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An L_p-Functional Busemann–Petty Centroid Inequality

Julián E. Haddad[1], Carlos Hugo Jiménez[2,*] and Letícia A. da Silva[2]

[1]Departamento de Matemática, ICEx, Universidade Federal de Minas Gerais, 30123-970, Belo Horizonte, Brasil and [2]Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, 22451-900 Rio de Janeiro, Brasil

Correspondence to be sent to: e-mail: hugojimenez@mat.puc-rio.br

For a convex body $K \subset \mathbb{R}^n$, let $\Gamma_p K$ be its L_p-centroid body. The L_p-Busemann–Petty centroid inequality states that $\mathrm{vol}(\Gamma_p K) \geq \mathrm{vol}(K)$, with equality if and only if K is an ellipsoid centered at the origin. In this work, we prove inequalities for a type of functional L_r-mixed volume for $1 \leq r < n$ and establish, as a consequence, a functional version of the L_p-Busemann–Petty centroid inequality.

1 Introduction

The study of affine isoperimetric inequalities on one side and affine Sobolev inequalities for functions on $\mathbb{R}^n$ on the other is connected to a great extent. The equivalence of the classical isoperimetric inequality and the classical L^1-Sobolev inequality has been known for quite some time (see, e.g., [1–7]). Following this path, Zhang in [8] established the equivalence of an affine L^1-Sobolev inequality with the Petty projection inequality for convex bodies. Some time after, along with Lutwak and Yang, they obtained L^p versions of this equivalence. Around the same time, these authors developed a rich theory of geometric inequalities for centroid bodies and established L_p extensions of many other fundamental notions from convex geometry, such as mixed volumes and surface area.

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On top of the strong connections mentioned above, other geometric inequalities of an isoperimetric flavor like the Busemann–Petty centroid inequality or Blaschke–Santaló inequality, among others, have been fundamental in the study of several inequalities of Sobolev type, like L^p-log-Sobolev, Gagliardo–Nirenberg, Sobolev trace, or weighted Sobolev inequalities (e.g., [9–14]). It is important to notice that in many of the works mentioned above, where the Busemann–Petty centroid inequality was used to recover some known results for Sobolev type inequalities, this inequality provided a more direct approach. This approach often avoided the use (in their original proofs) of other well-known tools in the area of convex geometric analysis like the solution to the Minkowski problem or the theory of mixed or dual mixed volumes.

In this work, we continue with this line of research. We obtain a family of inequalities for functions on $\mathbb{R}^n$, inequalities of Sobolev type, that in particular recover the L_p-Busemann–Petty centroid inequality for convex bodies in $\mathbb{R}^n$. Our main inequality is presented in the form of a functional mixed volume inequality.

Theorem 1.1. Let f be a C^∞ function and g a continuous non-negative function, both with compact support in $\mathbb{R}^n$. Then, for $1 \le r < n$, $q = \frac{nr}{n-r}$ and $\lambda \in \left(\frac{n}{n+p}, 1\right) \cup (1, \infty)$,

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(y) |\langle \nabla f(x), y \rangle|^p dy \right)^{r/p} dx \ge C_{n,p,\lambda} ||g||_1^{\frac{[(n+p)(\lambda-1)+p]r}{np(\lambda-1)}} ||g||_\lambda^{-\frac{\lambda r}{(\lambda-1)n}} ||f||_q^r. \qquad (1)$$

The sharp constant $C_{n,p,\lambda}$ is computed in Section 3. After a lengthy but straightforward computation, inequality (1) extends for f and g in the following Sobolev spaces:

$$g \in L^1_{1+|x|^p}(\mathbb{R}^n) = \left\{ h \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |h(x)|(1 + \|x\|_2^p) \mathrm{d}x < \infty \right\}$$

$$f \in W^{1,q,r}(\mathbb{R}^n) = \left\{ h \in L^q(\mathbb{R}^n) : \nabla h \in L^r(\mathbb{R}^n) \right\},$$

still with $g \ge 0$. These are the natural spaces to look for extremal functions of inequality (1), and equality is attained if and only if f and g have the following forms:

$$g(x) = a G_{p,\lambda}(||Ax||_2), \text{for a.e.} x$$

$$f(x) = b F_r(||Ax||_2)$$

for positive constants a, b, $A \in GL_n(\mathbb{R})$, $G_{p,\lambda} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$G_{p,\lambda}(t) = \begin{cases} (1 + t^p)^{\frac{1}{\lambda-1}} & \text{if } \lambda \in \left(\frac{n}{n+p}, 1\right) \\ (1 - t^p)_+^{\frac{1}{\lambda-1}} & \text{if } \lambda > 1, \end{cases}$$

and

$$F_r(t) = (1 + t^{\frac{r}{r-1}})^{1-\frac{r}{n}}.$$

2 Notions and Tools from Convex Geometry

In order to show the intrinsic geometric nature of inequality (1), and in particular, its relation to the L_p-Busemann-Petty centroid inequality, let us first recall some basic definitions. A convex body is a convex set $K \subset \mathbb{R}^n$, which is compact and has non-empty interior. For a convex body K, its support function h_K, which uniquely characterizes it, is defined as

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}.$$

If K contains the origin in the interior, then we also have the gauge $\|\cdot\|_K$ and radial $r_K(\cdot)$ functions of K defined, respectively, as

$$\|y\|_K := \inf\{\lambda > 0 : \ y \in \lambda K\}, \quad y \in \mathbb{R}^n \setminus \{0\},$$

$$r_K(y) := \max\{\lambda > 0 : \ \lambda y \in K\}, \quad y \in \mathbb{R}^n \setminus \{0\}.$$

Clearly, $\|y\|_K = \frac{1}{r_K(y)}$.

For a convex body $K \subset \mathbb{R}^n$ and $p \geq 1$, its L_p-moment and L_p-centroid bodies, denoted by M_pK and Γ_pK, are defined by their support functions

$$h_{M_pK}(x)^p = \int_K |\langle x, y\rangle|^p \mathrm{d}y, \quad \text{and} \quad h_{\Gamma_pK}(x)^p = \frac{1}{\mathrm{vol}(K)c_{n,p}} \int_K |\langle x, y\rangle|^p \mathrm{d}y, \tag{2}$$

respectively, where $c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}$ and ω_m is the m-dimensional volume of the unit ball B of $\mathbb{R}^m$. The L_p-Busemann–Petty centroid inequality states that

$$\mathrm{vol}(\Gamma_pK) \geq \mathrm{vol}(K) \quad \text{or} \quad \mathrm{vol}(M_pK) \geq c_{n,p}^{n/p}\, \mathrm{vol}(K)^{\frac{n+p}{p}}, \tag{3}$$

in terms of the moment body M_pK. Equality holds in (3) if and only if K is an 0-symmetric ellipsoid.

Centroid bodies for $p = 1$ can be found for the 1st time in a work of Blaschke [15], whereas the Busemann–Petty centroid inequality for $p = 1$ is due to Petty [16]. The L_p version of centroid bodies above was introduced by Lutwak and Zhang [17], while (3) was obtained by Lutwak *et al.* in [18]. For the history of the Busemann–Petty centroid inequality and a comprehensive introduction on centroid and moment bodies, we refer to Chapter 10 in [19].

The theory of mixed volumes, first developed by Minkowski [20, 21], is one of the pillars of the Brunn–Minkowski theory, it provides us with a unified approach to the study of several of the most important quantities from convex geometry, such as volume, mean width, surface area, among others. At the same time, it has been fundamental in many other problems ranging from characterization of special families of convex bodies to establish new isoperimetric inequalities, we refer to [19, 22] for a comprehensive introduction to the theory of mixed volumes. There are several extensions of the concept of mixed volume; in this work, we will focus mainly on the dual mixed volume and the L_p extension of the mixed volume, concepts belonging to the dual and L_p-Brunn–Minkowski theory, respectively. Regarding the latter, we have the following L_p extension of mixed volume; for some background on this, we refer to [23] and to [24] and the references therein.

For $r \geq 1$, the L_r-mixed volume $V_r(K, L)$ of convex bodies K and L is defined by

$$V_r(K,L) = \frac{r}{n} \lim_{\varepsilon \to 0} \frac{\mathrm{vol}(K +_r \varepsilon \cdot_r L) - \mathrm{vol}(K)}{\varepsilon},$$

where $K +_r \varepsilon \cdot_r L$ is the convex body defined by

$$h_{K +_r \varepsilon \cdot_r L}(x)^r = h_K(x)^r + \varepsilon h_L(x)^r, \quad \forall x \in \mathbb{R}^n.$$

One of the main aspects of mixed volumes is that they admit an integral representation. As in the classical case for the L_r version, it is known (see [23]) that there exists a unique finite positive Borel measure $S_r(K, .)$ on $\mathbb{S}^{n-1}$ such that

$$V_r(K,L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u)^r \mathrm{d}S_r(K,u), \tag{4}$$

for each convex body L.

If $1 \leq r < \infty$ and K and L are convex bodies in $\mathbb{R}^n$ containing the origin as interior point, then we can find also in [23] that

$$V_r(K,L) \geq \mathrm{vol}(K)^{\frac{n-r}{n}} \mathrm{vol}(L)^{\frac{r}{n}}, \tag{5}$$

with equality if and only if K and L are dilates of each other for $r > 1$ and if and only if K and L are homothetic if $r = 1$. Combining inequalities (5) and (3), we obtain

$$V_r(L, M_pK) \geq c_{n,p}^{r/p} \mathrm{vol}(L)^{\frac{n-r}{n}} \mathrm{vol}(K)^{\frac{(n+p)r}{np}}. \tag{6}$$

Taking $L = M_pK$ in (6), we recover (3); hence, (6) is an equivalent formulation for the L_p-Busemann–Petty centroid inequality. This and similar geometric inequalities for mixed volumes involving centroid and projection bodies were already considered in [25]. The main result, Theorem 1.1, is a functional version of inequality (6), replacing the sets $L$and K by functions f and g.

In order to establish a functional version of (6) and considering the integral representation of the geometric L_r-mixed volume (4), let us recall the following result obtained by Lutwak *et al.*, where they introduced the concept of surface area measure of a Sobolev function.

The L_r-surface area measure of a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^r-weak derivatives is given by

Lemma 2.1 (Lemma 4.1 of [24]). Given $1 \le r < \infty$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^r-weak derivatives, there exists a unique finite Borel measure $S_r(f,.)$ on $\mathbb{S}^{n-1}$ such that

$$\int_{\mathbb{R}^n} \phi(-\nabla f(x))^r \mathrm{d}x = \int_{\mathbb{S}^{n-1}} \phi(u)^r \mathrm{d}S_r(f,u), \tag{7}$$

for every non-negative continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree 1. If f is not equal to a constant function almost everywhere, then the support of $S_r(f,.)$ cannot be contained in any $n-1$ dimensional linear subspace.

Conversely, for a convex body L, the function $f_L(x) = F(\|x\|_L)$ satisfies $S_r(f,.) = S_r(L,.)$ if F is a smooth decreasing compactly supported function $F : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int_0^\infty t^{n-1}|F'(t)|^r \mathrm{d}t = 1$$

(see [24]). By the Sobolev inequality, we have

$$\int_{\mathbb{R}^n} f_L(x)^{\frac{nr}{n-r}} \mathrm{d}x \le c_s^{\frac{nr}{n-r}} (n\omega_n)^{\frac{n}{n-r}} \frac{\mathrm{vol}(L)}{\omega_n},$$

where c_s is the sharp constant in the Sobolev inequality on $\mathbb{R}^n$, and there is equality when $F(t) = aF_r(t)$ with $a, b > 0$, where

$$F_r(t) = (1 + t^{\frac{r}{r-1}})^{1-\frac{r}{n}}.$$

The function $F(||x||_2)$ is an extremal function of the euclidean L^r-Sobolev inequality on $\mathbb{R}^n$.

In view of identity (7), for any f and L such that $S_r(f,.) = S_r(L,.)$, we have

$$V_r(L,K) = \frac{1}{n}\int_{\mathbb{R}^n} h_K(-\nabla f(x))^r \mathrm{d}x.$$

This motivates the following definition:

Definition 2.2. Given $1 \leq r < \infty$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^r-weak derivatives, we define

$$V_r(f,K) = \frac{1}{n}\int_{\mathbb{R}^n} h_K(-\nabla f(x))^r \mathrm{d}x.$$

The L^p-Sobolev inequality for general norms was proved in [26] and [27] and can be stated as a mixed volume inequality for functions as follows:

Theorem 2.3. If f is a function with L^r-weak derivatives and compact support in $\mathbb{R}^n$ and K is an origin-symmetric convex body, then for $1 < r < n$ and $q = \frac{nr}{n-r}$

$$V_r(f,K) \geq c_1^r \|f\|_q^r \operatorname{vol}(K)^{\frac{r}{n}}, \tag{8}$$

where c_1 is the optimal constant and equality holds in (8) if and only if $f(x) = aF_r(b\|x\|_K)$ for some $a, b > 0$. Taking $f(x) = F_r(\|x\|_L)$, we recover inequality (5).

Theorem 2.3 was originally proved using an innovative approach based on optimal transportation of mass in [26] and in [27] using convex symmetrization.

In Section 4, we give an alternative, simpler, and elementary proof of this inequality using the tools developed in [28]. Some of the tools we are using here, specially those contained in [24], have been used in the study of Sobolev type inequalities. Their approach is often based on a functional extension of the so-called LYZ body and other known geometric inequalities for projection and polar projection bodies (see [19, Subsection 10.15] and references therein for more on this).

Let us go back to the definition of the moment body (2). It has been noticed that h_{M_pK} is a convex function regardless of the set K (see e.g., [29, Chapter 5]). The following definition has already appeared (in a slightly different way) in [30] and an asymmetric version of it in [31]. In both cases, it was used in the context of valuations to study the moment and centroid operators following the spirit of [32].

Definition 2.4. If g is a non-negative measurable function with compact support on $\mathbb{R}^n$, then we define the convex body $M_p g$ by

$$h_{M_p g}(\xi)^p = \int_{\mathbb{R}^n} g(x)|\langle x, \xi\rangle|^p \mathrm{d}x.$$

The left-hand side of (1) has the following geometric meaning:

$$V_r(f, M_p g) = \frac{1}{n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(y)|\langle \nabla f(x), y\rangle|^p \mathrm{d}y\right)^{r/p} \mathrm{d}x.$$

If K is a convex body and $g(x) = G(\|x\|_K)$ for any non-negative continuous function $G : \mathbb{R}_+ \to \mathbb{R}$ with compact support, it is not hard to verify using polar coordinates that

$$M_p g = \left((n+p)\int_0^\infty t^{n+p-1} G(t)\mathrm{d}t\right)^{1/p} M_p K.$$

Our main result (Theorem 1.1) is a consequence of Theorem 2.3 and Theorem 2.5 below:

Theorem 2.5. If g is a non-negative measurable function with compact support in $\mathbb{R}^n$, then, for each $\lambda \in \left(\frac{n}{n+p}, 1\right) \cup (1, \infty)$, we have that

$$\mathrm{vol}(M_p g)^{\frac{p}{n}} \geq c_{n,p} a_{n,p,\lambda} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} ||g||_\lambda^{-\frac{\lambda p}{(\lambda-1)n}}, \tag{9}$$

where $a_{n,p,\lambda}$ is given by the Lemma 3.4.

Let $G_{p,\lambda} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$G_{p,\lambda}(t) = \begin{cases} (1+t^p)^{\frac{1}{\lambda-1}} & \text{if } \lambda < 1 \\ (1-t^p)_+^{\frac{1}{\lambda-1}} & \text{if } \lambda > 1, \end{cases}$$

then taking $g(x) = G_{p,\lambda}(\|x\|_K)$ in (9), we recover (3).

Equality holds in (9) if and only if $g(x) = aG_{p,\lambda}(|Ax|_2)$ for any $a > 0$ and $A \in \mathrm{GL}_n(\mathbb{R}^n)$.

Even though Theorem 2.5 contains the geometric core of the main Theorem 1.1, the term $\mathrm{vol}(M_p g)$ cannot be expressed in terms of g in an elementary way, whereas this is possible for $V_r(f, M_p g)$. This is the reason why we need to combine it with Theorem 2.3 to obtain a purely functional inequality.

Let us note that Theorem 1.1 cannot be regarded as a functional mixed volume inequality in full generality since it can only be applied to a function f and the centroid/moment body of another function g. We refer the interested reader to review the works of Milman and Rotem [33, 34] where they have defined a functional extension of mixed volumes and have extended some of their main properties to a functional setting.

We should finally also mention other related extensions of the Busemann–Petty centroid inequality obtained by Paouris and Pivovarov in [35] where the authors obtained randomized versions of this and other important isoperimetric inequalities.

The rest of the paper is organized as follows: In Section 3, we shall prove some preliminary results, including an extension of the L_p-Busemann–Petty centroid inequality, to compact domains. Then, in Section 4, we prove Theorems 2.3 and 2.5.

We hope this work will shed some more light onto the deep connection between isoperimetric and functional inequalities.

3 Preliminary Results

In order to prove our main result, Theorem 1.1, we consider two cases: $r = 1$ and $1 < r < n$. For $r = 1$, inequality (5) holds for more general sets. As in [8], a compact domain is the closure of a bounded open set.

Lemma 3.1 ([8, Lemma 3.2]). If M is a compact domain with piecewise C^1 boundary and K a convex body in $\mathbb{R}^n$, then

$$V(M,K)^n \geq \mathrm{vol}(M)^{n-1}\,\mathrm{vol}(K),$$

with equality if and only if M and K are homothetic.

In the same spirit, the next lemma shows that the L_p-Busemann–Petty centroid inequality remains valid for a compact domain:

Lemma 3.2. If M is a compact domain in $\mathbb{R}^n$, then

$$\mathrm{vol}(\Gamma_p M) \geq \mathrm{vol}(M). \tag{10}$$

Equality holds in (10) if and only if M is an 0-symmetric ellipsoid.

Proof. For a compact domain M and $\xi \in \mathbb{S}^{n-1}$, we define the set

$$L_\xi = \{t \in [0,\infty) : t\xi \in M\}.$$

Consider $\delta(t) = \frac{t^n}{n}$, for $t \geq 0$, and the star set SM defined by its radial function

$$\rho_{SM}(\xi) = \delta^{-1}(\mu(\delta(L_\xi))),$$

where μ denotes the one-dimensional Lebesgue measure. It is easy to see that $\mathrm{vol}(SM) = \mathrm{vol}(M)$. Also, let $s = \delta(t) = \frac{t^n}{n}$, then $ds = t^{n-1}dt$. For $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\int_M |\langle x,y\rangle|^p \mathrm{d}y &= \int_{\mathbb{S}^{n-1}} \int_{L_\xi} |\langle x, t\xi\rangle|^p t^{n-1} \mathrm{d}t\mathrm{d}\xi \\ &= \int_{\mathbb{S}^{n-1}} \int_{L_\xi} |\langle x, \xi\rangle|^p t^p t^{n-1} \mathrm{d}t\mathrm{d}\xi \\ &= \int_{\mathbb{S}^{n-1}} \int_{\delta(L_\xi)} |\langle x, \xi\rangle|^p (ns)^{\frac{p}{n}} \mathrm{d}s\mathrm{d}\xi \\ &= n^{\frac{p}{n}} \int_{\mathbb{S}^{n-1}} |\langle x, \xi\rangle|^p \int_{\delta(L_\xi)} s^{\frac{p}{n}} \mathrm{d}s\mathrm{d}\xi.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{SM} |\langle x,y\rangle|^p \mathrm{d}y &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho_{SM}(\xi)} |\langle x, t\xi\rangle|^p t^{n-1} \mathrm{d}t\mathrm{d}\xi \\ &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho_{SM}(\xi)} |\langle x, \xi\rangle|^p t^p t^{n-1} \mathrm{d}t\mathrm{d}\xi \\ &= \int_{\mathbb{S}^{n-1}} \int_0^{\delta(\rho_{SM}(\xi))} |\langle x, \xi\rangle|^p (ns)^{\frac{p}{n}} \mathrm{d}s\mathrm{d}\xi \\ &= n^{\frac{p}{n}} \int_{\mathbb{S}^{n-1}} |\langle x, \xi\rangle|^p \int_0^{\mu(\delta(L_\xi))} s^{\frac{p}{n}} \mathrm{d}s\mathrm{d}\xi.\end{aligned}$$

By the Bathtub principle (see [36, Theorem 1.14, page 28]), we have

$$\int_{\delta(L_\xi)} s^{\frac{p}{n}} \mathrm{d}s \geq \int_0^{\mu(\delta(L_\xi))} s^{\frac{p}{n}} \mathrm{d}s.$$

Therefore,

$$\int_M |\langle x,y\rangle|^p \mathrm{d}y \geq \int_{SM} |\langle x,y\rangle|^p \mathrm{d}y. \tag{11}$$

Since $\text{vol}(SM) = \text{vol}(M)$, we obtain $h_{\Gamma_p M}(x)^p \geq h_{\Gamma_p SM}(x)^p$, whence $\Gamma_p M \supset \Gamma_p SM$ and $\text{vol}(\Gamma_p M) \geq \text{vol}(\Gamma_p SM)$. We conclude that

$$\text{vol}(\Gamma_p M) \geq \text{vol}(\Gamma_p SM) \geq \text{vol}(SM) = \text{vol}(M).$$

If M is a compact domain attaining equality in (10), then equality in (11) implies $\mu(\delta(L_\xi)) = \delta(L_\xi)$ for a.e. ξ, meaning that M is a star body. We conclude the proof recalling the equality case of (3). ■

Let f be a C^∞ function with compact support in $\mathbb{R}^n$. For $t > 0$, consider the level sets of f in $\mathbb{R}^n$:

$$N_{f,t} = \{x \in \mathbb{R}^n : |f(x)| \geq t\}$$

and

$$S_{f,t} = \{x \in \mathbb{R}^n : |f(x)| = t\}.$$

Since f is of class C^∞, by recalling Sard's theorem, $S_{f,t}$ is a C^∞ submanifold, which has non-zero normal vector ∇f, for almost all t. Denote by dS_t the surface area element of $S_{f,t}$. Then the co-area formula relates the area elements $dx = |\nabla f|^{-1} dS_t dt$.

We present a lemma, whose proof is inside of the proof of [8, Theorem 4.1]. It will be useful to prove Theorem 1.1 for the case $r = 1$.

Lemma 3.3. If f is a continuous function with compact support in $\mathbb{R}^n$, then

$$\int_0^\infty \text{vol}(N_{f,t})^{\frac{n-1}{n}} \mathrm{d}t \geq ||f||_{\frac{n}{n-1}}.$$

We observe that the proof of Lemma 3.3 carries over replacing $\frac{n-1}{n}$ by any $\eta \in (0, 1)$ but not for $\eta > 1$. We prove an analogous result for $\eta = \frac{n+p}{p} > 1$.

Lemma 3.4. If g is a continuous function with compact support in $\mathbb{R}^n$ and $\lambda \in \left(\frac{n}{n+p}, 1\right) \cup (1, \infty)$, then

$$\int_0^\infty \text{vol}(N_{g,t})^{\frac{n+p}{n}} \mathrm{d}t \geq a_{n,p,\lambda} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} ||g||_\lambda^{-\frac{\lambda p}{(\lambda-1)n}},$$

where

$$a_{n,p,\lambda} = \begin{cases} A_{n,p,\lambda}^{-\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} & \text{if } \lambda > 1 \\ B_{n,p,\lambda}^{\frac{p}{(\lambda-1)n}} & \text{if } \lambda \in \left(\frac{n}{n+p}, 1\right) \end{cases}$$

with

$$A_{n,p,\lambda} = ((\lambda-1)n+\lambda p)\left(\frac{\Gamma\left(\frac{\lambda}{\lambda-1}\right)(\lambda p)^{\frac{1}{1-\lambda}}((\lambda-1)(n+p))^{-\frac{n+p}{p}}\Gamma\left(\frac{n}{p}+2\right)}{\Gamma\left(\frac{n}{p}+\frac{1}{\lambda-1}+2\right)}\right)^{\frac{(\lambda-1)p}{(\lambda-1)n+\lambda p}}$$

and

$$B_{n,p,\lambda} = \lambda\frac{p}{n+p}\left(\lambda-\frac{n}{n+p}\right)^{\frac{(1-\lambda)(n+p)}{p}-1}\left(\frac{(1-\lambda)^{-\frac{n}{p}-2}\Gamma\left(\frac{n}{p}+2\right)\Gamma\left(\frac{\lambda}{1-\lambda}-\frac{n}{p}\right)}{\Gamma\left(\frac{\lambda-2}{\lambda-1}\right)}\right)^{1-\lambda}.$$

Proof. For $\lambda > 1$ and $t > 0$, let $p_\lambda(t) = (1-t^{\lambda-1})_+^{\frac{n}{p}}$ and $l(t) = \mathrm{vol}(N_{g,t})$. Then $p_\lambda(t/s) = (1-t^{\lambda-1}s^{1-\lambda})_+^{\frac{n}{p}}$ and

$$p_\lambda(t/s)^{\frac{p}{n}} \geq 1-t^{\lambda-1}s^{1-\lambda}. \tag{12}$$

Multiplying (12) by $l(t)$ and integrating, we obtain

$$\int_0^\infty l(t)p_\lambda(t/s)^{\frac{p}{n}}\,\mathrm{d}t \geq \int_0^\infty l(t)\mathrm{d}t - s^{1-\lambda}\int_0^\infty l(t)t^{\lambda-1}\mathrm{d}t,$$

whence

$$||g||_1 \leq \int_0^\infty l(t)p_\lambda(t/s)^{\frac{p}{n}}\,\mathrm{d}t + s^{1-\lambda}\int_0^\infty l(t)t^{\lambda-1}\mathrm{d}t.$$

By Hölder's inequality, we have

$$\int_0^\infty l(t)p_\lambda(t/s)^{\frac{p}{n}}\,\mathrm{d}t \leq \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{n}{n+p}}\left(\int_0^\infty p_\lambda(t/s)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{p}{n+p}}.$$

Write $u = t/s$ and $dt = sdu$. Thus,

$$\int_0^\infty l(t)p_\lambda(t/s)^{\frac{p}{n}}\,\mathrm{d}t \leq \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{n}{n+p}}\left(\int_0^\infty p_\lambda(u)^{\frac{n+p}{n}}\,\mathrm{d}u\right)^{\frac{p}{n+p}} s^{\frac{p}{n+p}}.$$

Now, observe that

$$\int_0^\infty l(t)t^{\lambda-1}\mathrm{d}t = \int_0^\infty \mathrm{vol}(N_{g^\lambda,t^\lambda})t^{\lambda-1}\mathrm{d}t.$$

Write $v = t^{\lambda}$, $dv = \lambda t^{\lambda-1}dt$, then $t^{\lambda-1}dt = \frac{1}{\lambda}dv$ and

$$\int_0^{\infty} l(t)t^{\lambda-1}\mathrm{d}t = \frac{1}{\lambda}\int_0^{\infty} \mathrm{vol}(N_{g^{\lambda},t})\mathrm{d}t = \frac{1}{\lambda}||g||_{\lambda}^{\lambda}.$$

Hence,

$$||g||_1 \leq \frac{1}{\lambda}||g||_{\lambda}^{\lambda}s^{1-\lambda} + \left(\int_0^{\infty} l(t)^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{n}{n+p}}\left(\int_0^{\infty} p_{\lambda}(t)^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{p}{n+p}} s^{\frac{p}{n+p}} = as^{-\alpha} + bs^{\beta}, \quad (13)$$

where $a = \frac{1}{\lambda}||g||_{\lambda}^{\lambda}$, $b = \left(\int_0^{\infty} l(t)^{\frac{n+p}{n}}dt\right)^{\frac{n}{n+p}}\left(\int_0^{\infty} p_{\lambda}(t)^{\frac{n+p}{n}}dt\right)^{\frac{p}{n+p}}$, $\alpha = \lambda - 1$, and $\beta = \frac{p}{n+p}$.

Notice that the right-hand side of (13) has a unique minimum for $s \in (0,\infty)$. Minimizing with respect to $s \in (0,\infty)$, we obtain

$$||g||_1 \leq A_{n,p,\lambda}||g||_{\lambda}^{\frac{\lambda p}{(n+p)(\lambda-1)+p}}\left(\int_0^{\infty} l(t)^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{(\lambda-1)n}{(n+p)(\lambda-1)+p}},$$

where $A_{n,p,\lambda}$ is given in the statement of the lemma.

Hence,

$$||g||_1 \leq A_{n,p,\lambda}||g||_{\lambda}^{\frac{\lambda p}{(n+p)(\lambda-1)+p}}\left(\int_0^{\infty} \mathrm{vol}(N_{g,t})^{\frac{n+p}{n}}\mathrm{d}t\right)^{\frac{(\lambda-1)n}{(n+p)(\lambda-1)+p}},$$

which proves the statement of the lemma for the case $\lambda > 1$.

For the case $\lambda \in \left(\frac{n}{n+p}, 1\right)$, we define $q_{\lambda}(t) = (t^{\lambda-1} - 1)_+^{\frac{n}{p}}$. Then, $q_{\lambda}(t)^{\frac{p}{n}} \geq t^{\lambda-1} - 1$ and $q_{\lambda}(t/s)^{\frac{p}{n}} \geq t^{\lambda-1}s^{1-\lambda} - 1$.

It follows that

$$\int_0^{\infty} l(t)q_{\lambda}(t/s)^{\frac{p}{n}}\mathrm{d}t \geq s^{1-\lambda}\int_0^{\infty} l(t)t^{\lambda-1}\mathrm{d}t - \int_0^{\infty} l(t)\mathrm{d}t.$$

Since $\int_0^{\infty} l(t)t^{\lambda-1}\mathrm{d}t = \frac{1}{\lambda}||g||_{\lambda}^{\lambda}$ and $\int_0^{\infty} l(t)\mathrm{d}t = ||g||_1$, we obtain

$$\frac{s^{1-\lambda}}{\lambda}||g||_{\lambda}^{\lambda} \leq ||g||_1 + \int_0^{\infty} l(t)q_{\lambda}(t/s)^{\frac{p}{n}}\mathrm{d}t.$$

By Hölder's inequality,

$$\int_0^\infty l(t) q_\lambda(t/s)^{\frac{p}{n}}\,\mathrm{d}t \le \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{n}{n+p}} \left(\int_0^\infty q_\lambda(t/s)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{p}{n+p}}$$
$$= \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{n}{n+p}} \left(\int_0^\infty q_\lambda(u)^{\frac{n+p}{n}}\,\mathrm{d}u\right)^{\frac{p}{n+p}} s^{\frac{p}{n+p}}.$$

Hence,

$$\frac{1}{\lambda}||g||_\lambda^\lambda \le s^{\lambda-1}||g||_1 + \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{n}{n+p}} \left(\int_0^\infty q_\lambda(t)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{p}{n+p}} s^{\frac{p}{n+p}+\lambda-1}. \tag{14}$$

For $\lambda \in \left(\frac{n}{n+p}, 1\right)$, the right-hand side of (14) has a unique minimum $s \in (0,\infty)$. Minimizing with respect to $s \in (0,\infty)$, we obtain

$$||g||_\lambda^\lambda \le B_{n,p,\lambda}||g||_1^{\frac{(n+p)(\lambda-1)+p}{p}} \left(\int_0^\infty l(t)^{\frac{n+p}{n}}\,\mathrm{d}t\right)^{\frac{(1-\lambda)n}{p}},$$

where $B_{n,p,\lambda}$ is given in the statement of the lemma.

Therefore,

$$\int_0^\infty \mathrm{vol}(N_{g,t})^{\frac{n+p}{n}}\,\mathrm{d}t \ge B_{c,d,\lambda}^{-\frac{p}{(1-\lambda)n}} ||g||_1^{-\frac{(n+p)(\lambda-1)+p}{(1-\lambda)n}} ||g||_\lambda^{\frac{p\lambda}{(1-\lambda)n}}.$$

■

Now, we present other tools for the case $1 < r < n$ of our main result, introduced by Lutwak *et al.* in [28]. Let $H^{1,r}(\mathbb{R}^n)$ denote the usual Sobolev space of real-valued functions of $\mathbb{R}^n$ with L^r partial derivatives. If $f \in H^{1,r}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and Q is a compact convex set that contains the origin in its relative interior, then they define

$$V_r(f,t,Q) = \frac{1}{n}\int_{S_{f,t}} h_Q(\nu(x))^r |\nabla f(x)|^{r-1}\,\mathrm{d}S_t(x),$$

where $\nu(x) = \frac{\nabla f(x)}{|\nabla f(x)|}$. They prove that for almost every $t > 0$, there exists an origin-symmetric convex body K_t such that, for each origin-symmetric convex body Q

$$V_r(K_t, Q) = V_r(f,t,Q). \tag{15}$$

The next lemma can be deduced from [28], inequalities (6.3), (5.3), (5.4), and (5.1).

Lemma 3.5. If $r \in (1, n)$, $f \in H^{1,r}(\mathbb{R}^n)$ and $q = \frac{nr}{n-r}$, then

$$\int_0^\infty \mathrm{vol}(K_t)^{\frac{n-r}{n}} \mathrm{d}t \geq n^{\frac{r-n}{n}} c_2^r \|f\|_q^r,$$

where

$$c_2 = n^{\frac{1}{q}} \left(\frac{n-r}{r-1}\right)^{\frac{r-1}{r}} \left(\frac{\Gamma\left(\frac{n}{r}\right)\Gamma\left(n+1-\frac{n}{r}\right)}{\Gamma(n)}\right)^{\frac{1}{n}}.$$

4 Proof of the Main Results

We present separate proofs for the cases $1 < r < n$ and $r = 1$.

4.1 Case $1 < r < n$:

Proof. of Theorem 2.3: By Sard's lemma, the co-area formula, (15), (5), and Lemma 3.5,

$$\begin{aligned}
V_r(f, K) &= \frac{1}{n} \int_{\mathbb{R}^n} h_K(-\nabla f(x))^r \mathrm{d}x \\
&= \frac{1}{n} \int_0^\infty \int_{S_{f,t}} h_K(n_x^{S_{f,t}})^r |\nabla f(x)|^{r-1} \mathrm{d}S_{f,t} \mathrm{d}t \\
&= \int_0^\infty V_r(f, t, K) \mathrm{d}t \\
&= \int_0^\infty V_r(K_t, K) \mathrm{d}t \\
&\geq \int_0^\infty \mathrm{vol}(K_t)^{\frac{n-r}{n}} \mathrm{vol}(K)^{\frac{r}{n}} \mathrm{d}t \\
&= \int_0^\infty \mathrm{vol}(K_t)^{\frac{n-r}{n}} \mathrm{d}t \, \mathrm{vol}(K)^{\frac{r}{n}} \\
&\geq n^{\frac{r-n}{n}} c_2^r \|f\|_q^r \mathrm{vol}(K)^{\frac{r}{n}}.
\end{aligned}$$

■

Proof of Theorem 2.5: We may observe that

$$\begin{aligned}
h_{M_p g}(\xi)^p &= \int_{\mathbb{R}^n} g(x) |\langle x, \xi \rangle|^p \mathrm{d}x \\
&= \int_0^\infty \int_{\{g \geq t\}} |\langle x, \xi \rangle|^p \mathrm{d}x \mathrm{d}t \\
&= \int_0^\infty h_{M_p N_{g,t}}(\xi)^p \mathrm{d}t.
\end{aligned}$$

In this sense, we regard $M_p g$ as a generalized p-sum of sets, where we replace finite p-sums by a p-integral of sets

$$M_p g = \int_p M_p N_{g,t} \mathrm{d}t$$

and clearly, for any convex body K,

$$V_p\left(K, \int_p M_p N_{g,t} \mathrm{d}t\right) = \int_0^\infty V_p(K, M_p N_{g,t}) \mathrm{d}t.$$

We compute

$$\begin{aligned}
\mathrm{vol}(M_p g) &= V_p(M_p g, M_p g) \\
&= V_p\left(M_p g, \int_p M_p N_{g,t} \mathrm{d}t\right) \\
&= \int_0^\infty V_p(M_p g, M_p N_{g,t}) \mathrm{d}t \\
&\geq \mathrm{vol}(M_p g)^{\frac{n-p}{n}} \int_0^\infty \mathrm{vol}(M_p N_{g,t})^{p/n} \mathrm{d}t.
\end{aligned}$$

Then using Lemmas 3.2 and 3.4, it follows that

$$\begin{aligned}
\mathrm{vol}(M_p g)^{\frac{p}{n}} &\geq \int_0^\infty \mathrm{vol}(M_p N_{g,t})^{\frac{p}{n}} \mathrm{d}t \\
&\geq c_{n,p} \int_0^\infty \mathrm{vol}(N_{g,t})^{\frac{n+p}{n}} \mathrm{d}t \\
&\geq c_{n,p} a_{n,p,\lambda} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} ||g||_\lambda^{-\frac{\lambda p}{(\lambda-1)n}},
\end{aligned}$$

where $a_{n,p,\lambda}$ is given by Lemma 3.4. ■

4.2 Proof of Theorem 1.1: Case $r = 1$

Proof. Let $V_1(f, M_p g) = \frac{1}{n} \int_{\mathbb{R}^n} \left(\int_{R^n} g(y) |\langle \nabla f(x), y\rangle|^p \mathrm{d}y\right)^{1/p} \mathrm{d}x$. Then,

$$V_1(f, M_p g) = \frac{1}{n} \int_0^\infty \int_{S_{f,t}} \left(\int_{R^n} g(y) \left|\left\langle \frac{\nabla f(x)}{|\nabla f(x)|}, y\right\rangle\right|^p \mathrm{d}y\right)^{1/p} \mathrm{d}S_t \mathrm{d}t.$$

We denote by $\eta_x^{S_t} = \frac{\nabla f(x)}{|\nabla f(x)|}$. Since

$$h_{M_p g}(\eta_x^{S_t}) = \left(\int_{R^n} g(y)\left|\left\langle \eta_x^{S_t}, y\right\rangle\right|^p dy\right)^{1/p} = \left(\int_0^\infty \int_{N_{g,s}} \left|\left\langle \eta_x^{S_t}, y\right\rangle\right|^p dy ds\right)^{1/p},$$

it follows that

$$V_1(f, M_p g) = \frac{1}{n}\int_0^\infty \int_{S_{f,t}} h_{M_p g}(\eta_x^{S_t}) dS_t dt = \frac{1}{n}\int_0^\infty \int_{S_{f,t}} \left(\int_0^\infty \int_{N_{g,s}} \left|\left\langle \eta_x^{S_t}, y\right\rangle\right|^p dy ds\right)^{1/p} dS_t dt.$$

Write $h_{M_p N_{g,s}}(\eta_x^{S_t})^p = \int_{N_{g,s}} \left|\left\langle \eta_x^{S_t}, y\right\rangle\right|^p dy$, then

$$V_1(f, M_p g) = \frac{1}{n}\int_0^\infty \int_{S_{f,t}} \left(\int_0^\infty h_{M_p N_{g,s}}(\eta_x^{S_t})^p ds\right)^{1/p} dS_t dt.$$

By the co-area formula, the Minkowski integral inequality and Lemmas 3.1, 3.2, 3.5, and 3.4,

$$\begin{aligned} V_1(f, M_p g) &\geq \frac{1}{n}\int_0^\infty \left(\int_0^\infty \left(\int_{S_{f,t}} h_{M_p N_{g,s}}(\eta_x^{S_t}) dS_t\right)^p ds\right)^{\frac{1}{p}} dt \\ &\geq \frac{1}{n}\left(\int_0^\infty \left(\int_0^\infty \left(\int_{S_{f,t}} h_{M_p N_{g,s}}(\eta_x^{S_t}) dS_t\right) dt\right)^p ds\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \left(\int_0^\infty V_1(N_{f,t}, M_p N_{g,s}) dt\right)^p ds\right)^{\frac{1}{p}} \\ &\geq \left(\int_0^\infty \left(\int_0^\infty \mathrm{vol}(N_{f,t})^{\frac{n-1}{n}} \mathrm{vol}(M_p N_{g,s})^{\frac{1}{n}} dt\right)^p ds\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \left(\int_0^\infty \mathrm{vol}(N_{f,t})^{\frac{n-1}{n}} dt\right)^p \mathrm{vol}(M_p N_{g,s})^{\frac{p}{n}} ds\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \mathrm{vol}(N_{f,t})^{\frac{n-1}{n}} dt\right)\left(\int_0^\infty \mathrm{vol}(M_p N_{g,s})^{\frac{p}{n}} ds\right)^{\frac{1}{p}} \\ &\geq c_{n,p}^{\frac{1}{p}}\left(\int_0^\infty \mathrm{vol}(N_{f,t})^{\frac{n-1}{n}} dt\right)\left(\int_0^\infty \mathrm{vol}(N_{g,s})^{\frac{n+p}{n}} ds\right)^{\frac{1}{p}} \\ &\geq c_{n,p}^{1/p} ||f||_{\frac{n}{n-1}} C_{n,p,\lambda}^{-\frac{(n+p)(\lambda-1)+p}{(\lambda-1)np}} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)np}} ||g||_\lambda^{-\frac{\lambda}{(\lambda-1)n}}. \end{aligned}$$

■

Remark 4.1. Let us point out that a simpler proof of Theorem 1.1 for the case $r = p$ can be deduced from the L^p-affine Sobolev inequality [28] and the equivalence between the L_p-Busemann–Petty centroid inequality and the L_p-Petty projection inequality (see [18]). The well-known identity for sets

$$V_p(L, \Gamma_p K) = \frac{\omega_n}{\text{vol}(K)} \tilde{V}_{-p}(K, \Pi_p^\circ L),$$

where $\tilde{V}_p(\cdot,\cdot)$ denotes the L_p-dual mixed volume and $\Pi_p^\circ L$, the L_p-polar projection body of L, can be extended to functions as

$$V_p(f, M_p g) = \tilde{V}_{-p}(g, \Pi_p^\circ f),$$

where we define

$$\tilde{V}_{-p}(g, L) = \int_{\mathbb{R}^n} ||x||_L^p g(x) \mathrm{d}x$$

and

$$h(\Pi_p^\circ f, \xi)^p = \int_{\mathbb{R}^n} |\langle \nabla f(x), \xi \rangle|^p \mathrm{d}x.$$

Then an application of the dual mixed volume inequality for functions ([37, Lemma 4.1]) and the L^p-Affine Sobolev inequality (which corresponds to the L_p-Petty projection inequality for functions) gives the result.

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