r_{X,a}^{-1} = \Delta_Y^{-1}\Delta_{Y\setminus\{a\}} = \Delta_{Y\setminus\{s\}}\Delta_Y^{-1}$, where $s = \Delta_Y^{-1}a\Delta_Y$. To see that there is a positive elementary ribbon of the form $\Delta_{Y\setminus\{s\}}^{-1}\Delta_Y$, let $T = (X \cup \{a\}) \setminus \{s\}$. Hence, Γ_Y is the component of $\Gamma_{T\cup\{s\}}$ that contains s and $r_{T,s} = \Delta_{Y\setminus\{s\}}^{-1}\Delta_Y$. □

Remark 9. In the above lemma, $r_{T,s} = \Delta_{Y\setminus\{s\}}^{-1}\Delta_Y$ and $r_{X,a}^{-1} = \Delta_{Y\setminus\{s\}}\Delta_Y^{-1}$ are (positive and negative) elementary (T, X)—ribbons. Similarly, $r_{X,a} = \Delta_{Y\setminus\{a\}}^{-1}\Delta_Y$ and $r_{T,s}^{-1} = \Delta_{Y\setminus\{a\}}\Delta_Y^{-1}$ are (positive and negative) elementary (X, T)—ribbons.

The next two lemmas help us understand how the conjugation by ribbons transforms the generators of standard parabolic subgroups:

Lemma 10. *Let A_S be an Artin group and $X \subset S$. Let $t \in S \setminus X$ and $Z \subset X \cup \{t\}$ be such that Γ_Z is the connected component of $\Gamma_{X\cup\{t\}}$ containing t and it is of spherical type. Let $X' \subseteq X$ denote any subset defining a connected component $\Gamma_{X'}$ of Γ_X. Then*

- *If $A_{X'}$ defines a component which is not of spherical type, then $r_{X,t}^{-1} s r_{X,t} = s$, for every $s \in X'$.*
- *If $A_{X'}$ is of spherical type and of type different from A, D, E_6 and $I_2(m)$, then $r_{X,t}^{-1} s r_{X,t} = s$, for every $s \in X'$.*
- *If $A_{X'}$ is of type either E_6 or $I_2(m)$, then either $r_{X,t}^{-1} s r_{X,t} = s$ for every $s \in X'$ or $r_{X,t}^{-1} s r_{X,t} = \Delta_{X'} s \Delta_{X'}^{-1}$ for every $s \in X'$.*
- *If $A_{X'}$ is of type either A or D, then $r_{X,t}^{-1} X' r_{X,t} \subset X \cup \{t\}$.*

Proof. If X' is not of spherical type, then X' cannot be contained in Z and X' and Z are not adjacent, so the conjugation by the elementary ribbon $r_{X,t}$ does not modify X'. Suppose that X' is different from A, D, E_6 and $I_2(m)$. If X' is not contained in Z, again X' and Z are not adjacent and therefore X' cannot be modified by a conjugation by $r_{X,t}$. If $X' \subset Z$, then $X' \subset Z \setminus \{t\}$ so $(X', Z) \in \{(B_{m_1}, B_{m_2}), (B_3, F_4), (H_3, H_4), (E_7, E_8)\}$, for $1 < m_1 < m_2$. In this case, both $\Delta_{X'}$ and Δ_Z are central in $A_{X'}$ and A_Z, respectively. This means that $r_{X,t}^{-1} s r_{X,t} = s$, for every $s \in X'$. If X' is of type E_6 or $I_2(m)$, we can suppose as before that $X' \subset Z$. In this case, $(X', Z) \in \{(E_6, E_7), (E_6, E_8)(I_2(5), H_3), (I_2(5), H_4)\}$, so Δ_Z is central in A_Z and $\Delta_{X'}$ is not central in $A_{X'}$. Thus, $r_{X,t}^{-1} s r_{X,t} = \Delta_Z^{-1} \Delta_{X'} s \Delta_{X'}^{-1} \Delta_Z = \Delta_{X'} s \Delta_{X'}^{-1}$ for every $s \in X'$. The last item follows by definition. □

Remark 11. By Lemma 8, the previous lemma works analogously if we replace the positive elementary $(X, -)$—ribbon $r_{X,t}$ by a negative elementary $(X, -)$—ribbon.

Lemma 12. *Let A_S be an Artin group, $X \subset S$, and α be an (X, X)—ribbon. Let $X' \subseteq X$ denote any subset defining a connected component $\Gamma_{X'}$ of Γ_X. Then,*

- *If $A_{X'}$ has not spherical type or has a spherical type different from A, D, E_6 and $I_2(m)$, then $\alpha^{-1} s \alpha = s$, for every $s \in X'$.*
- *If $A_{X'}$ is of type E_6 or $I_2(m)$, then either $\alpha^{-1} s \alpha = s$ for every $s \in X'$ or $\alpha^{-1} s \alpha = \Delta_{X'} s \Delta_{X'}^{-1}$ for every $s \in X'$.*
- *If $A_{X'}$ is of type A or D, then $\alpha^{-1} X' \alpha = X''$, where $\Gamma_{X''}$ is isomorphic to $\Gamma_{X'}$.*

Proof. By definition, α is a product $\prod_{i=1}^{k} r_i$ of elementary (X_i, Y_i)—ribbons, r_i, where $Y_i = X_{i+1}$ and $X_1 = Y_k = X$. When we conjugate $A_{X'}$ by an elementary X—ribbon, we obtain a parabolic subgroup of the same type. Therefore, by Lemma 10 and Remark 11 we can distinguish three cases. If $A_{X'}$ has non-spherical type or has a spherical type different from A, D, E_6 and $I_2(m)$, then all the conjugations by the elementary ribbons are trivial. If $A_{X'}$ is of type E_6 or $I_2(m)$, then $\Delta_{X'}^2$ is the smallest positive power of $\Delta_{X'}$ that is central and all conjugations are as indicated in the second item of Lemma 10. If $A_{X'}$ is of type A or D, the result is trivial. □

Now we define the two main properties that used ribbons that are conjectured to be true for every Artin group:

Definition 13. Given an Artin group A_S and $S' \subseteq S$, we say that a pair (X,Y), $X,Y \subseteq S'$, is *conjugate by ribbons* in $A_{S'}$ if, for any $g \in A_{S'}$,

$$g^{-1}A_X g = A_Y \text{ if and only if } g \in A_X \cdot (\mathrm{Ribb}(X,Y) \cap A_{S'}).$$

We say that A_S satisfies the *ribbon property* if, for any two sets of generators $X,Y \subset S$, the pair (X,Y) is conjugate by ribbons in A_Z for every $Z \in \{T \subseteq S \,|\, X,Y \subseteq T\}$.

Definition 14. Let A_S be an Artin group and $X,Y \subset S$. We say that the pair (X,Y) is *standardisable* in A_S if

$$\forall g \in A_S \text{ such that } g^{-1}A_Y g \subseteq A_X \text{there are } h \in A_X \text{ and } Z \subseteq X$$
$$\text{such that } h^{-1}g^{-1}A_Y gh = A_Z.$$

In particular, if there is no $g \in A_S$ such that $g^{-1}A_Y g \subseteq A_X$, then (X,Y) is standardisable. We say that A_S is *standardisable* if every pair (X,Y), $X,Y \subset S$, is standardisable.

Godelle conjectures that every Artin group is standardisable and has the ribbon property [14, Conjecture 1, Conjecture 4.2] after the first article by Paris, [21] showing the ribbon property and other results about normalizers for spherical-type Artin groups. Godelle proves that FC-type Artin groups satisfy the ribbon property in [13, Theorem 3.2] and in [14, Proposition 4.3] he uses the ribbon property to prove that they are also standardisable. He also shows that all two-dimensional Artin groups are standardisable and satisfy the ribbon property, and this is what we use in Cumplido et al. [10] to solve the conjugacy stability problem for large Artin groups.

3.1. Proof of Theorem A

To prove Theorem A we will first prove the following theorem:

Theorem 15. *Let A_S be an Artin group and let $X \subset S$. There is an algorithm that decides whether A_X is conjugacy stable if the three following properties hold:*

- *For any $Y \subset S$, the pair (X,Y) is standardisable;*
- *For any $X_1, X_2 \subseteq X$, the pair (X_1, X_2) is conjugate by ribbons in A_S and in A_X;*
- *Every element $\alpha \in A_X$ has a parabolic closure P_α in A_S.*

Let us see that the previous theorem implies Theorem A:

Proof of Theorem A. Let A_S be an Artin group. To give a solution to the conjugacy stability problem for parabolic subgroups of Artin groups, we shall notice that the property of being conjugacy stable is preserved under conjugation. Hence, it suffices to give an algorithm that tells if a standard parabolic subgroup A_X is conjugacy stable for every $X \subset S$. To satisfy the conditions of Theorem A, A_S need to be standardisable and conjugate by ribbons and every element $\alpha \in A_S$ has a parabolic closure P_α in A_S. In particular, we have the three conditions of Theorem 15 for every $X \subset S$, so we have the desired algorithm. □

Before showing Theorem 15, we will prove some lemmas:

Lemma 16. *Let A_S be any Artin group, $X \subset S$ and $\alpha, \beta \in A_S$. The parabolic closure $P_{\beta^{-1}\alpha\beta}$ exists if and only if P_α exists and in this case $P_{\beta^{-1}\alpha\beta} = \beta^{-1}P_\alpha\beta$.*

Proof. Suppose that P_α exists and let Q be a parabolic subgroup containing $\beta^{-1}\alpha\beta$. Then $\alpha \in \beta Q\beta^{-1}$, where $\beta Q\beta^{-1}$ is a parabolic subgroup. Hence $P_\alpha \subset \beta Q\beta^{-1}$ and $\beta^{-1}P_\alpha\beta \subset Q$. The converse is symmetric. □

Remark 17. Let $(*)$ be a property for parabolic subgroups that is preserved under conjugation, such as being of spherical type. Notice that the previous proof can be adapted to prove that, if $P(*)_\alpha$ is the unique minimal parabolic subgroup containing α and satisfying $(*)$, then $\beta^{-1}P(*)_\alpha\beta$ is the unique minimal parabolic subgroup containing $\beta^{-1}\alpha\beta$ and satisfying $(*)$.

The *support*, $\mathrm{supp}(g)$, of a positive element $g \in A_S$ is the set of all generators that appear in any positive word representing g. For Artin groups of spherical type, the parabolic closure of an element depends on the element support. In particular, for positive elements we have the following result:

Lemma 18. ([9, Proposition 6.8]) *Let A_S be an Artin group of spherical type. The parabolic closure of a positive element $g \in A_S$ is $A_{\mathrm{supp}(g)}$.*

Our strategy to know if two elements are conjugate inside a parabolic subgroup will be based on taking their parabolic closure and verifying if this parabolic closure are conjugate inside the parabolic subgroup. However, there several ways of sending a set of generators to another set of generators by conjugacy. Due to that, standard parabolic subgroups of type D_k will produce special cases that will need to be treated separately. The following three lemmas will help to deal with these cases.

Lemma 19. *Let A_S be a spherical-type Artin group. Let Δ^e be a central power of the Garside element and α and β be two elements of A_S. Then α and β are conjugate if and only if $\alpha^{-1}\Delta^e$ and $\beta^{-1}\Delta^e$ are conjugate.*

In particular, using the same numbering as in Fig. 1, the elements $(s_1s_3s_4\cdots s_n)^{-1}\Delta$ and $(s_2s_3s_4\cdots s_n)^{-1}\Delta$ are not conjugate in D_n when n is even.

Proof. The first statement is quite straightforward as $\gamma^{-1}\alpha\gamma = \beta$ if and only if $\gamma^{-1}\alpha^{-1}\gamma = \beta^{-1}$. Since Δ^e is central, this happens if and only if $\gamma^{-1}\alpha^{-1}\Delta^e\gamma = \gamma^{-1}\alpha^{-1}\gamma\Delta^e = \beta^{-1}\Delta^e$.

For the second statement, notice that by Lemma 18 the parabolic closures of $s_1s_3s_4\cdots s_n$ and $s_2s_3s_4\cdots s_n$ are respectively $A_{S\setminus\{s_2\}}$ and $A_{S\setminus\{s_1\}}$. We know by Algorithm 1 that these two parabolic subgroups are not conjugate in D_n if n is even. Therefore, by Lemma 16 and the previous statement, $(s_1s_3s_4\cdots s_n)^{-1}\Delta$ and $(s_2s_3s_4\cdots s_n)^{-1}\Delta$ are not conjugate. □

Lemma 20. *Let A_S be an Artin group and let A_X and A_T be standard parabolic subgroups so that $T \subset X \subset S$ such that (T, T) is conjugate by ribbons in A_X. Suppose that there is $s \in S \setminus X$ so that the connected component of $\Gamma_{T\cup\{s\}}$ containing s is of type D_{2k+1}, $k > 1$, and $s = s_5$ with the numbering established for D_{2k+1} in Fig. 1. If A_X is conjugacy stable, then at least one of the following situations applies:*

- *There is $t \in X$ adjacent to s_4 such that the connected component of $A_{T\cup\{t\}}$ containing t and X' is of type $D_{2k'+1}$, $k' > 1$.*
- *There are $t_1 \in X$ adjacent to s_1 and $t_2 \in X$ adjacent to s_2, such that the connected component of $\Gamma_{T\cup\{t_1\}}$ containing t_1 is of type D_{2k_1+1} and the connected component of $\Gamma_{T\cup\{t_2\}}$ containing t_2 is of type D_{2k_2+1}, $k_1, k_2 > 1$.*

Proof. Let Γ_Y be the connected component of $A_{T\cup\{s_5\}}$ containing s, which is of type D_{2k+1}, for some $k > 1$. Notice that $s = s_5$ can be adjacent to at most two connected components of Γ_T, one being generated the subset $X' = \{s_1, s_2, s_3, s_4\}$ with the numbering established for D_{2k+1} in Fig. 1. The element Δ_Y conjugates $a := (s_1s_3s_4)^{-1}\Delta_{X'}\Delta_{T\setminus X'} = s_2s_1s_3s_4s_2s_1s_3s_4s_2\Delta_{T\setminus X'}$ to $b := (s_2s_3s_4)^{-1}\Delta_{X'}\Delta_{T\setminus X'} = s_1s_2s_3s_4s_1s_2s_3s_4s_1\Delta_{T\setminus X'}$.

Since A_X is conjugacy stable, there is an element $x \in A_X$ such that $x^{-1}ax = b$. As a and b are positive and $\operatorname{supp}(a) = \operatorname{supp}(b) = T$, by Lemma 18 the parabolic closures of a and b are both A_T. Then, by Lemma 16, any element that conjugates a to b normalises A_T. In particular $x^{-1}A_Tx = A_T$ and, since (T, T) is conjugate by ribbons in A_X, we can write x as $x = x_1x_2$ where $x_1 \in A_T$ and $x_2 \in A_X$ is a (T, T)—ribbon. Equivalently, we can write $x = x_2x_3$, where $x_3 = x_2^{-1}x_1x_2 \in A_T$.

The non-trivial elementary $(T, -)$—ribbons in A_X are the ones written as $r_t := \Delta^{-1}_{Z\setminus\{t\}}\Delta_Z$ or $\Delta_{Z\setminus\{t\}}\Delta^{-1}_Z$ (Remark 9), where $t \in X$, Γ_Z is the connected component of $\Gamma_{T\cup\{t\}}$ that contains t and $\Gamma_{Z\cup\{t\}}$ has spherical type. If Γ_Z does not contain X', since $\Delta_{T\setminus X'}$ is central in $A_{T\setminus X'}$, the conjugation by all other elementary $(T, -)$—ribbons will commute with X' and will fix a. If Γ_Z contains X', as X' has type D_4, we have that $\Gamma_{Z\cup\{t\}}$ has type D_m. In the case of type D_m, m even, the conjugation by r_t centralises A_Z and, in particular, it fixes a. In the case of type D_m with m odd, t is adjacent to either s_1, or s_2, or s_4. If for that case we suppose that none of the items of the lemma are satisfied, then all $t \in X$ are all adjacent to s_1 or all adjacent to s_2. So suppose without loss of generality that all $t \in X$ are adjacent to s_1. Then, the conjugation by r_t normalises A_Z and permutes the elements in Z (it switches s_2 and s_4). Hence, r_t conjugates a to $c := (s_1s_3s_2)^{-1}\Delta_{X'}\Delta_{T\setminus X'}$ and it conjugates c to a. It follows that, as x_2 is a product of elementary ribbons, each one preserving T, one has $x_2^{-1}ax_2 \in \{a, c\}$. Then, since $x_3^{-1}(x_2^{-1}ax_2)x_3 = b$ either a or c are conjugate to b in A_T (and so in $A_{X'}$). However, we know by Lemma 19 that a is not conjugate to b in $A_{X'}$, and that c is not conjugate to b in $A_{X'}$. A contradiction. Hence some of the items of the statement must be satisfied. □

Lemma 21. *Let A_S be an Artin group and A_X and A_T, $T \subset X \subset S$, be standard parabolic subgroups such that (T, T) is conjugate by ribbons in A_X.*

Suppose that there is $s \in S \setminus X$ *so that the connected component of* $\Gamma_{T\cup\{s\}}$ *containing* s *is of type* D_{2k+1}, $k > 2$, *and* $s = s_{2i+1}$, $i > 2$, *with the numbering established for* D_{2k+1} *in Fig. 1. If* A_X *is conjugacy stable in* A_S, *then there is* $t \in X$ *adjacent to* s_{2i} *such that the connected component of* $\Gamma_{T\cup\{t\}}$ *containing* t *is of type* $D_{2k'+1}$, $k' > 2$.

Proof. The proof is analogous to the proof of the previous lemma. Notice that $s = s_{2i+1}$ can be adjacent to at most two connected components of Γ_T, one being generated the subset $X' = \{s_1, s_2, s_3, s_4, \cdots s_{2i}\}$ with the numbering established for D_{2k+1} in Fig. 1. Suppose also that there is no $t \in X$ adjacent to s_{2i} such that the connected component of $\Gamma_{T\cup\{t\}}$ containing X' is of type $D_{2k'+1}$ for some $k' > 2$. The element Δ_Y conjugates $a := (s_1 s_3 \cdots s_{2i})^{-1}\Delta_{X'}\Delta_{T\setminus X'}$ to $b := (s_2 s_3 \cdots s_{2i})^{-1}\Delta_{X'}\Delta_{T\setminus X'}$. Suppose that A_X is conjugacy stable. Then, there is $x \in A_X$ such that $x^{-1}ax = b$.

It is well-known by experts that $\Delta_{X'} = (s_2 s_3 \cdots s_{2i} s_1)^{2i-1} = (s_1 s_3 \cdots s_{2i} s_2)^{2i-1}$ (see [4] and [23]). So, by Lemma 18, the parabolic closure of both a and b is A_T. Then, by Lemma 16, x normalises A_T. As in the proof of the previous lemma, (T, T) being conjugate by ribbons in A_X means that we can write $x = x_2 x_3$, where $x_2 \in A_X$ is an (T, T)—ribbon and $x_3 \in A_T$. The non-trivial elementary $(T, -)$—ribbons belonging to A_X are $r_t := \Delta^{-1}_{Z\setminus\{t\}}\Delta_Z$ or $\Delta_{Z\setminus\{t\}}\Delta_Z^{-1}$ (Remark 9), where $t \in X$, Γ_Z is the connected component of $\Gamma_{T\cup\{t\}}$ that contains t and $\Gamma_{Z\cup\{t\}}$ has spherical type. If Γ_Z does not contain X', since $\Delta_{T\setminus X'}$ is central in $A_{T\setminus X'}$, the conjugation by all other elementary $(T, -)$—ribbons will commute with X' and will fix a. Otherwise, in Γ_Z has type E_7 or $D_{2k'+1}$ for some $k' > i$. In the E_7 case, the conjugation by r_t centralises A_Z and it fixes a. We have supposed that the $D_{2k'+1}$ case does not happens, so $x_2^{-1}ax_2 = a$. Hence $x_3^{-1}ax_3 = b$ with $x_3 \in A_{X'}$, which by Lemma 19 is a contradiction. □

The following lemma will help to deal with fact that an element in the stabilizer of a standard parabolic subgroup can permute its connected components.

Lemma 22. *Let* A_S *be an Artin group and* A_X, $X \subset S$, *a standard parabolic subgroup of* A_S *which is conjugacy stable. Suppose that for any* $X_1, X_2 \subseteq X$, *the pair* (X_1, X_2) *is conjugate by ribbons in* A_X. *Let* $X_1, X_2 \subset X$ *be such that* $g^{-1}X_1 g = X_2$, *for some* $g \in A_S$. *Then there is* $g' \in A_X$ *such that* $g^{-1}Yg = g'^{-1}Yg'$ *for every connected component* Γ_Y *of* Γ_{X_1}.

Proof. Denote by $\Gamma_{Y_1}, \Gamma_{Y_2}, \ldots, \Gamma_{Y_l}$, $Y_i \subset X_1$, the connected components of Γ_{X_1}. For each i, denote by $s_{i,j}$, $1 \le j \le |Y_i|$ the elements in Y_i. Now define $y_i := \left(\prod_{j=1}^{|Y_i|} s_{i,j}\right)^{k_i}$, where the k_i's are chosen to satisfy that the number of letters in $\left(\prod_{j=1}^{|Y_i|} s_{i,j}\right)^{k_i}$ is different from the number of letters in $\left(\prod_{j=1}^{|Y_{i'}|} s_{i',j}\right)^{k_{i'}}$ for $i' \neq i$.

By Lemma 18, we know that the element $a := y_1 y_2 \ldots y_l$ has parabolic closure A_{X_1} and that $g^{-1}ag$ has parabolic closure A_{X_2}. Since A_X is conjugacy stable in A_S, there must be $h \in A_X$ such that $h^{-1}ah = g^{-1}ag$. Then, by Lemma 16, $h^{-1}A_{X_1}h = A_{X_2}$. The ribbon hypothesis tells us that $h = h_1h_2$ with $h_1 \in A_{X_1}$ and $h_2 \in \mathrm{Ribb}(X_1, X_2) \cap A_X$. So h_2 is an element in A_X conjugating X_1 to X_2. Also, since $h_1 \in A_{X_1}$, the conjugation h_1 preserves each Y_i and one has that $h^{-1}A_{Y_i}h = h_2^{-1}A_{Y_i}h_2$ for $1 \le i \le l$—observe that conjugations by g and h_2 send letters to letters, but h and h_1 do not have to—.

Suppose that $g^{-1}Y_ig \neq h_2^{-1}Y_ih_2$ for some $1 \le i \le l$. By the previous discussion, one should have that $hg^{-1}(y_1y_2\dots y_l)gh^{-1} = y_1y_2\dots y_l$, but let us see that this is impossible: The conjugation by gh^{-1} permutes non trivially the components of A_{X_1} and every y_i is contained in a different component, so there is some y_i such that $hg^{-1}y_igh^{-1} = y_{i'}$, for $i' \neq i$. However, since the relations in an Artin group are homogeneous, two positive words representing two conjugate positive elements need to have the same number of letters. This is a contradiction and therefore we can set $g' = h_2$. □

Proof of Theorem 15. We are going to prove that a standard parabolic subgroup A_X is conjugacy stable in A_S if and only if the following conditions are fulfilled:

1. For every $X' \subset X$ such that $g^{-1}X'g \subset X$, for some $g \in A_S$, there is $h \in A_X$ such that $g^{-1}X''g = h^{-1}X''h$ for every connected component $\Gamma_{X''}$ of $\Gamma_{X'}$.
2. Let A_T, $T \subset X$, be a parabolic subgroup. Let $s \in S \setminus X$ be such that the connected component of $\Gamma_{T\cup\{s\}}$ containing s is of type D_{2k+1}, $k > 2$ with $s = s_{2i+1}$ following the numbering of Fig. 1, then there is $s' \in X$ adjacent to s_{2i} such that the connected component of $\Gamma_{T\cup\{s'\}}$ containing s' is of type $D_{2k'+1}$, $k' > 2$. This condition is checked by Algorithm 2.
3. Let A_T, $T \subset X$, be a parabolic subgroup. Let $s \in S \setminus X$ be such that the connected component of $\Gamma_{T\cup\{s\}}$ containing s is of type D_{2k+1}, $k > 1$ and $s = s_5$ with the numbering of Fig. 1. Then either there is $s' \in X$ adjacent to s_4 such that the connected component of $\Gamma_{T\cup\{s'\}}$ containing s' is of type $D_{2k'+1}$, $k > 1$, or there are $t_1 \in X$ adjacent to s_1 and $t_2 \in X$ adjacent to s_2 such that the connected component of $\Gamma_{T\cup\{t_1\}}$ containing t_1 is of type D_{2k_1+1}, $k_1 > 1$ and the connected component of $\Gamma_{T\cup\{t_2\}}$ containing t_2 is of type D_{2k_2+1}, $k_2 > 1$. This condition is checked by Algorithm 3.

Once this is proven, we will be able to construct an algorithm to solve the conjugacy stability problem, explained in Algorithm 4. This is a refinement of Algorithm 1 that considers permutations of components and the D_{2k} exceptions.

By Lemmas 20, 21 and 22, we know that if some of the items is not satisfied, then A_X cannot be conjugacy stable in A_S. So we need to prove that if all the items are satisfied, then A_X is conjugacy stable.

Let $\alpha, \beta \in A_X$ be such that there is $g \in A_S$ satisfying $g^{-1}\alpha g = \beta$. Let $P_\alpha, P_\beta \subset A_X$ be the minimal parabolic subgroups containing α and β respectively. Suppose that all of the items of the theorem are fulfilled. Thanks to the standardisation condition, we know that $P_\alpha = g_1^{-1}A_Yg_1$ and

$P_\beta = g_2^{-1} A_Z g_2$, with $g_1, g_2 \in A_X$ and $Y, Z \subseteq X$. Also, Lemma 16 implies that $P_\alpha = g^{-1} P_\beta g$. So, up to conjugacy by elements of A_X, we can suppose that A_Y and A_Z are the parabolic closures of α and β and are conjugate by g.

(Y, Z) being conjugate by ribbons in A_S tells us that $g = a_1 \cdot a_2$ where $a_1 \in A_Y$ and a_2 is a (Y, Z)—ribbon in A_S. Since the first item is satisfied and $a_1^{-1} Y a_1 = Z$, we know that there is $h \in A_X$ such that $g^{-1} Y_i g = h^{-1} Y_i h$ for every connected component Γ_{Y_i} of Γ_Y. As (Y, Z) is conjugate by ribbons in A_X, $h = b_1 \cdot b_2$ where $b_1 \in A_Y$ and b_2 is a (Y, Z)—ribbon in A_S. Also, note that b_1 cannot permute the connected components of Y: $b_1 = y_1 \cdots y_n$ where $y_i \in Y_i$, so every y_i commutes with Y_j, $j \neq i$. Hence $a_2^{-1} Y_i a_2 = b_2^{-1} Y_i b_2$ for every connected component Γ_{Y_i} of Γ_Y.

Notice that $a_2 b_2^{-1}$ is a (Y, Y)—ribbon. Since the connected components of Y are preserved, the conjugation by $a_2 b_2^{-1}$ induces an isomorphism of the subgraph Γ_{Y_i}. If A_{Y_i} has not spherical type or it has a spherical type and it is non-twistable, we know by Lemma 12 that $b_2 a_2^{-1} s a_2 b_2^{-1} = s$, for every $s \in Y_i$. If A_{Y_i} is of type A or D_{2k+1}, the only isomorphisms of graphs that we can have are the trivial one and a reflection switching the vertices corresponding to the two first generators. Then we have either $b_2 a_2^{-1} s a_2 b_2^{-1} = s$ or $\Delta_{Y_i}^{-1} b_2 a_2^{-1} s a_2 b_2^{-1} \Delta_{Y_i} = s$. This is also valid if A_{Y_i} has type E_6 or $I_2(m)$ by Lemma 12. If A_{Y_i} is of type D_{2k}, the only non-trivial isomorphism of Γ_{Y_i} is a switch of the two first vertices. But, since Δ_{Y_i} is central, the conjugation by Δ_{Y_i} cannot perform this isomorphism. Then, if the conjugation by $a_2 b_2^{-1}$ switches the vertices, there must be a $t \in S$ adjacent to Γ_{Y_i} such that the connected component of $\Gamma_{Y \cup \{t\}}$ containing Y_i is of type $D_{2k'+1}$ (notice that the conjugation by $\Delta_{2k'+1}$ does the desired switching). Then, by conditions 2 and 3 above, either there is $t \in X$ adjacent to Y_i such that $\Delta_{Y'}^{-1} b_2 a_2^{-1} s a_2 b_2^{-1} \Delta_{Y'} = s$, where $\Gamma_{Y'}$ is the connected component of $\Gamma_{Y \cup \{t\}}$ containing t; or there are $t_1, t_2 \in X$ adjacent to Y_i such that $\Delta_{Y''}^{-1} \Delta_{Y'}^{-1} b_2 a_2^{1} s a_2 b_2^{-1} \Delta_{Y'} \Delta_{Y''} = s$, where $\Gamma_{Y'}$ is the connected component of $\Gamma_{Y \cup \{t_1\}}$ containing $\Gamma_{Y'}$ is the connected component of $\Gamma_{Y \cup \{t_2\}}$ containing $\Gamma_{Y'}$.

The previous paragraph means that we can suppose that $a_2^{-1} s a_2 = b_2^{-1} s b_2$ up to conjugations by elements of the form $\Delta_{X'}$, $X' \subset X$. Then, up these conjugations, $g^{-1} \alpha g = a_1^{-1} b_2^{-1} \alpha b_2 a_1$. Since $b_2, a_1 \in A_X$, we have proven that under the three items, A_X is conjugacy stable in A_S. $\square$

4. FC-type Artin Groups

If every standard parabolic subgroup of an Artin group A_S that does not contain infinite relations has a spherical type, then A_S is said to be of FC-type. The aim of this section is to prove Theorem B, that is, to discuss whether we can apply the algorithm to solve the conjugacy stability problem in Artin groups of FC-type. From now on, suppose that A_S has FC-type.

It is shown in [1, Proposition 2] that, if $s, t \in S$ are such that $m_{s,t} = \infty$, then A_S is isomorphic to the amalgamated free product of $A_{S \setminus \{s\}}$ and $A_{S \setminus \{t\}}$

over $A_{S\setminus\{s,t\}}$, denoted by $A_{S\setminus\{s\}} *_{A_{S\setminus\{s,t\}}} A_{S\setminus\{t\}}$. Then, if we give an order to the ∞-labels of A_S, we will obtain a specific amalgamated product structure for A_S. The next two propositions about canonical forms in amalgamated free products can be found in [19, Section 4] and sometimes will be used without explicit reference:

Proposition 23. *(Canonical form for amalgamated free products) Given the amalgamated free product $G = G_1 *_H G_2$ of the groups G_1 and G_2 over H, we respectively denote by C_1 and C_2 the transversals of G_1/H and G_2/H that contain 1. Then, every $x \in G$ can be uniquely expressed as a product $x = x_1 x_2 \cdots x_r h$, where $h \in H$, $x_i \in C_1 \cup C_2$ is not trivial for $i = 1, \dots, r$, and x_i and x_{i+1} belong to different transversals if $r > 1$. This expression is called the* amalgam normal form *of x, and we denote it by $\rho(x)$.*

Proposition 24. *(Conjugacy in amalgamated free products) Given the previous amalgamated free product $G = G_1 *_H G_2$, any element $g \in G$ is conjugate to an element x with amalgam normal form $x_1 x_2 \cdots x_r h$, in which x_1 and x_r belong to different transversals. We say that such an element x is* cyclically reduced. *Moreover,*

- *if x is conjugate to a element written $y = p_1 p_2 p_3 \cdots p_k$, $k \geq 2$, where p_i,p_{i+1} as well as p_1,p_k belong to different transversals, then x is obtained from y by cyclically permuting $p_1 p_2 p_3 \cdots p_k$ and then conjugating by an element of H;*
- *if $H = \{1\}$ and x is conjugate to an element y in one of the factors (G_1 or G_2), then x and y belong to the same factor and are conjugate in that factor.*

The amalgamated free product structure of a standard parabolic subgroup A_X of A_S heavily relies on the structure of A_S. The next result is a consequence of [1, Theorem 2].

Lemma 25. ([13, Corollary 1.12]) *Let $A_S \simeq A_{S\setminus\{s\}} *_{A_{S\setminus\{s,t\}}} A_{S\setminus\{t\}}$ be a FC-type Artin group. Let $X \subset S$. If $w \in A_X$, then the amalgam normal form of w has its terms in A_X.*

4.1. Proof of Theorem B

In [20, Corollary 3.2], it is proven that if an element α of an FC-type Artin group is contained in a spherical-type parabolic subgroup, then there is a unique minimal (by inclusion) spherical-type parabolic subgroup Q_α containing α. We call Q_α the *spherical-type parabolic closure* of α. Since all parabolic subgroups of a spherical-type Artin group have a spherical type, we can use Remark 17 to easily adjust the proof of Theorem 15 and be able to apply Algorithm 4 to spherical-type parabolic subgroups.
The following lemma will allow us to complete the proof of the first part of Theorem B.

Lemma 26. *Let A_S be an Artin group of FC type and A_X, $X \subset S$, a standard parabolic subgroup of non-spherical type. If $\alpha \in A_X$ belongs to a spherical parabolic subgroup of A_S, then the spherical-type parabolic closure Q_α of α is contained in A_X.*

Proof. Choose an ∞-label $m_{s,t}$ in A_X and take the decompositions $A \simeq A_{S\setminus\{s\}} *_{A_{S\setminus\{s,t\}}} A_{S\setminus\{t\}}$ and $A_X \simeq A_{X\setminus\{s\}} *_{A_{X\setminus\{s,t\}}} A_{X\setminus\{t\}}$. We know by Lemma 25 that the amalgam normal form of α has their terms in A_X so it is also an amalgam normal form with respect to the structure of A_X. Then, we can obtain a cyclically reduced element $x \in A_X$ from α by conjugating by an element c of A_X. Also, we can write $Q_\alpha = \beta^{-1}A_Y\beta$, where A_Y is a spherical type standard parabolic subgroup of A_S. Then, $\beta\alpha\beta^{-1} \in A_Y$ and we can obtain a cyclically reduced element $y \in A_Y$ from $\beta\alpha\beta^{-1}$ by conjugating by an element of A_Y. We will show our lemma by induction on the number of ∞-labels of A_X.

Suppose that there is only one ∞-label $m_{s,t}$ in A_X. We first prove that x is contained in a spherical-type standard parabolic subgroup $A_{X'}$. In this case $A_{X\setminus\{s\}}$ and $A_{X\setminus\{t\}}$ have a spherical type, so if x is contained in any of them we done. Suppose then that x is not contained in any of these two subgroups. As x and y are conjugate and cyclically reduced, by Proposition 24 x is obtained from y by conjugating by an element in $A_{Y\cup(S\setminus\{s,t\})}$. Then, $x \in A_{X'} := A_{Y\cup(S\setminus\{s,t\})}$. Since A_Y has spherical type, Y cannot simultaneously contain s and t. By Van der Lek, [24], the intersection of standard parabolic subgroups is (the expected) standard parabolic subgroup, meaning that $A_X \cap A_{X'} = A_{X\cap X'}$. So x is contained in $A_{X\cap X'}$, which has a spherical type because it lies in A_X and cannot contain simultaneously s and t. Conjugating by c^{-1}, we have that α is in the spherical-type parabolic subgroup $cA_{X\cap X'}c^{-1} < A_X$, which must contain Q_α because the spherical-type parabolic closure is unique. This finishes the proof of the base case of our induction.

To prove the step case suppose that, if α is contained in a standard parabolic subgroup with less than k ∞-labels, then Q_α is contained in that parabolic subgroup. Let A_X have k labels. If x belongs to $A_{X\setminus\{s\}}$ or $A_{X\setminus\{t\}}$, then x belongs to the standard parabolic subgroup containing less than k ∞-labels. Otherwise, applying the same reasoning as in the initial case, $x \in A_X \cap A_{Y\cup(S\setminus\{s,t\})}$, which also has less than k ∞-labels. Thus, by hypothesis, the spherical-type parabolic closure Q_x of x is in A_X. Therefore, α is in the spherical-type parabolic subgroup $cQ_xc^{-1} < A_X$, which must contain Q_α. □

In the particular case in which A_S is a free product of spherical-type Artin groups, we can prove the existence of a parabolic closure, hence all the hypotheses of Theorem A will be fulfilled.

Lemma 27. *Suppose that A_S is an Artin group of FC-type such that $A_S \simeq A_{X_1} * A_{X_1} * \cdots * A_{X_k}$, where every A_{X_i} is a spherical-type Artin group. Let $\alpha \in A_S$. Then, any minimal parabolic subgroup containing α^m for any $m \in \mathbb{Z}$ contains also α.*

Proof. If α in contained in a single factor A_{X_i}, this is proven in [9, Theorem 8.2]. Suppose otherwise and let P be a minimal parabolic subgroup containing α^m. There is an element β such that $\beta^{-1}P\beta = A_X$ is standard. Notice that $\beta^{-1}\alpha^m\beta = (\beta^{-1}\alpha\beta)^m$. This means that the amalgam normal form of $(\beta^{-1}\alpha\beta)^m$ can be written using only letters in X (Lemma 25). By hypothesis, the length of the amalgam form of $\beta^{-1}\alpha\beta$ is bigger than 1, hence all the letters in the amalgam normal form of $\beta^{-1}\alpha\beta$ are letters that appear in the amalgam normal form of $(\beta^{-1}\alpha\beta)^m$. Therefore A_X contains $\beta^{-1}\alpha\beta$. Conjugating by β^{-1}, we have that P contains α. □

Proposition 28. *If A_S is an Artin group of FC-type such that $A_S \simeq A_{X_1} * A_{X_1} * \cdots * A_{X_k}$, where every A_{X_i} is a spherical-type Artin group, then every element α has a parabolic closure P_α.*

Proof. We will prove the proposition by induction on k. If $k = 1$, A_S has spherical type and the result is true by [9, Proposition 7.2]. Now suppose that the result is true for $k-1$ and consider the free product structure $A_{X_1} * B$ where $B = A_{X_2} * A_{X_3} * \cdots * A_{X_k}$. Suppose there are two minimal parabolic subgroups $P_1 = \beta^{-1}A_Y\beta$, $P_2 = \gamma^{-1}A_Z\gamma$ containing α. By [20, Theorem 3.1], if P_1 and P_2 have spherical type, then α is contained in $P_1 \cap P_2$, so by minimality $P_1 = P_2$. Suppose then that P_1 has non-spherical type. Then, A_Y is a minimal parabolic subgroup containing $\alpha' := \beta\alpha\beta^{-1}$ and A_Z is a minimal parabolic subgroup containing $\alpha'' := \gamma\alpha\gamma^{-1}$. Applying Algorithm 1Algorithm implies that if A_Y and A_Z are different, they cannot be conjugate.

Let $\alpha_1\alpha_2\alpha_3\cdots\alpha_r$ be the amalgam normal form of α' with respect to $A_{X_1} * B$. We also know that α' and α'' are conjugate and that all α_i's are contained in A_Y (Lemma 25). If $r = 1$, then by Proposition 24 we have that α' and α'' belong to the same factor F of the free product and are conjugate by an element f in that factor. By inductive hypothesis, A_Y is the parabolic closure of α' in F and A_Z is the parabolic closure of α'' in F, so by Lemma 16 f has to conjugate A_Y to A_Z, which is only possible if $A_Y = A_Z$. Since $\alpha'' = \gamma\beta^{-1}\alpha'\beta\gamma^{-1}$, we can apply again Lemma 16 to obtain $\gamma\beta^{-1}A_Y\beta\gamma^{-1} = A_Y$, so $P_1 = P_2$. If $r \geq 2$, then α'' is obtained from α' by cyclically permuting the α_i's. This means that α', α'' belong $A_Y \cap A_Z$, which by Van der Lek,[24] is a parabolic subgroup contained in A_Y and A_Z. As A_Y and A_Z are minimal, we have that $A_Y = A_Y \cap A_Z = A_Z$. It remains to show that this implies $P_1 = P_2$. Notice that P_2 can be obtained from P_1 by using conjugation by an element that centralizes α, namely $c := \beta g\gamma^{-1}$, where g is the element that conjugates α' to α''. Now, by [19, Corollary 4.1.6], either c and α are in the same factor (this would be the case $r = 1$) or α and c are a power of the same element h. By Lemma 27, $h \in P_1$, hence $P_2 = c^{-1}P_1c = P_1$. □

Proof of Theorem B. Thanks to [13, Theorem 3.2] and [14, Proposition 4.3], we know that A is standardisable and has the ribbon property. If A has a free product structure, then every element has a parabolic closure (Proposition 28) and we can apply Algorithm 4Algorithm. Now suppose that A is any FC-type Artin group and that A_X is standard parabolic subgroup of A. We need to prove that there is an algorithm that takes as input A and A_X and decides whether for every two elements $x, y \in A_X$, with x, y contained in (possibly different) spherical-type parabolic subgroups, and such that $g^{-1}xg = y$ with $g \in A$, there is $g' \in H$ such that $g'^{-1}xg' = y$. Lemma 26 proves that the spherical-type parabolic closures Q_x and Q_y, are contained in A_X. This last condition and the existence of a spherical-type parabolic closure suffice to reproduce the proof of Theorem 15—just replacing parabolic closures by spherical-type parabolic closures—and show that running Algorithm 4Algorithm will do the job—notice that the only distinct irreducible standard parabolic subgroups that can be conjugate are the spherical-type ones—. □

Algorithm 2: Algorithm to check the D_{2k}, $k > 2$, exceptions described in the proof of Theorem 15

Input : The Coxeter graph Γ_S of an Artin group A_S and three subgraphs $\Gamma_X \subset \Gamma_S$, $\Gamma_{Y'} \subset \Gamma_Y \subset \Gamma_X$ such that A_X and A_S satisfies the hypotheses of Theorem 15 and Γ'_Y is a connected component of Γ_Y of type D_{2k}.

Output: 1 (if we know that A_X is not conjugacy stable) or 0.

Label the elements $s_1, s_2, \ldots, s_{2k}$ of Y as in Fig. 1.

for $t \in \mathrm{Adj}(\{s_{2k}\}) \cap (S \setminus X)$ **do**
- **if** *the connected component of* $\Gamma_{Y\cup\{t\}}$ *containing* Y' *(and* t*) is of type* D_{2m+1}*, for some* m **then**
  - **for** $t' \in \mathrm{Adj}(\{s_{2k}\}) \cap X$ **do**
    - **if** *the connected component of* $\Gamma_{Y\cup\{t'\}}$ *containing* Y' *(and* t'*) is of type* $D_{2m'+1}$*, for some* m' **then**
      - **return** 0;
  - **return** 1;

return 0

Algorithm 3: Algorithm to check the D_4 exceptions described in the proof of Theorem 15

Input : The Coxeter graph Γ_S of an Artin group A_S and two subgraphs $\Gamma_X \subset \Gamma_S$, $\Gamma_{Y'} \subset \Gamma_Y \subset \Gamma_X$ such that A_X and A_S satisfy the hypotheses of Theorem 15 and $\Gamma_{Y'}$ is a connected component of Γ_Y of type D_4.

Output: 1 (if we know that A_X is not conjugacy stable) or 0.

Label the elements s_1, s_2, s_3, s_4 of Y as in Fig. 1.

$Z = \{s_1, s_2, s_3\}$;

for $s \in Z$ **do**
- **for** $t \in \text{Adj}(\{s\}) \cap (S \setminus X)$ **do**
  - $p = 0$; $q = 0$;
  - **if** *the connected component of* $\Gamma_{Y \cup \{t\}}$ *containing* Y' *(and* t*) is of type* D_{2m}*, for some* m **then**
    - $p = 1$; $q = 1$;
    - **for** $t' \in \text{Adj}(\{s\}) \cap X$ **do**
      - **if** *the connected component of* $\Gamma_{Y \cup \{t'\}}$ *containing* Y' *(and* t'*) is of type* $D_{2m'+1}$*, for some* m' **then**
        - $p = 0$; break loop;
    - **if** $p = 1$ **then**
      - **for** $t_1 \in \text{Adj}(Z \setminus \{s\}) \cap X$ **do**
        - **if** *the connected component of* $\Gamma_{Y \cup \{t_1\}}$ *containing* Y' *(and* t_1*) is of type* D_{2m_1+1}*, for some* m_1 **then**
          - **for** $t_2 \in \text{Adj}(Z \setminus \{s, t_1\}) \cap X$ **do**
            - **if** *the connected component of* $\Gamma_{Y \cup \{t_2\}}$ *containing* Y' *(and* t_2*) is of type* D_{2m_2+1}*, for some* m_1 **then**
              - $p = 0$; break loop;
        - **if** $p = 0$ **then**
          - break loop;
  - **if** $p = 1$ **then**
    - **return** 1;
  - **if** $q = 1$ **then**
    - break loop;

return 0

Algorithm 4: Algorithm that tell us if a parabolic subgroup is conjugacy stable or not.

Input : The Coxeter graph Γ_S of an Artin group A_S and a $\Gamma_X \subset A_S$ such that A_X and A_S satisfy the hypotheses of Theorem 15.
Output: "A_X is conjugacy stable" or "A_X is not conjugacy stable".

for $(X_1, X_2) \subset (X, X)$ *such that* $|X_1| = |X_2|$ **do**
  if Γ_{X_1} *is of type* D_{2k} **then**
    if $k > 2$ **then**
      run algorithm 2;
      if *algorithm 2 returns* 1 **then**
        return "A_X is not conjugacy stable";
    if $k = 2$ **then**
      run algorithm 3;
      if *algorithm 3 returns* 1 **then**
        return "A_X is not conjugacy stable";
  $\Gamma_{X'_1}, \Gamma_{X'_2}, \dots, \Gamma_{X'_m}$:= components of Γ_{X_1};
  $C := \{(X'_1, X'_2, \dots, X'_m)\}$;
  if $X_1 = X_2$ **then**
    $D := \{(X'_1, X'_2, \dots, X'_m)\}$;
  else
    $D := \{\emptyset\}$;
  for $(Y_1, Y_2, \dots, Y_m) \in C$ **do**
    $Y := Y_1 \cup Y_2 \cup \dots \cup Y_m$;
    for $t \in X \cap \mathrm{Adj}(Y)$ **do**
      if *the connected component* $\Gamma_{Y'}$ *of* $\Gamma_{Y \cup \{t\}}$ *containing* t *is twistable* **then**
        $Z = \Delta_{Y'}^{-1} Y \Delta_{Y'}$;
        $T = (\Delta_{Y'}^{-1} Y_1 \Delta_{Y'}, \Delta_{Y'}^{-1} Y_2 \Delta_{Y'}, \dots, \Delta_{Y'}^{-1} Y_m \Delta_{Y'})$;
        if $T \notin C$ **then**
          $C = C \cup \{T\}$;
          if $Z = X_2$ *and* $T \notin D$ **then**
            $D = D \cup \{T\}$;
  for $(Y_1, Y_2, \dots, Y_m) \in C$ **do**
    $Y := Y_1 \cup Y_2 \cup \dots \cup Y_m$;
    for $t \in \mathrm{Adj}(Y)$ **do**
      if *the connected component* $\Gamma_{Y'}$ *of* $\Gamma_{Y \cup \{t\}}$ *containing* t *is twistable* **then**
        $Z = \Delta_{Y'}^{-1} Y \Delta_{Y'}$
        $T = (\Delta_{Y'}^{-1} Y_1 \Delta_{Y'}, \Delta_{Y'}^{-1} Y_2 \Delta_{Y'}, \dots, \Delta_{Y'}^{-1} Y_m \Delta_{Y'})$
        if $T \notin C$ **then**
          $C = C \cup \{T\}$;
          if $Z = X_2$ *and* $T \notin D$ **then**
            return *"A_X is not conjugacy stable";*
return *"A_X is conjugacy stable";*

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References

[1] Altobelli, J.A.: The word problem for Artin groups of FC type. J. Pure Appl. Algebra **129**, 1–22 (1998)

[2] Artin, E.: Theory of Braids. Ann. Math. **2**(48), 101–126 (1947)

[3] Blufstein, M.A.: Parabolic subgroups of two-dimensional Artin groups and systolic-by-function complexes. To appear in Bull. Lond. Math. Soc. https://doi.org/10.1112/blms.12697

[4] Brieskorn, E., Saito, K.: Artin-Gruppen und Coxeter-Gruppen. Invent. Math. **17**(4), 245–271 (1972)

[5] Calvez, M., de la Cruz, C., Bruno, A., Cumplido, M.: Conjugacy stability of parabolic subgroups of Artin-Tits groups of spherical type. J. Algebra **556**, 621–633 (2020)

[6] Charney, R., Davis, M.W.: The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups. J. Am. Math. Soc. **8**(3), 597–627 (1995)

[7] Charney, R., Paris, L.: Convexity of parabolic subgroups in Artin groups. Bull. Lond. Math. Soc. **46**(6), 1248–1255 (2014)

[8] Coxeter, H.S.M.: The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$. J. Lond. Math. Soc. **1–10**(1), 21–25 (1935)

[9] Cumplido, M., Gebhardt, V., González-Meneses, J., Wiest, B.: On parabolic subgroups of Artin-Tits groups of spherical type. Adv. Math. **352**, 572–610 (2019)

[10] Cumplido, M., Martin, A., Vaskou, N.: The poset of parabolic subgroups for Artin groups of large type. To appear in Math. Proc. Cambridge Philos. Soc. https://doi.org/10.1017/S0305004122000342

[11] Deligne, P.: Les immeubles des groupes de tresses généralisés. Invent. Math. **17**, 273–302 (1972)

[12] Godelle, E.: Normalisateur et groupe d'Artin de type sphérique. J. Algebra **269**(1), 263–274 (2003)

[13] Godelle, E.: Parabolic subgroups of Artin groups of type FC. Pacific J. Math. **208**(2), 243–254 (2003)

[14] Godelle, E.: Artin-Tits groups with CAT(0) Deligne complex. J. Pure Appl. Algebra **208**(1), 39–52 (2007)

[15] Godelle, E.: On fusion control in FC type Artin-Tits groups. J. Algebra To Appear (2021)

[16] González-Meneses, J.: Geometric embeddings of braid groups do not merge conjugacy classes. Bol. Soc. Mat. Mex. **20**(2), 297–305 (2014)

[17] Haettel, T.: Lattices, injective metrics and the $K(\pi, 1)$ conjecture. arXiv:2109.07891 (2021)

[18] Krammer, D.: The conjugacy problem for Coxeter groups. Groups Geom. Dyn. 71–171 (2009)

[19] Magnus, W., Karrass, A., Solitar, D.: Combinatorial Group Theory. Wiley, New York (1966)

[20] Morris-Wright, R.: Parabolic subgroups in FC-type Artin groups. J. Pure Appl. Algebra **225**(1), 106468 (2021)

[21] Paris, L.: Parabolic Subgroups of Artin Groups. J. Algebra **196**(2), 369–399 (1997)

[22] Paris, L.: Artin monoids inject in their groups. Comment. Math. Helv. **77**(3), 609–637 (2002)

[23] Paris, L.: Artin groups of spherical type up to isomorphism. J. Algebra **281**(2), 666–678 (2004)

[24] Van der Lek, H.: The Homotopy Type of Complex Hyperplane Complements. Ph.D. thesis, Nijmegen (1983)

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