



Pseudo-Fubini Real-Entire Functions on the Plane

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Abstract. In this note, it is proved the existence of a c -dimensional vector space of real-entire functions all of whose nonzero members are non-integrable in the sense of Lebesgue but yet their two iterated integrals exist as real numbers and coincide. Moreover, it is shown that this vector space can be chosen to be dense in the space of all real C^∞ -functions on the plane endowed with the topology of uniform convergence on compacta for all derivatives of all orders. If the condition of being entire is dropped, then a closed infinite dimensional subspace satisfying the same properties can be obtained.

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1. Introduction

According to Fubini's theorem (see e.g. [11, Chapter 17]), if a real function f , which is defined on a measure space $X \times Y$ that is the product of two σ -finite product spaces, is integrable, then its two iterated integrals exist as real numbers and coincide, and in fact their common value is the integral of f on the product space. In [2] it is analyzed –among other questions– the algebraic size of the set of those measurable functions $f : X \times Y \rightarrow \mathbb{R}$ being non-integrable but still satisfying the conclusion of Fubini's theorem, that is, its two iterated integrals exist as real numbers and are equal. These functions are called *pseudo-Fubini functions*, and they abound in a topological-algebraic sense.

Specifically, it is shown in [2] that, under appropriate soft conditions on the measure spaces X and Y , there exists a c -dimensional vector space all of

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whose nonzero members are pseudo-Fubini that is dense in the space of all measurable functions $X \times Y \rightarrow \mathbb{R}$ when endowed with its natural metrical topology. Here \mathfrak{c} denotes the cardinality of the continuum.

Turning to the more familiar setting of the Lebesgue measure (dx on the real line \mathbb{R} , and $dx dy$ on the plane \mathbb{R}^2) as well as to functions with richer properties, in [3] it is exhibited an explicit \mathfrak{c} -dimensional vector space of *analytic* functions $\mathbb{R}^2 \rightarrow \mathbb{R}$ all of whose nonzero elements are pseudo-Fubini that is dense in the Fréchet space of all real continuous functions on \mathbb{R}^2 when endowed with the compact-open topology. Recall that, if Ω is an open subset of \mathbb{R}^2 , then a function $f : \Omega \rightarrow \mathbb{R}$ is said to be analytic on Ω whenever, given $(x_0, y_0) \in \Omega$, there are a neighborhood U of (x_0, y_0) with $U \subset \Omega$ and a double sequence $\{a_{jk}\}_{j,k \geq 0} \subset \mathbb{R}$ such that $f(x, y) = \sum_{j,k \geq 0} a_{jk}(x-x_0)^j(y-y_0)^k$ for every $(x, y) \in U$, the convergence being absolute (see, e.g., [5, Chap. 4] for background on real or complex analytic functions of several variables).

Going one step further, in [3] it is posed the problem of the existence of *entire* functions $\mathbb{R}^2 \rightarrow \mathbb{R}$ that are pseudo-Fubini. Recall that a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be entire if an absolutely convergent expansion $f(x, y) = \sum_{j,k \geq 0} a_{jk}x^jy^k$ is valid on the whole plane. Note that any entire function on \mathbb{R}^2 is analytic but the converse is false: consider, for instance, the function $f(x, y) = \frac{1}{1+x^2+y^2}$.

In this short note, we solve in the affirmative the last problem. In fact, it is proved that the family of pseudo-Fubini entire functions is, again, rather large in both algebraic and topological senses. This will be done in Sect. 3. Section 2 will be devoted to fix some notation and provide a number of preliminary results. In Sect. 4 the problem of existence of closed infinite dimensional spaces made up of pseudo-Fubini smooth functions is considered.

2. Preliminaries and Notation

Let $k, N \in \mathbb{N} := \{1, 2, \dots\}$ and Ω be a nonempty open subset of \mathbb{R}^N . The case $N = 2$, $\Omega = \mathbb{R}^2$ will be mostly considered. Throughout this note, we shall use the following –mostly standard– notation:

- $C(\Omega)$ will stand for the set of all real continuous functions on Ω . This set becomes a Fréchet space when endowed with the topology of uniform convergence in compacta, see [9].
- $C^k(\Omega)$ denotes the vector space of all functions $\Omega \rightarrow \mathbb{R}$ that are k times differentiable on Ω .
- $C^\infty(\Omega)$ represents the set of all real functions on Ω that are infinitely times differentiable on Ω . This set becomes a Fréchet space when endowed with the topology of uniform convergence in compacta for all partial derivatives of all orders, see [9].
- $C^\omega(\Omega)$ will stand for the vector space of all functions $\Omega \rightarrow \mathbb{R}$ that are analytic on Ω .
- \mathcal{E}_N denotes the vector space of all real entire functions on \mathbb{R}^N . Then $\mathcal{E}_N \subset C^\omega(\mathbb{R}^N) \subset C^\infty(\mathbb{R}^N)$.

- $\mathcal{L}^1(\Omega)$ represents the vector space of all functions $\Omega \rightarrow \mathbb{R}$ that are Lebesgue integrable on Ω .
- For each multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$, we set $|\alpha| := \alpha_1 + \dots + \alpha_N$.

In addition, we denote by $p\mathcal{F}(\mathbb{R}^2)$ the vector space of all pseudo-Fubini functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, meaning that each of such functions is Lebesgue measurable but not Lebesgue integrable and, in addition, both iterated integrals $\int_{\mathbb{R}} (\int_{\mathbb{R}} f(x, y) dx) dy, \int_{\mathbb{R}} (\int_{\mathbb{R}} f(x, y) dy) dx$ exist as real numbers and have the same value.

We now turn to a different setting. If E is a vector space, we can study the algebraic size of a subset, which becomes more interesting if such a subset is not a vector space in itself (see [1] for background on this line of research, called lineability). The basic concepts that are relevant to this note are contained in the following definition.

Definition 2.1. Assume that E is a vector space. Let $\mathcal{A} \subset E$ and α be a cardinal number. Then \mathcal{A} is called *lineable* if $\mathcal{A} \cup \{0\}$ contains some infinite dimensional vector space. If, in addition, E is a topological vector space and $\mathcal{A} \cup \{0\}$ contains some dense (some (dense) α -dimensional, some closed infinite dimensional, resp.) vector subspace of E , then \mathcal{A} is said to be *dense lineable* (α -(dense) lineable, *spaceable*, resp.) in E . And \mathcal{A} is called *maximal dense lineable* in E if it is $\dim(E)$ -dense lineable.

Under the previous terminology, the main result in [3] can be stated as follows.

Theorem 2.2. *The set $p\mathcal{F} \cap C^\omega(\mathbb{R}^2)$ is maximal dense lineable in $C(\mathbb{R}^2)$.*

The main result in this note (Theorem 3.1) tells us that the same assertion holds when one replaces $C^\omega(\mathbb{R}^2)$ by the much smaller family \mathcal{E}_2 , and the space $C(\mathbb{R}^2)$ by the space $C^\infty(\mathbb{R}^2)$, whose topology is much stronger than that inherited from the former space. With this aim, we shall make use of the next two assertions. The first one enables us to extract dense lineability from mere lineability and can be found in [1, Chapter 7], while the second one is an approximation result due to Frih and Gauthier [6], which is a strengthening of corresponding results due to Carleman [4], Scheinberg [12] and Hoischen [8] [cases $(k = 0, N = 1)$, $(k = 0, N \text{ arbitrary})$ and $(k \text{ arbitrary}, N = 1)$, resp.].

Theorem 2.3. *Assume that E is a metrizable separable topological vector space, that κ is an infinite cardinal number and that \mathcal{A}, \mathcal{B} are subsets of E satisfying the following properties:*

- \mathcal{A} is κ -lineable,
- \mathcal{B} is dense lineable,
- $\mathcal{A} + \mathcal{B} \subset \mathcal{A}$, and
- $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Then \mathcal{A} is κ -dense lineable in E .

Theorem 2.4. *Assume that $N, k \in \mathbb{N}$, that $f \in C^k(\mathbb{R}^N)$ and that $\varepsilon : \mathbb{R}^N \rightarrow (0, +\infty)$ is a continuous function. Then there is an entire function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ such that*

$$|D^\alpha f(x) - D^\alpha g(x)| < \varepsilon(x)$$

for all $x \in \mathbb{R}^N$ and all $\alpha \in (\mathbb{N} \cup \{0\})^N$ with $|\alpha| \leq k$.

3. A Large Vector Space of Pseudo-Fubini Entire Functions

This section is devoted to prove the following theorem.

Theorem 3.1. *The set $p\mathcal{F} \cap \mathcal{E}_2$ is maximal dense lineable in $C^\infty(\mathbb{R}^2)$.*

Proof. It is well-known (an easy proof follows, for instance, from the Baire category theorem) that $\dim C^\infty(\mathbb{R}^2) = \mathfrak{c}$. Hence, we have to prove that $p\mathcal{F} \cap \mathcal{E}_2$ is \mathfrak{c} -dense lineable in $C^\infty(\mathbb{R}^2)$.

Our first task is to locate a pseudo-Fubini C^∞ -function on \mathbb{R}^2 . For this, denote by I_0 the open unit interval $(0, 1)$ and consider the function $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\varphi_0(t) = \begin{cases} e^{\frac{1}{t(t-1)}} & \text{if } t \in I_0 \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $\varphi_0 \in C^\infty(\mathbb{R})$. For each $a \in \mathbb{R}$, set $I_a := a + I_0 = (a, a+1)$, and let $\varphi_a : \mathbb{R} \rightarrow \mathbb{R}$ be the C^∞ -function given by $\varphi_a(t) = \varphi_0(t-a)$. Observe that, for every pair a, b of reals, the function $(x, y) \mapsto \varphi_a(x)\varphi_b(y)$ belongs to $C^\infty(\mathbb{R}^2)$ and vanishes exactly outside $I_a \times I_b$. Now, define the function $\Phi_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} \Phi_0(x, y) = & \varphi_1(x)\varphi_1(y) + \sum_{n=1}^{\infty} \varphi_{2n}(x)\varphi_{2n+1}(y) + \sum_{n=1}^{\infty} \varphi_{2n+1}(x)\varphi_{2n}(y) \\ & - \sum_{n=1}^{\infty} \varphi_{2n}(x)\varphi_{2n-1}(y) - \sum_{n=1}^{\infty} \varphi_{2n-1}(x)\varphi_{2n}(y). \end{aligned}$$

Observe that Φ_0 is well defined because the products of intervals $I_a \times I_b$ where the functions $\varphi_a(x)\varphi_b(y)$ participating in the sum do not vanish are mutually disjoint and so, given a point of \mathbb{R}^2 , there is at most one term of the series that is not zero at it. The same argument and the fact that differentiability is a local property yields that $\Phi_0 \in C^\infty(\mathbb{R}^2)$.

Since $\varphi_0 > 0$ on I_0 , we get

$$\alpha := \int \int_{I_0^2} \varphi_0(x)\varphi_0(y) \, dx dy = \left(\int_{I_0} \varphi_0(t) \, dt \right)^2 > 0,$$

and the change of variables rule together with Fubini's theorem gives

$$\begin{aligned} \int \int_{\mathbb{R}^2} \varphi_a(x)\varphi_b(y) \, dx dy &= \int \int_{I_a \times I_b} \varphi_a(x)\varphi_b(y) \, dx dy \\ &= \left(\int_{I_a} \varphi_a(x) \, dx \right) \left(\int_{I_b} \varphi_b(x) \, dx \right) = \left(\int_{I_0} \varphi_0(x) \, dx \right)^2 \\ &= \alpha. \end{aligned}$$

Therefore (note that for nonnegative functions it is always possible to interchange the integral with the series) we get

$$\begin{aligned} \int \int_{\mathbb{R}^2} |\Phi_0(x, y)| \, dx dy &\geq \int \int_{\mathbb{R}^2} \left(\sum_{n=1}^{\infty} \varphi_{2n}(x) \varphi_{2n+1}(y) \right) \, dx dy \\ &= \sum_{n=1}^{\infty} \int \int_{\mathbb{R}^2} \varphi_{2n}(x) \varphi_{2n+1}(y) \, dx dy = \sum_{n=1}^{\infty} \alpha = +\infty, \end{aligned}$$

which shows that $\Phi_0 \notin \mathcal{L}^1(\mathbb{R}^2)$.

To see that $\Phi_0 \in p\mathcal{F}$, observe that if $y_0 \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} I_n$ then $\Phi_0(\cdot, y_0) \equiv 0$, and so $\int_{\mathbb{R}} \Phi_0(x, y_0) \, dx = 0$. And if y_0 belongs to some (necessarily unique) I_n , then $\Phi_0(\cdot, y_0) = c_{y_0} (\varphi_p \chi_{I_p} - \varphi_q \chi_{I_q})$ for some different $p, q \in \mathbb{N}$ and a constant $c_{y_0} \in \mathbb{R}$. Therefore

$$\begin{aligned} \int_{\mathbb{R}} \Phi_0(x, y_0) \, dx &= c_{y_0} \left(\int_{I_p} \varphi_p(x) \, dx - \int_{I_q} \varphi_q(x) \, dx \right) \\ &= c_{y_0} \left(\int_{I_0} \varphi_0(x) \, dx - \int_{I_0} \varphi_0(x) \, dx \right) = 0 \end{aligned}$$

too. Thus, the iterated integral $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi_0(x, y) \, dx \right) dy$ exists in the Lebesgue sense as a real number, with value 0. Now the symmetry property $\Phi_0(x, y) = \Phi_0(y, x)$ yields the same conclusion for $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi_0(x, y) \, dy \right) dx$, which shows that Φ_0 is pseudo-Fubini.

Next, consider for each $k \in \mathbb{N}$ the translation

$$\tau_k : (x, y) \in \mathbb{R}^2 \mapsto (x + 4k, y + k) \in \mathbb{R}^2.$$

For every $a, b \in \mathbb{R}$ we have $\tau_k(I_a \times I_b) = I_{a+4k} \times I_{b+k}$. Observe that if $k, l \in \mathbb{N}$ (with $k \neq l$) and $I_m \times I_n$ and $I_p \times I_q$ are two different products of intervals where Φ_0 does not vanish, then we have not only $(I_{m+4k} \times I_{n+k}) \cap (I_{p+4k} \times I_{q+k}) = \emptyset$ but also $(I_{m+4k} \times I_{n+k}) \cap (I_{p+4l} \times I_{q+l}) = \emptyset$. In particular, the supports of the functions

$$\Phi_k : (x, y) \in \mathbb{R}^2 \mapsto (\Phi_0 \circ \tau_k^{-1})(x, y) = \Phi_0(x - 4k, y - k) \in \mathbb{R} \quad (k = 1, 2, \dots)$$

are mutually disjoint. The fact that the mappings $x \mapsto x - 4k, y \mapsto y - k$ are C^∞ -smooth on \mathbb{R} together with the rule of change of variables (for integrals on \mathbb{R} and \mathbb{R}^2) shows that $\Phi_k \in C^\infty(\mathbb{R}^2) \cap p\mathcal{F}$ for every $k \in \mathbb{N}$.

Let us choose an *almost disjoint family* \mathcal{N} of subsets of \mathbb{N} , that is, \mathcal{N} satisfies the following properties (see [10, Chapter 8]): each $S \in \mathcal{N}$ is infinite, $\text{card}(\mathcal{N}) = \mathfrak{c}$ and $S \cap S'$ is finite for every pair of different $S, S' \in \mathcal{N}$. Let us define, for every $S \in \mathcal{N}$, the function $f_S : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f_S := \sum_{k \in S} \Phi_k.$$

From the facts that the supports of the Φ_k 's are mutually disjoint and $\Phi_k \in C^\infty(\mathbb{R}^2)$ we obtain $f_S \in C^\infty(\mathbb{R}^2)$. It is also obtained that $|f_S| = \sum_{k \in S} |\Phi_k|$. Hence selecting any $k_0 \in S$ we get

$$\int_{\mathbb{R}^2} |f_S(x, y)| \, dx dy \geq \int_{\mathbb{R}^2} |\Phi_{k_0}(x, y)| \, dx dy = \int_{\mathbb{R}^2} |\Phi_0(x, y)| \, dx dy = +\infty,$$

and so $f_S \notin \mathcal{L}^1(\mathbb{R}^2)$. Finally, the fact that the supports of the Φ_k 's are steadily moving up and right when $k \rightarrow \infty$ yields that, given $x_0 \in \mathbb{R}$, the function $y \in \mathbb{R} \mapsto f_S(x_0, y) \in \mathbb{R}$ is a (possibly empty) sum of finitely many functions of the form $c(\varphi_a \chi_{I_a} - \varphi_b \chi_{I_b})$ (with $c \in \mathbb{R}$ and $a, b \in \mathbb{N}$, $a \neq b$), whose integrals over \mathbb{R} are zero. An analogous property happens if we fix $y_0 \in \mathbb{R}$ and consider the function $x \in \mathbb{R} \mapsto f_S(x, y_0) \in \mathbb{R}$. Therefore both iterates integrals of f_S exist in the sense of Lebesgue and share the value 0. Summarizing, we have $f_S \in C^\infty(\mathbb{R}^2) \cap p\mathcal{F}$ for each $S \in \mathcal{N}$.

Now, it is the turn of approximation. Thanks to Theorem 2.4, for every $S \in \mathcal{N}$ we can find $g_S \in \mathcal{E}_2$ such that

$$|f_S(x, y) - g_S(x, y)| < \frac{1}{(1+x^2)(1+y^2)} \text{ for all } (x, y) \in \mathbb{R}^2. \tag{3.1}$$

We define the family \mathcal{M} as

$$\mathcal{M} := \text{span} \{g_S : S \in \mathcal{N}\}.$$

Then \mathcal{M} is a vector space which is contained in \mathcal{E}_2 . If we showed that any nontrivial finite linear combination of functions from \mathcal{M} is pseudo-Fubini (and, hence, in particular non-integrable) then we would prove at one stroke that the g_S 's are linearly independent (hence $\dim(\mathcal{M}) = \mathfrak{c}$) and that $\mathcal{M} \subset (p\mathcal{F} \cap \mathcal{E}_2) \cup \{0\}$, which would show the desired maximal lineability.

Assume that $G \in \mathcal{M}$ is one of such combinations, so that there are $p \in \mathbb{N}$, reals $\alpha_1, \dots, \alpha_p$ and pairwise different sets $S_1, \dots, S_p \in \mathcal{N}$ such that $G = \alpha_1 g_{S_1} + \dots + \alpha_p g_{S_p}$ and not all the α_i 's are zero (we may assume without loss of generality that $\alpha_1 \neq 0$). From (3.1), the comparison test and the fact that $\frac{1}{(1+x^2)(1+y^2)} \in \mathcal{L}^1(\mathbb{R}^2)$, it follows that the functions

$$h_i := g_{S_i} - f_{S_i} \quad (i = 1, \dots, p)$$

are integrable C^∞ -functions from \mathbb{R}^2 to \mathbb{R} . Then every h_i satisfies the conclusion of Fubini's theorem and, as f_{S_i} also does, the linearity of the integral shows that for the function g_{S_i} and, consequently, for the function G , both iterated integrals exist as real numbers in the sense of Lebesgue and coincide.

Now, since the family \mathcal{N} is almost disjoint, the set $S_1 \setminus (\bigcup_{j=2}^p S_j)$ is not empty, so that we can select an element k_0 in it. Consider the set $A := \bigcup_{n=1}^\infty I_{2n+4k_0} \times I_{2n+1+k_0}$, which has empty intersection with each of the supports of the functions f_{S_i} ($i = 2, \dots, p$). Therefore, G is not Lebesgue integrable on \mathbb{R}^2 because if G were Lebesgue integrable on \mathbb{R}^2 , it would be integrable on A , but we have

$$\begin{aligned}
 \int \int_A |G(x, y)| \, dx dy &= \int \int_A \left| \sum_{i=1}^p \alpha_i g_{S_i}(x, y) \right| \, dx dy \\
 &\geq \int \int_A |\alpha_1| |g_{S_1}(x, y)| \, dx dy - \int \int_A \sum_{i=2}^p |\alpha_i| |g_{S_i}(x, y)| \, dx dy \\
 &\geq \int \int_A |\alpha_1| (|f_{S_1}(x, y)| - |g_{S_1}(x, y) - f_{S_1}(x, y)|) \, dx dy \\
 &\quad - \int \int_A \sum_{i=2}^p |\alpha_i| (|f_{S_i}(x, y)| + |g_{S_i}(x, y) - f_{S_i}(x, y)|) \, dx dy \\
 &= \int \int_A |\alpha_1| |f_{S_1}(x, y)| \, dx dy - \sum_{i=1}^p |\alpha_i| \int \int_A |g_{S_i}(x, y) - f_{S_i}(x, y)| \, dx dy \\
 &\geq \int \int_A |\alpha_1| |f_{S_1}(x, y)| \, dx dy - \sum_{i=1}^p |\alpha_i| \int \int_A \frac{1}{(1+x^2)(1+y^2)} \, dx dy \\
 &= |\alpha_1| \int \int_A |\Phi_{k_0}(x, y)| \, dx dy - C \\
 &= |\alpha_1| \sum_{n=1}^{\infty} \int \int_{I_{2n+4k_0} \times I_{2n+1+k_0}} |\Phi_{k_0}(x, y)| \, dx dy - C \\
 &= |\alpha_1| \sum_{n=1}^{\infty} \int \int_{I_{2n} \times I_{2n+1}} |\Phi_0(x, y)| \, dx dy - C \\
 &= |\alpha_1| \sum_{n=1}^{\infty} \int \int_{I_{2n} \times I_{2n+1}} \varphi_{2n}(x) \varphi_{2n+1}(y) \, dx dy - C \\
 &= |\alpha_1| \sum_{n=1}^{\infty} \alpha - C = +\infty,
 \end{aligned}$$

since $\alpha > 0$ and $C := \sum_{i=1}^p |\alpha_i| \int \int_A \frac{1}{(1+x^2)(1+y^2)} \, dx dy < \infty$, that is absurd. Consequently, $G \in p\mathcal{F}$ and so the family $\mathcal{A} := p\mathcal{F} \cap \mathcal{E}_2$ is maximal lineable in $C^\infty(\mathbb{R}^2)$.

Next, we consider the family

$$\mathcal{B} := \{P \cdot \Psi_1 : P \text{ is a polynomial of two real variables with coefficients in } \mathbb{R}\},$$

where Ψ_1 is the Gaussian function $\Psi_1(x, y) = e^{-x^2-y^2}$, that is integrable on the plane (with integral equal to π) together with every function $\Psi_\lambda(x, y) := e^{-\lambda(x^2+y^2)}$ with $\lambda > 0$. Plainly, \mathcal{B} is a vector space contained in \mathcal{E}_2 . On the one hand, each function $h := P \cdot \Psi_1 \in \mathcal{B}$ belongs to $\mathcal{L}^1(\mathbb{R}^2)$. Indeed, since P is a polynomial, there are a constant $M \in (0, +\infty)$ and an $N \in \mathbb{N}$ such that $|P(x, y)| \leq M(1 + (x^2 + y^2)^N)$ for all $(x, y) \in \mathbb{R}^2$. And since $\lim_{t \rightarrow +\infty} \frac{M(1+t^N)}{e^{t/2}} = 0$, we derive the existence of a constant $\gamma \in (0, +\infty)$ such that $|P(x, y)| \leq \gamma \cdot e^{\frac{x^2+y^2}{2}}$ on \mathbb{R}^2 , and therefore $|h| \leq \gamma \cdot \Psi_{1/2}$ on \mathbb{R}^2 . The comparison test yields the integrability of h . In particular, Fubini's theorem implies that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x, y) \, dx \right) dy = \int \int_{\mathbb{R}^2} h(x, y) \, dx dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x, y) \, dy \right) dx$$

for all $h \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{L}^1(\mathbb{R}^2)$, we have $\mathcal{A} \cap \mathcal{B} = \emptyset$. On the other hand, we also have $\mathcal{A} + \mathcal{B} \subset \mathcal{A}$. This is a consequence of the facts that the sum of an integrable function with a non-integrable one is non-integrable and that both finiteness and coincidence of the iterated integrals are preserved under finite summations.

Take $E := C^\infty(\mathbb{R}^2)$ and $\kappa := \mathfrak{c}$. According to Theorem 2.3, and taking into account that \mathcal{B} is itself a vector space, it only remains to prove that \mathcal{B} is dense in E . To this end, observe first that the formula

$$\|f\|_k := \max_{0 \leq i+j \leq k} \sup_{(x,y) \in [-k,k]^2} |D^{ij}f(x,y)| \quad (f \in C^\infty(\mathbb{R}^2); k = 1, 2, \dots) \tag{3.2}$$

defines an increasing sequence of seminorms generating the natural Fréchet topology of our space E . In fact, $C^\infty(\mathbb{R}^2)$ endowed with multiplication of functions is a Fréchet algebra (see, e.g., [7, Chapter 1] for background about this structure) because the Leibniz rule for the derivative of a product gives the existence of a sequence $\{C_k\}_{k \geq 1} \subset (0, +\infty)$ such that

$$\|f \cdot g\|_k \leq C_k \|f\|_k \|g\|_k \quad (f, g \in E; k \in \mathbb{N}). \tag{3.3}$$

Now, fix a nonempty open subset U of E . Then there are $\varepsilon > 0$, $f \in E$ and $k \in \mathbb{N}$ such that $U \supset \{h \in E : \|h - f\|_k < \varepsilon\}$. Define $\varepsilon(x, y) := \varepsilon/2$. By Theorem 2.4, there is $g \in \mathcal{E}_2$ such that $|D^{ij}f(x, y) - D^{ij}g(x, y)| < \varepsilon/2$ for all $(i, j) \in (\mathbb{N} \cup \{0\})^2$ with $i + j \leq k$ and all $(x, y) \in \mathbb{R}^2$. Consider the sequence $\{P_n\}_{n \geq 0}$ of Taylor polynomials of g at the origin, that is,

$$P_n(x, y) = \sum_{0 \leq i+j \leq n} \frac{D^{ij}g(0)}{i!j!} x^i y^j \quad (n = 0, 1, 2, \dots).$$

Then (see, e.g., [5]) for each 2-index $(i, j) \in (\mathbb{N} \cup \{0\})^2$ the sequence $\{D^{ij}P_n\}_{n \geq 0}$ converges uniformly to $D^{ij}g$ on every compact subset of \mathbb{R}^2 . In particular, we can find $n_0 \in \mathbb{N}$ such that $|D^{ij}P_{n_0}(x, y) - D^{ij}g(x, y)| < \varepsilon/2$ for all $(i, j) \in (\mathbb{N} \cup \{0\})^2$ with $i + j \leq k$ and all $(x, y) \in [-k, k]^2$. Let us set $h := P_{n_0}$. Thus, according to (3.2) we obtain

$$\|h - f\|_k \leq \|P_{n_0} - g\|_k + \|f - g\|_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, $h \in U$. This tells us that the set \mathcal{P} of polynomials in two real variables with real coefficients is dense in E . Finally, consider the multiplication operator

$$S : f \in E \mapsto \Psi_1 \cdot f \in E,$$

that is linear and surjective (because Ψ_1 never vanishes and $1/\Psi_1 \in E$). Thanks to (3.3), we have $\|S(f)\|_k \leq C_k \|\Psi_1\|_k \|f\|_k$ for all $f \in E$ and all $k \in \mathbb{N}$, which proves that S is continuous. Then $\mathcal{B} = S(\mathcal{P})$ is dense in $S(E) = E$. The proof is complete. \square

4. Pseudo-Fubini Smooth Functions: Spaceability

In this final section, we want to raise the question of existence of closed infinite dimensional spaces of entire pseudo-Fubini functions.

Question. Is $p\mathcal{F} \cap \mathcal{E}_2$ spaceable in $C^\infty(\mathbb{R}^2)$? Or at least: Is $p\mathcal{F} \cap C^\omega(\mathbb{R}^2)$ spaceable in $C^\infty(\mathbb{R}^2)$?

We have not been able to give an answer to it. Therefore, we shall for the moment content ourselves with establishing the next assertion, which puts an end to this note.

Theorem 4.1. *The set $p\mathcal{F} \cap C^\infty(\mathbb{R}^2)$ is spaceable in $C^\infty(\mathbb{R}^2)$.*

Proof. Consider the functions $\Phi_k \in C^\infty(\mathbb{R}^2) \cap p\mathcal{F}$ ($k \in \mathbb{N}$) defined in the proof of Theorem 3.1. Since their supports S_k are pairwise disjoint, they are linearly independent. By the same reason, for each sequence $\alpha = (\alpha_k) \in \mathbb{R}^\mathbb{N}$ the function $\sum_{k=1}^\infty \alpha_k \Phi_k$ is well defined and belongs to $C^\infty(\mathbb{R}^2)$. Let us define the family

$$\mathcal{M} := \left\{ \sum_{k=1}^\infty \alpha_k \Phi_k : \alpha \in \mathbb{R}^\mathbb{N} \right\},$$

which is plainly a vector subspace of $C^\infty(\mathbb{R}^2)$. Moreover, it is infinite dimensional because \mathcal{M} contains every Φ_k . The fact that every $f \in \mathcal{M} \setminus \{0\}$ belongs to $p\mathcal{F}$ can be seen as in the proof of Theorem 3.1, where one uses the property that the supports of the Φ_k 's move steadily up and right as $k \rightarrow \infty$. Hence, our unique task is to show that \mathcal{M} is closed in $C^\infty(\mathbb{R}^2)$.

To this end, assume that $f_j \rightarrow f$ as $j \rightarrow \infty$ in the topology of $C^\infty(\mathbb{R}^2)$, where $f_j = \sum_{k=1}^\infty \alpha_{j,k} \Phi_k \in \mathcal{M}$ ($k = 1, 2, \dots$). It should be shown that $f \in \mathcal{M}$. Since convergence in $C^\infty(\mathbb{R}^2)$ implies pointwise convergence, we get $\lim_{k \rightarrow \infty} f_j(x) = f(x)$ for every $x \in \mathbb{R}^2$. If $x \notin S := \bigcup_{k \in \mathbb{N}} S_k$ then $\Phi_k(x) = 0$ for all k , and so $f_j(x) = 0$. Hence $f(x) = 0$ in this case. If $x \in S$ then there is a (unique) $k = k(x) \in \mathbb{N}$ with $x \in S_k$. Then $f_j(x) = \alpha_{j,k} \Phi_k(x) \rightarrow f(x)$ as $j \rightarrow \infty$. Therefore $\lim_{j \rightarrow \infty} \alpha_{j,k} = \frac{f(x)}{\Phi_k(x)}$. But this limit (say, β_k) must be independent of x . Thus, $f(x) = \beta_k \Phi_k(x)$ for all $x \in S_k$. Consequently, from the disjointness of the S_k 's, we get $f = \sum_{k=1}^\infty \beta_k \Phi_k \in \mathcal{M}$, as required. \square

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