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Can trans-*S*-manifolds be defined from the Gray-Hervella classification for almost Hermitian manifolds?

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Abstract
Recently, trans-*S*-manifolds have been defined as a wide class of metric *f*-manifolds which includes, for instance, *f*-Kenmotsu manifolds, *S*-manifolds and *C*-manifolds and generalize well-studied trans-Sasakian manifolds. The definition of trans-*S*-manifolds is formulated using the covariant derivative of the tensor *f* and although this formulation coincides with the characterization of trans-Sasakian manifolds in such a particular case, this latter type of manifolds were not initially defined in this way but using the Gray-Hervella classification of almost Hermitian manifolds. The aim of this paper is to study how (almost) trans-*S*-manifolds relate with the Gray-Hervella classification and to establish both similarities and differences with the trans-Sasakian case.

Keywords Almost trans-*S*-manifold · Trans-*S*-manifold · Gray-Hervella's classification

Mathematics Subject Classification 53C15 · 53C25 · 53C99

1 Introduction

Since its beginning, manifolds and Riemannian geometry has been an intensive area of research. The discovery of general relativity, which would have been impossible without this theory, was the greatest proof of its power to explain phenomena which could not be treated before. While relations of this new geometry with other fields of mathematics and science were found, the study of some structures gained importance, for instance, complex and symplectic structures in even dimensions and contact structures in odd dimensions. In

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particular, a Kaehler manifold is a Riemannian manifold with both complex and symplectic structure and a Sasakian manifold is a contact manifold M such that $M \times \mathbb{R}$ is conformal to a Kaehler manifold. Despite all these objects having desirable properties, it is a difficult task to find out when they can be defined on a particular manifold.

This is one of the reasons why almost complex and almost contact structures were found so convenient at first. An almost complex structure is a (1, 1) tensor field J which satisfies $J^2X = -X$ and an almost contact structure is a triple formed by a (1, 1) tensor field ϕ, a vector field ξ and its dual 1-form η such that $\phi^2 X = -X + \eta(X)\xi$. Both are much easier to define than a proper complex or contact structure and, if these tensors satisfy some properties, they actually induce not only complex and contact structures, but more rigid ones like Kaehler and Sasakian structures. Moreover, one can prove that in both cases, there exists a Riemannian metric compatible with the structure.

Moreover, the existence of these two weaker structures provides some properties to the manifolds which make them of interest by their own right. Keeping this idea in mind, K. Yano introduced in [9] the f-structures, tensors fields of type (1, 1) satisfying $f^3 + f = 0$. This means that almost complex and almost contact structures are particular cases of f-structures. So, an $f.pK$-manifold, briefly named f-manifold, is a manifold endowed with an f-structure with parallelizable Kernel and satisfying $f^2 = -I + \sum_{i=1}^{s} \eta_i \otimes \xi_i$ where ξ_i are global vector fields and η_i their dual 1-forms. The study of these tensors and how they relate with the already known results involving almost contact and almost complex structures is an active area of research. Again, it is possible to find a compatible Riemannian metric with an f-structure, and then, we deal with metric f-manifolds.

In [5], Gray and Hervella studied how the covariant derivative of the associated 2-form can be used to create 16 classes of almost complex manifolds. The four basic classes are denoted by W_1, W_2, W_3, W_4 and the rest are given by their direct sum. Kaehler manifolds are precisely the intersection of these four classes. In fact, if (N, J, G) is a $2n$-dimensional almost Hermitian manifold, these four classes are shown in the following table [5]:

Class	Defining conditions
W_1	$3\nabla\Omega = d\Omega$
W_2	$d\Omega = 0$
W_3	$\delta\Omega = N_J = 0$ (or $(\nabla_X\Omega)(Y,Z) - (\nabla_{JX}\Omega)(JY,Z) = 0$)
W_4	$(\nabla_X\Omega)(Y,Z) = \frac{-1}{2(n-1)}\{G(X,Y)\delta\Omega(Z) - G(X,Z)\delta\Omega(Y) - G(X,JY)\delta\Omega(JZ) + G(X,JZ)\delta\Omega(JY)\}$

where Ω is the fundamental 2-form $\Omega(X,Y) = G(X,JY)$ and N_J is the Nijenhuis tensor field of J given by

$$N_J(X,Y) = -[X,Y] - J[JX,Y] - J[X,JY] + [JX,JY],$$

for any two vector fields X, Y tangent to N.

In 1985, Oubiña [7] defined trans-Sasakian manifolds as almost contact manifolds M such that $M \times \mathbb{R}$ is an almost complex manifold in the class W_4, containing locally conformal Kaehler manifolds. In addition, he proved that being trans-Sasakian is equivalent to being normal and satisfying

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\} \quad (1)$$

for some differentiable functions α, β on M. Almost Sasakian manifolds set up another class of almost contact metric manifolds [7]. If M is almost Sasakian, then it is not normal, because $M \times \mathbb{R}$ falls in the class $W_2 \oplus W_4$.

The objective of Oubiña was to generalize Kenmotsu, cosymplectic and Sasakian manifolds. The relation between these manifolds and trans-Sasakian manifolds is given by the following table, where $\Phi(X, Y) = g(X, \phi Y)$ is the fundamental 2-form.

Kenmotsu: $d\Phi = 2\eta \wedge \Phi$, $d\eta = 0$, normal		
Cosymplectic: $d\Phi = 0, \quad d\eta = 0$, normal	Quasi-Sasakian: $d\Phi = 0$, normal	Trans-Sasakian: $d\Phi = 2\beta(\Phi \wedge \eta)$, $d\eta = \alpha\Phi$, $\phi^*(\delta\Phi) = 0$, normal
Sasakian: $\Phi = d\eta$, normal		

Analogously, P. Alegre, L. M. Fernández and A. Prieto introduced in [1] a new class of metric f-manifold called tran-S-manifolds. They are defined as normal metric f-manifolds satisfying

$$\left(\nabla_X f\right)Y = \sum_{i=1}^{s} \left[\alpha_i\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\} + \beta_i\{g(fX, Y)\xi_i - \eta_i(Y)fX\}\right], \tag{2}$$

where α_i, β_i are differentiable functions on the manifold. If the normality condition is removed, the metric f-manifold is called an almost trans-S-manifold. We can resume the situation showed in [1] by using the following table,

f-Kenmotsu [2, 8]: **Definition:** $dF = 2\sum_{i=1}^{s} \eta_i \wedge F$, $d\eta_i = 0$ for any i, normal		
C-manifold [3]: **Definition:** $dF = 0$, $d\eta_i = 0$ for any i, normal	K-manifold [3]: **Definition:** $dF = 0$, normal	Trans-S-manifold [1]: **Necessary Condition:** $dF = 2F \wedge \sum_{i=1}^{s} \beta_i \eta_i$, $d\eta_i = \alpha_i F$ for any i, $f^*(\delta F) = 0$, normal
S-manifold [3]: **Definition:** $F = d\eta_i$ for any i, normal		

where $F(X, Y) = g(X, fY)$ is the fundamental 2-form. It should be noted that there exist K-manifolds which are not trans-S-manifolds (see [1]). In other words, trans-S-manifolds cannot be defined using $dF, d\eta_i$ and $f^*(\delta F)$. On the other hand f-Kenmotsu, S-manifolds and C-manifolds are always trans-S-manifolds. Moreover, there are more types of metric f-manifolds which actually are trans-S-manifolds: for instance, the homothetic s-th Sasakian manifolds [6] and f-manifolds of Kenmotsu type introduced by Falcitelli and Pastore [4]. Consequently, the new class of trans-S-manifolds deserves to be studied.

Then, one question remains: Can we use the Gray-Hervella classification to define (almost) trans-S-manifolds?

This paper tries to clarify this question and compare the results with the trans-Sasakian case. In Sect. 2, we recall some results about f-structures for later use. In Sect. 3, we study the product manifold of a $(2n+s)$-dimensional trans-S-manifold and $\mathbb{R}^s$ equipped with the Euclidean metric in order to classify it in the Gray-Hervella classification. Section 4 is devoted to prove the main result:

Theorem *If (M, f, η_i, ξ_i, g) is a trans-S-manifold of dimension $2n+s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ is an Hermitian manifold and*

1. *If $s = 1$, $\bar{M}$ lies in W_4.*
2. *If $s > 1$, $\bar{M}$ lies in $W_3 \oplus W_4$.*

We also show that the situation in the case $s > 1$ is the best possible. In Sect. 5, we present a manifold in the class $W_3 \oplus W_4$ of the form $M \times \mathbb{R}^s$, where M is a metric f-manifold but not a trans-S-manifold. Finally, in Sect. 6, we give an example of a manifold $M \times \mathbb{R}^s$ in only the most general class $W_1 \oplus W_2 \oplus W_3 \oplus W_4$, where M is an almost trans-S-manifold.

Therefore, we will conclude that although the Gray-Hervella classification cannot be used to define (almost) trans-S-manifold, there is still an interesting relation given by the previous theorem.

2 *f*-Structures

A $(2n+s)$-dimensional Riemannian manifold (M, g) endowed with an f-structure f (that is, a tensor field of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$ [9]) is said to be a *metric f-manifold* if there exist s global vector fields $\xi_1, \dots, \xi_s$ on M (called *structure vector fields*) and s 1-foms $\eta_1, \dots, \eta_s$ such that,

$$\begin{gathered} \eta_i(\xi_j) = \delta_i^j,\ f\xi_i = 0;\ \eta_i \circ f = 0; \\ f^2 = -I + \sum_{i=1}^{s} \eta_i \otimes \xi_i; \\ g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta_i(X)\eta_i(Y), \end{gathered} \tag{3}$$

for any $X, Y \in \mathcal{X}(M)$ and $i, j = 1, \dots, s$. If $s = 0$, f is an almost complex structure and M an almost Hermitian manifold. When $s = 1$, f is an almost contact structure and M an almost contact manifold.

If we fix $l = -f^2$ and $m = f^2 + I$, it is clear that:

$$l + m = I, \quad l^2 = l, \quad m^2 = m,$$
$$fl = lf = f, \quad mf = fm = 0.$$

In other words, both the operators are projection onto two mutually orthogonal distributions $\mathcal{L} = \mathrm{Im}(f)$, $\mathcal{M} = span(\xi_1, \dots, \xi_s)$ and $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. Because of (3), these two distributions are complementary, and in any neighborhood, we can find an orthonormal local basis of $\mathcal{L}$, $\{X_1, \dots, X_n, fX_1, \dots, fX_n\}$ which together with $\{\xi_1, \dots, \xi_s\}$ forms a orthonormal local basis for $\mathcal{X}(M)$.

Let F be the 2-form on M defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$. Since f is of rank $2n$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0$ and, in particular, M is orientable.

The f-structure f is said to be *normal* if

$$[f, f] + 2 \sum_{i=1}^{s} \xi_i \otimes d\eta_i = 0,$$

where $[f, f]$ denotes the Nijenhuis tensor of f given by

$$[f, f](X, Y) = f^2[X, Y] - f[fX, Y] - f[X, fY] + [fX, fY],$$

for any $X, Y \in \mathcal{X}(M)$.

A metric f-manifold is said to be a K-manifold [3] if it is normal and $dF = 0$. In a K-manifold M, the structure vector fields are Killing vector fields [3]. A K-manifold is called an S-manifold if $F = d\eta_i$, for any i and a C-manifold if $d\eta_i = 0$, for any i. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geq 2$, non-trivial examples can be found in [3, 6]. Moreover, a K-manifold M is an S-manifold if and only if

$$\nabla_X \xi_i = -fX, X \in \mathcal{X}(M), \;\; i = 1, \dots, s, \tag{4}$$

and it is a C-manifold if and only if

$$\nabla_X \xi_i = 0, X \in \mathcal{X}(M), \;\; i = 1, \dots, s. \tag{5}$$

It is easy to show that in an S-manifold,

$$(\nabla_X f)Y = \sum_{i=1}^{s} \left\{ g(fX, fY)\xi_i + \eta_i(Y) f^2 X \right\}, \tag{6}$$

for any $X, Y \in \mathcal{X}(M)$ and in a C-manifold,

$$\nabla f = 0. \tag{7}$$

3 Almost Complex Manifolds induced by a trans-*S*-manifold

A $(2n + s)$-dimensional metric f-manifold $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$ is said to be an *almost trans-S-manifold* [1] if it satisfies

$$(\nabla_X f)Y = \sum_{i=1}^{s} \left[\alpha_i\{g(fX,fY)\xi_i + \eta_i(Y)f^2X\} + \beta_i\{g(fX,Y)\xi_i - \eta_i(Y)fX\}\right], \tag{8}$$

where α_i, β_i are smooth functions on M called *structure functions*. If M is normal, it is called a *trans-S-manifold*.

When $s = 1$, trans-S-manifolds are actually trans-Sasakian manifolds. Some more relations studied in [1] are as follows:

1. M is a K-manifold if $\beta_i = 0$ for all i.
2. M is a S-manifold if and only if it is a K-manifold and $\alpha_i = 1$ for all i.
3. M is a C-manifolds if and only if $\alpha_i = \beta_i = 0$ for all i.
4. Generalized Kenmotsu manifolds are trans-S with $\alpha_i = 0$ and $\beta_i = 1$ for all i.

Our goal is to prove that this type of manifold can be embedded canonically into an almost complex manifold $(\bar{M}, \bar{J}, \bar{G})$ which is in the Gray-Hervella class $W_3 \oplus W_4$ [5].

Consider a metric f-manifold (M, f, η_i, ξ_i, g) of dimension $2n + s$, $i = 1, \dots, s$. Let $\bar{M}$ be the product manifold $\bar{M} = M \times \mathbb{R}^s$ of dimension $2n + 2s$, where $\mathbb{R}^s$ is the Euclidean space of dimension s, equipped with the product metric $\bar{G} = \pi^*(g) + \sigma^*(g_e)$, being π and σ the projections of $\bar{M}$ onto M and $\mathbb{R}^s$, respectively, and g_e is the Euclidean metric of $\mathbb{R}^s$. From now on, $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ will be tensor fields of $\bar{M}$.

We extend the tensors in M to $\bar{M}$ as follows

$$\begin{aligned} \bar{f}\bar{X} &= \left(f\pi_*\bar{X}\right)^*, \\ \bar{\eta}_i\left(\bar{X}\right) &= \eta_i\left(\pi_*\bar{X}\right), \end{aligned} \tag{9}$$

for any $\bar{X}$ and $i = 1, \dots, s$. Then, if $\{x_1, \dots, x_s\}$ denote the Euclidean coordinates of $\mathbb{R}^s$, from the definitions, we deduce

$$\begin{aligned} &\bar{\eta}_i\left(\frac{\partial}{\partial x_j}\right) = 0, \ \bar{f}\left(\frac{\partial}{\partial x_j}\right) = 0, \\ &\mathrm{d}x_i(\xi_j) = 0, \qquad \mathrm{d}x_i \circ \bar{f} = 0, \\ &\bar{\eta}_i \circ \bar{f} = 0. \end{aligned} \tag{10}$$

for all i, j. Next, if ∇^1 and ∇^2 denote the Levi-Civita connections of M and $\mathbb{R}^s$, respectively, then, the Levi-Civita of $\bar{M}$ is defined by the sum of ∇^1 and ∇^2:

$$\nabla_{\bar{Y}}\bar{X} = \left(\nabla^1_{\pi_*\bar{Y}}\pi_*\bar{X}\right)^* + \left(\nabla^2_{\sigma_*\bar{Y}}\sigma_*\bar{X}\right)^*. \tag{11}$$

This implies that we can extend by linearity the formulas involving derivatives from M and $\mathbb{R}^s$ to $\bar{M}$. Therefore, $\bar{M}$ is also a metric f-manifolds with structure tensors:

$$\left(\bar{f}, \bar{\eta}_i, \mathrm{d}x_i, \xi_i, \frac{\partial}{\partial x_i}, \bar{G}\right), \ i = 1, \dots, s.$$

Moreover, if M is a trans-S-manifold with structure functions (α_i, β_i), $i = 1, \dots, s$, from (8), we obtain that $\bar{M}$ is also a trans-S-manifold with functions

$$(\alpha_1, \dots, \alpha_s, 0, \overset{s)}{\dots}, 0, \beta_1, \dots, \beta_s, 0, \overset{s)}{\dots}, 0),$$

where, of course, we are also denoting $\alpha_i = \alpha_i \circ \pi$ and $\beta_i = \beta_i \circ \pi$, for any i (see Corollary 4.7 of [1]).

Next, we define the (1,1) tensor field $\bar{J}$ on $\bar{M}$ by

$$\bar{J}\bar{X} = \bar{f}\bar{X} - \sum_i \mathrm{d}x_i(\bar{X})\xi_i + \sum_j \bar{\eta}_j(\bar{X})\frac{\partial}{\partial x_j}. \tag{12}$$

Using (9), we can extend the formulas involving these tensors fields from M and $\mathbb{R}^s$ to $\bar{M}$. Moreover, it is straightforward to show that $\bar{J}$ is an almost complex structure and $\bar{M}$ an almost Hermitian manifold. We define the 2-forms associated to $\bar{J}$ and $\bar{f}$ as follows:

$$\begin{aligned} \bar{\Omega}(\bar{X},\bar{Y}) &= \bar{G}(\bar{X},\bar{J}\bar{Y}), \\ \bar{F}(\bar{X},\bar{Y}) &= \bar{G}(\bar{X},\bar{f}\bar{Y}). \end{aligned} \tag{13}$$

Obviously, $\bar{F}$ restricted to M coincides with the 2-form associated to f and its restriction to $\mathbb{R}^s$ vanishes. A direct expansion using the definitions shows that $\bar{\Omega}$ can be expressed as the sum of more simple 2-forms where $\mathrm{d}x_i$ are the coordinate 1-forms in $\mathbb{R}^s$.

$$\bar{\Omega}(\bar{X},\bar{Y}) = \bar{F}(\bar{X},\bar{Y}) + \sum_{i=1}^{s}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)(\bar{X},\bar{Y}) \tag{14}$$

Moreover, due to the linearity of the covariant derivative, (14) can be easily rewritten as follows:

$$\left(\nabla_{\bar{X}}\bar{\Omega}\right)(\bar{Y},\bar{Z}) = \left(\nabla_{\bar{X}}\bar{F}\right)(\bar{Y},\bar{Z}) + \sum_i \nabla_{\bar{X}}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)(\bar{Y},\bar{Z}). \tag{15}$$

In this context, we can prove the following proposition.

Proposition 1 *If M is a trans-S-manifold of dimension $2n + s$, with structure functions α_i, β_i and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the 2-form $\bar{\Omega}$ in the product manifold $\bar{M} = M \times \mathbb{R}^s$ satisfies*

$$\begin{aligned} \left(\nabla_{\bar{X}}\bar{\Omega}\right)(\bar{Y},\bar{Z}) = \sum_i \Big[&\alpha_i\{\bar{\eta}_i(\bar{Y})\bar{G}(\bar{f}\bar{X},\bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{G}(\bar{f}\bar{Y},\bar{f}\bar{X}) \\ &+ \mathrm{d}x_i(\bar{Z})\bar{G}(\bar{Y},\bar{f}\bar{X}) - \mathrm{d}x_i(\bar{Y})\bar{G}(\bar{Z},\bar{f}\bar{X})\} \\ &+ \beta_i\{\bar{\eta}_i(\bar{Y})\bar{G}(\bar{f}\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{G}(\bar{Y},\bar{f}\bar{X}) \\ &+ \mathrm{d}x_i(\bar{Y})\bar{G}(\bar{f}\bar{Z},\bar{f}\bar{X}) - \mathrm{d}x_i(\bar{Z})\bar{G}(\bar{f}\bar{Y},\bar{f}\bar{X})\}\Big]. \end{aligned} \tag{16}$$

Proof Firstly, since $\left(\nabla_{\bar{X}}\bar{F}\right)(\bar{Y},\bar{Z}) = \bar{G}\left(\bar{Y},\left(\nabla_{\bar{X}}\bar{f}\right)\bar{Z}\right)$, for any $\bar{X},\bar{Y},\bar{Z}$, and since $\bar{M}$ is also a trans-S-manifold with structure functions

$$\left(\alpha_1,\ldots,\alpha_s,0,\overset{s)}{\ldots},0,\beta_1,\ldots,\beta_s,0,\overset{s)}{\ldots},0\right),$$

taking into account (3) and using $g(\bar{Y},\xi_i) = \bar{\eta}_i(Y)$, for any i, we get

$$
\begin{aligned}
\left(\nabla_{\bar{X}}\bar{F}\right)\left(\bar{Y},\bar{Z}\right) = \sum_{i=1}^{s} & \left[\alpha_i\{\bar{\eta}_i\left(\bar{Y}\right)\bar{G}\left(\bar{f}\bar{X},\bar{f}\bar{Z}\right) - \bar{\eta}_i\left(\bar{Z}\right)\bar{G}\left(\bar{f}\bar{Y},\bar{f}\bar{X}\right)\}\right. \\
& \left. + \beta_i\left\{\bar{\eta}_i\left(\bar{Y}\right)\bar{G}\left(\bar{f}\bar{X},\bar{Z}\right) - \bar{\eta}_i\left(\bar{Z}\right)\bar{G}\left(\bar{Y},\bar{f}\bar{X}\right)\right\}\right].
\end{aligned}
\tag{17}
$$

Moreover, we also have $\left(\nabla_{\bar{X}}\mathrm{d}x_i\right)\left(\bar{Y}\right) = \nabla_{\bar{X}}\left(\mathrm{d}x_i\left(\bar{Y}\right)\right) - \mathrm{d}x_i\left(\nabla_{\bar{X}}\bar{Y}\right) = 0$ and, consequently,

$$
\begin{aligned}
\nabla_{\bar{X}}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)\left(\bar{Y},\bar{Z}\right) &= \left(\left(\nabla_{\bar{X}}\mathrm{d}x_i\right) \wedge \bar{\eta}_i\right)\left(\bar{Y},\bar{Z}\right) + \left(\mathrm{d}x_i \wedge \left(\nabla_{\bar{X}}\bar{\eta}_i\right)\right)\left(\bar{Y},\bar{Z}\right) \\
&= \left(\mathrm{d}x_i \wedge \left(\nabla_{\bar{X}}\bar{\eta}_i\right)\right)\left(\bar{Y},\bar{Z}\right).
\end{aligned}
$$

But, we know that [1]

$$
\left(\nabla_{\bar{X}}\bar{\eta}_i\right)\bar{Y} = -\alpha_i\bar{G}\left(\bar{Y},\bar{f}\bar{X}\right) + \beta_i\bar{G}\left(\bar{f}\bar{Y},\bar{f}\bar{X}\right),
\tag{18}
$$

and, so

$$
\begin{aligned}
\nabla_{\bar{X}}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)\left(\bar{Y},\bar{Z}\right) &= \mathrm{d}x_i\left(\bar{Y}\right)\left(-\alpha_i\bar{G}\left(\bar{Z},\bar{f}\bar{X}\right) + \beta_i\bar{G}\left(\bar{f}\bar{Z},\bar{f}\bar{X}\right)\right) \\
&\quad - \mathrm{d}x_i\left(\bar{Z}\right)\left(-\alpha_i\bar{G}\left(\bar{Y},\bar{f}\bar{X}\right) + \beta_i\bar{G}\left(\bar{f}\bar{Y},\bar{f}\bar{X}\right)\right) \\
&= \alpha_i\left(\mathrm{d}x_i\left(\bar{Z}\right)\bar{G}\left(\bar{Y},\bar{f}\bar{X}\right) - \mathrm{d}x_i\left(\bar{Y}\right)\bar{G}\left(\bar{Z},\bar{f}\bar{X}\right)\right) \\
&\quad + \beta_i\left(\mathrm{d}x_i\left(\bar{Y}\right)\bar{G}\left(\bar{f}\bar{Z},f\bar{X}\right) - \mathrm{d}x_i\left(\bar{Z}\right)\bar{G}\left(\bar{f}\bar{Y},\bar{f}\bar{X}\right)\right).
\end{aligned}
\tag{19}
$$

Putting together (15), (17) and (19), we complete the proof. □

4 A Gray-Hervella class for $\bar{M}$

After computing the covariant derivative of $\bar{\Omega}$, we can calculate the class of $\bar{M}$. This is done in the following proposition.

Proposition 2 *If M is a trans-S-manifold of dimension $2n + s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ lies in $W_3 \oplus W_4$.*

Proof Remember from [5] that a manifold $\bar{M}$ is in $W_3 \oplus W_4$ if and only if

$$
\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right) - \left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{Y},\bar{Z}\right) = 0.
\tag{20}
$$

Therefore, we just have to compute $\left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{Y},\bar{Z}\right)$ and see if it is equal to $\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right)$. Using (12), we obtain

$$
\begin{aligned}
\bar{\eta}_i\left(\bar{J}\bar{X}\right) &= -\mathrm{d}x_i\left(\bar{X}\right), \\
\mathrm{d}x_i\left(\bar{J}\bar{X}\right) &= \bar{\eta}_i\left(\bar{X}\right), \\
\bar{J}\bar{f}\bar{X} &= \bar{f}^2\bar{X}.
\end{aligned}
\tag{21}
$$

Since, from the definition of $\bar{G}$,

$$\begin{aligned}\bar{G}(\bar{f}\bar{X},\bar{Y}) &= -\bar{G}(\bar{X},\bar{f}\bar{Y}),\\ \bar{G}\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) &= \bar{G}(\xi_i,\xi_j) = \delta_i^j,\\ \bar{G}(\bar{f}\bar{X},\xi_i) &= \bar{G}\left(\bar{f}\bar{X},\frac{\partial}{\partial x_i}\right) = \bar{G}\left(\xi_i,\frac{\partial}{\partial x_j}\right) = 0,\end{aligned}$$

from (16) together with (3), we deduce

$$\begin{aligned}(\nabla_{\bar{J}\bar{X}}\bar{\Omega})(\bar{J}\bar{Y},\bar{Z}) = \sum_i \Big[\alpha_i\Big\{ & -\mathrm{d}x_i(\bar{Y})\bar{G}(\bar{f}\bar{X},\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{G}(\bar{f}\bar{Y},\bar{f}\bar{X})\\ & + \mathrm{d}x_i(\bar{Z})\bar{G}(\bar{Y},\bar{f}\bar{X}) + \bar{\eta}_i(\bar{Y})\bar{G}(\bar{f}\bar{Z},\bar{f}\bar{X})\}\\ & + \beta_i\{+\mathrm{d}x_i(\bar{Y})\bar{G}(\bar{f}\bar{X},\bar{f}\bar{Z}) - \bar{\eta}_i(\bar{Z})\bar{G}(\bar{Y},\bar{f}\bar{X})\\ & + \bar{\eta}_i(\bar{Y})\bar{G}(\bar{Z},\bar{f}\bar{X}) - \mathrm{d}x_i(\bar{Z})\bar{G}(\bar{f}\bar{Y},\bar{f}\bar{X})\Big\}.\end{aligned} \tag{22}$$

Sorting the addends, we see that (22) is equal to (16), and then, (20) is satisfied. □

We recall that an equivalent defining condition of the class $W_3 \oplus W_4$ is $N_J = 0$ [5]. Therefore, we can formulate the following corollary.

Corollary 1 *If M is a trans-S-manifold of dimension $2n+s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar{M} = M \times \mathbb{R}^s$ is an Hermitian manifold.*

As $\bar{M}$ lies in $W_3 \oplus W_4$, we could ask if it is actually either in W_3 or in W_4. The answer is, in general, negative. To verify this assertion, we need to compute $\delta\bar{\Omega}$. Since

$$\delta\bar{\Omega} = \delta\bar{F} + \sum_i \delta(\mathrm{d}x_i \wedge \bar{\eta}_i),$$

the following lemma will be enough.

Lemma 1 *Let M be a trans-S-manifold of dimension $2n+s$ and $\mathbb{R}^s$ equipped with the Euclidean metric. Then, for the product manifold $\bar{M} = M \times \mathbb{R}^s$, the following formulas are satisfied*

$$\begin{aligned}\delta\bar{F}(\bar{X}) &= 2n\sum_i \alpha_i\bar{\eta}_i(\bar{X})\\ \delta(\mathrm{d}x_i \wedge \bar{\eta}_i)(\bar{X}) &= 2n\beta_i\mathrm{d}x_i(\bar{X}).\end{aligned} \tag{23}$$

Proof The first equality can be found in [1]. For the second equality, consider a local orthonormal basis of tangent vector fields in M given by

$$\{X_1, \dots X_n, fX_1, \dots, fX_n, \xi_1, \dots, \xi_s\} \tag{24}$$

where $X_1, \dots, X_n \in \mathcal{L}$. Then, together with

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \right\}$$

we can form a local orthonormal basis for tangent vector fields in $\bar{M}$. Thus, it follows:

$$\begin{aligned} \delta\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)\left(\bar{X}\right) = - \sum_j \Bigg[& \left(\mathrm{d}x_i \wedge \left(\nabla_{X_j}\bar{\eta}_i\right)\right)\left(X_j, \bar{X}\right) + \left(\mathrm{d}x_i \wedge \left(\nabla_{\bar{f}X_j}\bar{\eta}_i\right)\right)\left(\bar{f}X_j, \bar{X}\right) \\ & + \left(\mathrm{d}x_i \wedge \left(\nabla_{\xi_j}\bar{\eta}_i\right)\right)\left(\xi_j, \bar{X}\right) + \left(\mathrm{d}x_i \wedge \left(\nabla_{\frac{\partial}{\partial x_j}}\bar{\eta}_i\right)\right)\left(\frac{\partial}{\partial x_j}, \bar{X}\right)\Bigg]. \end{aligned}$$

Now, using the formula (18) and taking into account (3), the orthonormality of the basis,

$$\mathrm{d}x_i(X_j) = \mathrm{d}x_i\left(\bar{f}X_j\right) = \mathrm{d}x_i\left(\xi_j\right) = 0$$

and

$$\bar{f}\xi_i = \bar{f}\frac{\partial}{\partial x_i} = 0, \quad \bar{f}^2\bar{X}_j = -\bar{X}_j,$$

we complete the proof. □

Using the linearity of the codifferential we have

Proposition 3 *Let M be a trans-S-manifold of dimension $2n + s$ and $\mathbb{R}^s$ equipped with the Euclidean metric. Then, the 2-form $\bar{\Omega}$ of the product manifold $\bar{M} = M \times \mathbb{R}^s$ satisfies*

$$\delta\bar{\Omega} = 2n \sum_i \left(\alpha_i\bar{\eta}_i + \beta_i \mathrm{d}x_i\right). \tag{25}$$

This allows us to establish that if $\bar{M}$ is in W_3, then it is a C-manifold, because in [5] is proved that in case of being $\bar{M}$ in W_3, then $\delta\bar{\Omega} = 0$. Furthermore, we also can assert that if M is a trans-S-manifold of dimension $2n + s$ with $s > 1$, the product manifold $\bar{M} = M \times \mathbb{R}^s$ is not, in general, in the class W_4. In fact, since the real dimension of $\bar{M}$ is $2(n + s)$, the condition for it to be in W_4 is [5]:

$$\begin{aligned} &\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y}, \bar{Z}\right) + \frac{1}{2(n+s-1)}\left\{\bar{G}\left(\bar{X}, \bar{Y}\right)\delta\bar{\Omega}\left(\bar{Z}\right) - \bar{G}\left(\bar{X}, \bar{Z}\right)\delta\bar{\Omega}\left(\bar{Y}\right)\right. \\ &\left. -\bar{G}\left(\bar{X}, \bar{J}\bar{Y}\right)\delta\bar{\Omega}\left(\bar{J}\bar{Z}\right) + \bar{G}\left(\bar{X}, \bar{J}\bar{Z}\right)\delta\bar{\Omega}\left(\bar{J}\bar{Y}\right)\right\} = 0. \end{aligned} \tag{26}$$

By using the definition of $\bar{G}$, (3) for $\bar{f}$, (16) and (25), a straightforward computation gives that

$$
\begin{aligned}
&\left(\nabla_{\bar X}\bar\Omega\right)\left(\bar Y,\bar Z\right)+\frac{1}{2(n+s-1)}\Big\{\bar G\left(\bar X,\bar Y\right)\delta\bar\Omega\left(\bar Z\right)-\bar G\left(\bar X,\bar Z\right)\delta\bar\Omega\left(\bar Y\right)\\
&-\bar G\left(\bar X,\bar J\bar Y\right)\delta\bar\Omega\left(\bar J\bar Z\right)+\bar G\left(\bar X,\bar J\bar Z\right)\delta\bar\Omega\left(\bar J\bar Y\right)\Big\}\\
&=\frac{s-1}{n+s-1}\Big\{\bar G\left(\bar f\bar X,\bar f\bar Z\right)\sum_{i=1}^{s}\left(\alpha_i\bar\eta_i\left(\bar Y\right)+\beta_i\mathrm{d}x_i\left(\bar Y\right)\right)\\
&-\bar G\left(\bar f\bar X,\bar f\bar Y\right)\sum_{i=1}^{s}\left(\alpha_i\bar\eta_i\left(\bar Z\right)+\beta_i\mathrm{d}x_i\left(\bar Z\right)\right)\\
&-\bar G\left(\bar X,\bar f\bar Y\right)\sum_{i=1}^{s}\left(\alpha_i\mathrm{d}x_i\left(\bar Z\right)-\beta_i\bar\eta\left(\bar Z\right)\right)\\
&+\bar G\left(\bar X,\bar f\bar Z\right)\sum_{i=1}^{s}\left(\alpha_i\mathrm{d}x_i\left(\bar Y\right)-\beta_i\bar\eta\left(\bar Y\right)\right)\Big\}\\
&+\frac{n}{n+s-1}\Big\{\sum_{i=1}^{s}\left(\bar\eta_i\left(\bar X\right)\bar\eta_i\left(\bar Y\right)+\mathrm{d}x_i\left(\bar X\right)\mathrm{d}x_i\left(\bar Y\right)\right)\sum_{j=1}^{s}\left(\alpha_j\bar\eta_j\left(\bar Z\right)+\beta_j\mathrm{d}x_j\left(\bar Z\right)\right)\\
&-\sum_{i=1}^{s}\left(\bar\eta_i\left(\bar X\right)\bar\eta_i\left(\bar Z\right)+\mathrm{d}x_i\left(\bar X\right)\mathrm{d}x_i\left(\bar Z\right)\right)\sum_{j=1}^{s}\left(\alpha_j\bar\eta_j\left(\bar Y\right)+\beta_j\mathrm{d}x_j\left(\bar Y\right)\right)\\
&-\sum_{i=1}^{s}\left(\mathrm{d}x_i\left(\bar X\right)\bar\eta_i\left(\bar Y\right)-\bar\eta_i\left(\bar X\right)\mathrm{d}x_i\left(\bar Y\right)\right)\sum_{j=1}^{s}\left(-\alpha_j\mathrm{d}x_j\left(\bar Z\right)+\beta_j\bar\eta_j\left(\bar Z\right)\right)\\
&+\sum_{i=1}^{s}\left(\mathrm{d}x_i\left(\bar X\right)\bar\eta_i\left(\bar Z\right)-\bar\eta_i\left(\bar X\right)\mathrm{d}x_i\left(\bar Z\right)\right)\sum_{j=1}^{s}\left(-\alpha_j\mathrm{d}x_j\left(\bar Y\right)+\beta_j\bar\eta_j\left(\bar Y\right)\right)\Big\},
\end{aligned}
\tag{27}
$$

which is not zero in general. For instance, let (S,ϕ,ξ,η,g_S) be a $(2n+1)$-dimensional Sasakian manifold. Consider the Euclidean space $\mathbb{R}$ with coordinate t. From Corollary 4.7 and Corollary 4.8 in [1], we know that $M=S\times\mathbb{R}$ is a $(2n+2)$-dimensional trans-S-manifold with structure functions $(1,0,0,0)$. Finally, let $\bar M = M\times\mathbb{R}^2$ and denote by x_1, x_2 the coordinates of $\mathbb{R}^2$. If we put

$$
\bar X=\frac{\partial}{\partial t}+\frac{\partial}{\partial x_1},\ \bar Y=\frac{\partial}{\partial t}+\frac{\partial}{\partial x_2}\ \text{and}\ \bar Z=\xi,
$$

then (27) reduces to

$$
\frac{n}{n+1}\neq 0.
$$

Note that, if $s=1$, M is a trans-Sasakian manifold. In this case, we can consider $i=j=1$ and then (27) is zero and $\bar M$ lies in W_4. So, we can summarize the results of this section in the following theorem.

Theorem 1 *If M is a trans-S-manifold of dimension $2n+s$ and $\mathbb{R}^s$ is equipped with the Euclidean metric, then the product manifold $\bar M = M\times\mathbb{R}^s$ is an Hermitian manifold and*

1. *If* $s=1$, $\bar{M}$ *lies in* W_4.
2. *If* $s>1$, $\bar{M}$ *lies in* $W_3 \oplus W_4$.

5 Trans-*S* manifolds cannot be defined using $W_3 \oplus W_4$

In this section, we will prove there are manifolds $M \times \mathbb{R}^s$ in $W_3 \oplus W_4$ with M a metric f-manifold which are not trans-S-manifolds.

Firstly, since from (15), we have

$$\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right)=\left(\nabla_{\bar{X}}\bar{F}\right)\left(\bar{Y},\bar{Z}\right)+\sum_i \nabla_{\bar{X}}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)\left(\bar{Y},\bar{Z}\right),$$

using (5) and (7), we can easily state

Proposition 4 *The almost complex manifold induced by a C-manifold is a Kaelher manifold.*

Next, we prove the following theorem.

Theorem 2 *There exist almost complex manifolds* $\bar{M}=M\times\mathbb{R}^s$ *in* W_3 *where* M *is metric f-manifold but not a trans-S-manifold.*

Proof Let (N, J) be a complex manifold which lies in the class W_3 and suppose that it is not Kaehler. If ∇^N is its connection, the associated 2-form Φ satisfies

$$\left(\nabla^N_X\Phi\right)(Y,Z)=\left(\nabla^N_{JX}\Phi\right)(JY,Z) \text{ and } (\delta\Phi)X=0. \tag{28}$$

Now, we construct a metric f-manifold $M=N\times\mathbb{R}^s$ with

$$fX=\left(J\iota_{1*}X\right)^*,\ \eta_i(X)=\frac{\partial}{\partial x_i}\left(\iota_{2*}(X)\right), \text{ and } \xi_i=\partial/\partial x_i,$$

for any $i=1,\dots,s$, where ι_1 and ι_2 denote the projections of M onto N and $\mathbb{R}^s$, respectively. We can easily check that

$$\nabla_X f=\nabla^N_{\iota_{1*}X}J.$$

This formula is independent of either ξ_i or η_i, $i=1,\dots,s$ and then, M cannot be a trans-S-manifold.

Define now $\bar{M}=M\times\mathbb{R}^s=N\times\mathbb{R}^s\times\mathbb{R}^s$, with complex structure $\bar{J}$ as in (12). Using (11), we obtain

$$\nabla_{\bar{X}}\bar{J}=\nabla^N_{\left(\iota_{1*}\pi_*\bar{X}\right)}J \text{ and } \nabla_{\bar{X}}\bar{\eta}_i=\nabla^2_{\iota_{2*}\sigma_*X}\mathrm{d}x_i=0,\ i=1,\dots,s.$$

From (15), we have

$$\begin{aligned}\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right) =&\left(\nabla_{\bar{X}}\bar{F}\right)\left(\bar{Y},\bar{Z}\right) + \sum_i \nabla_{\bar{X}}\left(\mathrm{d}x_i \wedge \bar{\eta}_i\right)\left(\bar{Y},\bar{Z}\right)\\ =&\left(\nabla^N_{\iota_{1*}\pi_*\bar{X}}\Phi\right)\left(\iota_{1*}\pi_*\bar{Y},\iota_{1*}\pi_*\bar{Z}\right)\end{aligned}$$

and finally, using (28):

$$\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right) = \left(\nabla^N_{J\iota_{1*}\pi_*\bar{X}}\Phi\right)\left(J\iota_{1*}\pi_*\bar{Y},\iota_{1*}\pi_*\bar{Z}\right) = \left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{Y},\bar{Z}\right).$$

This formula means there are manifolds in W_3 induced by no trans-*S* metric *f*-manifolds.

□

Since $W_3 \subset W_3 \oplus W_4$, we obtain a more general result

Corollary 2 *There exists almost complex manifolds $\bar{M} = M \times \mathbb{R}^s$ in $W_3 \oplus W_4$ where M is metric f-manifold but not a trans-S-manifold.*

Therefore, we cannot use the class $W_3 \oplus W_4$ to define trans-*S* in contrast to the almost contact case, where the class W_4 can be used to characterize trans-Sasakian manifolds.

6 Almost trans-*S*-manifolds and the Gray-Hervella classification

By Proposition 1, it follows that if M is a trans-*S*-manifold, then $M \times \mathbb{R}^s$ lies neither in the class $W_1 \oplus W_2 \oplus W_3$ nor in the class $W_1 \oplus W_2 \oplus W_4$. In this section, starting by a suitable $(2n + s + s_1)$-dimensional almost trans-*S*-manifold M, we prove that $\bar{M} = M \times \mathbb{R}^{s+s_1}$ falls neither in $W_2 \oplus W_3 \oplus W_4$ nor in $W_1 \oplus W_2 \oplus W_4$. First, we remind the following expression for almost trans-*S*-manifolds which appears in [1].

$$\nabla_X \xi_i = -\alpha_i fX - \beta_i f^2 X + \sum_{j=1}^{s} \eta_j\left(\nabla_X \xi_i\right)\xi_j.$$

We also have that

$$\begin{aligned}\left(\nabla_{\bar{X}}\bar{\eta}_i\right)\left(\bar{Y}\right) =&\bar{G}\left(\nabla_{\bar{X}}\xi_i,\bar{Y}\right)\\ =&-\alpha_i\bar{G}\left(\bar{f}\bar{X},\bar{Y}\right) - \beta_i\bar{G}\left(\bar{f}^2\bar{X},\bar{Y}\right) + \sum_{j=1}^{s}\eta_j\left(\nabla_X\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right) = &\left(\nabla_{\bar{X}}\bar{F}\right)\left(\bar{Y},\bar{Z}\right) + \sum_{i=1}^{s}\mathrm{d}x_i \wedge \nabla_{\bar{X}}\bar{\eta}_i\left(\bar{Y},\bar{Z}\right) = A(X,Y,Z)\\ &+ \sum_{i,j=1}^{s}\mathrm{d}x_i\left(\bar{Y}\right)\eta_j\left(\nabla_X\xi_i\right)\bar{\eta}_j\left(\bar{Z}\right) - \mathrm{d}x_i\left(\bar{Z}\right)\eta_j\left(\nabla_X\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right),\end{aligned} \tag{29}$$

where $A\left(\bar{X},\bar{Y},\bar{Z}\right)$ is the second part of the equality in Eq. (22). Remember we have proved in Proposition 2, without using the normality condition,

$$A\left(\bar{X},\bar{Y},\bar{Z}\right)-A\left(\bar{J}\bar{X},\bar{J}\bar{Y},\bar{Z}\right)=0, \tag{30}$$

for any $\bar{X},\bar{Y},\bar{Z}$.

From [5], the condition that an almost Hermitian manifold needs to satisfy for being in $W_2 \oplus W_3 \oplus W_4$ is as follows:

$$\sum_{cyc}\{\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right)-\left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{Y},\bar{Z}\right)\}=0,$$

where $\sum_{cyc}$ is the cyclic sum of $\left(\bar{X},\bar{Y},\bar{Z}\right)$. Then, from (29), (30) and taking into account the definition of $\bar{J}$, it follows that

$$\begin{aligned}
&\sum_{cyc}\left\{\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{Y},\bar{Z}\right)-\left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{Y},\bar{Z}\right)\right\}\\
&\quad=\sum_{cyc}\sum_{i,j=1}^{s}\left\{\mathrm{d}x_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Z}\right)-\mathrm{d}x_i\left(\bar{Z}\right)\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right)\right.\\
&\qquad\left.-\bar{\eta}_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Z}\right)-\mathrm{d}x_i\left(\bar{Z}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\mathrm{d}x_j\left(\bar{Y}\right)\right\}.
\end{aligned} \tag{31}$$

Analogously, for $W_1 \oplus W_3 \oplus W_4$, the defining condition is

$$\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{X},\bar{Y}\right)-\left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{X},\bar{Y}\right)=0,$$

for any $\bar{X},\bar{Y},\bar{Z}$. Again, by using (29), (30) and the definition of $\bar{J}$, the above equation is equal to

$$\begin{aligned}
&\left(\nabla_{\bar{X}}\bar{\Omega}\right)\left(\bar{X},\bar{Y}\right)-\left(\nabla_{\bar{J}\bar{X}}\bar{\Omega}\right)\left(\bar{J}\bar{X},\bar{Y}\right)=\sum_{i,j=1}^{s}\left\{\mathrm{d}x_i\left(\bar{X}\right)\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right)\right.\\
&\quad-\mathrm{d}x_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{X}\right)-\bar{\eta}_i\left(\bar{X}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right)\\
&\quad\left.-\mathrm{d}x_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\mathrm{d}x_j\left(\bar{X}\right)\right\}.
\end{aligned} \tag{32}$$

Actually, we can find an explicit example where these formulas are not zero. Indeed, from Theorem 5 in [1], if we consider N a $(2n+s_1)$-dimensional trans-S-manifold with functions α_i, β_i, with f, ξ_i coming from the f-structure of N, $\{\partial/\partial t_i\}$ a basis for $\mathbb{R}^s$, $\imath$ the inclusion of N into the warped product $M=\mathbb{R}^s\times_h N$ and f^* the following metric f-structure in M,

$$f^*(X)=\imath_*(f(\imath^*X)),$$

with structure vector fields

$$\xi_i^*=\begin{cases}\dfrac{\partial}{\partial t_i} & 1\leq i\leq s\\ \dfrac{1}{h}\xi_{i-s} & s+1\leq i\leq s+s_1,\end{cases}$$

then, we have that M is a $(2n+s+s_1)$ almost trans-S-manifold with functions

$$\alpha_i^* = \begin{cases} 0 & 1 \le i \le s \\ \dfrac{\alpha_{i-s}}{h} & s+1 \le i \le s+s_1, \end{cases}$$

$$\beta_i^* = \begin{cases} \dfrac{h^{i)}}{h} & 1 \le i \le s \\ \dfrac{\beta_{i-s}}{h} & s+1 \le i \le s+s_1, \end{cases}$$

where $h^{i)}$ are denoting the components of the gradient of the function h and

$$\nabla_X \xi_i^* = \frac{h^{i)}}{h} \imath^* X.$$

Now, consider the induced manifold $\bar{M} = M \times \mathbb{R}^{s+s_1} = \mathbb{R}^s \times_h N \times \mathbb{R}^{s+s_1}$. Putting $\bar{X} = \bar{Y} = -\partial/\partial x_{s+1}$, then $\bar{J}\bar{Y} = \bar{J}\bar{X} = \xi_{s+1}^* = (1/h)\xi_1$ and putting $\bar{Z} = -\partial/\partial x_1$, then $\bar{J}\bar{Z} = \xi_1^* = \partial/\partial t_1$. So, we have

$$\begin{aligned}
&\nabla_{\bar{X}} \xi_i^* = \nabla_{\bar{Y}} \xi_i^* = \nabla_{\bar{Z}} \xi_i^* = 0, \\
&\quad \nabla_{\bar{J}\bar{X}} \xi_i^* = \nabla_{\bar{J}\bar{Y}} \xi_i^* = \nabla_{\xi_{s+1}^*} \xi_i^* = \frac{h^{i)}}{h} \imath^* \xi_{s+1}^* = \frac{h^{i)}}{h} \xi_{s+1}^*, \\
&\quad \nabla_{\bar{J}\bar{Z}} \xi_i^* = \frac{h^{i)}}{h} \imath^* \xi_1^* = 0, \\
&\quad \bar{\eta}_i(\bar{X}) = \bar{\eta}_i(\bar{Y}) = \bar{\eta}_i(\bar{Z}) = 0, \\
&\quad \mathrm{d}x_i(\bar{X}) = \mathrm{d}x_i(\bar{Y}) = -\delta_{s+1}^i, \ \mathrm{d}x_i(\bar{Z}) = \delta_1^i.
\end{aligned}$$

Evaluating these vector fields at the formula (31), we obtain

$$\begin{aligned}
\sum_{cyc} \left(\left(\nabla_{\bar{X}} \bar{\Omega}\right)\left(\bar{Y}, \bar{Z}\right) - \left(\nabla_{\bar{J}\bar{X}} \bar{\Omega}\right)\left(\bar{J}\bar{Y}, \bar{Z}\right) \right) &= -\mathrm{d}x_1\left(\bar{Z}\right)\bar{\eta}_{s+1}\left(\nabla_{\bar{J}\bar{X}}\xi_1\right)\mathrm{d}x_{s+1}\left(\bar{Y}\right) \\
- \mathrm{d}x_1\left(\bar{Z}\right)\bar{\eta}_{s+1}\left(\nabla_{\bar{J}\bar{Y}}\xi_1\right)\mathrm{d}x_{s+1}\left(\bar{X}\right) &= -2\frac{h^{1)}}{h} \neq 0
\end{aligned}$$

and, in (32)

$$\begin{aligned}
&\left(\nabla_{\bar{X}} \bar{\Omega}\right)\left(\bar{X}, \bar{Y}\right) - \left(\nabla_{\bar{J}\bar{X}} \bar{\Omega}\right)\left(\bar{J}\bar{X}, \bar{Y}\right) \\
&\quad = \sum_{i,j=1}^{s+1} \left\{ \mathrm{d}x_i(\bar{X})\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right) - \mathrm{d}x_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{X}\right) \right. \\
&\quad\quad \left. -\bar{\eta}_i\left(\bar{X}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\bar{\eta}_j\left(\bar{Y}\right) - \mathrm{d}x_i\left(\bar{Y}\right)\bar{\eta}_j\left(\nabla_{\bar{J}\bar{X}}\xi_i\right)\mathrm{d}x_j\left(\bar{X}\right) \right\} \\
&\quad = -\mathrm{d}x_{s+1}\left(\bar{Y}\right)\bar{\eta}_{s+1}\left(\nabla_{\bar{J}\bar{X}}\xi_{s+1}^*\right)\mathrm{d}x_{s+1}\left(\bar{X}\right) = -\frac{h^{s+1)}}{h} \neq 0,
\end{aligned}$$

as expected. It follows that $\bar{M}$ does not fall in any proper subclass of the total class $W_1 \oplus W_2 \oplus W_3 \oplus W_4$.

7 Conclusions

Given a trans-S-manifold M, we have proved that the product manifold $M \times \mathbb{R}^s$ is a Hermitian manifold and lies in the class $W_3 \oplus W_4$. This fact generalizes the trans-Sasakian case in which $M \times \mathbb{R}$ lies in the class W_4. However, there exist metric f-manifolds M such that $M \times \mathbb{R}^s$ lies in the class $W_3 \oplus W_4$ without M being trans-S-manifolds. Consequently, this class cannot be used to define trans-S manifolds. Finally, we have presented an example of an almost trans-S-manifold M such that the product manifold $M \times \mathbb{R}^s$ lies only in the most general class $W_1 \oplus W_2 \oplus W_3 \oplus W_4$.

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