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RESEARCH

Approaching Developable Surfaces Through Shadow and Penumbra

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Abstract

This paper enquires into the relationship between the rigorous graphic construction of penumbra and shadow and the idea of a developable surface circumscribed to two cores (whether these are curves or surfaces). The first rigorously constructed graphic representations of shadow and penumbra appearing in different historical treatises are contextualised and contrasted with the first works addressing the mathematical description of such surfaces. We describe how artisans and artists have employed physical and mechanical models to construct circumscribed developable surfaces, examining both their formal exploration and constructive formalisation. Next we extrapolate the mechanisms for generating circumscribed surfaces to digital and parametric models. The two procedures presented here evince the existing geometric limits of this way of generating a surface, thus influencing its constructive applications. Lastly, several examples of ephemeral architecture based on such procedures are offered, in which the digital design and manufacturing of experimental pavilions with circumscribed developable surfaces are addressed.

Keywords Descriptive geometry · Shadow · Penumbra · Developable surfaces · Circumscribed surfaces · Digital manufacturing · Experimental pavilions · Augmented graphic thinking

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The Rigorous Graphic Control of Penumbra and Shadow

Shadow has been studied since Antiquity. According to Pliny, outlining the shadow of a man with lines (*umbra hominis lineis circumducta*) is associated with the mythical origin of painting.[1] Since then, the importance of shadow, as one of the major problems of painting, has been evidenced by the works of painters and theoreticians of painting, as found in Leonardo's notebooks or in Alberti's *De pictura*. However, it is in treatises on perspective that the evolution of the geometric mechanism for controlling and representing shadow can be fully appreciated.

Dürer's *Underweysung* (1525) offers the first example of controlling shadow in perspective by means of double orthogonal projection. This development was contextualised in the sixteenth century, above all in the treatise by Vignola-Danti (1583), in which shadow was represented with indefinite edges.

In the seventeenth century there was a renewed interest in the study of shadow and the rigorous definition of its edges.[2] This change should be attributed to the work of Guidobaldo Del Monte (1600) and the observations made by Galileo with his telescope, which had a notable impact on Florentine academic and artistic circles at the beginning of the century.[3]

We know that penumbra is a natural phenomenon addressed in optics and studied in astronomy. The issue of indefinite edges, or half-light, is commented on in the works of Alhacen, Maurolico and Kepler, to whom we owe the current term "penumbra" (Raynauld 2019). Leonardo also speculated on half-light, or penumbra, as reflected in the drawings appearing in many different codices, although in all these cases, his study was restricted to spherical lumens and their manifestation in a dark room.

Curiously, and despite being a well-known phenomenon, the graphic control of penumbra is a topic that has received a lot less attention than shadows (Fig. 1). It was not addressed by the most prominent sixteenth- and seventeenth-century perspective authors who had indeed defined shadow projection with precision, such as François d'Aguillon, Samuel Marolois, Jean Dubreuil, Bosse and Desargues, among others. The pioneers in the rigorous representation of penumbra were a small group of Florentine authors very closely related to Galileo's circle, such as Ludovico Cardi and Pietro Accolti. It was in this cultural context in which special attention was paid not only to shadow, but also to penumbra, which ceased to be an astronomical problem to become a graphic one, as can be seen precisely in the work of Cardi and Accolti.

The first case of rigorous representation of penumbra is to be found in the treatise by the painter Ludovico Cardi (c. 1613). In his manuscript, he rigorously

[1] As regards the huge implications of shadows in Western culture and art, see Gombrich 1995; Ronchi 1983. A first geometric study is found in Kaufmann (1975).

[2] The shadow cast by a point of light (a candle or torch) was the first case study, while the projection of a shadow produced by the sun came about after half a century of controversies. For a comprehensive study of this subject, see Martín-Pastor et al. (2017).

[3] Ludovico Cardi, a painter and disciple of Galileo, represented his Virgin of the Immaculate Conception (Roma, Sta. Maria Maggiore) standing on a moon defined by craters and shadows. His treatises have been comprehensively studied in Camerota (2010).

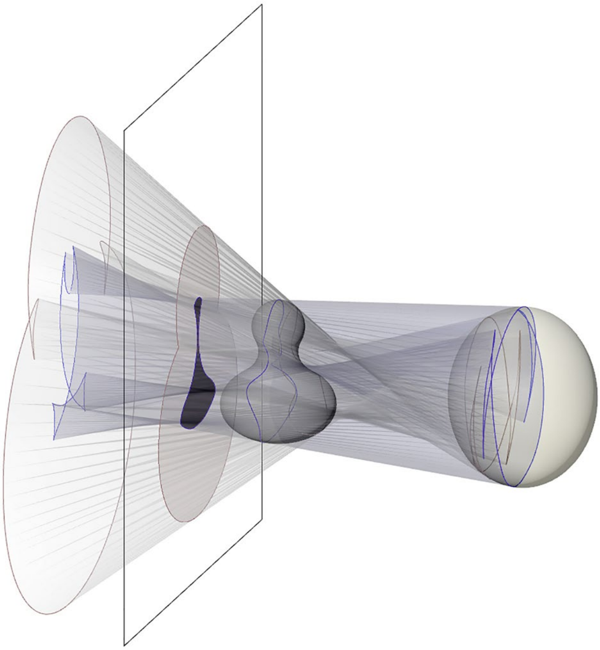

Fig. 1 Penumbra and shadow as developable surfaces. Image: authors

defines the outline of penumbra and shadow cast by a prismatic body when illuminated by diffuse light coming through a window. The graphic construction used by Cardi (Fig. 2 Left) allows for defining the projection using points on the perimeter of the window, in a sort of minimal envelope of luminous rays. This same problem was also addressed in a treatise by Accolti (1625) (Fig. 2 Centre) and subsequently by Níçeron (1646) (Fig. 2 Right).

The graphic model proposed by Cardi, Accolti and Níçeron might seem evident in the current context. It warrants noting, however, that the proposed solution was a milestone in the history of graphic expression and the seed of the synthetic conception of the problem of developable surfaces which, as will be seen further on, consists in finding a surface circumscribed to both bodies.

The rectangular form of a window and the prismatic body reveal their limitations when the intention is to resolve more complex cases of penumbra and shadow. For this reason, the geometric approach seems to have been designed for multifaceted bodies. However, the assumption that these first illustrations served as a source of inspiration for mathematicians at the end of the eighteenth century for linking penumbra and shadow to the problem of developable surfaces is not entirely outlandish.

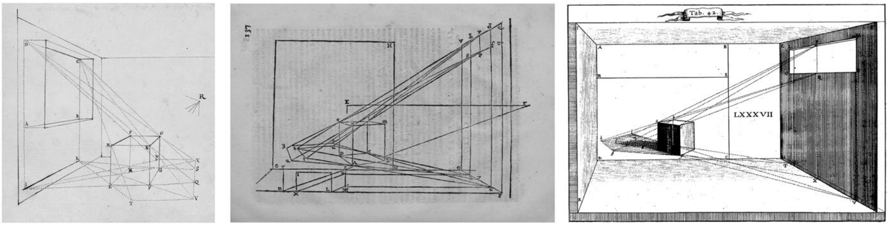

Fig. 2 **Left** Ludovico Cardi, *Prospettiva pratica*, "Libro Secondo. Quinta Parte/Terza Regola/Degli sbattimenti del sole" (ca. 1613: fol. 83 V); **Centre** Pietro Accolti, *Lo inganno de gl'occhi: prospettiua pratica di Pietro Accolti...* (1625: 137); **Right** Jean-François Níçeron, *Thaumaturgus Opticus* (1646: Table 42, Pl. 87)

Developable Surfaces as Penumbra and Shadow: Mathematics and Representation

The history of developable surfaces is well known since its beginnings (Lawrence 2011). According to Hachette (1818 VII), the first geometrician to address the theory of these surfaces was Euler, in his *De solidus quorum superficem in planum explicare licet* (*Of solids whose [entire] surface can be unfolded onto a plane*) (1772). In addition to its being a text revolving around mathematical analysis with original results in the field of algebra, it should be stressed that it also proposes the first approach to the problem from the perspective of "the question of light and shadow":

> THIRD SOLUTION OF THE PRINCIPAL PROBLEM DERIVED FROM THE THEORY OF LIGHT AND SHADOW. §39 But what especially extends to our undertaking from this theory, is that the shapes of the shadows are always of such a nature that their surface can be unfolded onto a plane; ... (Euler 1772: 27; trans. Alexander Aycock).

The next mathematical treatise was published by Gaspard Monge a few years afterwards. In the prologue to the *Mémoire sur les propriétés de plusieurs genres de surfaces courbes, particulièrtement sur celles des surfaces d'eveloppables* (*Dissertation on the properties of several kinds of curved surfaces, particularly on those of developable surfaces*) (1780), Monge stated:

> Having addressed this issue, on the occasion of a dissertation that M. Euler delivered on developable surfaces in his book of 1771, of the Academy of St Petersburg, in which the illustrious geometrician provides formulas for recognising whether a given curved surface has the attribute of being able to be unfolded on a plane, I have obtained results that seem to be much simpler and much easier to use for the same purpose. From this I deduced the resolution of the penumbra and shadow of a body of any shape, illuminated by a luminous body in any way (1780: 382; our trans.).

It is important to stress that Monge proposed the generalisation of the problem to all cases, as can be seen in the "Problem V. Find the equations of the surfaces that envelop the penumbra and shadow of any opaque body, illuminated by any luminous body, giving the two bodies figures and positions in space" (1780:407).

The essential idea underlying Monge's text is that of a mobile plane rolling over two bodies, while remaining tangential to both. In this way, the problem of the relationship between developable surfaces and penumbra and shadow is perfectly expressed from a synthetic point of view:

> From the foregoing it can be inferred that the required surfaces should be generated by the continuous intersection, with itself, of a plane that revolves around the two bodies while always remaining tangential to them, with the sole difference that the plane should touch the two bodies on the same side to generate the surface of pure shadow. Therefore, let us first find the equation of any plane tangential to the surfaces of both bodies (Monge 1780: 407; our trans.).

The next important milestone is found in Jules de la Gournerie's *Traité de Géométrie descriptive* (*Treatise on descriptive geometry*) (1860). La Gournerie represents, for the first time and in a rigorous fashion, a developable surface in an orthographic projection system (Figs. 3 and 4).

As with Monge, La Gournerie defines developable surfaces "as those that, supposedly flexible and inextensible, can be unfolded onto a plane without breaks or duplications" (La Gournerie 1860: 50; our trans.). In his treatise, he also expounds on other procedures for determining developable surfaces on the basis of the envelope of the motion of a cone and as a surface of constant slope (La Gournerie 1860: 53, 104), which still appear in descriptive geometry texts (Izquierdo-Asensi 1985).

The same synthetic formulation of the problem of developable surfaces, associated with the movement of a mobile plane put forward by Monge is also to be found in the treatise by Leroy (1842), which ended up becoming one of the most important and most read in the field of descriptive geometry. Leroy has the following to say in this respect:

> So, for example, when rolling, a mobile plane can be affixed to fixed surfaces, constantly remaining tangential to these two surfaces, provided that none of them are developable" (Leroy 1842: Chap. 184).

Since then and to this day, there have been extraordinary mathematical advances in the state of the knowledge of developable surfaces. A compilation of the results of the geometrical design of nondegenerate developable surfaces can be found in Krivoshapko and Shambina (2012).

Circumscribed Developable Surfaces in Trades and Handicrafts

The control of developable surfaces is directly related to trades and handicrafts employing materials that are manufactured flat and which can be curved without breaks or distortions, whether of leather, fabric, metal, wood or any other flat material. These traditional trades include tailoring, shoemaking, shipbuilding and

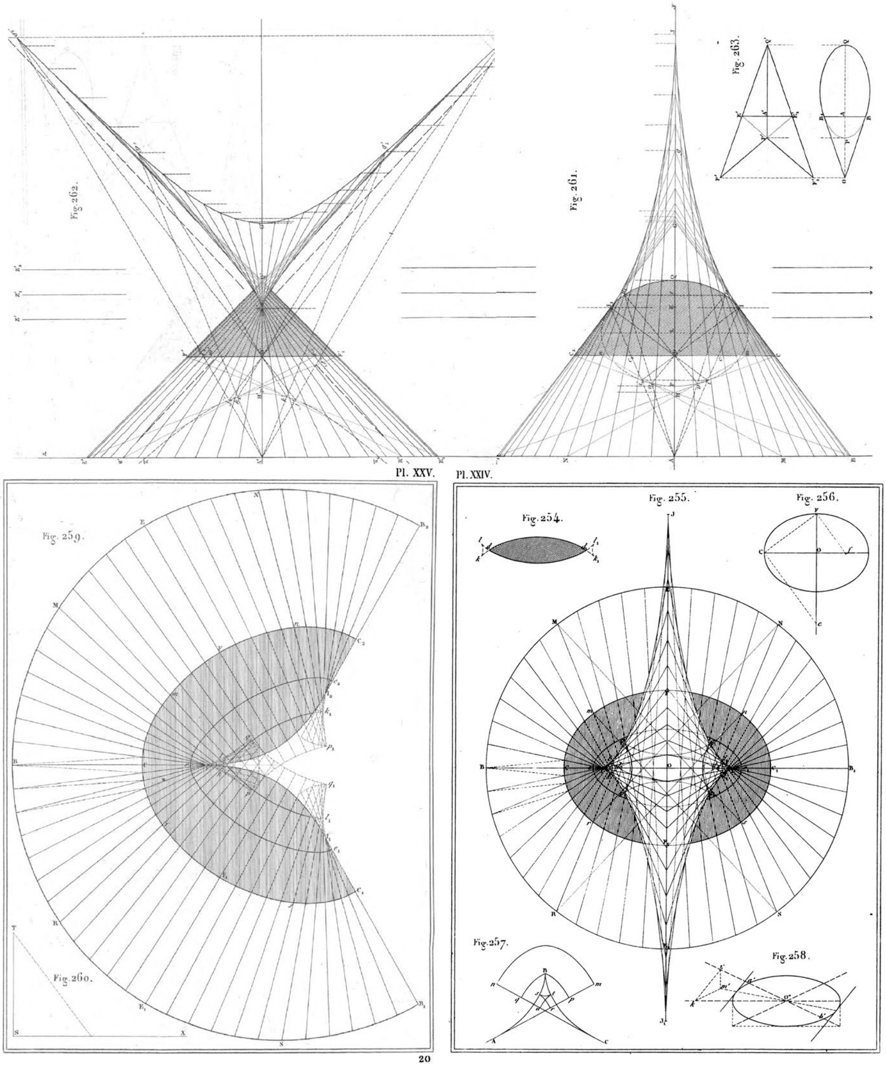

Fig. 3 Determining the shadow of an ellipse illuminated by a circle. Plan, elevation, section and flat development of a developable surface. Image: La Gournerie (1860: Pl. XXIV–XXVII)

so forth. Treatises summarising geometric knowledge have been written on nearly all of them, providing exceedingly valuable information for their pursuit.[4]

The Spanish translation of Leroy's treatise by Andrés Antonio de Gorbea y Gancedo (1845) offers a subtle example of the relationship between the theory of developable surfaces and handicrafts:

[4] In Spain alone there are the treatises on tailoring by Juan de Alcega (1580), Diego Freyle (1588), and Albayzeta (1720), among others. Information retrieved from Gentil-Baldrich (2021).

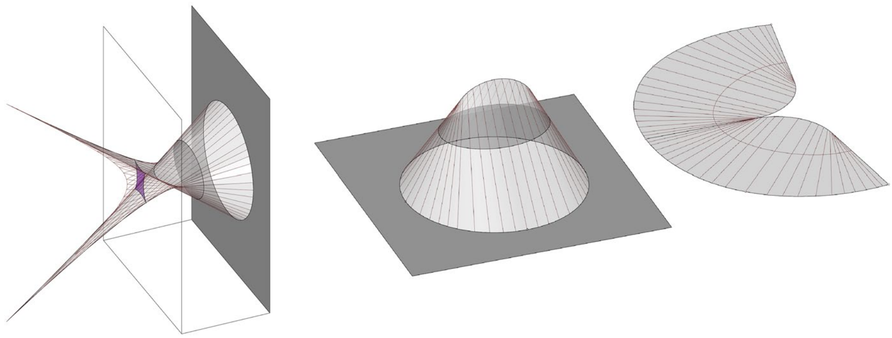

Fig. 4 3D representation of Fig. 3 appearing in the treatise by La Gournerie. Image: authors

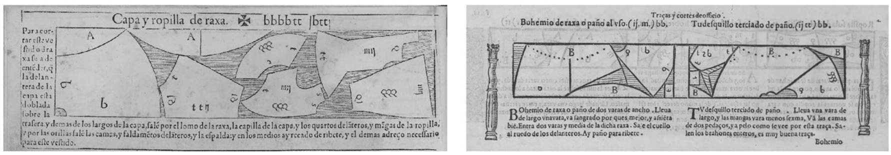

Fig. 5 **Left** Juan de Alcega, *Libro de geometria, practica y traça…* (1580: fol. 31r); **Right** Diego Freyle, *Geometria y traça para el oficio de los sastres…* (1588: Fol. 16v)

> It is also with the help of the envelopes that tinsmiths make developable surfaces; because they make use of a cylindrical or conical anvil to double, bit by bit, the sheet of tin along a series of lines drawn on their plan (Gorbea y Gancedo 1845: 117; our trans.).

It is interesting to see how in the different pattern papers appearing in sixteenth-century treatises on tailoring (Fig. 5 Left) and (Fig. 6 Right), the problems that arose are the same as those faced now in digital manufacturing. Specifically, they have to do with the optimisation of material usage, on the one hand, and that of the alphanumeric coding of the cuts or machined parts on flat pieces, with a view to placing or assembling them in their ultimate spatial form, on the other.

Circumscribed Developable Surfaces in the Arts and Architecture

It is well known that the twentieth-century artistic avant-gardes discovered an inexhaustible source of inspiration in geometry and mathematics. In the field of sculpture, Antoine Pevsner, one of the fathers of Russian Constructivism, was the pioneer in the formal exploration of developable surfaces (Farinella and Baglioni 2017). Since then, there has been an endless stream of new contributions, including the work of Ilhan Koman (Tevfik et al. 2006).

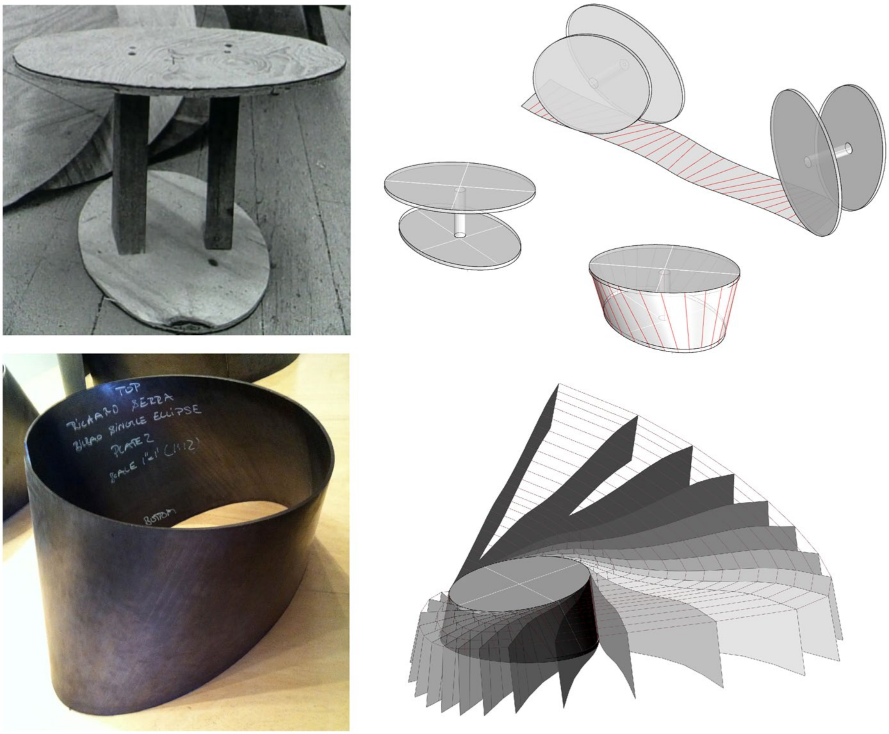

Fig. 6 **Top left** Rotary mechanism by Richard Serra. Image retrieved from the video Charlie Rose Conversation (2001), Time 33:32, reproduced by permission; **Top right** Calculating the developable surface on a plane with the rotary mechanism. Source: author; **Bottom left** *Torqued Ellipse in Brancusi-Serra Exhibition at the Guggenheim Museum Bilbao between 2011 and 2012.* Photo: authors; **Bottom right** Movement of the sheet as it is unrolled, going from its flat to spatial form. Images: authors

The sculpture series of the minimalist artist Richard Serra, *Torqued ellipses*, has a special relationship with the object of study.[5] It involves the generation of a surface circumscribed to two rotated ellipses (Fig. 6 Bottom left). As is apparent from the many interviews that the artist has given, both the process of generating the form and the curving of the huge steel sheets with heavy machinery are directly related to the geometric processes analysed above. In one of his interviews, he has the following to say in this respect:

> They [the artists] constantly come up with ways of informing themselves by inventing tools or techniques or processes that allow them to see into a material manifestation in the way that you would not if you dealt with standardized or academic ways of thinking. These ellipses only came about because we had invented a wheel to make these things in order for us to understand what we

[5] On this topic and the geometric errors detected in the explanatory posters of the Guggenheim Museum in Bilbao, see González-Quintial et al. (2022).

were doing. And albeit it's a small invention and in another sense, it has never come up in the history of form-making before (Art21 2013).

The wheel formed by two interconnected ellipses that are rotating on the same horizontal plane (Fig. 6 Top left) uses gravity to find a geometric solution to an analytically complex problem.[6] As the ellipses rotate, the horizontal plane on which the set rests acts as the tangent plane common to both. On completing a full rotation, the wheel draws the outline of the developable surface circumscribed to the two ellipses (Fig. 6 Top right). In a way, Serra's "invention" is the mechanical extrapolation of the synthetic procedure set out by Monge, represented by La Gournerie and exemplified by Leroy and Gorbea y Gancedo in the case of tinsmiths (Fig. 6 Bottom right).

The application of developable surfaces in the field of architecture is currently being comprehensively revised and updated (Glaeser and Gruber 2007). The team of engineers of Frank Gehry paved the way for the formal control of these surfaces for their architectural use.[7] Since then, the list of significant contributions from a mathematical and computational perspective has grown far too long to address here.[8] From a more modest standpoint, the widespread use of digital and parametric graphic tools is also favouring the geometric control of these surfaces on the basis of graphic thinking (Martín-Pastor 2019).

Penumbra, Shadow and the Limits of the Rotary Mechanism

It is important to understand that parallel or conical illumination, in which there is only shadow and no penumbra, are simpler—and more exceptional—cases of a physical problem. The presence of penumbra is a natural phenomenon that manifests itself when the light is not a point but covers a certain surface or volume (Fig. 7). The model of the window through which diffuse light enters clearly exemplifies this phenomenon. Always provided that there are two closed curves, one can act as a window and the other as an opaque object.

At this point, it is should be noted that the rotary mechanism does not allow for the complete construction of a surface circumscribed to two cores (curves or surfaces) in all cases. In Fig. 8 Left, the mechanism would only roll on the outer edge of the two curves, leaving an occluded inner area. The model based on the shadow cast by a window (Fig. 8 Centre) provides a complete solution (Fig. 8 Right).

The field of application of the rotary mechanism as a procedure for finding the circumscribed surface is limited to two surfaces (or curves) with a positive curvature, as can be seen in Fig. 9.

[6] The description of the mechanism of the double ellipse is found Cooke et al. (1997) and commented on in Mattsson (2000). This idea is also described in the video Art21 (2013): https://www.youtube.com/watch?v=G-mBR26bAzA and Charlie Rose Conversation (2001), https://charlierose.com/videos/18060.

[7] Gehry's approach to developable surfaces has been studied in Shelden (2002).

[8] Special mention should go to Yang et al. (2006) and the works of Pottmann et al. (2007, 2008, 2010), Jung et al. (2015), Tang et al. (2016), Bo et al. (2019) and Hoffmann et al. (2022).

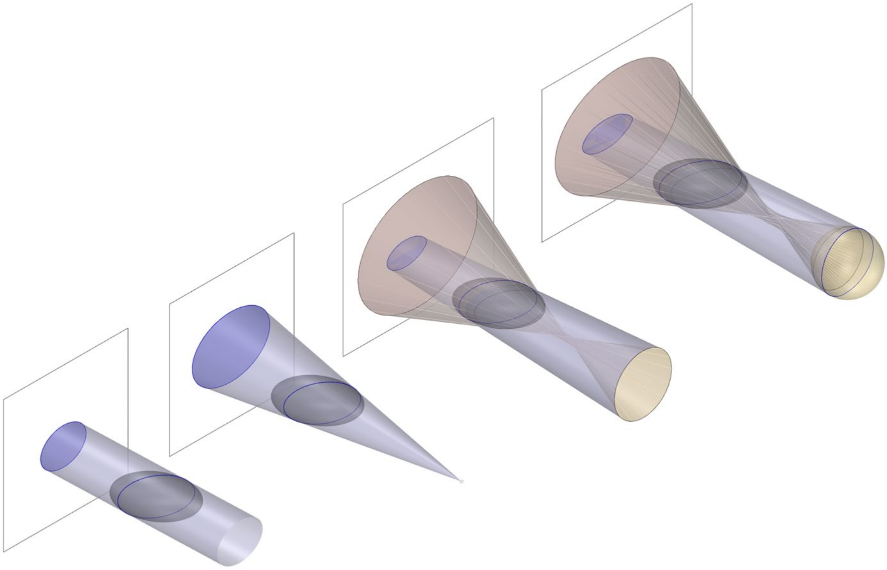

Fig. 7 **a** Shadow cast by parallel rays; **b** cast by conical rays; **c** cast by a flat light; **d** cast by a solid light. Image: authors

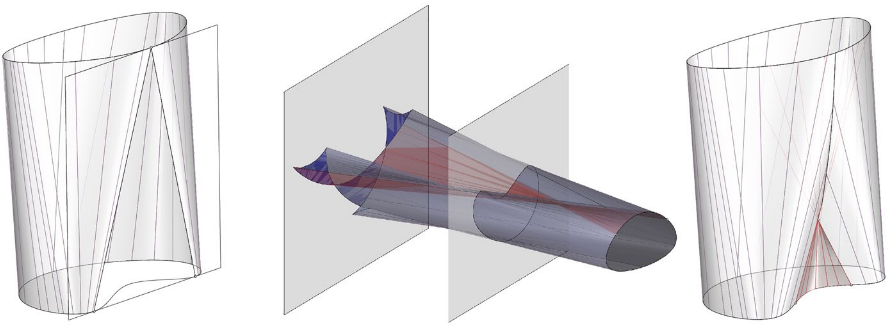

Fig. 8 **Left** Occluded inner area. **Centre** Model based on the shadow cast by a window; **Right** Complete solution. Images: authors

From the Physical Model to the Digital and Parametric Model

The idea of a mobile plane that rolls tangentially to two cores can be extrapolated from its geometric fundamentals to a parametric logic.[9] The aim is to create an automatism suitable for the graphic design and control of these surfaces.[10]

[9] There is the Grasshopper plugin that operates under the Rhinoceros.

[10] Other graphic procedures for generating developable surfaces in Martín-Pastor et al. (2019, 2020).

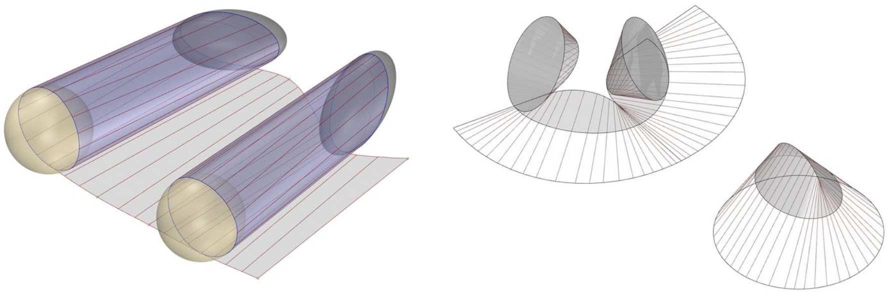

Fig. 9 Rotary mechanism applied to surfaces with a positive curvature. Image: authors

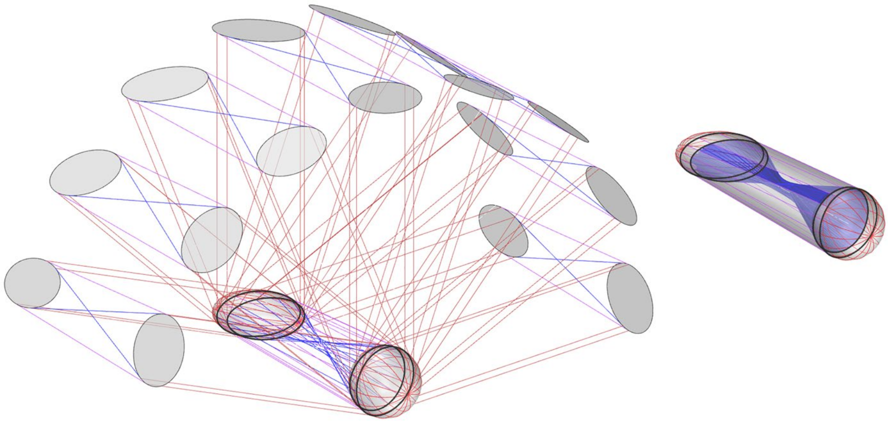

Fig. 10 Method of projections. Image: authors

Method of Projections

This procedure allows for calculating the sequence of tangent planes common to two interconnected cores, surfaces or curves (Fig. 10). In a way, it is the extrapolation of the mobile mechanism, but with several additional advantages: (1) it makes it possible to find the tangent planes in the areas of occlusion, if one of the cores has a positive and negative curvature; (2) it is operative for determining penumbra and shadow, namely, the sequence of both outer and inner planes; and (3) it is operative for both open and closed cores.

The sequence of graphic thinking employed in the algorithm is as follows.

Let two cores be joined by a straight segment. Let a circumference be constructed perpendicular to that segment and divided into a number of parts. Let a series of vectors be constructed from the centre of the circumference to their points of division, which define a series of directions of projection. This operation results in two flat curves. Let the outer and inner tangent lines be drawn to those projected flat curves, finding the points of tangency. Let the two points of tangency be taken from the flat projection to their spatial position. Those points define a ruling of the developable surface. Let the

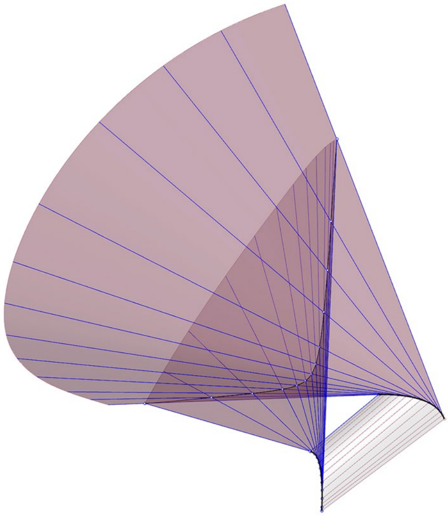

Fig. 11 Method of *convolutas*. Image: authors

operation be repeated for each one of the directions until completing a full circuit. Let the boundary of light and shade be defined in each core.

Method of Geometric Places or Locus

This procedure was conceived specifically for finding a developable surface circumscribed to two open or closed curves (Fig. 11), which is described in full in González-Quintial et al (2022). With this procedure it is determined graphically when it is not possible to find a developable surface between the two curves.

Given two curves C1 and C2, let the tangent developable surface of both be found. Let the intersection G of both surfaces be calculated. At each point of G, let tangent vectors to C1 and C2, be determined. The pair of points of tangency in C1 and C2 define the ruling of the developable surface circumscribed to C1 and C2.

Results and Discussion

If penumbra and shadow are phenomena that manifest themselves at all times, irrespective of the form of light or the opaque body, is it always possible to construct a developable surface between any two curves? An attempt is made here to demonstrate that this is not possible. The limitations reveal themselves in those areas where it is impossible to find a tangent plane common to both curves.

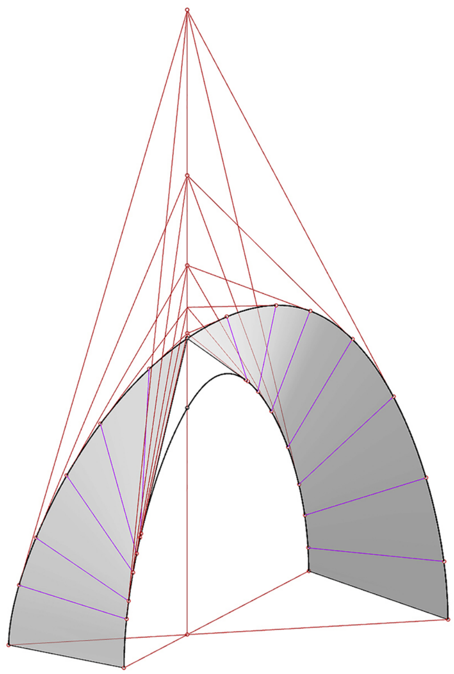

Fig. 12 Discontinuity of a developable surface supported between two skew curves. Image: authors

Two skew curves (Fig. 12). In this situation it is easy to confirm that there will always be an area of discontinuity in which it will be impossible to find a plane tangential to both curves.

Two flat and closed curves. In this case, it will always be possible to find a developable surface circumscribed to both, because it has a direct relationship with the problem of the shadow cast by a window (Fig. 13). The occasional presence of a regression edge would be an obstacle for manufacturing and constructing a surface as a sheet. These determinants are not geometric but constructive.

It can be observed that skew lines (in red) appear in the vicinity of the regression edge. This case can be reconstructed by substituting the affected area (Fig. 14 Left) with a conical surface of transition (Fig. 14 Centre). However, the surface strictly circumscribed to the two curves should be understood considering the possible crests resulting from the self-intersections of the surface (Fig. 8 Right) and (Fig. 14 Right).

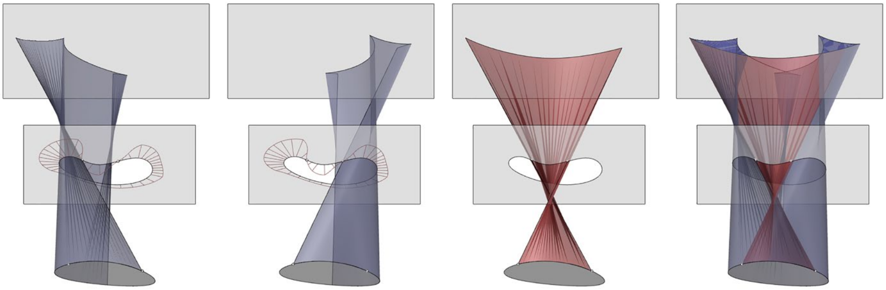

Fig. 13 The area in red. Image: authors

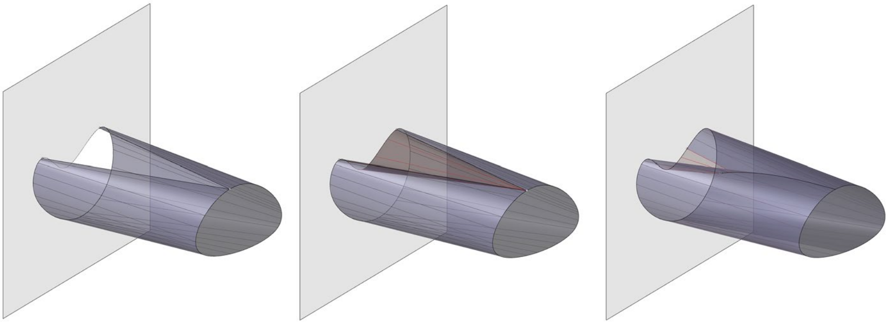

Fig. 14 **Left** Affected area; **Centre** Conical surface of transition; **Right** The complete solution that presents a self-intersection. Image: authors

Two flat and open curves. In this case there is no guarantee that the developable surface will exist on all the points of the two curves, it being possible that the surface may show a discontinuity (Fig. 15 Right). In order to prevent this from occurring, it is essential to guarantee the existence of a common tangent plane at the initial and end point of both curves and at the possible inflection points of each one of them (Fig. 15 Left).

If the idea is to generate developable surfaces free from discontinuities, self-intersection and creases between the two flat curves defining them, the following conditions must be met: (1) the two curves have the same curvature sign; and (2) they have common tangent planes in each pair of initial, final and inflection points (Fig. 16 Left).

When working with two non-planar curves, predicting the final surface on the basis of the curves is harder. Nonetheless, for there to be continuity on a developable surface, there should always be a unique ruling – associated with the tangent plan—which separates the area of positive curvature from the area of negative curvature of the surface (Fig. 16 Right).

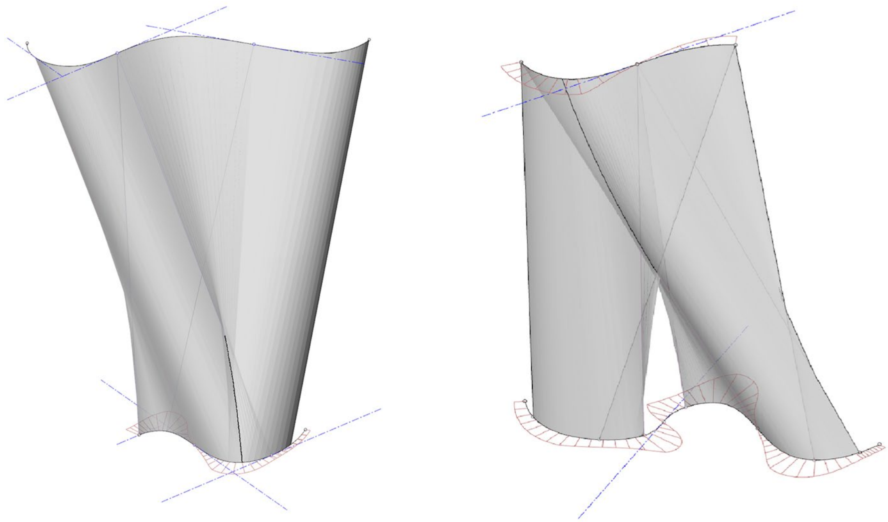

Fig. 15 **Left** There are tangent planes to the two curves at each one of their inflection points; **Right** Discontinuity due to the fact that there is no common tangent plane to the two curves at each one of their inflection points. Image: authors

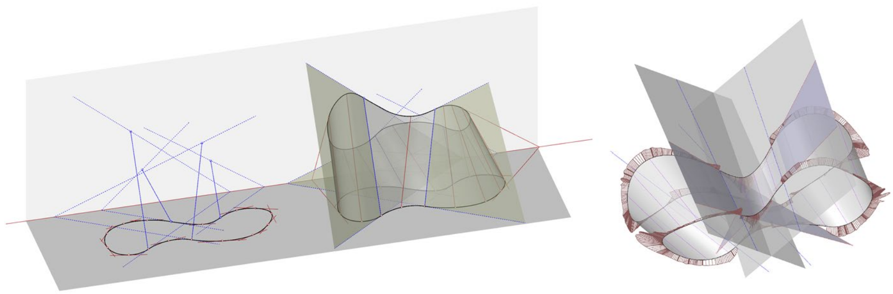

Fig. 16 **Left** Developable surface between two planar curves with a common tangent plane in each pair of inflection points of them; **Right** Developable surface free of creases and self-intersections between two non-planar curves. Images: authors

The developable strips between two open and spatial curves are often used in the discretisation of doubly curved surfaces.[11] When studying the discretisation of surfaces by means of rectifying strips on the geodesic curves of the surface (…) we noted the presence of discontinuities in the proximity of certain points of the geodesic in anticlastic surfaces (Fig. 17 Left). From this we deduced that not all the geodesic lines of a surfaces allow for the placement of rectifying strips free of discontinuities.

[11] On discretisation by means of developable surfaces, see Rose et al (2007); González-Quintial (2012, 2016).

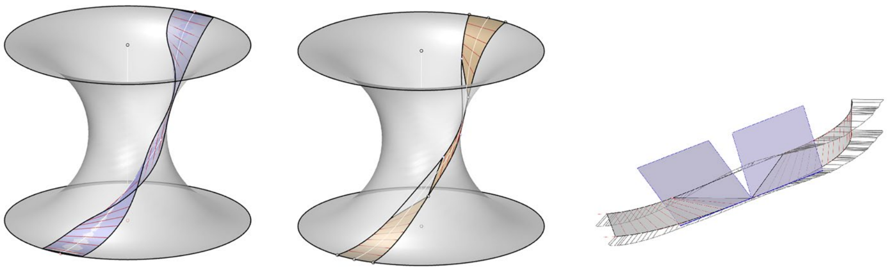

Fig. 17 **Left** Discontinuities in the rectifying strip on anticlastic surfaces; **Centre** Discontinuities in the strip circumscribed to the two equidistant curves; **Right** Amplified area of discontinuity. Images: authors

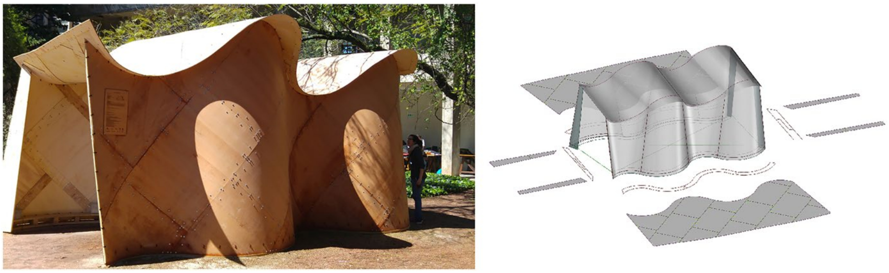

Fig. 18 Dieste Pavilion. Image: authors

It can be observed how similar results can be obtained by resolving this same problem with the placement of a developable strip circumscribed to two equidistant curves of the geodesic (Fig. 17 Centre) and (Fig. 17 Right).

Applications

As has been seen, not all surfaces circumscribed to two cores can be used for construction purposes. The parametric tools that we have generated allow us to trace in a rigorous fashion the penumbra and shadow of all types of bodies, as well as to explore and define developable surfaces on the basis of two curves or surfaces. We will now offer several examples in which we have considered and put into practice the issues addressed above.

Dieste Pavilion

The proposal is an installation formed by three surfaces of plywood with a thickness of between 3 and 4 mm which, after being assembled on the ground, acquires its spatial position by folding and tying together the three surfaces. The shape was inspired by the work of the Uruguayan engineer Eladio Dieste (Fig. 18).

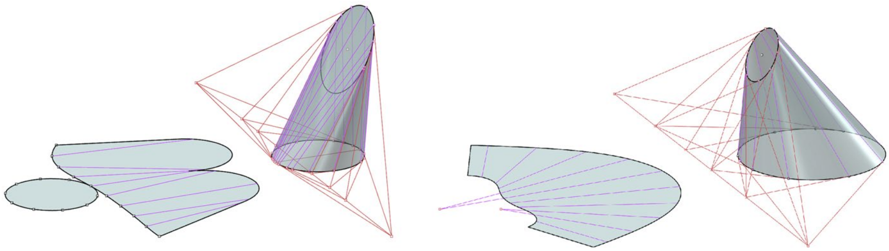

Fig. 19 Developable surface between two circumferences. **Left** Type 1; **Right** Type 2. Image: authors

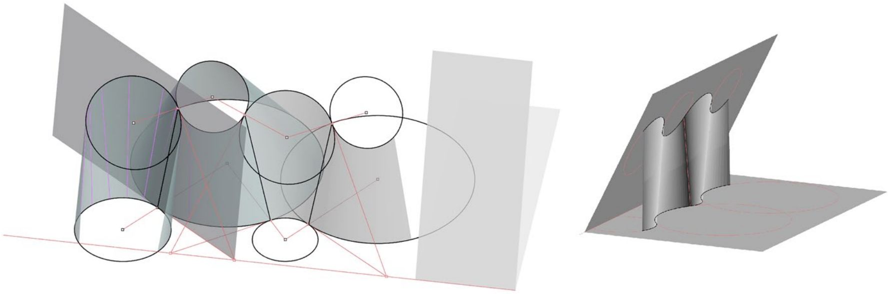

Fig. 20 Developable surfaces tangentially linked by a common tangent plane. Image: authors

Starting from any two circumferences located in the space, the surface circumscribed to both is a generic developable surface.[12] Based on this idea, it is possible to generate two types of surfaces depending on whether the top plane intersects the bottom circumference (Type 1, Fig. 19 left) or not (Type 2, Fig. 19 right). The proposal involves the alternate concatenation of these two types of surfaces on a sole sheet. As has been seen, the condition for guaranteeing the continuity and constructability of the surface is that there are common tangent planes in each pair of inflection points of both curves (Fig. 20).

Dieste Pavilion is an experimental pavilion made in collaboration with different universities, in which a large work team was involved.[13]

Cactus Pavilion

This pavilion was designed as an itinerant interpretation centre in the framework of the project 'Architectural & Natural Heritage Rescue Project of Santiago de Anaya'. It is a biomimetic architectural proposal in the shape of a hollow wooden cactus, made from developable strips formed by plywood panels of a thickness of between

[12] In the event that the two circles were co-spherical sections, it would be a cone.

[13] Department of Graphic Engineering -ETSIE- University of Seville; DepInfo and Fablab MVD, FADU, University of the Republic, UdelaR.

Fig. 21 Cactus Pavilion. Image: authors

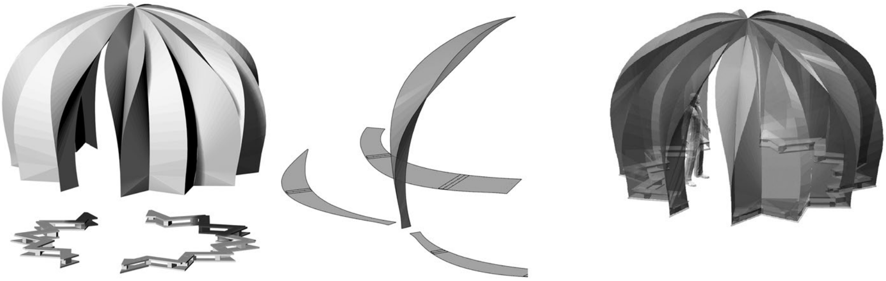

Fig. 22 Segments of the pavilion based on surfaces circumscribed to two flat curves. Image: authors

3 and 4 mm. The shape as a whole serves structurally as a sole self-supporting skin (Fig. 21).

The geometric structure of the cactus is formed by segments comprising three types of developable surfaces. Each one of these three surfaces has been generated as a surface circumscribed to two open flat curves.

The exploration of the shape of the pavilion was possible thanks to the development of the method of geometric places, which made it possible to manipulate those developable surfaces on the basis of the flat curves on which they rested, the crests and the valleys of the segments of the cactus (Fig. 22).

Cactus Pavilion is a multidisciplinary project with the participation of many entities.[14]

[14] The Department of Graphic Engineering -ETSIE- University of Seville; Fablab Donostia ETSA-AGET UPV-EHU; The Master's Program and Doctorate in Architecture of The National Autonomous University of Mexico and the Laboratory of Conservation of the Natural and Cultural Heritage, UNAM. The project received an honourable mention in Laka Competition 2018. Credits and project at https://lakareacts.com/winners/cactus-pavilion/.

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