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Multiple q-Integral and Records from Geometrically Distributed Sequences

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Abstract: We study the distribution theory of some statistics related to records from a sequence of geometrically distributed random variables. The main novelty in this study is the introduction of q-calculus techniques. We obtain representations for the probability functions of some variables of interest, such as, for instance, record times, number of records, inter-record times, record indicators, etc., in terms of multiple q-integrals. It is remarkable to note that the expressions thus obtained are q-analogues of the corresponding well-known results in the absolutely continuous case. We also explore a duality between the results in the case of weak and ordinary records.

Keywords: weak and ordinary records; geometric distributions; Jackson integral; multiple q-integral; q-Stirling numbers

MSC: 05A30; 60E05; 62E15

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1. Introduction

The origins of q-calculus date back to the works of [1] in connection to infinite series in his famous book *Introductio in Analysin Infinitorum*. In the last few decades, there has been a considerable interest in the topic due to its numerous applications in different areas of mathematics and physics, such as number theory, special functions, orthogonal polynomials, operator theory, combinatorics, probability, quantum mechanics, relativity, string theory, etc. An extensive treatment on q-calculus, its history and applications can be found in [2].

In this paper, we explore a connection between q-calculus and the distribution theory of records from sequences of random variables geometrically distributed.

Ref. [3] started the probabilistic study of record values and gave many of their basic distributional properties. Let us recall some definitions and results. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed (iid) random variables with the common distribution function F. The sequence of (upper) *ordinary record times*, $\{T_m\}_{m\geq 0}$, is defined as $T_m = \min\{j > T_{m-1} : X_j > X_{T_{m-1}}\}$, $m \geq 1$ with $T_0 = 1$. The m-th *(ordinary) record* of the sequence is the random variable $R_m = X_{T_m}$, $m \geq 0$. For $n \geq 2$, the *record indicators* are the binary random variables

$$I_n = \begin{cases} 1, & \text{if } X_n > \max\{X_1, \dots, X_{n-1}\} \\ 0, & \text{otherwise} \end{cases}$$

and $I_1 = 1$ with probability one. In the case in which F is continuous, the basic distribution theory of the statistics above defined is very well-known; see, for instance, chapter 2 of [4]. In the continuous case, the distributions of the record indicators, the record times and the inter-record times do not depend on the parent distribution, and we have results like the following (for further references, see [5]):

- The record indicators are independent Bernoulli random variables with $\mathbb{P}(I_n = 1) = 1/n$, $n \geq 1$. Then
$$\mathbb{P}(I_1 = 1,\, I_2 = 1, \ldots, I_n = 1) = \frac{1}{n!}. \tag{1}$$
- The joint probability mass function (pmf) of the first m (non-trivial) record times is
$$\mathbb{P}(T_1 = t_1, T_2 = t_2, \ldots, T_m = t_m) = \frac{1}{(t_1 - 1)(t_2 - 1)\cdots(t_m - 1)t_m}, \tag{2}$$
for integers $1 < t_1 < t_2 < \cdots < t_m$; also
$$\mathbb{P}(T_m = n) = \frac{|s_{n-1,m}|}{n!}, \tag{3}$$
with $s_{n,k}$ being the Stirling numbers of the first kind defined by the equality
$$x(x-1)\cdots(x-n+1) = \sum_{k\geq 0} s_{n,k} x^k.$$
- The pmf of the inter-record times, $S_m = T_m - T_{m-1}$, $m \geq 1$ satisfies (see [6])
$$\mathbb{P}(S_m > s) = \frac{1}{(m-1)!}\int_0^\infty x^{m-1}e^{-x}(1-e^{-x})^s\,dx, \quad s \geq 0,$$
so then, for $s \geq 1$,
$$\begin{aligned}\mathbb{P}(S_m = s) = \mathbb{P}(S_m > s-1) - \mathbb{P}(S_m > s) &= \frac{1}{(m-1)!}\int_0^\infty x^{m-1}e^{-2x}(1-e^{-x})^{s-1}\,dx \\ &= \frac{1}{(m-1)!}\sum_{\ell=0}^{s-1}\binom{s-1}{\ell}(-1)^\ell\int_0^\infty x^{m-1}e^{-(\ell+2)x}\,dx = \sum_{\ell=0}^{s-1}\binom{s-1}{\ell}\frac{(-1)^\ell}{(\ell+2)^m}. \end{aligned} \tag{4}$$
- The number of records in a sequence of length n, denoted N_n, has pmf
$$\mathbb{P}(N_n = m) = \frac{|s_{n,m}|}{n!}, \; m = 1, \ldots, n,$$
and its expected value is $\mathbb{E}N_n = \sum_{j=1}^n \frac{1}{j}$.

If the parent distribution, F, is discrete, the previous results are no longer true, and in general, the distribution of record times, inter-record times, record indicators and number of records depend on F. Thus, the distribution theory for records from discrete parents is more complicated than in the continuous case. In fact, there exist in the literature two definitions of records, ordinary and weak records, which are equivalent in the case of continuous parents, but not so in the discrete case. The *weak record time* sequence is defined as $\widetilde{T}_m = \min\{j > \widetilde{T}_{m-1} : X_j \geq X_{\widetilde{T}_{m-1}}\}$, $m \geq 1$, and $\widetilde{T}_0 = 1$. Then, the *mth weak record* of the sequence $\{X_n\}_{n\geq 1}$ (or from the parent distribution F) is the random variable $\widetilde{R}_m = X_{\widetilde{T}_m}$, $m \geq 0$. Note that the difference with ordinary records is that ties with the previous records are also considered as records. Obviously, for continuous distributions, weak records coincide (almost surely) with ordinary records. In the case of discrete distributions, ordinary and weak records lead to different definitions.

We say that a random variable, X, follows a geometric distribution with parameter $p \in (0,1)$ if its pmf is $\mathbb{P}(X = x) = q^x p$ for $x = 0, 1, \ldots$ and $q = 1 - p$. We denote $X \sim \mathrm{Ge}(p)$.

The number of records in geometrically distributed sequences has been studied in connection with a data structure called skiplist; see [7,8] or [9]. Related studies can be found in [10], which established the asymptotic normality of the number of records from geometric parents and also studied large deviations, local limit theorems and approximations. Some of these results have been extended to other discrete parents, so [11] established a CLT for the number of records from discrete distributions; rates of growth for the number of records and laws of large numbers can be found in [12,13].

Most of the results above deal with limit properties, and not very much attention has been paid to the exact distribution of statistics related to the number of records. Our main contribution in this paper is the use of techniques of q-calculus for the study of the distribution theory of records from geometric parents.

In Section 2, we present the Jackson integral or q-integral and the extension of this concept to multiple q-integration in a convex polytope; see [14]. We also present in this section some useful lemmas for the calculation of multiple q-integrals (proofs are deferred to the Appendix A) and the non-central q-Stirling numbers; see [15]. No previous knowledge of q-calculus is necessary to understand this section. In Section 3, we study some statistics related to the sequence of weak records from a sequence of geometrically distributed random variables. Theorem 1 is a central result in this section. It gives a multiple q-integral representation of the joint probability function of the (weak) record times. As a consequence of this result, we show that the distribution of record times and that of the number of records in a sample of size n can be written in terms of the central and non-central q-Stirling numbers. We also study the distribution of inter-(weak) record times and we prove its asymptotic log-normal behavior. In Section 4, it becomes clear that the results obtained for a geometric parent, $\mathrm{Ge}(p)$, are q-analogues of the corresponding results in the case of an absolutely continuous distribution, so we can say that the absolutely continuous case is the limit case of geometric parents when $q = 1 - p \to 1$. Finally, in Section 5, we consider the case of ordinary records. We show that there is the following duality: the formulas in the case of ordinary records can be obtained from the corresponding ones in the case of weak records by replacing $q = 1 - p$ with $Q := 1/q$.

Throughout this paper, we will use the usual conventions with sums and products: for $a > b$, $\sum_a^b \cdot = 0$ and $\prod_a^b \cdot = 1$, and more generally, the summation (product) over an empty set of indexes is equal to zero (one). Additionally, $0^0 = 1$.

2. The Jackson Integral and q-Analogues

In the following, we present some basic concepts of q-calculus. The interested reader can find more details in [16], or, for a deeper treatment of the subject, see [2]. Regardless, this section contains all the results of q-calculus that we need in this paper.

The most basic concept in q-calculus is perhaps that of the q-number. Let $q \in \mathbb{R} - \{0\}$. The q-numbers are defined as

$$\{x\}_q := \begin{cases} (1-q^x)/(1-q), & q \neq 1 \\ x, & q = 1 \end{cases}$$

for $x \in \mathbb{R}$. For a positive integer, m, its q-factorial is $\{m\}_q! = \prod_{j=1}^m \{j\}_q$ and $\{0\}_q! = 1$. Throughout this paper, $Q := 1/q$. Thus, for instance, for $x \in \mathbb{R}$, we have $\{x\}_Q = \{x\}_q / q^{x-1}$.

Given a function f defined on the non-negative semi-axis of the real line, a, $b > 0$ and $q \in (0,1)$, the Jackson integral or q-integral of f in the interval (a,b) is defined as

$$\int_a^b f(y) d_q y := (1-q) \sum_{\ell=0}^{\infty} \Big(f(q^\ell b) b q^\ell - f(q^\ell a) a q^\ell \Big) \tag{5}$$

provided that the series in the right hand side (rhs) of (5) is absolutely convergent.

The Jackson integral has some resemblances to the Riemann integral; for instance, for $\alpha > -1$,

$$\int_0^x y^\alpha d_q y = \frac{x^{\alpha+1}}{\{\alpha+1\}_q}. \tag{6}$$

Among the differences with respect to the ordinary Riemann integrals, we mention that the formula of the change of variables is no longer valid, so, for natural α,

$$\frac{1}{\{\alpha+1\}_q} \neq \int_0^1 (1-y)^\alpha d_q y = \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \frac{1}{\{j+1\}_q}.$$

The definition of a q-integral can be extended to a multiple q-integral in a convex polytope; see [14]. Let us denote as $\mathcal{S}_m$ the set of permutations of $\{1,\ldots,m\}$. Let $\pi = (\pi_1 \cdots \pi_m) \in \mathcal{S}_m$. For a function of m variables, $f(y_1, \ldots, y_m)$ and a convex polytope $C \subset \mathbb{R}^m$, the q-integral of f over C with respect to the order π of integration is defined as

$$\int_C f(y_1,\ldots,y_m) d_q y_{\pi_1} \cdots d_q y_{\pi_m} = \int_{\min(C_m)}^{\max(C_m)} \left(\cdots \left(\int_{\min(C_1)}^{\max(C_1)} f(y_1,\ldots,y_m) d_q y_{\pi_1} \right) \cdots \right) d_q y_{\pi_m} \quad (7)$$

where C_i, $i = 1,\ldots,m-1$ is the set of real numbers depending on the values of $y_{\pi_{i+1}},\ldots,y_{\pi_m}$ with

$$C_i := C_i(y_{\pi_{i+1}},\ldots,y_{\pi_m}) = \{x_{\pi_i} : (x_1,\ldots,x_m) \in C,\ x_{\pi_j} = y_{\pi_j},\ \text{for } j > i\},$$

and $C_m := \{x_{\pi_m} : \exists (x_1,\ldots,x_m) \in C\}$ (implicitly, hereafter we assume the absolute convergence of the involved series).

It follows from the definition that when the polytope C is a hyper-rectangle, i.e., $C = [a_1, b_1] \times \cdots \times [a_m, b_m]$, the ordering π is irrelevant; that is to say, we can change the order of integration, and so, for any $\pi \in \mathcal{S}_m$,

$$\int_{[0,1]^m} f(y_1,\ldots,y_m) d_q y_{\pi_1} \cdots d_q y_{\pi_m} := (1-q)^m \sum_{\ell_1,\ldots,\ell_m \geq 0} f(q^{\ell_1},\ldots,q^{\ell_m}) q^{\sum_{i=1}^m \ell_i}, \quad (8)$$

for f, a function in m variables, such that the series in the rhs of (8) is absolutely convergent. Thus, for example,

$$\int_{[0,1]^m} \prod_{j=1}^m y_j^{\alpha_j} d_q y_{\pi_1} \cdots d_q y_{\pi_m} = \int_{[0,1]^m} \prod_{j=1}^m y_j^{\alpha_j} d_q y_1 \cdots d_q y_m = \prod_{j=1}^m \frac{1}{\left\{\alpha_j + 1\right\}_q} \quad (9)$$

for any $\pi \in \mathcal{S}_m$ and $\alpha_j > -1$, $j = 1,\ldots,m$.

The next Lemmas are useful in connection with iterated and multiple Jackson integrals.

Lemma 1. *Let f be a real function of m variables such that the series appearing below are absolutely convergent. Then, for any $\pi \in \mathcal{S}_m$,*

(a)
$$(1-q)^m \sum_{0 \leq x_1 \leq \cdots \leq x_m < \infty} f(q^{x_1}, q^{x_2}, \ldots, q^{x_m})\, q^{\sum_{i=1}^m x_i}$$
$$= \int_{[0,1]^m} f(y_1, y_1 y_2, \ldots, y_1 \cdots y_m) \prod_{j=1}^m y_j^{m-j} d_q y_{\pi_1} \cdots d_q y_{\pi_m}.$$

(b)
$$(1-q)^m \sum_{0 \leq x_1 < \cdots < x_m < \infty} f(q^{x_1}, q^{x_2}, \ldots, q^{x_m})\, q^{\sum_{i=1}^m x_i}$$
$$= q^{\binom{m}{2}} \int_{[0,1]^m} f(y_1, q y_1 y_2, \ldots, q^{m-1} y_1 \cdots y_m) \prod_{j=1}^m y_j^{m-j} d_q y_{\pi_1} \cdots d_q y_{\pi_m}.$$

In general, when the polytope of integration is not a hyper-rectangle, the order of integration is important. We illustrate this fact with the following example:

$$\int_{0 \leq y_2 \leq y_1 \leq 1} d_q y_2\, d_q y_1 = \int_0^1 \left(\int_0^{y_1} d_q y_2 \right) d_q y_1 = \int_0^1 y_1\, d_q y_1 = \frac{1}{\{2\}_q},$$

but,

$$\int_{0 \leq y_2 \leq y_1 \leq 1} d_q y_1\, d_q y_2 = \int_0^1 \left(\int_{y_2}^1 d_q y_1 \right) d_q y_2 = \int_0^1 (1-y_2)\, d_q y_2 = 1 - \frac{1}{\{2\}_q}.$$

Throughout this paper we will use the reverse ordering, $d_q\mathbf{y}^{(m)} := d_q y_m d_q y_{m-1} \cdots d_q y_1$. For $q \in (0,1]$, we define the polytopes

$$D_{m,q} := \{(y_1, \ldots, y_m) \in \mathbb{R}^n : 0 \le y_1 \le 1;\ 0 \le y_j \le q y_{j-1},\ j = 2, \ldots, m\},$$
$$D_m := D_{m,1} = \{(y_1, \ldots, y_m) \in \mathbb{R}^n : 0 \le y_m \le y_{m-1} \le \cdots \le y_1 \le 1\},$$

and from the definition of a multiple q-integral given in (7),

$$\int_{D_m} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)} = \int_0^1 \int_0^{y_1} \cdots \int_0^{y_{m-1}} f(y_1, \ldots, y_m) d_q\mathbf{y}^{(m)}, \tag{10}$$
$$\int_{D_{m,q}} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)} = \int_0^1 \int_0^{qy_1} \cdots \int_0^{qy_{m-1}} f(y_1, \ldots, y_m) d_q\mathbf{y}^{(m)}. \tag{11}$$

Lemma 2. *Let f be a real function of m variables such that the series appearing below are absolutely convergent. Then,*

(a) $$(1-q)^m \sum_{0 \le x_1 \le \cdots \le x_m < \infty} f(q^{x_1}, \ldots, q^{x_m})\, q^{\sum_{j=1}^m x_j} = \int_{D_m} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)}.$$

(b) $$(1-q)^m \sum_{0 \le x_1 < \cdots < x_m < \infty} f(q^{x_1}, \ldots, q^{x_m})\, q^{\sum_{j=1}^m x_j} = \int_{D_{m,q}} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)}.$$

Comparing the results in Lemmas 1 and 2, we obtain

$$\int_{D_m} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)} = \int_{[0,1]^m} f(y_1, y_1y_2, \ldots, y_1 \cdots y_m) \prod_{j=1}^m y_j^{m-j}\, d_q\mathbf{y}^{(m)} \tag{12}$$
$$\int_{D_{m,q}} f(y_1, \ldots, y_m)\, d_q\mathbf{y}^{(m)} = q^{\binom{m}{2}} \int_{[0,1]^m} f(y_1, qy_1y_2, \ldots, q^{m-1}y_1 \cdots y_m) \prod_{j=1}^m y_j^{m-j}\, d_q\mathbf{y}^{(m)}, \tag{13}$$

and using these results in combination with (6), it can be confirmed that for $m \ge 1$ and $\beta_1, \ldots, \beta_m$, real numbers such that $\beta_j + \cdots + \beta_m + m - j > -1$, for $j = 1, \ldots, m$,

$$\int_{D_m} \prod_{j=1}^m y_j^{\beta_j} d_q\mathbf{y}^{(m)} = \prod_{j=1}^m \frac{1}{\left\{\beta_j + \cdots + \beta_m + m - j + 1\right\}_q}, \tag{14}$$

$$\int_{D_{m,q}} \prod_{j=1}^m y_j^{\beta_j} d_q\mathbf{y}^{(m)} = q^{\binom{m}{2}} \prod_{j=1}^m \frac{q^{(j-1)\beta_j}}{\left\{\beta_j + \cdots + \beta_m + m - j + 1\right\}_q}$$
$$= Q^{\beta_1 + \cdots + \beta_m} \prod_{j=1}^m \frac{1}{\left\{\beta_j + \cdots + \beta_m + m - j + 1\right\}_Q}, \tag{15}$$

and in particular, for $\beta_1 = \cdots = \beta_m = 0$,

$$\int_{D_m} d_q\mathbf{y}^{(m)} = \frac{1}{\{m\}_q!}, \tag{16}$$
$$\int_{D_{m,q}} d_q\mathbf{y}^{(m)} = \frac{1}{\{m\}_Q!}, \tag{17}$$

and for $\beta_1 = \cdots = \beta_{m-1} = -1$ and $\beta_m = b > -1$,

$$\int_{D_m} \frac{y_m^b}{\prod_{j=1}^{m-1} y_j} d_q\mathbf{y}^{(m)} = \frac{1}{\{b+1\}_q^m}, \tag{18}$$
$$\int_{D_{m,q}} \frac{y_m^b}{\prod_{j=1}^{m-1} y_j} d_q\mathbf{y}^{(m)} = \frac{Q^{b-m+1}}{\{b+1\}_Q^m}. \tag{19}$$

Given a (formal) power series $A(z) = \sum_{k\geq 0} a_k z^k$, we denote $[z^k]A(z) = a_k$, $k \geq 0$. For non-negative integers, n and r, the non-central q-Stirling numbers of the first kind (see [15]) are defined as

$$s_q(n,k;r) = [z^k] \prod_{\ell=r}^{n+r-1} (z - \{\ell\}_q), \;\; k = 0, \ldots, n,$$

and by convention $s_q(n,k;r) = 0$ if $k < 0$ or $k > n$ or $n < 0$. The case $r = 0$ corresponds to the q-Stirling numbers of the first kind; see [2], Ch. 5. We denote $s_q(n,k) := s_q(n,k;0)$. The non-central q-Stirling numbers satisfy the recurrence

$$\begin{aligned} &s_q(n+1,k;r) = s_q(n,k-1;r) - \{r+n\}_q s_q(n,k;r), \;\; n \geq 0; \; k = 0,\ldots,n+1, \\ &s_q(0,0;r) = 1. \end{aligned}$$

Some useful relationships for our purposes are the following:

$$|s_q(n,k;r)| = (-1)^{n-k} s_q(n,k;r), \tag{20}$$

$$s_q(k,k;r) = 1, \; k \geq 0, \tag{21}$$

$$s_q(n,k;1) = s_q(n-1,k;2) - s_q(n-1,k;2), \tag{22}$$

$$s_q(n,k) = s_q(n-1,k-1;1). \tag{23}$$

By equating coefficients of z^ℓ in both sides of

$$\left(1 + \frac{z}{\{r\}_q}\right) \cdots \left(1 + \frac{z}{\{n+r-1\}_q}\right) = \frac{\{r-1\}_q!}{\{n+r-1\}_q!} (z + \{r\}_q) \cdots (z + \{n+r-1\}_q)$$

we obtain

$$\sum_{r \leq c_1 < \cdots < c_\ell \leq n+r-1} \prod \frac{1}{\left\{c_j\right\}_q} = \frac{(-1)^{n-\ell} \{r-1\}_q! \, s_q(n,\ell;r)}{\{n+r-1\}_q!}. \tag{24}$$

Lemma 3. *For the integers $m \geq 1$, $k \geq 0$ and $q \in (0,1)$,*

(a) $$\sum_{\substack{\alpha_1,\ldots,\alpha_m \geq 0 \\ \alpha_1 + \cdots \alpha_m = k}} \int_{D_m} \prod_{j=1}^m y_j^{\alpha_j} d_q \mathbf{y}^{(m)} = \frac{(-1)^k s_q(m+k-1, m-1; 2)}{\{m+k+1\}_q!}.$$

(b) $$\sum_{\substack{\alpha_1,\ldots,\alpha_m \geq 0 \\ \alpha_1 + \cdots \alpha_m = k}} \int_{D_{m,q}} \prod_{j=1}^m y_j^{\alpha_j} d_q \mathbf{y}^{(m)} = \frac{(-1)^k s_Q(m+k-1, m-1; 2)}{\{m+k+1\}_Q!}, \text{ where } Q := 1/q.$$

Lemma 4. *For $m > 1$ and f, a non-negative function defined in the non-negative real numbers,*

(a) $$\sum_{s_1,\ldots,s_{m-1} \geq 1} \int_{D_m} f(y_m) \prod_{j=1}^{m-1} (1 - y_j)^{s_j - 1} d_q \mathbf{y}^{(m)} = \int_{D_m} \frac{f(y_m)}{y_1 \cdots y_{m-1}} d_q \mathbf{y}^{(m)}.$$

(b) $$\sum_{s_1,\ldots,s_{m-1} \geq 1} \int_{D_{m,q}} f(y_m) \prod_{j=1}^{m-1} (1 - qy_j)^{s_j - 1} d_q \mathbf{y}^{(m)} = \frac{1}{q^{m-1}} \int_{D_{m,q}} \frac{f(y_m)}{y_1 \cdots y_{m-1}} d_q \mathbf{y}^{(m)}.$$

Many variants of Lemma 4 (all of them with similar proofs) are possible; for instance, for a non-negative function f_i, $i = 1, \ldots, m$,

$$\sum_{s \geq 1} \int_{D_m} f_1(y_1) \cdots f_m(y_m) (1 - y_m)^{s-1} d_q \mathbf{y}^{(m)} = \int_{D_m} \frac{f_1(y_1) \cdots f_m(y_m)}{y_m} d_q \mathbf{y}^{(m)}. \tag{25}$$

Note that results like (25) or those in Lemma 4 justify the formal interchange of the summation and integration symbols.

Suppose that $A(q)$ is an expression involving the parameter q. Then, it is said that $A(q)$ is a q-analogue of A if $A(q) \to A$ when $q \to 1$. q-analogues appear in different areas of mathematics and physics, such as, for instance, in quantum groups, string theory, fractals, dynamical systems, elliptic curves, etc. It is clear that q-numbers are q-analogues of the corresponding real numbers. Expression (6) is a q-analogue of the well-known formula for Riemann integrals $\int_0^x y^\alpha \, dy = x^{\alpha+1}/(\alpha + 1)$, $\alpha > -1$. Similarly, (9), (14)–(17) are q-analogues of the corresponding expressions obtained by substituting q-integrals for Riemann integrals and q-numbers for real numbers. Then, the following lemma is immediate.

Lemma 5. *If $R(y_1, \ldots, y_m)$ is a polynomial with real coefficients in the variables $y_1, \ldots, y_m$, then the multiple q-integral*

$$\int_{[0,1]^m} R(y_1, \ldots, y_m) d_q \mathbf{y}^{(m)} \quad \left(or \int_{D_m} R(y_1, \ldots, y_m) d_q \mathbf{y}^{(m)} \right)$$

is a q-analogue of the multiple Riemann integral

$$\int_{[0,1]^m} R(y_1, \ldots, y_m) dy_m\, dy_{m-1} \ldots dy_1 \quad \left(or \int_{D_m} R(y_1, \ldots, y_m) dy_m\, dy_{m-1} \ldots dy_1 \right).$$

Finally, we show a connection between multiple q-integrals and expected values involving geometric samples. Let $X_i \sim \mathrm{Ge}(p)$, $i \geq 1$ be independent random variables with geometric distributions of the parameter $p \in (0,1)$ (i.e., $\mathbb{P}(X_i = k) = (1-p)^k p$, $k = 0, 1, \ldots$). Let $q = 1 - p$. Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a function such that $\mathbb{E}\left| f(q^{X_1}, \ldots, q^{X_m}) \right| < +\infty$, and thus, from (8) we get

$$\mathbb{E}\Big(f(q^{X_1}, \ldots, q^{X_m}) \Big) = \int_{[0,1]^m} f(y_1, \ldots, y_m) d_q \mathbf{y}^{(m)}. \tag{26}$$

3. Weak Records from Geometric Parent

Let $\mathbf{X} = \{X_n\}_{n \geq 1}$ be a sequence of iid random variables with a common distribution $\mathrm{Ge}(p)$ with $p \in (0,1)$ and $q = 1 - p$. The index n in the process $\mathbf{X} = \{X_n\}_{n \geq 1}$ can be regarded as discrete time. A trajectory of the process $\mathbf{X}$ is a sequence of non-negative integers. A trajectory up to time n is a finite sequence of length n of non-negative integers. We will denote a finite trajectory of length $n \geq 1$ by a sequence $(k_1, \ldots, k_n)$ where k_i, $i \geq 1$ are non-negative integers. Consecutive equal integers in a trajectory are denoted with exponents; for instance, $(k)^t$ is the sequence $(k, \ldots, k)$ of length t. A sequence of length 1 formed with a non-negative integer less than k is denoted by $(<k)$. Similar notations are used in the cases $\leq, >, \geq$, etc. As an example of these notations, consider the trajectory (3, 1, 1, 5, 4, 2, 3, 1, 8). Whenever it is convenient, this trajectory could be denoted as: $(3)(1)^2(\leq 5)^5(8)$ or $(\leq 3)^3(\geq 3)^2(\leq 8)^4$, or $(3)(\geq 1)^8$,... etc.

Theorem 1. *For $m \geq 1$,*

(a) *For $1 = t_0 < t_1 < \cdots < t_m$, the joint pmf of $(\widetilde{T}_1, \ldots, \widetilde{T}_m)$ from $\mathbf{X}$ admits the following q-integral representation:*

$$\mathbb{P}\Big(\widetilde{T}_1 = t_1, \ldots, \widetilde{T}_m = t_m\Big) = \int_{D_m} y_m \prod_{j=1}^{m} (1 - y_j)^{t_j - t_{j-1} - 1} d_q \mathbf{y}^{(m)}. \tag{27}$$

(b) *The joint probability generating function of $(\widetilde{T}_1, \ldots, \widetilde{T}_m)$ is*

$$G(z_1, \ldots, z_m) = \int_{D_m} y_m \prod_{j=1}^{m} \frac{z_j^{j+1}}{1 - (1 - y_j) z_j \cdots z_m} d_q \mathbf{y}^{(m)},$$

for $|z_j| \leq 1$, $j = 1, \ldots, m$.

Proof. (a) For the integers $1 = t_0 < t_1 < \cdots < t_m$, $m \geq 1$, the event $\{\widetilde{T}_1 = t_1, \ldots, \widetilde{T}_m = t_m\}$ corresponds to finite trajectories of length $t_m \geq m + 1$ of the process $\mathbf{X}$, which are of the form:

$$(x_1)(< x_1)^{t_1 - 2}(x_2)(< x_2)^{t_2 - t_1 - 1} \cdots (x_m)(< x_m)^{t_m - t_{m-1} - 1}(x_{m+1}), \tag{28}$$

with $0 \le x_1 \le x_2 \le \cdots \le x_{m+1}$. The probability associated with the trajectory given in (28) is $\left(\prod_{j=1}^m q^{x_j} p(1-q^{x_j})^{t_j - t_{j-1}-1}\right) q^{x_{m+1}} p$. Then, using Lemma 2(a),

$$
\begin{aligned}
\mathbb{P}\left(\widetilde{T}_1 = t_1, \ldots, \widetilde{T}_m = t_m\right) &= \sum_{0 \le x_1 \le x_2 \le \cdots \le x_{m+1} < \infty} \left(\prod_{j=1}^m q^{x_j} p(1-q^{x_j})^{t_j - t_{j-1}-1}\right) q^{x_{m+1}} p \\
&= \int_{D_{m+1}} \prod_{j=1}^m (1-y_j)^{t_j - t_{j-1}-1} d_q \mathbf{y}^{(m+1)} = \int_{D_m} \prod_{j=1}^m (1-y_j)^{t_j - t_{j-1}-1} \left(\int_0^{y_m} d_q y_{m+1}\right) d_q \mathbf{y}^m \\
&= \int_{D_m} y_m \prod_{j=1}^m (1-y_j)^{t_j - t_{j-1}-1} d_q \mathbf{y}^{(m)}.
\end{aligned}
$$

(b) Firstly, observe that for the integers $s_1, \ldots, s_m$, and $m \ge 1$,

$$
\prod_{j=1}^m z_j^{s_1 + \cdots + s_j + 1} = z_1^2 z_2^3 \cdots z_m^{m+1} \prod_{j=1}^m \left(\prod_{l=j}^m z_\ell\right)^{s_j - 1} \tag{29}
$$

For $|z_j| \le 1$ and $s_j = t_j - t_{j-1}$, $j = 1, \ldots, m$, $(t_0 = 1)$. Using Theorem 1(a) and (29), we have

$$
\begin{aligned}
G(z_1, \ldots, z_m) &= \sum_{1 = t_0 < t_1 < \cdots < t_m < +\infty} P\left(\widetilde{T}_1 = t_1, \ldots, \widetilde{T}_m = t_m\right) \prod_{j=1}^m z_j^{t_j} \\
&= \sum_{s_1, \ldots, s_m \ge 0} P\left(\widetilde{T}_1 = s_1 + 1,\ \widetilde{T}_2 = s_1 + s_2 + 1, \ldots, \widetilde{T}_m = s_1 + \cdots + s_m + 1\right) \prod_{j=1}^m z_j^{s_1 + \cdots + s_j + 1} \\
&= z_1^2 z_2^3 \cdots z_m^{m+1} \sum_{s_1, \ldots, s_m \ge 0} \int_{D_m} y_m \prod_{j=1}^m \left((1-y_j) \prod_{\ell=j}^m z_\ell\right)^{s_j - 1} d_q \mathbf{y}^{(m)} \\
&= \left(\prod_{j=1}^m z_j^{j+1}\right) \int_{D_m} y_m \sum_{s_1, \ldots, s_m \ge 0} \prod_{j=1}^m \left((1-y_j) z_j \cdots z_m\right)^{s_j - 1} d_q \mathbf{y}^{(m)} \\
&= \int_{D_m} y_m \prod_{j=1}^m \frac{z_j^{j+1}}{1 - (1-y_j) z_j \cdots z_m} d_q \mathbf{y}^{(m)},
\end{aligned}
$$

where the interchange of the summation and integral symbols can be justified with a similar argument to the one in the proof of Lemma 4. □

Theorem 2. *For $m \ge 1$ and $n \ge m+1$,*

(a) $\displaystyle \mathbb{P}(\widetilde{T}_m = n) = \sum_{\ell = m-1}^{n-2} \binom{n-2}{\ell} \frac{s_q(\ell, m-1; 2)}{\{\ell + 2\}_q!}$

(b) $\displaystyle \mathbb{P}(\widetilde{T}_m \le n) = \sum_{\ell = m-1}^{n-2} \binom{n-1}{\ell + 1} \frac{s_q(\ell, m-1; 2)}{\{\ell + 2\}_q!}$

Proof. (a) From Theorem 1(b), the probability-generating function of $\widetilde{T}_m$ is

$$
G_{\widetilde{T}_m}(z) = G(1, \ldots, 1, z) = z^{m+1} \int_{D_m} y_m \prod_{j=1}^m \frac{1}{1 - (1-y_j) z} d_q \mathbf{y}^{(m)}, \quad |z| \le 1. \tag{30}
$$

If $|z| < 1/2$, then $|z/(1-z)| < 1$. Therefore, if $0 < y_j < 1$, $i = 1, \ldots, m$ then we have that $\left| y_j \frac{z}{1-z} \right| < 1$, and the following expansion holds:

$$\begin{aligned}\prod_{j=1}^{m}\frac{1}{1-(1-y_j)z} &= (1-z)^{-m}\prod_{j=1}^{m}\left(1+y_j\frac{z}{1-z}\right)^{-1}\\ &= (1-z)^{-m}\prod_{j=1}^{m}\sum_{\alpha_j\ge 0}(-1)^{\alpha_j}\left(y_j\frac{z}{1-z}\right)^{\alpha_j}\\ &= \sum_{k=0}^{\infty}(-1)^k\left(\sum_{\substack{\alpha_1,\dots,\alpha_m\ge 0\\ \alpha_1+\cdots\alpha_m=k}}\prod_{j=1}^{m}y_j^{\alpha_j}\right)z^k(1-z)^{-m-k}. \end{aligned}\tag{31}$$

On the other hand, it is not difficult to check that

$$[z^{n-m-1}]z^k(1-z)^{-m-k} = \binom{n-2}{m+k-1},\tag{32}$$

for $n \ge 2$, $m \ge 1$ and $0 \le k \le n-m-1$.
Thus, for $n \ge m+1$, using (30), (31), Lemma 3 and (32),

$$\begin{aligned}\mathbb{P}(\widetilde{T}_m = n) &= [z^n]G_{\widetilde{T}_m}(z) = [z^{n-m-1}]\int_{D_m} y_m\prod_{j=1}^{m}\frac{1}{1-(1-y_j)z}d_q\mathbf{y}^{(m)}\\ &= \sum_{k=0}^{\infty}(-1)^k\left(\sum_{\substack{\alpha_1,\dots,\alpha_m\ge 0\\ \alpha_1+\cdots\alpha_m=k}}\int_{D_m} y_m\prod_{j=1}^{m}y_j^{\alpha_j}d_q\mathbf{y}^{(m)}\right)[z^{n-m-1}]z^k(1-z)^{-m-k}\\ &= \sum_{k=0}^{n-m-1}\binom{n-2}{m+k-1}\frac{s_q(m+k-1,m-1;2)}{\{m+k+1\}_q!} = \sum_{\ell=m-1}^{n-2}\binom{n-2}{\ell}\frac{s_q(\ell,m-1;2)}{\{\ell+2\}_q!}.\end{aligned}$$

(b) For $n \ge m+1$, using Theorem 2(a),

$$\begin{aligned}\mathbb{P}(\widetilde{T}_m \le n) &= \sum_{k=m+1}^{n}\mathbb{P}(\widetilde{T}_m = k) = \sum_{k=m+1}^{n}\sum_{\ell=m-1}^{k-2}\binom{k-2}{\ell}\frac{s_q(\ell,m-1;2)}{\{\ell+2\}_q!}\\ &= \sum_{\ell=m-1}^{n-2}\left(\sum_{k=\ell+2}^{n}\binom{k-2}{\ell}\right)\frac{s_q(\ell,m-1;2)}{\{\ell+2\}_q!} = \sum_{\ell=m-1}^{n-2}\binom{n-1}{\ell+1}\frac{s_q(\ell,m-1;2)}{\{\ell+2\}_q!}.\end{aligned}$$

□

Some other statistics related to weak record indicators can be expressed in terms of q-integrals and q-numbers as shown below.

Theorem 3. *For $n \ge 2$,*

(a) $\mathbb{P}(\widetilde{I}_n = 1) = \dfrac{1}{q}\displaystyle\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_q}$.

(b) $\mathbb{P}(\widetilde{I}_2 = 1,\dots,\widetilde{I}_n = 1) = \dfrac{1}{\{n\}_q!}$.

(c) $\mathbb{P}(\widetilde{I}_2 = 0,\dots,\widetilde{I}_n = 0) = \displaystyle\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_q}$.

Proof. (a) For $n \ge 2$, firstly observe that

$$0 = (1-1)^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^\ell = \frac{1}{1-q}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^\ell\frac{1-q^{\ell+1}}{\{\ell+1\}_q}$$

Then,

$$\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^\ell\frac{1}{\{\ell+1\}_q} = q\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^\ell\frac{q^\ell}{\{\ell+1\}_q}.\tag{33}$$

The event $\{\widetilde{I}_n = 1\}$ consists of all the trajectories of the process **X** with a weak record at position n, i.e., trajectories of the form $(\leq x)^{n-1}x\cdots$; then, using the definition of the Jackson integral (see (5) and (33))

$$\mathbb{P}(\widetilde{I}_n = 1) = \sum_{x\geq 0}(1-q^{x+1})^{n-1}q^x p = \int_0^1 (1-qy)^{n-1}d_q y = \sum_{\ell=0}\binom{n-1}{\ell}(-1)^\ell q^\ell \int_0^1 y^\ell d_q y$$
$$= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^\ell \frac{q^\ell}{\{\ell+1\}_q} = \frac{1}{q}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_q}.$$

(b) The trajectories corresponding to the event $\{\widetilde{I}_1 = 1, \widetilde{I}_2 = 1, \ldots, \widetilde{I}_n = 1\}$ are of the form $(x_1)(x_2)\cdots(x_n)\cdots$, with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n < +\infty$. Then, using Lemma 2 and (16),

$$\mathbb{P}(\widetilde{I}_1 = 1, \widetilde{I}_2 = 1, \ldots, \widetilde{I}_n = 1) = \sum_{0\leq x_1\leq x_2\leq\cdots\leq x_n<+\infty}\prod_{j=1}^{n} q^{x_j}p = \int_{D_n} d_q\mathbf{y}^{(n)} = \frac{1}{\{n\}_q!}.$$

(c) The trajectories corresponding to the event $\{\widetilde{I}_2 = 0, \ldots, \widetilde{I}_n = 0\}$ are of the form $(x)(<x)^{n-1}\cdots$, with $1 \leq x < +\infty$. Then

$$\mathbb{P}(\widetilde{I}_2 = 0, \ldots, \widetilde{I}_n = 0) = \sum_{x\geq 1} q^x p(1-q^x)^{n-1} = q\sum_{\ell\geq 0} q^\ell p(1-q^{\ell+1})^{n-1}$$
$$= q\int_0^1 (1-qy)^{n-1}d_q y = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_q},$$

(see the proof of *(a)* for the last step). □

The weak record indicators from geometric parents, as opposed to the continuous case, are not independentl for instance,

$$P(\widetilde{I}_2 = 1, \widetilde{I}_3 = 1) = \frac{1}{(1+q)(1+q+q^2)} \neq \frac{1}{1+q} \times \frac{1+q^2}{(1+q)(1+q+q^2)} = P(\widetilde{I}_2 = 1) \times P(\widetilde{I}_3 = 1).$$

The number of weak records in a sample of size n is $\widetilde{N}_n = \sum_{m=1}^n \widetilde{I}_m$ (note that the 0-th record, $R_0 = X_1$, is counted).

Theorem 4. *For $n \geq 1$,*

(a) $$\mathbb{P}(\widetilde{N}_n \geq m) = \sum_{\ell=m-1}^{n-1}\binom{n-1}{\ell}\frac{s_q(\ell-1, m-2; 2)}{\{\ell+1\}_q!}, \ m = 2, \ldots, n.$$

(b) $$\mathbb{P}(\widetilde{N}_n = m) = \sum_{\ell=m}^{n}\binom{n-1}{\ell-1}\frac{s_q(\ell, m)}{\{\ell\}_q!}, \ m = 1, \ldots, n.$$

(c) $$\mathbb{E}\widetilde{N}_n = \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}q^{\ell-1}}{\{\ell\}_q}.$$

Proof. (a) Observe that $\widetilde{N}_n \geq m$ if $\widetilde{T}_{m-1} \leq n$, $m = 2, \ldots, n$. Then, the result is immediate using Theorem 2(b).

(b) For $m = 2, \ldots, n$, using (a) and the properties (22) and (23)

$$\mathbb{P}(\widetilde{N}_n = m) = \mathbb{P}(\widetilde{N}_n \geq m) - \mathbb{P}(\widetilde{N}_n \geq m+1)$$
$$= \sum_{j=m-1}^{n-1}\binom{n-1}{j}\frac{s_q(j-1, m-2; 2)}{\{j+1\}_q!} - \sum_{j=m}^{n-1}\binom{n-1}{j}\frac{s_q(j-1, m-1; 2)}{\{j+1\}_q!}$$
$$= \binom{n-1}{m-1}\frac{s_q(m-2, m-2; 2)}{\{m\}_q!} + \sum_{j=m}^{n-1}\binom{n-1}{j}\frac{s_q(j-1, m-2; 2) - s_q(j-1, m-1; 2)}{\{j+1\}_q!}$$
$$= \binom{n-1}{m-1}\frac{s_q(m-1, m-1; 1)}{\{m\}_q!} + \sum_{j=m}^{n-1}\binom{n-1}{j}\frac{s_q(j, m-1; 1)}{\{j+1\}_q!} =$$
$$= \sum_{j=m-1}^{n-1}\binom{n-1}{j}\frac{s_q(j, m-1; 1)}{\{j+1\}_q!} = \sum_{\ell=m}^{n}\binom{n-1}{\ell-1}\frac{s_q(\ell, m)}{\{\ell\}_q!}.$$

For $m = 1$, note that if in a sample of size n, there is only one weak record. Then, it is in the first position, so, using Theorem 3(c),

$$\mathbb{P}(\widetilde{N}_n = 1) = \mathbb{P}(\widetilde{I}_2 = 0, \ldots, \widetilde{I}_n = 0) = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{\{j+1\}_q} = \sum_{\ell=1}^{n} \binom{n}{\ell-1} \frac{s_q(\ell,1)}{\{\ell\}_q!},$$

where in the last equality we have used the property $s_q(\ell,1) = (-1)^{\ell-1}\{\ell-1\}_q!$.

(c) In the proof of Theorem 3(a), we obtained the integral representation

$$\mathbb{P}(\widetilde{I}_m = 1) = \int_0^1 (1-qy)^{m-1} d_q y, \quad m \geq 2.$$

Clearly, this formula is also valid for $m = 1$. Therefore,

$$\begin{aligned}\mathbb{E}\widetilde{N}_n &= \sum_{m=1}^{n} \mathbb{E}\widetilde{I}_m = \sum_{m=1}^{n} \mathbb{P}(\widetilde{I}_m = 1) = \int_0^1 \sum_{m=1}^{n} (1-qy)^{m-1} d_q y = \int_0^1 \frac{1-(1-qy)^n}{qy} d_q y \\ &= \int_0^1 \sum_{\ell=1}^{n} \binom{n}{\ell} (-1)^{\ell-1} (qy)^{\ell-1} d_q y = \sum_{\ell=1}^{n} \binom{n}{\ell} \frac{(-1)^{\ell-1} q^{\ell-1}}{\{\ell\}_q}.\end{aligned}$$

□

The inter-(weak)-record times are defined as $\widetilde{S}_m = \widetilde{T}_m - \widetilde{T}_{m-1}$, $m \geq 1$.

Theorem 5. *For $m \geq 1$,*

(a) The joint pmf of the inter-record times from a Ge(p) parent is

$$\mathbb{P}\Big(\widetilde{S}_1 = s_1, \widetilde{S}_2 = s_2, \ldots, \widetilde{S}_m = s_m\Big) = \int_{D_m} y_m \prod_{j=1}^{m} (1-y_j)^{s_j-1} d_q \mathbf{y}^{(m)} \tag{34}$$

for $s_j \geq 1$, $j = 1, \ldots, m$.

(b) The pmf of the m-th inter-weak record is

$$\mathbb{P}(\widetilde{S}_m = s) = \sum_{\ell=0}^{s-1} \binom{s-1}{\ell} \frac{(-1)^\ell}{\{\ell+2\}_q^m}, \; s \geq 1. \tag{35}$$

(c) $\mathbb{P}(\widetilde{S}_m > s) = \int_{[0,1]^m} (1 - y_1 \cdots y_m)^s \, d_q \mathbf{y}^{(m)}, s \geq 1$

Proof. (a) It is straightforward from Theorem 1(a).
(b) For $s \geq 1$ (integer), from (34), using Lemma 4(a),

$$\begin{aligned}\mathbb{P}(\widetilde{S}_m = s) &= \sum_{s_1,\ldots,s_{m-1} \geq 1} \mathbb{P}\Big(\widetilde{S}_1 = s_1, \widetilde{S}_2 = s_2, \ldots, \widetilde{S}_m = s\Big) \\ &= \sum_{s_1,\ldots,s_{m-1} \geq 1} \int_{D_m} y_m (1-y_m)^{s-1} \prod_{j=1}^{m-1} (1-y_j)^{s_j-1} d_q \mathbf{y}^{(m)} \\ &= \int_{D_m} \frac{y_m (1-y_m)^{s-1}}{y_1 \cdots y_{m-1}} d_q \mathbf{y}^{(m)} \\ &= \sum_{\ell=0}^{s-1} \binom{s-1}{\ell} (-1)^\ell \int_{D_m} y_1^{-1} \cdots y_{m-1}^{-1} y_m^{\ell+1} \, d_q \mathbf{y}^{(m)} = \sum_{\ell=0}^{s-1} \binom{s-1}{\ell} \frac{(-1)^\ell}{\{\ell+2\}_q^m},\end{aligned} \tag{36}$$

where the last equality follows from (18).
(c) Using (36) and (12), for $s \geq 1$,

$$\begin{aligned}\mathbb{P}(\widetilde{S}_m > s) &= \sum_{t>s} \mathbb{P}(\widetilde{S}_m = t) = \sum_{t>s} \int_{D_m} \frac{y_m (1-y_m)^{t-1}}{y_1 \cdots y_{m-1}} d_q \mathbf{y}^{(m)} \\ &= \int_{D_m} \frac{(1-y_m)^s}{y_1 \cdots y_{m-1}} d_q \mathbf{y}^{(m)} = \int_{[0,1]^m} (1 - y_1 \cdots y_m)^s \, d_q \mathbf{y}^{(m)}.\end{aligned}$$

□

The next result proves the asymptotic log-normality of the inter-(weak) record times from geometric parents.

Theorem 6. *The inter-weak record times from a* $Ge(p)$ *parent,* $p \in (0,1)$*, satisfy*

$$\frac{\log \widetilde{S}_m + \frac{mq}{p}\log q}{-\frac{\sqrt{mq}}{p}\log q} \xrightarrow{d} Z \sim N(0,1), \quad as\ m \to \infty. \tag{37}$$

Proof. Let $X_i \sim \mathrm{Ge}(p)$, $i \geq 1$, be independent random variables. The CLT yields,

$$Z_m := \frac{\sum_{i=1}^{m} X_i - \frac{mq}{p}}{\frac{\sqrt{mq}}{p}} \xrightarrow{d} Z \sim \mathrm{N}(0,1), \quad \text{as } m \to \infty.$$

For $x \in \mathbb{R}$, let us define $v_m(x) := \frac{mq}{p} + x\frac{\sqrt{mq}}{p}$. From Theorem 5(c) and (26),

$$\mathbb{P}(\widetilde{S}_m > s) = \mathbb{E}\Big(1 - q^{\sum_{i=1}^m X_i}\Big)^s = \mathbb{E}\Big(1 - q^{v(Z_m)}\Big)^s$$

for $s = 0, 1, \ldots$.

Let us define the functions $g_m(z,t) := (1 - q^{v_m(z)})^{1/q^{v_m(t)}}$, $z, t \in \mathbb{R}$, $m \geq 1$. These functions satisfy the following properties:

(a) $$\lim_{m\to+\infty} g_m(z,t) = \begin{cases} 0, & \text{if } z < t \\ e^{-1}, & \text{if } z = t \\ 1, & \text{if } z > t \end{cases}$$

(b) For a fixed value of $t > -\sqrt{mq}$, the function $z \in (-\sqrt{mq}, +\infty) \mapsto g_m(z,t)$ is non-decreasing.

Then, for $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left(\frac{\log \widetilde{S}_m + \frac{mq}{p}\log q}{-\frac{\sqrt{mq}}{p}\log q} > t\right) &= \mathbb{P}\Big(\widetilde{S}_m > q^{-v_m(t)}\Big) = \mathbb{E}\Big(1 - q^{v_m(Z_m)}\Big)^{q^{-v_m(t)}} \\ &= \mathbb{E}(g_m(Z_m,\, t)) = \sum_{k\geq 0} g_m(z_{m,k}, t)\, \mathbb{P}(Z_m = z_{m,k}), \end{aligned}$$

where $z_{m,k} = (k - mq/p)/\big(\sqrt{mq}/p\big)$.

Let Φ be the cumulative distribution function of a standard normal random variable. If we show that

$$\lim_{m\to+\infty} \mathbb{E}|\mathbf{I}(Z_m > t) - g_m(Z_m,\, t)| = 0, \quad t \in \mathbb{R} \tag{38}$$

then we can conclude

$$\lim_{m\to+\infty} \mathbb{E}(g_m(Z_m,\, t)) = \lim_{m\to+\infty} P(Z_m > t) = 1 - \Phi(t), \quad t \in \mathbb{R},$$

and this proves (37).

In order to prove (38), for $\eta > 0$, we have

$$\begin{aligned} \mathbb{E}|\mathbf{I}(Z_m > t) - g_m(Z_m,\, t)| &= \sum_{k\geq 0} \big|\mathbf{I}(z_{m,k} > t) - g_m(z_{m,k}, t)\big|\, \mathbb{P}(Z_m = z_{m,k}) \\ &= \left(\sum_{\substack{k\geq 0 \\ z_{m,k} > t+\eta}} + \sum_{\substack{k\geq 0 \\ z_{m,k} < t-\eta}} + \sum_{\substack{k\geq 0 \\ |z_{m,k} - t| \leq \eta}}\right) \big|\mathbf{I}(z_{m,k} > t) - g_m(z_{m,k}, t)\big|\, \mathbb{P}(Z_m = z_{m,k}) \\ &= (I) + (II) + (III), \end{aligned}$$

where

$$\begin{aligned}(I) &= \sum_{\substack{k\geq 0\\ z_{m,k}>t+\eta}} (1-g_m(z_{m,k},t))\mathbb{P}(Z_m = z_{m,k}) \leq (1-g_m(t+\eta,t))\mathbb{P}(Z_m > t+\eta)\\ &\leq (1-g_m(t+\eta,t)) \longrightarrow 0, \text{ as } m\to+\infty;\end{aligned}$$

$$\begin{aligned}(II) &= \sum_{\substack{k\geq 0\\ z_{m,k}<t-\eta}} g_m(z_{m,k},t)\mathbb{P}(Z_m = z_{m,k}) \leq g_m(t-\eta,t)\mathbb{P}(Z_m < t-\eta)\\ &\leq g_m(t-\eta,t) \longrightarrow 0, \text{ as } m\to+\infty;\end{aligned}$$

$$\begin{aligned}(III) &= \sum_{\substack{k\geq 0\\ |z_{m,k}-t|\leq\eta}} \left|\mathbf{I}(z_{m,k}>t) - g_m(z_{m,k},t)\right| \mathbb{P}(Z_m = z_{m,k})\\ &\leq 2\mathbb{P}(t-\eta\leq Z_m\leq t+\eta) \overset{m\uparrow\infty}{\longrightarrow} 2(\Phi(t+\eta)-\Phi(t-\eta)),\end{aligned}$$

and this quantity can be made arbitrarily small choosing an $\eta > 0$ yet small enough. □

A by-product of Theorem 6 is the following weak law of large numbers:

$$\widetilde{S}_m^{1/m} \overset{\mathbb{P}}{\longrightarrow} q^{-\frac{q}{p}}, \quad m\uparrow\infty. \tag{39}$$

When $q\to 1$, $q^{-q/p}\to e$, and this is the limit in probability for $\Delta_m^{1/m}$ as $m\to+\infty$, where Δ_m denotes the waiting-time between the $(m-1)$-th and the m-th upper record in a sequence of iid random variables with a continuous distribution; see [6]. Then, (39) can be seen as a q-analogue of Theorem 1 in [6]. Similarly, our Theorem 6 is a q-analogue of Theorem 2 in [6].

4. Weak Records from Geometric Parents Are q-Analogues of Records from Absolutely Continuous Parents

In this section, we show that the distributions and statistics related to weak records from a geometric parent, $\mathrm{Ge}(p)$ are q-analogues ($q = 1-p$) of the corresponding distributions and statistics related to ordinary records from absolutely continuous parents.

Theorem 7. *The joint pmf of $\left(\widetilde{T}_1,\ldots,\widetilde{T}_m\right)$ from a Ge(p) parent given in (27) is a q-analogue of the joint pmf of $(T_1^*,\ldots,T_m^*)$ from an absolutely continuous parent given in (2).*

Proof. Without loss of generality, suppose that $(T_1^*,\ldots,T_m^*)$ are the record times of a sequence of iid random variables uniformly distributed in the interval $(0,1)$. For integers $t_0 = 1 < t_1 < \cdots < t_m$, the event $\{T_1^* = t_1,\ldots,T_m^* = t_m\}$ corresponds to trajectories of length $t_m \geq m+1$ of the form

$$(x_1)(<x_1)^{t_1-2}(x_2)(<x_2)^{t_2-t_1-1}\cdots(x_m)(<x_m)^{t_m-t_{m-1}-1}(x_{m+1}),$$

with $0 < x_1 < x_2 < \cdots < x_{m+1} < 1$. Then,

$$\begin{aligned}\mathbb{P}(T_1^* = t_1,\ldots,T_m^* = t_m) &= \int_0^1\int_{x_1}^1\cdots\int_{x_m}^1 \prod_{j=1}^m x_j^{t_j-t_{j-1}-1}\,dx_{m+1}\cdots dx_2\,dx_1\\ &= \int_0^1\int_{x_1}^1\cdots\int_{x_{m-1}}^1 (1-x_m)\prod_{j=1}^m x_j^{t_j-t_{j-1}-1}\,dx_m\cdots dx_2\,dx_1\\ &= \int_0^1\int_0^{y_1}\cdots\int_0^{y_{m-1}} y_m\prod_{j=1}^m (1-y_j)^{t_j-t_{j-1}-1}dy_m\cdots dy_2\,dy_1,\end{aligned} \tag{40}$$

where the last step is the result of the change in variables $y_j = 1-x_j$, $j = 1,\ldots,m$.

Finally, the result follows by comparing (40) with the q-integral expression given in (27) and by using Lemma 5. □

The addition of q-analogues is a q-analogue of the corresponding sum. As an application of this property, as a corollary of Theorem 7, we have that the pmf of $\widetilde{T}_m$ (given in Theorem 2) is a q-analogue of (3). Clearly, the pmf of inter-weak-record times from geometric parents (see (35)) is a q-analogue of (4). Similarly, as $q \to 1$, it can be checked that

$$\begin{aligned}
\mathbb{P}(\widetilde{I}_n = 1) &= \frac{1}{q}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_q} \longrightarrow \frac{1}{n} \\
\mathbb{P}(\widetilde{I}_2 = 1, \ldots, \widetilde{I}_n = 1) &= \frac{1}{\{n\}_q!} \longrightarrow \frac{1}{n!}. \\
\mathbb{P}(\widetilde{N}_n = m) &= \sum_{\ell=m}^{n}\binom{n-1}{\ell-1}\frac{s_q(\ell,m)}{\{\ell\}_q!} \longrightarrow \frac{|s_{n,m}|}{n!}. \\
\mathbb{E}\widetilde{N}_n &= \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}q^{\ell-1}}{\{\ell\}_q} \longrightarrow \sum_{j=1}^{n}\frac{1}{j},
\end{aligned}$$

so all of these quantities in the case of a geometric parent are q-analogues of the corresponding quantities in the absolutely continuous case.

5. Ordinary Records from Geometric Parents

Following similar arguments as in the previous sections, it is possible to derive results for ordinary records from a $\mathrm{Ge}(p)$ parent, $p \in (0,1)$. As before, let $q = 1-p$ and $Q = 1/q$. We quote without proof the following results.

Joint distribution of (ordinary) record times:

$$\mathbb{P}(T_1 = t_1, T_2 = t_2, \ldots, T_m = t_m) = \int_{D_{m,q}} q y_m \prod_{j=0}^{m}(1-qy_j)^{t_j - t_{j-1}-1} d_q\mathbf{y}^{(m)}, \tag{41}$$

for $m \geq 1$ and $1 = t_0 < t_1 < \ldots < t_m$. The random vector $(T_1, \ldots, T_m)$ has a probability generating function

$$G(z_1, \ldots, z_m) = \int_{D_{m,q}} q y_m \prod_{j=1}^{m}\frac{z_j^{j+1}}{1-(1-qy_j)z_j\cdots z_m} d_q\mathbf{y}^{(m)}.$$

Probability function of T_m:

$$\mathbb{P}(T_m = n) = \sum_{\ell=m-1}^{n-2}\binom{n-2}{\ell}\frac{s_Q(\ell, m-1; 2)}{\{\ell+2\}_Q!} \tag{42}$$

for $m \geq 1$ and $n \geq m+1$.

Joint distribution of inter-record times:

$$\mathbb{P}(S_1 = s_1, S_2 = s_2, \ldots, S_m = s_m) = \int_{D_{m,q}} \prod_{j=0}^{m}(1-qy_j)^{s_j} d_q\mathbf{y}^{(m)}, \tag{43}$$

for $s_j \geq 0$, $j = 1, \ldots, m$.

Probability function of inter-record times:

$$\mathbb{P}(S_m = s) = \sum_{\ell=0}^{s-1}\binom{s-1}{\ell}\frac{(-1)^\ell}{\{\ell+2\}_Q^m},\ s \geq 1. \tag{44}$$

for $m \geq 1$, $s \geq 0$.

Record indicators:

$$\mathbb{P}(I_n = 1) = \frac{1}{Q}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_Q}. \tag{45}$$

$$\mathbb{P}(I_2 = 1, \dots, I_n = 1) = \frac{1}{\{n\}_Q!}. \tag{46}$$

$$\mathbb{P}(I_2 = 0, \dots, I_n = 0) = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\frac{(-1)^\ell}{\{\ell+1\}_Q}. \tag{47}$$

Number of ordinary records:

$$\mathbb{P}(N_n = m) = \sum_{\ell=m}^{n}\binom{n-1}{\ell-1}\frac{s_Q(\ell,m)}{\{\ell\}_Q!},\ m = 1,\dots,n,$$

$$\mathbb{P}(N_n \geq m) = \sum_{\ell=m-1}^{n-1}\binom{n-1}{\ell}\frac{s_Q(\ell-1,m-2;2)}{\{\ell+1\}_Q!}m = 1,\dots,n.$$

$$\mathbb{E}N_n = \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}Q^{\ell-1}}{\{\ell\}_Q}.$$

Comparing these results with those in Section 3, we observe that the formulas for ordinary records are obtained by changing q by $Q := 1/q$ in the formulas for weak records. Additionally, as in the case of weak records, the above quantities are Q-analogues of the corresponding ones in the absolutely continuous case.

6. Conclusions

In the case of a geometric parent, we have obtained representations of the joint distribution of weak and ordinary record times and inter-record times in terms of Jackson integrals. These representations are useful for obtaining the marginal distribution of record times, inter-record times, record indicators, number of records, etc. In particular, the marginal distribution of record times can be expressed in terms of non-central q-Stirling numbers. Additionally, we show that statistics related to weak records from geometric parents, $\mathrm{Ge}(p)$, behave as q-analogues ($q = 1-p$) of the corresponding statistics in the case of absolutely continuous parents. There is also a duality between weak and ordinary records from geometric parents: it is possible to obtain results from ordinary records by changing q by $Q = 1/q$ in the corresponding formulas or results for weak records.

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Appendix A

Proof of Lemma 1. (a) By changing the summation variables, $\ell_1 = x_1$, $\ell_2 = x_2 - x_1$, …, $\ell_m = x_m - x_{m-1}$, then

$$\begin{aligned}
&(1-q)^m \sum_{0\leq x_1\leq\cdots\leq x_m<\infty} f(q^{x_1}, q^{x_2}, \dots, q^{x_m})\, q^{\sum_{i=1}^m x_i} \\
&= (1-q)^m \sum_{\ell_1,\dots,\ell_m\geq 0} f(q^{\ell_1}, q^{\ell_1+\ell_2}, \dots, q^{\ell_1+\cdots,\ell_m}) \prod_{j=1}^{m}\left(q^{\ell_j}\right)^{m-j}\prod_{i=1}^{m} q^{\ell_i} \\
&= \int_{[0,1]^m} f(y_1, y_1y_2, \dots, y_1\cdots y_m)\prod_{j=1}^{m} y_j^{m-j} d_q y_{\pi_1}\cdots d_q y_{\pi_m}
\end{aligned}$$

where the last equality is justified by (8).

(b) By using the change of variables $\ell_1 = x_1$, $\ell_2 = x_2 - x_1 - 1, \ldots, \ell_m = x_m - x_{m-1} - 1$, we obtain

$$
\begin{aligned}
&(1-q)^m \sum_{0 \le x_1 < \cdots < x_m < \infty} f(q^{x_1}, q^{x_2}, \ldots, q^{x_m})\, q^{\sum_{i=1}^m x_i} \\
&= q^{\binom{m}{2}} (1-q)^m \sum_{\ell_1, \ldots, \ell_m \ge 0} f(q^{\ell_1}, q^{\ell_1+\ell_2+1}, \ldots, q^{\ell_1+\cdots,\ell_m+(m-1)}) \prod_{j=1}^m \left(q^{\ell_j}\right)^{m-j} \prod_{i=1}^m q^{\ell_i} \\
&= q^{\binom{m}{2}} \int_{[0,1]^m} f(y_1, qy_1y_2, \ldots, q^{m-1}y_1\cdots y_m) \prod_{j=1}^m y_j^{m-j} d_q y_{\pi_1} \cdots d_q y_{\pi_m}.
\end{aligned}
$$

□

Proof of Lemma 2. (a) If $m = 1$, the result is immediate by using the definition of Jackson integral given in (5). Thus, let us suppose in the following that $m > 1$. Let us define

$$
\begin{aligned}
A_m(x_1, \ldots, x_{m-1}) &:= (1-q) \sum_{x_m = x_{m-1}}^{\infty} f(q^{x_1}, \ldots, q^{x_m}) q^{x_m}, \\
A_j(x_1, \ldots x_{j-1}) &:= (1-q) \sum_{x_j = x_{j-1}}^{\infty} q^{x_j} A_{j+1}(x_1, \ldots, x_j),\ j = m-1, \ldots, 2 \\
A_1 &:= (1-q) \sum_{x_1=0}^{\infty} q^{x_1} A_2(x_1).
\end{aligned}
\tag{A1}
$$

First of all, observe that by construction, A_1 coincides with the left-hand side of the statement in Lemma 2(a). On the other hand, the definition of the Jackson integral yields

$$
\begin{aligned}
A_m(x_1, \ldots, x_{m-1}) &= (1-q) q^{x_{m-1}} \sum_{\ell=0}^{\infty} f(q^{x_1}, \ldots, q^{x_{m-1}}, q^{x_{m-1}} q^{\ell}) q^{\ell} \\
&= \int_0^{q^{x_{m-1}}} f(q^{x_1}, \ldots, q^{x_{m-1}}, y_m) d_q y_m,
\end{aligned}
$$

and by (back) recurrence, it is not difficult to show that

$$
A_j(x_1, \ldots x_{j-1}) = \int_0^{q^{x_{j-1}}} \int_0^{y_j} \cdots \int_0^{y_{m-1}} f(q^{x_1}, \ldots, q^{x_{j-1}}, y_j, \ldots, y_m) d_q y_m \cdots d_q y_j, \tag{A2}
$$

for $j = m, \ldots, 2$. Finally, from (A1) and (A2),

$$
\begin{aligned}
A_1 &= (1-q) \sum_{x_1=0}^{\infty} q^{x_1} \int_0^{q^{x_1}} \int_0^{y_j} \cdots \int_0^{y_{m-1}} f(q^{x_1}, y_2, \ldots, y_m) d_q y_m \cdots d_q y_2 \\
&= \int_0^1 \int_0^{y_1} \cdots \int_0^{y_{m-1}} f(y_1, \ldots, y_m) d_q \mathbf{y}^{(m)}
\end{aligned}
$$

and the result follows from (10).
(b) The proof is similar to the one in (a), so we omit it. □

Proof of Lemma 3. We only prove (a). The proof of (b) is similar. For $m \ge 1$ and $k = 0$, using (14), the change of variables $c_j = \alpha_j + \cdots + \alpha_m + m - j + 2$, $j = 1, \ldots, m-1$ and (24), we obtain

$$
\begin{aligned}
&\sum_{\substack{\alpha_1, \ldots, \alpha_m \ge 0 \\ \alpha_1 + \cdots \alpha_m = k}} \int_{D_m} y_m \prod_{j=1}^m y_j^{\alpha_j} d_q \mathbf{y}^{(m)} \\
&= \sum_{\substack{\alpha_1, \ldots, \alpha_m \ge 0 \\ \alpha_1 + \cdots \alpha_m = k}} \prod_{j=1}^m \frac{1}{\left\{\alpha_j + \cdots + \alpha_m + m - j + 2\right\}_q} \\
&= \frac{1}{\{m+k+1\}_q} \sum_{2 \le c_1 < \cdots < c_{m-1} \le m+k} \prod_{i=1}^{m-1} \frac{1}{\left\{c_j\right\}_q} = \frac{(-1)^k s_q(m+k-1, m-1; 2)}{\{m+k+1\}_q!}
\end{aligned}
$$

where the last equality follows from (24). □

Proof of Lemma 4. For simplicity, we only give the proof in the case that $m = 2$; for larger values the proof is similar but tedious. Thus, we have

$$\sum_{s_1\geq 1}\int_0^1\int_0^{y_1} f(y_2)(1-y_1)^{s_1-1}d_qy_2\,d_qy_1 = \sum_{s_1\geq 1}\int_0^1 (1-y_1)^{s_1-1}\left(\int_0^{y_1} f(y_2)\,d_qy_2\right)d_qy_1$$
$$= \sum_{s_1\geq 1}(1-q)\left(\sum_{\ell=0}^{\infty}(1-q^\ell)^{s_1-1}\left(\int_0^{q^\ell} f(y_2)\,d_qy_2\right)\right)q^\ell$$
$$= (1-q)\sum_{\ell=0}^{\infty}\left(\int_0^{q^\ell} f(y_2)\,d_qy_2\right)q^\ell\sum_{s_1\geq 1}(1-q^\ell)^{s_1-1}$$
$$= (1-q)\left(\int_0^1 f(y_2)\,d_qy_2 + \sum_{\ell=1}^{\infty}\left(\int_0^{q^\ell} f(y_2)\,d_qy_2\right)q^\ell\sum_{s_1\geq 1}(1-q^\ell)^{s_1-1}\right)$$
$$= (1-q)\left(\int_0^1 f(y_2)\,d_qy_2 + \sum_{\ell=1}^{\infty}\int_0^{q^\ell} f(y_2)\,d_qy_2\right) = (1-q)\sum_{\ell=0}^{\infty}\int_0^{q^\ell} f(y_2)\,d_qy_2$$
$$= (1-q)\sum_{\ell=0}^{\infty}\left(\frac{1}{q^\ell}\int_0^{q^\ell} f(y_2)d_q(y_2)\right)q^\ell = \int_0^1\left(\frac{1}{y_1}\int_0^{y_1} f(y_2)\,d_qy_2\right)d_qy_1$$
$$= \int_0^1\int_0^{y_1}\frac{f(y_2)}{y_1}\,d_qy_2\,d_qy_1.$$

□

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