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Parabolic subgroups of large-type Artin groups

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Abstract

We show that the geometric realisation of the poset of proper parabolic subgroups of a large-type Artin group has a systolic geometry. We use this geometry to show that the set of parabolic subgroups of a large-type Artin group is stable under arbitrary intersections and forms a lattice for the inclusion. As an application, we show that parabolic subgroups of large-type Artin groups are stable under taking roots and we completely characterise the parabolic subgroups that are conjugacy stable.

We also use this geometric perspective to recover and unify results describing the normalisers of parabolic subgroups of large-type Artin groups.

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1. *Introduction*

Artin groups are a class of groups strongly related to Coxeter groups, and defined as follows: Let S be a finite set, and for every distinct $s, t \in S$, choose an integer $m_{st} \in \{2, 3, \ldots, \infty\}$. The associated **Artin group** is given by the following presentation:

$$A_S := \langle S \mid \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \text{ when } m_{st} \neq \infty \rangle.$$

If we add the relations $s^2 = 1$ for all $s \in S$, we obtain the associated **Coxeter group** W_S. Every Artin group A_S has an associated **Coxeter graph** Γ_S defined as follows:

(i) the set of vertices of Γ_S is S;

(ii) there is an edge connecting s and t if and only if $m_{s,t} \neq \infty$. This edge is labelled with $m_{s,t}$.

Many questions remain open for general Artin groups, such as whether they are torsion-free, whether they have a soluble word problem, or whether the satisfy the celebrated $K(\pi, 1)$-conjecture. However, several classes of Artin groups are better understood, for instance: **right-angled** Artin groups ($m_{st} = 2$ or ∞ for all $s, t \in S$), Artin groups **of spherical type** (such that the associated Coxeter group W_S is finite) and Artin groups of **large type** ($m_{ab} \geq 3$ for all $a, b \in S$).

The aim of this paper is to investigate the structure of certain subgroups of large-type Artin groups. For a general Artin group A_S with generating set S, it is a theorem of [23] that the subgroup generated by a subset $S' \subset S$ is isomorphic to the Artin group $A_{S'}$. The various subgroups $A_{S'}$, for subsets S' of S, are called the **standard parabolic subgroups** of A_S, and their conjugates are the **parabolic subgroups** of A_S. A parabolic subgroup conjugated to a standard parabolic subgroup $A_{S'}$ will be said to be of **type** S'. An Artin group that does not decompose as the direct product of two of its standard parabolic subgroups is called **irreducible**. Since a parabolic subgroup can naturally be viewed as an Artin group by the above, one defines similarly the notion of **irreducible** parabolic subgroup.

Parabolic subgroups form a natural class of subgroups that has been playing an increasing role in the geometric study of Artin groups in recent years. Indeed, several complexes have been associated to Artin groups using the combinatorics of parabolic subgroups. For instance, Deligne complexes and their variants are built out of (cosets of) standard parabolic subgroups of spherical type [6], and have been used to study various aspects of Artin groups: $K(\pi, 1)$-conjecture [6, 21], acylindrical hyperbolicity [7, 18, 24], Tits alternative [17], etc. More recently, using the connections between braid groups and mapping class groups, the irreducible parabolic subgroups have been used to define a possible analogue of the complex of curves for Artin groups of spherical type [9, 19]. The geometry of this complex is currently being intensively studied.

The combinatorics of the set of parabolic subgroups of Coxeter groups are well understood. For instance, it is known that the intersection of any subset of parabolic subgroups of a Coxeter group is itself a parabolic subgroup [22]. This implies in particular that the set of parabolic subgroups is a lattice for the inclusion. By contrast, the analogous problem is open for general Artin groups:

Question. Let A_S be a general Artin group. Is the set of parabolic subgroups stable under arbitrary intersections?

The answer to this question is known for braid groups: a braid group can be seen as the mapping class group of an n-punctured disc $\mathcal{D}_n$. In this situation, parabolic subgroups are in bijection with isotopy classes of non-degenerated simple closed multicurves, each of them defining a disjoint union of (at least 2-punctured) discs in $\mathcal{D}_n$. An intersection between these families of discs can be defined (see Farb and Margali [10, section 1] to get an idea of the construction). This corresponds to the intersection of parabolic subgroups of the braid group and gives us an affirmative answer to our question. This answer was recently generalised to all Artin groups of spherical type by [9] using Garside theory. For so-called Artin groups **of type FC**, it was shown that the intersection of two parabolic subgroups **of spherical type** is again a parabolic subgroup of spherical type [19]. However, the case of general parabolic subgroups remains open.

Besides being interesting in their own right, such results about the poset of parabolic subgroups can be valuable tools in studying the structure of Artin groups. For instance, the

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analogue of Theorem A for Artin groups of spherical type was a key ingredient in the proof that Artin groups of type FC satisfy the Tits alternative [17].

In this paper, we solve this problem for Artin groups of large-type:

THEOREM A. *Let A_S be a large-type Artin group. Then the intersection of an arbitrary subset of parabolic subgroups of A_S is itself a parabolic subgroup. Moreover, the set of parabolic subgroups of A_S is a lattice for the inclusion.*

Note that a consequence of this theorem is that every subset of A_S is contained in a unique minimal parabolic subgroup. This generalises to large-type Artin groups the notion of **parabolic closure** known for Coxeter groups [22] and Artin groups of spherical type [9].

The approach in this paper is geometric in nature. We associate to each Artin group A_S a simplicial complex X_S, called its **Artin complex**, whose first barycentric subdivision is exactly the geometric realisation of the poset of proper parabolic subgroups of A_S. In essence, the Artin complex X_S is the complex obtained by modifying the construction of the Deligne complex in order to allow *all* proper standard parabolic subgroups instead of those of spherical type (see section 2 for more details). The advantage in considering this complex is that all the parabolic subgroups of A_S arise as stabilisers of simplices of X_S and can thus be studied geometrically. In particular, studying intersections of parabolic subgroups can be done if we have a sufficiently strong control over the (combinatorial) geodesics of X_S between two simplices. This is possible for large-type Artin groups, as we show that these complexes are non-positively curved in an appropriate sense. The key geometric result of this article is the following:

THEOREM B. *Let A_S be a large-type Artin group on at least 3 generators. Then its Artin complex X_S is systolic.*

Large-type Artin groups were recently shown by [15] to be systolic groups. However, we emphasise that the systolic geometry appearing here is of a rather different nature: the systolic complex associated to A_S considered by Huang–Osajda is essentially a (thickened) Cayley graph of A_S for the standard generating set, and as such is quasi-isometric to A_S. By contrast, the Artin complex X_S studied here is quasi-isometric to the Cayley graph of A_S with respect to all its proper parabolic subgroups, and in particular the action of A_S on X_S is cocompact but far from being proper.

As an application, we solve the conjugacy stability problem for parabolic subgroups of large-type Artin groups. A subgroup H of a group G is **conjugacy stable** if for every pair of elements $a, b \in H$ such that $a = \alpha^{-1} b \alpha$ there is $\beta \in H$ such that $a = \beta^{-1} b \beta$. A natural question to ask is which parabolic subgroups of an Artin group are conjugacy stable. This problem had already been solved for parabolic subgroups of spherical Artin groups [5], generalising pre-existing results for braids of [13]. We answer this question for large-type Artin groups:

THEOREM C. *Let A_X be a standard parabolic subgroup of a large-type Artin group A_S. Then A_X is not conjugacy stable in A_S if and only if there exist vertices a and b of Γ_X that are connected by an odd-labelled path in Γ_S and that are not connected by an odd-labelled path in Γ_X.*

Notice that conjugacy stability is preserved under subgroup conjugation, hence the previous theorem classifies all parabolic subgroups of a large-type Artin group under conjugacy stability.

As another application, we show that parabolic subgroups of large-type Artin groups are stable under taking roots, whose analogue for Artin groups of spherical type was proved in [9, corollary 8·3]

THEOREM D. *Let A_S be a large-type Artin group, let P be a parabolic subgroup of A_S, and let $g \in A_S$. If $g^n \in P$ for some non-zero integer n, then $g \in P$.*

Beside the intersection properties of parabolic subgroups, the previous result relies on understanding the fixed-point sets and normalisers of parabolic subgroups. Their structure has been studied by various authors as we explain below, but the results are a bit hidden in the literature. In the case of large-type Artin groups, our approach provides a unifying perspective that allows us to recover all these results within a single framework. We mention the full result here for ease of reference and we re-prove it with our techniques, as we believe such results on normalisers of parabolic subgroups are of independent interest:

THEOREM E. *Let A_S be a large-type Artin group and let P be a parabolic subgroup of type S'.*

(i) *If $|S'| \geq 2$, then $N(P) = P$.*

(ii) *If $|S'| = 1$, then $N(P)$ splits as a direct product of the form*

$$N(P) = P \times F,$$

where F is a finitely-generated free group. Moreover, there is an explicit description of a basis of F (see Corollary 34 for details).

The structure of normalisers of parabolic subgroups in Artin groups of large type had already been investigated by Luis Paris and Eddy Godelle, although it is a bit hidden in their papers. Recall that an Artin group that cannot be decomposed as the direct product of two of its standard parabolic subgroups is called irreducible. In [20, section 4], the conjugation of standard parabolic subgroups is described by an algorithm. In particular, we know that the only pairs of different irreducible standard parabolic subgroups that can be conjugated are the spherical ones. In the large case, as all parabolic subgroups are irreducible and the only spherical parabolic subgroups are the **dihedral** ones (i.e. the parabolic subgroups on two generators), the situation is as follows: A_X and $A_{X'}$ are conjugate if and only if $X = X'$ or $X = \{a\}, X' = \{b\}$ and a and b are connected in Γ_S by an odd-labelled path. [12, definition 4·1, corollary 4·12] tell us that the conjugating elements between two (possibly equal) standard parabolic subgroups A_X and $A_{X'}$ must be the product of an element in A_X and an element associated to the previous path. If $|X| > 1$, such a path does not exists and then $N(A_X) = A_X$. If $|X| = 1$, the description of the normaliser is similar to the one given in Corollary 34. However, the description Godelle gives there is set-theoretic and does not describe the direct product structure.

The structure of the normaliser of cyclic parabolic subgroups for large-type Artin groups (and more generally two-dimensional Artin groups) had been obtained, albeit under a different name, in [18, proposition 4·5]. Moreover, a basis of the corresponding free group had been stated as a remark, but without details.

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The paper is organised as follows. In Section 2, we introduce the Artin complex of a general Artin group, and show that its local structure is particularly well-behaved: The links of simplices are themselves (smaller) Artin complexes, see Lemma 6. In Section 3, we use this local structure to prove Theorem B. Section 4 exploits the systolic geometry of the Artin complex to prove Theorem 11. In Section 5, we study the geometry of fixed-point sets of parabolic subgroups in order to prove Theorem E. Finally, we prove Theorem C and Theorem D in Section 6.

2. *The Artin complex*

The goal of this section is to introduce our main geometric object: the Artin complex associated to an Artin group. Later on, we present some of its basic properties. When talking about complexes of groups, we will use the notations of [**3**, chapter II·12].

Definition 1. Consider an Artin group A_S with $|S| \geq 2$, and a simplex K of dimension $|S|-1$. We define a simplex of groups over K as follows. The simplex K is given a trivial local group. There is a one-to-one correspondance between the elements $s_i \in S$ and the codimension 1 faces of K, and we denote by Δ_{s_i} these codimension 1 faces. In particular, Δ_{s_i} is given the local group $\langle s_i \rangle$. Changing the codimension, there is a bijection between the strict subsets of S and the faces of K. Every face of K of codimension k can be written uniquely as the intersection

$$\Delta_{S'} := \bigcap_{s_i \in S'} \Delta_{s_i} \text{ for some } S' \subsetneq S \text{ with } |S'| = k.$$

The face $\Delta_{S'}$ is then given the local group $A_{S'}$.

The morphism associated to an inclusion of faces $K_{S''} \subset K_{S'}$ is the natural inclusion $\psi_{S'S''} : A_{S''} \hookrightarrow A_{S'}$. Let $\mathcal{P}$ be the poset of standard parabolic subgroups of A_S ordered with natural inclusion. As each $A_{S'}$ is itself an Artin group [**23**], there is a simple morphism, $\varphi : G(\mathcal{P}) \hookrightarrow A_S$, given by inclusion, from the complex of groups to the Artin group. The complex $X_S := D_K(\mathcal{P}, \varphi)$ obtained by development of $\mathcal{P}$ over K along φ is called the **Artin complex** associated to A_S (see [**3**, theorem II·12·18] for the definition of development, see also the remark below).

Note that the action of A_S on X_S is without inversions and cocompact, with strict fundamental domain a single simplex which is isomorphic to K. To avoid any confusion, we will from now on denote by $\overline{K}$ the quotient space and by $\overline{\Delta}_{S'}$ its faces, and we will denote by K a chosen fundamental domain of X_S and by $\Delta_{S'}$ its faces. For every simplex Δ of X_S, there is a unique subset $S' \subsetneq S$ such that Δ is the same orbit as $\Delta_{S'}$. We say that such a simplex is **of type S'**.

Remark 2. In [**3**, proof of theorem II·12·18], the authors give a topological description of the spaces obtained by development of such complexes of groups. In light of this, the Artin complex X_S can also be described by the following:

$$X_S := D_K(\mathcal{P}, \varphi) := A_S \times K \,/\!\sim,$$

where $(g,x) \sim (g',x') \iff x = x'$ and $g^{-1}g'$ belongs to the local group of the smallest simplex of K containing x.

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Remark 3. Another perhaps more intuitive way to look at X_S is the following. Consider the poset of proper parabolic subgroups of A_S and its geometric realisation P_S, defined as follows:

(i) the vertex set of P_S is the set of *proper* parabolic subgroups of A_S;

(ii) there is a $(n-1)$-simplex between vertices of P_S corresponding to proper parabolic subgroups $P_1, \ldots, P_n$ whenever there is a sequence of inclusions $P_n \subsetneq \cdots \subsetneq P_1$. This happens if and only if there is an element $g \in A_S$ and a proper subsets $S^{(n)} \subsetneq \cdots \subsetneq S^{(1)}$ of S such that $P_i = gA_{S^{(i)}}$.

Then P_S is exactly the barycentric subdivision of X_S.

LEMMA 4. *Let A_S be an Artin group and let X_S be its Artin complex. Then X_S is connected. Additionally, if $|S| \geq 3$, then X_S is simply-connected.*

Note that in the case where $|S| = 2$, then X_S is a graph that is not a tree. (It contains for instance loops corresponding to relations of the form $aba\cdots = bab\cdots$)

Proof. This is a direct consequence of [3, chapter II·12, proposition 12·20]. X_S is connected because the Artin group A_S is generated by its standard parabolic subgroups. Moreover, if $|S| \geq 3$, then A_S is the colimit of its standard parabolic subgroups, by [23], and thus $X_S = D_\Delta(\mathcal{P}, \varphi)$ is the universal cover of the complex of groups $G(\mathcal{P})$, hence is simply-connected.

Definition 5. Let Y be a simplex in a simplicial complex X. The **link** of Y in X is the simpicial complex $Lk_X(Y)$ consisting of the simplices of X that are disjoint from Y and which together with Y span a simplex of X.

LEMMA 6. *Let A_S be an Artin group with Artin complex X_S. Then the link of a simplex of type S' is isomorphic to the Artin complex $X_{S'}$ associated to the Artin group $A_{S'}$.*

Proof. By [3, chapter II·12, construction 12·24], it is possible to describe the link of a simplex in the development of a complex of groups as the development of an appropriate subcomplex of groups. We explain below how this applies to X_S.

The link of $\overline{\Delta}_{S'}$ in $\overline{K}$ is a simplex of dimension $|S'| - 1$, whose poset of faces is isomorphic to the poset of proper subsets of S' ordered with the inclusion. The complex of groups $G(\overline{K})$ induces a complex of groups on the link $Lk_{\overline{K}}(\overline{\Delta}_{S'})$. Moreover, there is a simple morphism $\varphi_{S'} : G(Lk_{\overline{K}}(\overline{\Delta}_{S'})) \to A_{S'}$ given by the family of homomorphisms

$$(\varphi_{S'})_{S''} : A_{S''} \xrightarrow[\psi_{S'S''}]{} A_{S'}.$$

It follows from the construction described in [3, chapter II·12, construction 12·24] that the link of $Lk_{X_S}(\Delta_{S'})$ is isomorphic to the development $D(Lk_{\overline{K}}(\overline{\Delta}_{S'}), \varphi_{S'})$. Note that the induced complex of groups on $Lk_{\overline{K}}(\overline{\Delta}_{S'})$ is naturally isomorphic to the complex of groups associated to $A_{S'}$ in Definition 1. Moreover, the simple morphism $\varphi_{S'}$ coincides with the simple morphism used in Definition 1 to define the Artin complex $X_{S'}$. Putting everything together, it now follows that the link $Lk_{X_S}(\Delta_{S'})$ is isomorphic to $X_{S'}$.

This argument generalises in a straightforward way to any simplex $g\Delta_{S'}$ of X_S of type S'.

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3. *Systolicity*

The goal of this section is to prove Theorem B. Recall that a subcomplex Y of a simplicial complex X is **full** if every simplex of X spanned by vertices of Y is a simplex of Y. If γ is a combinatorial path in the 1-skeleton of X, then the **simplicial length** of γ is the number $|\gamma|$ of edges contained in γ. We will denote by $\mathrm{Stab}(T)$ or $\mathrm{Stab}_{X_S}(T)$ the stabiliser of a set of points T in X_S. We introduce a few more definitions from systolic geometry [16]:

Definition 7. The **systole** of a simplicial complex X is

$$\mathrm{sys}(X) := \min\{|\gamma| \mid \gamma \text{ is an embedded full cycle of } X\} \in \{3, 4, \ldots, \infty\}.$$

For $k \in \{3, \ldots, \infty\}$, we say that X is **locally k-large** if $\mathrm{sys}(Lk_X(Y)) \geq k$ for all simplices $Y \subseteq X$. We say that X is **k-large** if it is locally k-large and $\mathrm{sys}\,(X) \geq k$. X is **k-systolic** if it is connected, simply-connected and locally k-large. Finally, X is called **systolic** if it is 6-systolic.

The main result of this section is the following:

THEOREM 8. *Let A_S be an Artin group with $|S| \geq 3$. If all coefficients in A_S are at least $k \in \{3, \ldots, \infty\}$, then its Artin complex X_S is $2k$-systolic. In particular, if A_S is of large type, then X_S is systolic.*

In order to prove this theorem, we need the following lemma:

LEMMA 9. *Let A_S be an Artin group on two generators a, b with coefficient $m_{ab} \in \{3, \ldots, \infty\}$ and Artin complex X_S. Then* $\mathrm{sys}\,(X_S) = 2m_{ab}$.

Proof. If $m_{ab} = \infty$, it follows directly from the definition of the Artin complex that X_S is the Bass–Serre tree associated to the splitting $\langle a \rangle * \langle b \rangle$. The result is then immediate.

Let us now assume that $m_{ab} < \infty$. Let e be the edge in X whose vertices x, y correspond to the cosets $\langle a \rangle$ and $\langle b \rangle$. Let γ be a non-backtracking loop in X_S. Since X_S is a bipartite graph coloured by the cosets of $\langle a \rangle$ and $\langle b \rangle$ respectively, the length of γ is even. Denote by $e_0, e_1, \ldots, e_k$ the edges of γ. Since the action of A_S on X_S is transitive on edges, let us assume that $e_0 = e$.

Note that the action of $\langle a \rangle$ is transitive on the set of edges around x, and so is the action of $\langle b \rangle$ on the edges around y. Assume without loss of generality that γ first goes through x, i.e. e_1 and e_0 share the vertex x. Then e_1 must be of the form $a^{r_1}e$, for some $r_1 \in \mathbf{Z}\backslash\{0\}$. Note that the edges e_1 and e_2 then share the vertex $a^{r_1}y$. The action of $a^{r_1}\langle b \rangle a^{-r_1}$ is transitive on the set of edges around $a^{r_1}y$, thus e_2 must of the form $a^{r_1}b^{r_2}e$, for some $r_2 \in \mathbf{Z}\backslash\{0\}$. We continue this process by induction until γ stops. In particular, the final edge e_k is of the form

$$a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}$$

for non-zero integers $r_1, \ldots, r_k$. But since $e_k = e$ as γ is a loop, we get $a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}e = e$. Since $\mathrm{Stab}(e) = \{1\}$, it follows that $a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}$ must be trivial in A_S. But it is also a non-trivial word, as γ is not homotopically trivial. By [1, lemma 6], we must have $k \geq 2m_{ab}$. Hence, the combinatorial length of γ is $|\gamma| = k \geq 2m_{ab}$.

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We can now prove the main theorem:

Proof of Theorem 8. We will prove by induction on the number $|S|$ of generators of the Artin groups A_S that their associated Artin complexes X_S are $2k$-systolic.

If $|S| = 3$, we know from Lemma 4 that X_S is connected and simply connected. It only remains to show that for all $g \in A_S$, for all $S' \subsetneq S$, the simplex $g \cdot \Delta_{S'}$ is such that $Lk_{X_S}(g \cdot \Delta_{S'})$ is $2k$-large. If $|S'| = 2$, then the link $Lk_{X_S}(g \cdot \Delta_{S'})$ is isomorphic to the Artin complex $X_{S'}$ associated to the Artin group $A_{S'}$ (Lemma 6), and the latter is $2k$-large by Lemma 9. The cases $|S'| = 0$ or 1 are trivial.

Let us now assume that $|S| > 3$ and that every Artin complex $A_{S'}$ with $S' \subsetneq S$ is $2k$-systolic. Again, we know from Lemma 4 that X_S is connected and simply connected, so it only remains to show that for all $g \in A_S$, for all $S' \subsetneq S$, the simplex $g \cdot \Delta_{S'}$ is such that $Lk_{X_S}(g \cdot \Delta_{S'})$ is $2k$-large. If $|S'| \geq 2$, then $Lk(g \cdot \Delta_{S'}, X_S)$ is isomorphic to the Artin complex $X_{S'}$ associated to the Artin group $A_{S'}$ (Lemma 6). The latter is $2k$-systolic by the induction hypothesis, hence is $2k$-large as well [**16**, proposition 1·4]. Once again, the cases $|S'| = 0$ or 1 are trivial.

4. *Intersection of parabolic subgroups*

The aim of this section is to use the systolicity of the Artin complex of an Artin group of large type to prove Theorem A. We will do it by proving the following theorem:

Definition 10. Let P_1 and P_2 be two parabolic subgroups of an Artin group A_S such that $P_1 \subseteq P_2$. We say that P_1 is a parabolic subgroup of P_2 if $P_1 \subseteq P_2$ is conjugate to an inclusion of standard parabolic subgroups $A_{S''} \subseteq A_{S'}$, $S'' \subseteq S'$.

THEOREM 11. *Let A_S be an Artin group of large-type.*

1. *The intersection of two parabolic subgroups of A_S is again a parabolic subgroup of A_S.*
2. *If P_1 and P_2 are two parabolic subgroups of A_S such that $P_1 \subseteq P_2$, then P_1 is a parabolic subgroup of P_2.*

Note that the second item in the previous theorem is already a result of [**12**, theorem 3]. However, we believe the reader may be interested in recovering this result directly from our perspective.

In all this section, A_S denotes an Artin group on at least 3 generators. First notice the Artin complex allows us to understand geometrically the parabolic subgroups of A_S, via the following correspondence:

LEMMA 12. *Let A_S be an Artin group on at least 3 generators and let X_S be its associated Artin complex. Then:*

(i) *the parabolic subgroups of A_S are exactly the stabilisers of simplices of X_S;*

(ii) *let Δ be a simplex of X_S. The parabolic subgroups of* $\mathrm{Stab}_{X_S}(\Delta)$ *are exactly the stabilisers of the simplices that contain Δ.*

Proof. By construction, every standard parabolic subgroup $A_{S'}$ is precisely the stabiliser of some simplex $\Delta_{S'}$ lying on the fundamental domain K of X_S, and viceversa. Moreover,

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any parabolic subgroup of the form $gA_{S'}g^{-1}$ is the stabiliser of the simplex $g \cdot \Delta_{S'}$, $g \in A$. To prove the first claim, notice that any simplex of X_S can be expressed as $g' \cdot \Delta'$, where Δ' is in K and $g' \in A$.

Let us now prove the second claim. On the one hand, let P be a parabolic subgroup of $\mathrm{Stab}_{X_S}(\Delta)$. Up to conjugation, we can suppose that Δ lies in K of X_S, and that P is the stabiliser of a simplex Δ' that also lies in K. Now notice that, by construction of the fundamental domain, this implies that Δ' contains Δ, as we desired. On the other hand, note that if Δ'' is a simplex that contains Δ, then we can find an element $g \in A_S$ such that $g \cdot \Delta''$ belongs to K. Hence $g'\mathrm{Stab}_{X_S}(\Delta'')g'^{-1} \subseteq g'\mathrm{Stab}_{X_S}(\Delta)g'^{-1}$ is an inclusion of standard parabolic subgroups, as we wanted to prove.

Remark 13. The previous correspondence is not a bijection between the parabolic subgroups of A_S and the simplices of its Artin complex, as two distinct simplices may have the same stabiliser.

Secondly, we mention the following result from systolic geometry, well known to experts, that will be used in our proof:

LEMMA 14. *Let G be a group acting without inversions on a systolic complex Y, and let H be a subgroup of G. Suppose that H fixes two vertices v and v' of Y. Then H fixes pointwise every combinatorial geodesic between v and v_0.*

Proof. We prove the result by induction on the combinatorial distance between v and v'. If $d(v, v') = 1$, the result is immediate, as there is unique edge between v and v'. Suppose by induction that the result is true for vertices at distance at most $n \geq 1$, and let v, v' be two vertices of Y at distance $n+1$. Since Y is systolic, it follows from [**16**, corollary 7·5] that the combinatorial ball of radius n around v', denoted $B(v', n)$, is a convex subset of Y in the sense of [**16**, definition 7·1]. Moreover, by [**16**, lemma 7·7], this combinatorial ball intersects the combinatorial ball $B(v,1)$ along a single simplex. This implies that there exists a simplex Δ of Y containing v, and such that every combinatorial geodesic from v to v' starts with an edge of Δ. In particular, we define Δ' as the simplex of Y spanned by the first edges of *all* the combinatorial geodesics from v to v'. Since H fixes v and v', H preserves the set of combinatorial geodesics from v to v', and in particular H stabilises Δ'. Since G acts on Y without inversion, it follows that H fixes Δ' pointwise.

Let γ be a combinatorial geodesic from v to v'. By the above, H fixes the first edge e of γ. Let v_1 be the vertex of e distinct from v. We have that H fixes v_1 and v', and these two vertices are at combinatorial distance n. By the induction hypothesis, H fixes pointwise the portion of γ between v_1 and v', and it now follows that H fixes pointwise all of γ. This concludes the induction.

We proceed now to the proof of the main theorem of this section:

Proof of Theorem 11. We will prove the theorem by induction on the number n of generators of A_S. If $n = 2$, A_S is an Artin group on two generators a, b and there are two cases to consider. If $m_{ab} < \infty$, then A_S is a spherical Artin group, so item 1 follows from [**9**, theorem 9·5] and item 2 follows from [**11**, theorem 0·2]. If $m_{ab} = \infty$, then A_S is a free group on two generators a, b. Moreover, the proper parabolic subgroups are either trivial or infinite cyclic.

Since the action of A_S on the Bass–Serre tree associated to the splitting $\langle a\rangle * \langle b\rangle$ has trivial edge stabilisers, it follows that two distinct proper parabolic subgroups intersect trivially. Thus, item 1 and item 2 follow immediately.

Let us now assume that the result is known for Artin groups of large type on at most n generators with $n \geq 2$, and let A_S be an Artin group of large type on $n+1$ generators. Let X_S be its associated Artin complex.

Claim 1. Let $e_1, \ldots, e_k$ be a combinatorial path p in X_S. Then there exists a simplex Δ of X_S containing the edge e_k such that

$$\bigcap_{1\leq i\leq k} \mathrm{Stab}_{X_S}(e_i) = \mathrm{Stab}_{X_S}(\Delta).$$

Proof of Claim 1. We will prove the claim by induction on k. If $k=1$, p is just the edge e_1 and the proof is trivial. Now suppose that the claim is true for k and let us prove it for $k+1$. By applying the induction hypothesis to the subpath $e_1, \ldots, e_k$, we will then have

$$\bigcap_{1\leq i\leq k+1} \mathrm{Stab}_{X_S}(e_i) = \mathrm{Stab}_{X_S}(\Delta') \cap \mathrm{Stab}_{X_S}(e_{k+1}),$$

where Δ' is a simplex containing the edge e_k. Let v be a vertex contained in both e_k and e_{k+1}. By Lemma 12, this means that both $\mathrm{Stab}_{X_S}(\Delta')$ and $\mathrm{Stab}_{X_S}(e_{k+1})$ are parabolic subgroups of $\mathrm{Stab}_{X_S}(v)$. Also, up to conjugacy, $\mathrm{Stab}(v)$ is an Artin group on n generators. Therefore, by the induction hypothesis on n, $\mathrm{Stab}_{X_S}(\Delta') \cap \mathrm{Stab}_{X_S}(e_{k+1})$ is a parabolic subgroup of $\mathrm{Stab}(v)$ contained in $\mathrm{Stab}_{X_S}(e_{k+1})$, so it is a parabolic subgroup of $\mathrm{Stab}_{X_S}(e_{k+1})$. Geometrically, $\mathrm{Stab}_{X_S}(\Delta') \cap \mathrm{Stab}_{X_S}(e_{k+1})$ is the stabiliser of some simplex containing e_{k+1}. This completes the proof of Claim 1.

Claim 2. Let Δ_1 and Δ_2 be two simplices of X_S. Then there exists a simplex Δ of X_S containing Δ_2 such that $\mathrm{Stab}_{X_S}(\Delta_1) \cap \mathrm{Stab}_{X_S}(\Delta_2) = \mathrm{Stab}_{X_S}(\Delta)$.

Proof of Claim 2. Let Δ' be any simplex of X_S and let $V_{\Delta'}$ be the set of vertices of Δ'. As the action of A_S on X_S is without inversions, we have that $\mathrm{Stab}_{X_S}(\Delta') = \cap_{w\in V_{\Delta'}} \mathrm{Stab}(w)$. Define a combinatorial path p that is the concatenation of the three following paths: a combinatorial path p_1 that travels along every vertex in V_{Δ_1}; a combinatorial geodesic p_2 between the endpoint of p_1 and V_{Δ_2}; and a combinatorial path that starts in the endpoint of p_2 and travels along every vertex in V_{Δ_2}. Denote the endpoint of p by v and let E_p be the set of edges of p. Then, by Claim 1 and Lemma 14,

$$\mathrm{Stab}_{X_S}(\Delta_1) \cap \mathrm{Stab}_{X_S}(\Delta_2) = \bigcap_{w\in V_{\Delta_1}\cup V_{\Delta_2}} \mathrm{Stab}_{X_S}(w) = \bigcap_{e\in E_p} \mathrm{Stab}_{X_S}(e) = \mathrm{Stab}_{X_S}(\Delta),$$

for some simplex Δ containing v. Now we need to show that Δ contains also Δ_2. Notice that $\mathrm{Stab}_{X_S}(\Delta_2)$ contains $\mathrm{Stab}_{X_S}(\Delta)$ and both $\mathrm{Stab}_{X_S}(\Delta_2)$ and $\mathrm{Stab}_{X_S}(\Delta)$ are parabolic subgroups of $\mathrm{Stab}_{X_S}(v)$. This group is, up to conjugacy, an Artin group on n generators. So by using the induction hypothesis on n, $\mathrm{Stab}_{X_S}(\Delta)$ is a parabolic subgroup of $\mathrm{Stab}_{X_S}(\Delta_2)$, which means that we can choose Δ to contain Δ_2. This finishes the proof of Claim 2.

In particular, note that Claim 2 together with Lemma 12 implies that the parabolic subgroups of A_S are stable under intersection, proving item 1.

Claim 3. For every pair of simplices Δ_1 and Δ_2 of X_S such that $\mathrm{Stab}_{X_S}(\Delta_1) \subseteq \mathrm{Stab}_{X_S}(\Delta_2)$, there exists a simplex Δ of X_S containing Δ_2 such that $\mathrm{Stab}_{X_S}(\Delta_1) = \mathrm{Stab}_{X_S}(\Delta)$.

Proof of Claim 3. Just notice that $\mathrm{Stab}_{X_S}(\Delta_1) = \mathrm{Stab}_{X_S}(\Delta_1) \cap \mathrm{Stab}_{X_S}(\Delta_2)$, so by Claim 2 there is a simplex Δ containing Δ_2 such that $\mathrm{Stab}_{X_S}(\Delta_1) = \mathrm{Stab}_{X_S}(\Delta)$. This completes the proof of the claim.

We now explain why this claim implies that A_S satisfies item 2. Let P_1 and P_2 be two parabolic subgroups of A_S such that $P_1 \subseteq P_2$. By Lemma 12 there are simplices Δ_1 and Δ_2 of A_S such that $P_1 = \mathrm{Stab}_{X_S}(\Delta_1)$ and $P_2 = \mathrm{Stab}_{X_S}(\Delta_2)$. By Claim 3, there exists a simplex Δ of X_S containing Δ_2 such that $\mathrm{Stab}_{X_S}(\Delta_1) = \mathrm{Stab}_{X_S}(\Delta)$. Again by Lemma 12, this means that P_1 is a parabolic subgroup of P_2, as we wanted to prove.

Remark 15. Notice that the only place where the systolic geometry was used in the previous proof is the following argument coming from Lemma 14: if an element fixes two simplices, then it fixes pointwise a combinatorial path between these simplices. Therefore, a strong enough requirement to prove Theorem 11 for any Artin group A_S is to have this fixing-path condition in its Artin complex X_S.

Question. Let X_S be the Artin complex of any Artin group A_S and let $g \in A_S$ be an element fixing Δ_1 and Δ_2. Is there a combinatorial path between Δ_1 and Δ_2 fixed by g pointwise?

Following the release of this paper, Blufstein generalised this approach to a larger class of two-dimensional Artin groups [**2**].

We can generalise some interesting results concerning parabolic results that were previously shown for spherical Artin groups [**9**, section 10]:

COROLLARY 16. *Let A_S be an Artin group of large type. Then an arbitrary intersection of parabolic subgroup of A_S is a parabolic subgroup. In particular:*

(i) *for a subset $B \subset A_S$, there is a unique minimal parabolic subgroup of A_S (with respect to the inclusion) containing B*;

(ii) *the set of parabolic subgroups of A_S is lattice with respect to the inclusion.*

The strategy will be the same standard argument used in [**9**, proposition 10·1]. We can find the generalised FC version of the first statement for spherical parabolic subgroups in [**19**, corollary 3·2].

Proof. Let $\mathcal{P}$ be an arbitrary set of parabolic subgroups of A_S and let $Q = \cap_{P \in \mathcal{P}} P$. Q is contained in every parabolic subgroup in $\mathcal{P}$, so by Theorem 11, we just need to prove that Q is equal to a finite intersection of parabolic subgroups. Notice that every parabolic subgroup is expressed as the conjugate of some standard parabolic subgroup. Since A_S is a countable group and there are only finitely many standard parabolic subgroups of A_S, the

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set of parabolic subgroups of A_S is countable. In particular, $\mathcal{P}$ is countable. Enumerate the elements in $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ and let

$$Q_m = \bigcap_{1 \leq i \leq m} P_i.$$

By Theorem 11, all Q_m's belong to $\mathcal{P}$. As $Q = \cap_{i \in \mathbb{N}} Q_m$, we need to show that the set $\{Q_m \mid m \in \mathbb{N}\}$ is finite.

Let X_S be the Artin complex of A_S. Notice that we have a descending chain

$$Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \cdots$$

By doing an induction on the Claim 3 in the proof of Theorem 11, one can easily see that if $\mathrm{Stab}_{X_S}(\Delta_1) \supsetneq \mathrm{Stab}_{X_S}(\Delta_2) \supsetneq \mathrm{Stab}_{X_S}(\Delta_3) \dots$, the dimension of Δ_i has to be bigger than the dimension of Δ_{i-1}. As the dimension of X_S is finite, the chain cannot be infinite. Therefore, Q is the minimal parabolic subgroup containing every element of $\mathcal{P}$.

To see the first statement, just assume that $\mathcal{P} = \{P \mid B \subset P\}$. For the second statement let P_1 and P_2 be any two parabolic subgroups of A_S. We need a maximal parabolic subgroup R_1 contained in P_1 and P_2 and a minimal parabolic subgroup R_2 containing P_1 and P_2. By all the previous discussion, $R_1 = P_1 \cap P_2$ and R_2 is the minimal parabolic subgroup in $\mathcal{P}$ when $\mathcal{P} = \{P \mid P_1 \cup P_2 \subseteq P\}$.

5. *Normalisers and fixed-point sets of parabolic subgroups*

The aim of this section is to prove Theorem E. In all this section we consider an Artin group A_S with $|S| \geq 3$. For a parabolic subgroup P of A_S, we denote by $\mathrm{Fix}(P)$ (or $\mathrm{Fix}_{X_S}(P)$ if we wish to highlight the ambient complex) the fixed-point set of P in X_S. Since A_S acts on X_S without inversions, $\mathrm{Fix}(P)$ is a subcomplex of X_S. The connection between the normaliser $N(P)$ of a parabolic subgroup P and its fixed-point set $\mathrm{Fix}(P)$ is given by the following:

LEMMA 17. *Let P be a parabolic subgroup of A_S. Then the normaliser $N(P)$ of P satisfies*

$$N(P) = \mathrm{Stab}(\mathrm{Fix}(P)).$$

In addition, an element of A_S belongs to $N(P)$ if and only if it sends some maximal simplex of $\mathrm{Fix}(P)$ *to some maximal simplex of* $\mathrm{Fix}(P)$.

Proof. ($\subseteq$) Let $g \in N(P)$, that is, $gP = Pg$, and let $v \in \mathrm{Fix}(P)$. Then

$$P \cdot (g \cdot v) = g \cdot (P \cdot v) = g \cdot v.$$

In particular, $g \cdot v \in \mathrm{Fix}(P)$ and thus $g \in \mathrm{Stab}(\mathrm{Fix}(P))$.

($\supseteq$) Let $g \in \mathrm{Stab}(\mathrm{Fix}(P))$ and let $\Delta \subseteq \mathrm{Fix}(P)$ be a maximal simplex in the sense that $\mathrm{Stab}(\Delta) = P$. Then $g \cdot \Delta \subseteq \mathrm{Fix}(P)$, thus

$$P \cdot (g \cdot \Delta) = g \cdot \Delta.$$

In particular, gPg^{-1} fixes Δ, hence $gPg^{-1} \subseteq P$. In other words, $g \in N(P)$.

The key geometric result to prove Theorem E by means of studying fixed-point sets is the following:

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PROPOSITION 18. *Let A_S be a large-type Artin groups, and let P be a parabolic subgroup of A_S of type S'.*

(i) *If $|S'| \geq 2$, then* Fix(P) *is a single simplex.*

(ii) *If $|S'| = 1$, then* Fix(P) *is a subcomplex whose dual graph is a simplicial tree (see Definition 24 for the terminology).*

The proof of this proposition will be split into two cases. We first mention a useful observation that will allow for proofs by induction:

LEMMA 19. *For a simplex Δ of* Fix(P) *of type S'', the link $Lk_{\mathrm{Fix}(P)}(\Delta)$ is isomorphic to* $\mathrm{Fix}_{X_{S''}}(P)$.

Proof. We have $Lk_{\mathrm{Fix}(P)}(\sigma) = \mathrm{Fix}(P) \cap Lk_{X_S}(\sigma)$. Since $Lk_{X_S}(\sigma)$ is equivariantly isomorphic to $X_{S''}$ by Lemma 6, the previous intersection is thus isomorphic to $\mathrm{Fix}_{X_{S''}}(P)$.

Parabolic subgroups on at least two generators. We start with the case of a parabolic subgroup P of type S' with $|S'| \geq 2$.

LEMMA 20. *If $|S'| \geq 2$ then* Fix$(A_{S'})$ *is a single simplex Δ such that* $\mathrm{Stab}(\Delta) = A_{S'}$.

Proof. We will be using the following claim:

Claim. If a subcomplex Y of X_S is such that all of its links are simplices or empty, then Y itself is a simplex.

Indeed, if Y is not a simplex, then it contains a combinatorial path u, v, w that forms a geodesic of X_S. The two vertices u, w define two vertices of $Lk_Y(v)$ at distance at least 2 by assumption, hence $Lk_Y(v)$ is not a simplex, which proves the claim.

Recall from Lemma 19 that for a simplex Δ of Fix(P) corresponding to a simplex of type S'', the link $Lk_{\mathrm{Fix}(P)}(\Delta)$ is isomorphic to $\mathrm{Fix}_{X_{S''}}(P)$. If $|S - S'| = 1$, then Fix(P) must be a single vertex v: if it weren't, it would follow from the convexity of Fix(P) (Lemma 14) that P fixes an edge of X_S, which is impossible since in that case P is a maximal proper parabolic subgroup of A_S. Fix$(A_{S'})$ being a single simplex now follows by induction on $|S - S'| \geq 1$ by applying the above Claim. The dimension of Fix$(A_{S'})$ is $|S - S'| - 1$, so by maximality its stabiliser has to be $A_{S'}$.

COROLLARY 21. *If P is a parabolic subgroup of A_S of type S' with $|S'| \geq 2$, then $N(P) = P$.*

Proof. By Lemma 17 we know that $N(P) = \mathrm{Stab}(\mathrm{Fix}(P))$. Moreover, we know from Lemma 20 that there is a simplex Δ in X_S such that $\mathrm{Fix}(P) = \Delta$ and $\mathrm{Stab}(\Delta) = P$. In particular,

$$N(P) = \mathrm{Stab}(\mathrm{Fix}(P)) = \mathrm{Stab}(\Delta) = P.$$

Parabolic subgroups on one generator. We now move to the case of a parabolic subgroup of type S' with $|S'| = 1$. We start with the following general remark:

LEMMA 22. *Let P be a parabolic subgroup of A_S. Then* Fix(P) *is contractible.*

The proof of this lemma will rely on the following notion of convexity from [**16**]:

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Definition 23. A subcomplex Y of a simplicial complex X is **3-convex** if it is full and every combinatorial geodesic of length 2 with endpoints in Y is contained in Y. It is **locally 3-convex** if for every simplex σ of Y, the link $Lk_Y(\sigma)$ is 3-convex in $Lk_X(\sigma)$.

Proof of Lemma 22. By Lemma 14, Fix(P) contains every geodesic between two vertices of Fix(P). In particular, it is connected and 3-convex, hence locally 3-convex by [**16**, fact 3·3·1]. By [**16**, lemma 7·2], Fix(P) is thus contractible.

It turns out that such fixed-point sets have a very simple geometry. We introduce the following:

Definition 24. The **dual graph** T_P of Fix(P) is defined as follows:

(i) vertices of T_P correspond to the simplices of Fix(P) of type $S' \subsetneq S$ with $|S'| = 1$ (called **type 1 vertices**) or $|S'| = 2$ (called **type 2 vertices**);
(ii) we put an edge between a type 1 vertex Δ and a type 2 vertex Δ' whenever $\Delta' \subset \Delta$;
(iii) finally, T_P is the subgraph obtained by removing the type 2 vertices that have valence 1.

We think of T_P as a subgraph of the first barycentric subdivision of Fix(P).
We have the following:

LEMMA 25. *The dual graph T_P is a simplicial tree.*

In a nutshell, the proof of Lemma 25 goes as follows: we construct a sequence of subcomplexes

$$X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_k,$$

where X_0 is the first barycentric subdivision of Fix(P) and $X_k = T_P$, and such that for each $0 \leq i \leq k-1$, X_{i+1} is a deformation retract of X_i. Since X_0 is contractible by Lemma 22, it will then follow that the graph T_P is also contractible, hence is a tree.

We will need the following standard result from algebraic topology to construct deformation retractions:

LEMMA 26. *Let X be a simplicial complex, and let v be a vertex of X whose link $Lk_X(v)$ is contractible. Then the subcomplex spanned by $X - v$ is a deformation retract of X.*

Proof. Since the star $\mathrm{Star}_X(v)$ is isomorphic to a cone over $Lk_X(v)$, we first notice that X is obtained from $X - v$ by coning-off the contractible link $Lk_X(v)$. Recall that for a simplicial complex Y and a contractible subcomplex Z, the quotient map $Y \to Y/Z$ obtained by collapsing Z to a point is a homotopy equivalence, see [**14**, proposition 0·17]. We thus have the following commutative diagram:

$$\begin{array}{ccc} X - v & \hookrightarrow & X \\ \downarrow & & \downarrow \\ (X-v)/Lk_X(v) & = & X/\mathrm{Star}_X(v), \end{array}$$

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where both vertical arrows are homotopy equivalences since $Lk_X(v)$ and its cone $\mathrm{Star}_X(v)$ are contractible. Thus, the inclusion $X - v \hookrightarrow X$ is a homotopy equivalence, and it follows from [**14**, corollary 0·20] that the subcomplex spanned by $X - v$ is a deformation retract of X.

Proof of Lemma 25. Consider the barycentric subdivision $\mathrm{Fix}(P)'$ of $\mathrm{Fix}(P)$. A vertex v of $\mathrm{Fix}(P)'$ corresponds to a simplex of $\mathrm{Fix}(P)$; We will call the dimension of the corresponding simplex the *height* of v. For every $0 \le k \le |S|-2$, we define the subcomplex X_k of $\mathrm{Fix}(P)'$ spanned by the vertices of height at least k. In particular, $X_0 = \mathrm{Fix}(P)'$ and $X_{|S|-2}$ is a subgraph of $\mathrm{Fix}(P)'$ containing T_P.

We now show that for every $0 \le k \le |S|-3$, X_{k+1} is a deformation retract of X_k. Notice that X_k is obtained from X_{k+1} by adding for every vertex v of height k the star $\mathrm{Star}_{X_k}(v)$, which is isomorphic to a simplicial cone over the link $Lk_{X_k}(v)$. Let v be a vertex of height $0 \le k \le |S|-3$. This vertex corresponds to a simplex Δ of $\mathrm{Fix}(P)$ of type S' for some subset $S' \subsetneq S$ with $|S'| \ge 3$. Note that a vertex of X_k adjacent to v must have height greater than k by construction, hence the link $Lk_{X_k}(v)$ is isomorphic to the first barycentric subdivision of $Lk_{\mathrm{Fix}(P)}(\Delta)$. In particular, $Lk_{X_k}(v)$ is isomorphic to the first barycentric subdivision of $\mathrm{Fix}_{X_{S'}}(P)$ by Lemma 19, and hence is contractible by Lemma 22. It thus follows from Lemma 26 that X_{k+1} is a deformation retract of $X_{k+1} \cup \mathrm{Star}_{X_k}(v)$. Since for two distinct vertices v, v' of height k, the subcomplexes $X_{k+1} \cup \mathrm{Star}_{X_k}(v)$ and $X_{k+1} \cup \mathrm{Star}_{X_k}(v')$ intersect along X_{k+1}, we can glue the various deformation retractions into a deformation retraction of

$$X_k = X_{k+1} \cup \bigcup_{\mathrm{height}(v)=k} \mathrm{Star}_{X_k}(v)$$

onto X_{k+1}. Thus, for every $0 \le k \le |S|-3$, X_{k+1} is a deformation retract of X_k. Thus, the graph $X_{|S|-2}$ is a deformation retract of $X_0 = \mathrm{Fix}(P)'$. Since the latter complex is contractible by Lemma 22, so is the graph $X_{|S|-2}$, and it follows that $X_{|S|-2}$ is a tree. Finally, T_P is obtained from $X_{|S|-2}$ by removing the type 2 vertices that have valence 1. Thus, T_P is a deformation retract of $X_{|S|-2}$, hence T_P is a tree.

Note that since $N(P) = \mathrm{Stab}(\mathrm{Fix}(P))$ by Lemma 17, N(P) acts on $\mathrm{Fix}(P)$, hence on the dual tree T_P. We will use this action to prove the following:

LEMMA 27. *The normaliser N(P) of P splits as a direct product $P \times F$, where F is a finitely generated free group.*

Remark 28. It can be shown that the tree T_P is N(P)-equivariantly isomorphic to the standard tree associated to P as considered in [**18**, definition 4·1]. In particular, the proof of Lemma 27 is essentially the same as the proof of [**18**, lemma 4·5]. We however include a proof formulated in our setting for the sake of self-containment.

Since P is a normal subgroup of $N(P)$ acting trivially on T_P by construction of $\mathrm{Fix}(P)$, we can look at the induced action of $N(P)/P$ on T_P. We will use this action to completely describe the normaliser $N(P)$. We first need the following result:

LEMMA 29. *For the action of $N(P)/P$ on T_P we have*:

(i) *type 1 vertices of T_P have a trivial stabiliser*;

(ii) *type 2 vertices of T_P have an infinite cyclic stabiliser.*

Before starting this proof, let us recall a standard result about dihedral Artin groups:

LEMMA 30 ([4]). *Let A_{ab} be a dihedral Artin group with $2 < m_{ab} < \infty$, and let δ_{ab} be its* ***Garside element****, defined as follows*:

$$\delta_{ab} = \underbrace{abab\cdots}_{m_{ab}}.$$

Then the centre of A_{ab} is infinite cyclic and equal to $\langle \delta_{ab} \rangle$ if m_{ab} is even, and $\langle \delta_{ab}^2 \rangle$ otherwise.

Proof of Lemma 29. A type 1 vertex v of T_P corresponds to a maximal simplex of $\mathrm{Fix}(P)$. Such a simplex has stabiliser P by construction, hence $\mathrm{Stab}_{N(P)/P}(v)$ is trivial.

Let v be a type 2 vertex of T_P of type $\{c, d\}$. This vertex corresponds to a simplex with associated coset gA_{cd} for some $g \in A_\Gamma$. It follows from [18, lemma 4·5] and Lemma 30 that we have:

(i) if m_{cd} is even, then

$$\mathrm{Stab}_{N(P)/P}(v) = gZ(A_{cd})g^{-1} = \langle g\delta_{cd}g^{-1} \rangle;$$

(ii) if m_{cd} is odd, then

$$\mathrm{Stab}_{N(P)/P}(v) = gZ(A_{cd})g^{-1} = \langle g\delta_{cd}^2 g^{-1} \rangle,$$

We are now ready to prove Lemma 27.

Proof of Lemma 27. Since two type 1 vertices of T_P corresponding to cosets of the same standard parabolic subgroup are in the same $N(P)$-orbit, hence in the same $N(P)/P$ orbit, it follows that the action of $N(P)/P$ on T_P is cocompact.

Thus, $N(P)$ acts cocompactly and without inversion on a simplicial tree. By Lemma 29 the stabilisers of type 1 vertices are trivial (hence so are the stabilisers of edges) and the stabilisers of type 2 vertices are infinite cyclic. It thus follows from Bass–Serre theory that $N(P)/P$ is a finitely-generated free group, and thus $N(P)$ splits as a semi-direct product $P \rtimes F$, where F is a finitely generated free group. To see that this product is a direct product, it is enough to show that P is central in $N(P)$. Let a^k (with $k \in \mathbb{Z}$) be an element of $P = \langle a \rangle$ and let $h \in N(P)$. Since h normalises P, there exists an integer $\ell \in \mathbb{Z}$ such that $ha^k h^{-1} = a^\ell$. By applying to this equality the homomorphism $A_S \to \mathbb{Z}$ that sends every generator to 1, it follows that $k = \ell$, hence a^k is central in $N(P)$. This concludes the proof.

An explicit basis of the normaliser. Finding an explicit basis for the free subgroup appearing in Theorem E is now a standard application of Bass–Serre theory, which was stated as a remark without further justification in [18, remark 4·6]. We first start by describing a fundamental domain for the action, as well as the quotient space $T_P/N(P)$.

Definition 31. Let Γ' be the first barycentric subdivision of the Coxeter graph Γ_S. A vertex of Γ' corresponding to a generator a of A_S will be denoted v_a and will be said to be of **type 1**, while a vertex of Γ' corresponding to an edge of Γ between generators a and b will be denoted v_{ab} and will be said to be of **type 2**.

Let a be a generator of A_S and let $P = \langle a \rangle$ be the corresponding standard parabolic subgroup. Let $\Gamma_{a,\mathrm{odd}}$ denote the maximal connected subgraph of Γ that contains the vertex a

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and only odd-labelled edges. Let Γ_P be the graph obtained from the disjoint union of all the edges of Γ' that contain a vertex of $\Gamma_{a,\text{odd}}$, by the following identification: if such an edge e (e' respectively) of Γ' contains a vertex v (v' respectively) such that v, v' correspond to the same vertex of $\Gamma_{a,\text{odd}}$, then v and v' are identified and define the same vertex of Γ_P.

Some examples of the graph Γ_P are given in Figure 1, when the underlying Coxeter graph is a triangle.

Definition 32. Let e be an edge of Γ_P between a type 1 vertex v_c and a type 2 vertex v_{cd}, for c,d spanning an edge of Γ. We denote by $\widetilde{e}$ the edge of T_P between the vertex A_c and the vertex A_{cd}. Choose an orientation of each edge of Γ. For each oriented path of Γ_P based at v_a, we denote by $e_1, \ldots, e_n$ the oriented sequences of edges of Γ crossed by γ, and we define

$$g_\gamma := \delta_{e_1}^{\pm 1} \cdots \delta_{e_n}^{\pm 1},$$

where the sign for each Garside element δ_{e_i} depends on whether γ follows the orientation of e_i.

We now choose a spanning tree τ of Γ_P, which we think of as being based at v_a. For a vertex v of Γ_P, we denote γ_v the oriented geodesic of τ from v_a to v. Let e be an edge of Γ_P. If e is contained in τ, let v be the vertex of e closest to v_a in τ. If e is not contained in τ, let v be the vertex of e closest to v_a in Γ_P (as Γ_P is bipartite). We denote $g_v := g_{\gamma_v}$, and we set

$$Y_P := \bigcup_{e \subset \Gamma_P} g_v \widetilde{e}.$$

This defines a connected subtree of T_P. To see that Y_P is connected, note that if e, e' are two adjacent edges of Γ_P contained in τ, then by construction of the various elements g_γ, we have that $g_v\widetilde{e}$ and $g_{v'}\widetilde{e}'$ are adjacent in Y_P. Moreover, if e is an edge of Γ_P not contained in τ and if e' is an edge of τ meeting e at the vertex of e closest to v_a in Γ_P, then $g_v\widetilde{e}$ and $g_{v'}\widetilde{e}'$ are adjacent in Y_P.

LEMMA 33. *The subtree Y_P is a fundamental domain for the action of $N(P)$ on T_P, and the quotient $T_P/N(P)$ is isomorphic to Γ_P.*

Proof. An edge of T_P corresponds to a pair consisting of a maximal simplex of T_P (of type c for some $c \in V(\Gamma)$) and one of its codimension 1 faces (of type cd for some $d \in V(\Gamma)$ adjacent to c). We thus mention the following useful fact, which is an immediate consequence of Lemma 17:

Fact. Two edges of T_P in the same A_S-orbit are also in the same $N(P)$-orbit.

Let us first show that Y_P is a fundamental domain for the action of $N(P)$ (and hence $N(P)/P$) on T_P. The fact that Y_P is connected, hence a subtree of T_P, is a consequence of the construction. By construction of the various edges $\widetilde{e}$, it thus follows that the edges of Y_P are in different A_S-orbits, and in particular in different $N(P)$-orbits. Now let e be an edge of T_P. Its type 1 vertex is of type c, for some $c \in V(\Gamma)$ such that $\langle c\rangle$ and $\langle a\rangle$ are conjugated. It thus follows from [20] that $c \in V(\Gamma_{a,odd})$, and it then follows that e is in the A_S-orbit, hence the $N(P)$-orbit, of an edge of Y_P. Thus, Y_P is a fundamental domain for the action of $N(P)$ (and hence $N(P)/P$) on T_P.

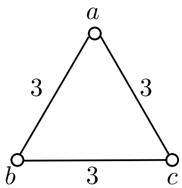
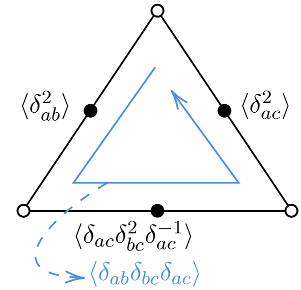
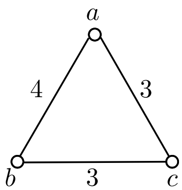
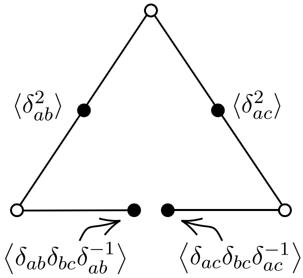
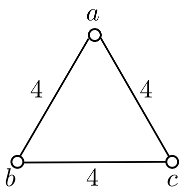
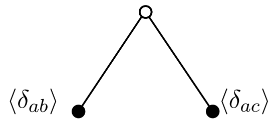
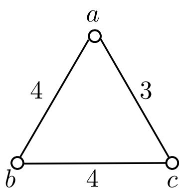
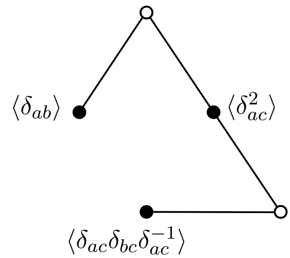
Coxeter graph Γ	Induced graph of groups on Γ_p	Rank and basis of F, with $N(P) \simeq P \times F$
		The rank is 4 and a basis is $\{\delta_{ab}^2, \delta_{ac}^2, \delta_{ac}\delta_{bc}^2\delta_{ac}^{-1}, \delta_{ab}\delta_{bc}\delta_{ac}\}$
		The rank is 4 and a basis is $\{\delta_{ab}^2, \delta_{ac}^2, \delta_{ab}\delta_{bc}\delta_{ab}^{-1}, \delta_{ac}\delta_{bc}\delta_{ac}^{-1}\}$
		The rank is 2 and a basis is $\{\delta_{ab}, \delta_{ac}\}$
		The rank is 3 and a basis is $\{\delta_{ab}, \delta_{ac}^2, \delta_{ac}\delta_{bc}\delta_{ac}^{-1}\}$

Fig. 1. Examples of computations of normalisers of the parabolic subgroup $P = \langle a\rangle$, for various large-type triangular Artin groups. Type 2 vertices of Γ_P are indicated in bold in the second column and come with their infinite cyclic stabilisers. The group element in blue corresponds to the element of a basis of F coming from the fundamental group of Γ_P. Note that the structure of the normaliser for large-type triangular Artin groups depends only on the parity of the labels and not on the labels themselves, so the above cases cover all possible cases.

We now want to study the quotient space $T_P/N(P)$. Let us analyse the action of $N(P)/P$ on T_P at a local level.

Let v be a vertex of T_P of type $c \in V(\Gamma)$. By the above remark, we will assume up to the action of $N(P)$ that this vertex corresponds to the codimension 1 simplex of X_S corresponding to $g_v A_c$. By construction of T_P, the codimension 1 faces of Δ that correspond to a type 2

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vertex of T_P adjacent to v are the simplices corresponding to the parabolic subgroups g_vA_{cd} with d connected to c in Γ.

Let v be a vertex of T_P of type $\{c, d\}$ where c, d span an edge of Γ. Up to the action of $N(P)$, we will assume that this vertex corresponds to the simplex with associated coset g_vA_{cd}. Then it follows from Lemma 29 that we have:

(i) if m_{cd} is even, then all the edges of T_P containing v are in the same $\langle\delta_{cd}\rangle$-orbit;

(ii) if m_{cd} is odd, then there are exactly two $N(P)$-orbits of edges of T_P containing v, corresponding to the $\langle\delta_{cd}^2\rangle$-orbits of the maximal simplices of type $\{c\}$ and $\{d\}$ respectively.

The description of the quotient $T_P/N(P)$ now follows from this local description.

As mentioned earlier, the fundamental group $N(P)/P$ of this graph of groups over Γ_P is a free group, and by Bass–Serre theory a basis for it is obtained by choosing a generator of each (infinite cyclic) stabiliser of vertex of dihedral type, as well as a family of elements corresponding to a basis of the fundamental group of Γ_P. We now explain how to construct explicitly these elements.

(1) For each vertex v of Y_P of type $\{c, d\}$, a generator of

$$\mathrm{Stab}_{N(P)/P}(v) = g_vZ(A_{cd})g_v^{-1}$$

is given by

$$\begin{cases} g_v \cdot \delta_{cd}^2 \cdot g_v^{-1} & \text{if } m_{cd} \text{ is odd,} \\ g_v \cdot \delta_{cd} \cdot g_v^{-1} & \text{otherwise.} \end{cases}$$

(2) A basis of $\pi_1(\Gamma_P)$ is in bijection with the edges of $\Gamma_P - \tau$. Let e be such an edge, joining a type 1 vertex v_c and a type 2 vertex v_{cd}, and let e' be the edge joining v_d and v_{cd}. Then the edges $g_{v_c}\delta_{cd}^{\pm1}\widetilde{e}$ and $g_{v_d}\widetilde{e}'$ of Y_P contain two type 2 vertices in the same $N(P)$-orbit, and the geodesic of Y_P between these two vertices project to a loop of Γ_P crossing e exactly once that represents the element

$$g_{v_c} \cdot \delta_{cd}^{\pm1} \cdot g_{v_d}^{-1} \in N(P).$$

Note that this element is of the form g_γ, for some combinatorial γ containing e. Thus, a family of elements for item 2) is given by the family of elements g_γ when γ runs over a basis of $\pi_1(\Gamma_P)$.

We thus get the following:

COROLLARY 34. *The normaliser $N(P)$ splits as a direct product $N(P) = P \times F$, where F is a finitely-generated free group with a basis given by the following family of elements*:

(i) *for every vertex v of Γ_P of dihedral type $\{c, d\}$, the element*

$$\begin{cases} g_v \cdot \delta_{cd}^2 \cdot g_v^{-1} & \textit{if } m_{cd} \textit{ is odd,} \\ g_v \cdot \delta_{cd} \cdot g_v^{-1} & \textit{otherwise;} \end{cases}$$

(ii) *for each combinatorial loop γ based at v_a in a chosen basis of $\pi_1(\Gamma_P)$, the element g_γ.*

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In Figure 1, we give examples for various Artin groups associated to a triangular Coxeter graph of the normalisers of standard generators.

6. *Conjugacy stability and root stability*

We are now ready to prove Theorem C and Theorem D. In this section, A_S denotes as usual an Artin group of large type on at least three generators.

By Corollary 16, we can define the following subgroups of A_S:

Definition 35. Let $g \in A_S$. The minimal parabolic subgroup P_g containing g is called the **parabolic closure** of g.

This subgroup behaves well under conjugacy as illustrated by the following result (which generalises an analogous statement for spherical Artin groups [9, lemma 8·1]):

LEMMA 36. *Let $g \in A_S$ and $\alpha \in A_S$. Then*

$$P_{\alpha^{-1}g\alpha} = \alpha^{-1}P_g\alpha.$$

In particular, if a and b are conjugate, their parabolic closures correspond to stabilisers of simplices of X_S with the same dimension.

Proof. It is obvious that $\alpha^{-1}P_g\alpha$ contains $\alpha^{-1}g\alpha$. We need to prove that this parabolic subgroup is the minimal one containing $\alpha^{-1}g\alpha$. Let Q be any parabolic subgroup containing $\alpha^{-1}g\alpha$. As $\alpha Q\alpha^{-1}$ contains g, $P_g \subseteq \alpha Q\alpha^{-1}$. Therefore, $\alpha^{-1}P_g\alpha \subseteq Q$.

We are finally able to prove the conjugacy stability theorem:

Proof of Theorem C. Let g and g' be two elements of A_X that are conjugated by an element $\alpha \in A_S$. As $P_g, P_{g'} \subset A_X$, by Theorem 11 there must be $Y, Y' \in X$ and $\beta, \beta' \in A_X$ such that $P_g = \beta^{-1}A_Y\beta$ and $P_g = \beta'^{-1}A_{Y'}\beta'$. Since P_g and $P_g{}'$ are conjugate by Lemma 36, A_Y and $A_{Y'}$ have to be conjugate. At the beginning of this section, we have seen that if $|Y| > 1$, then $Y = Y'$. Also, if $|Y| = 1$, then either $Y = Y'$, or Y and Y' are single generators connected by an odd-labelled path in Γ_S. Thus, there are two possibilities:

(i) suppose that $P_g = \beta^{-1}A_Y\beta$ and $P_{g'} = \beta'^{-1}A_Y\beta'$, with $Y \subseteq X$ and $\beta, \beta' \in A_X$. Then $(\beta\alpha)^{-1}A_Y(\beta\alpha) = \beta'^{-1}A_Y\beta'$ and $\beta\alpha\beta'^{-1}$ normalises A_Y. If the dimension of A_Y is bigger than 1, then by Corollary 21, $N(A_Y) = A_Y \subseteq A_X$, so $\alpha \in A_X$. If the dimension of A_Y is 1, $g = \beta^{-1}a\beta$, $g' = \beta'^{-1}a\beta'$, for some $a \in X$, and they are conjugate by $\beta^{-1}\beta' \in A_X$;

(ii) suppose that $g = \gamma^{-1}a^n\gamma$ and $g' = \gamma'^{-1}b^n\gamma'$, $\gamma, \gamma' \in A_X$, where a and b are Artin generators that are connected in Γ_S by an odd-labelled path. Then, there is an element of A_S conjugating a to b. If there is an odd-labelled path in Γ_X connecting a to b, then there is an element c in A_X that conjugates a to b. Thus, $\gamma^{-1}c\gamma'$ conjugates g to g'.

On the contrary, if there is no such a path in Γ_X, there is no element in A_X conjugating a to b. Since the parabolic closures of g and g' are respectively $\gamma^{-1}\langle a\rangle\gamma$ and $\gamma'^{-1}\langle b\rangle\gamma'$, by Lemma 36 there is no element in A_X conjugating g to g'. This is then the only case in which A_X is not conjugacy stable in A_S.

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We also prove that the parabolic closure of an element g is stable when taking roots and powers of g. This is a generalisation of [**9**, corollary 8·3].

PROPOSITION 37. *Let A_S be a large-type Artin group of rank at least 2, and let $g \in A_S$. Then for every $n \in \mathbf{Z}\backslash\{0\}$ we have $P_g = P_{g^n}$.*

Before coming to the proof of this proposition, we first introduce the following Lemma. Note that this result and its proof are analogous to [**8**, theorem 7·3]:

LEMMA 38. *Let G be a group acting by simplicial automorphisms on a systolic complex X. Suppose that there is a vertex $v \in X$ whose orbit Gv is finite. Then there exists a simplex of X that is invariant under the action of G.*

Proof. The statement of [**8**, theorem 7·3] is given for a finite group G. However, their proof only uses the finiteness of G to obtain a finite G-orbit, out of which they construct an invariant simplex. In particular, their proof generalises without any change to the case of an infinite group G with a finite G-orbit.

We now come to the proof of Proposition 37:

Proof of Proposition 37. We show by induction on $|S|$ that $P_g = P_{g^n}$. If $|S| = 2$, A_S is a dihedral Artin group. In particular, it is spherical, and the result follows from [**9**, corollary 8·3]. Let now $|S| \geq 3$, and suppose that $P_g \neq P_{g^n}$. We have that $P_{g^n} \subseteq P_g$, and then there is a chain of inclusions of the form

$$P_{g^n} \subsetneq P_g \subseteq A_S.$$

Claim. We have $P_g \subsetneq A_S$.

Indeed, since $P_{g^n} \subsetneq A_S$, the set $\mathrm{Fix}_{X_S}(P_{g^n})$ is non-empty. In particular, g^n is elliptic, and thus g has finite orbits, as for every point $v \in \mathrm{Fix}(g^n)$,

$$\langle g \rangle \cdot v = \{v, gv, g^2v, \ldots, g^{n-1}v\}.$$

By Lemma 38, g must stabilise some simplex Δ in X_S. Because the action of A_S on X_S is without inversions, g must fix Δ pointwise. In other words, $\mathrm{Fix}(g)$ is non-empty, hence $P_g \subsetneq A_S$. This finishes the proof of our claim.

Also, we have $P_g = hA_{S'}h^{-1}$ for some $h \in A_S$ and $S' \subsetneq S$. Now notice that

$$h^{-1}P_{g^n}h \subsetneq h^{-1}P_g h = A_{S'},$$

and thus $P_{h^{-1}g^nh} \subsetneq P_{h^{-1}gh} = A_{S'}$ by Lemma 36. As $|S'| < |S|$, we can use the induction hypothesis on $X_{S'}$. This yields $P_{h^{-1}gh} = P_{h^{-1}g^nh}$. In particular, one has $P_g = P_{g^n}$ by Lemma 36, which is a contradiction.

As an immediate consequence, we have the following result:

COROLLARY 39. *Let A_S be a large-type Artin group of rank at least 2, and let P be a parabolic subgroup of A_S. If $g^n \in P$ for some $n \in \mathbf{Z}\backslash\{0\}$, then $g \in P$.*

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