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Neoclassical toroidal plasma viscosity in bounce-transit and drift resonance regimes in tokamaks

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Abstract

Neoclassical toroidal plasma viscosity in the bounce-transit and drift resonance regimes is calculated using a version of the drift kinetic equation that encompasses the physics of the nonlinear trapping and quasilinear plateau regimes in tokamaks. It is demonstrated that the mirror-force like term controls the transition between these two regimes. When the effective collision frequency is larger than the mirroring or the nonlinear bounce frequency, the quasilinear regime prevails; otherwise, the nonlinear trapping regime reigns. The demonstration is accomplished by using the Eulerian approach and is beyond the grasp of the method of the integration along the unperturbed orbit in solving the drift kinetic equation. The neoclassical toroidal plasma viscosity in the quasilinear plateau regime is calculated. Approximate analytic expressions for the neoclassical toroidal plasma viscosity that include the asymptotic limits of the nonlinear trapping and quasilinear regimes are presented to facilitate thermal and energetic alpha particle transport modeling in tokamaks.

Keywords: neoclassical toroidal plasma viscosity, tokamak modeling, bounce-transit and drift resonance

1. Introduction

Neoclassical toroidal plasma viscous force is one of the key physics mechanisms that control toroidal plasma rotation when toroidal symmetry is broken in tokamaks [1, 2]. The physics in the drift frequency range is well understood [1, 3–8]. However, in the bounce-transit frequency range, the theory is still evolving because of the subtlety resulting from the radial drift motion of the bounce (i.e. bananas) and transit (i.e. circulating particles) orbits [9–14]. It is now understood that the best choice of the independent variables in solving the drift kinetic equation is to replace poloidal flux χ with the toroidal component of the canonical momentum p_ζ for the radial coordinate [15–19]. With this choice, the equation can be solved by adopting the Eulerian approach without the need to treat explicitly the poloidal mode coupling [18, 19]. When the collision frequency decreases, the transport process associated with the broken toroidal symmetry or neoclassical toroidal plasma viscosity [1, 2] is dominated by the bounce-transit and drift resonance in the bounce-transit time scale. For the equilibrium banana particles, the relevant resonance condition is [9, 10, 20–23]

$$l\omega_b + n\omega_d = 0; \tag{1}$$

and for the equilibrium circulating particles, it is [15–19]

$$\left[l - nq\left(p_\zeta\right)\right]\sigma_t\omega_t + n\omega_d = 0, \tag{2}$$

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where ω_b is the bounce frequency of the equilibrium bananas, ω_t is the transit frequency of the equilibrium circulating particles, σ_t denotes the sign of the parallel motion of the circulating particles, l is the poloidal mode number, n is the toroidal mode number, q is the safety factor and ω_d is the toroidal drift frequency. Note that it is $q\left(p_\zeta\right)$ not $q\left(\chi\right)$ that appears in the transit and drift resonance as a result of the equilibrium radial drift. Even though $q\left(p_\zeta\right)$ does not appear explicitly in the bounce and drift resonance condition, it still affects the resonance. Because $\omega_d \sim \omega_{b,\,t}\rho_p/L$, the bounce-transit and drift resonance conditions can be satisfied for those bananas and circulating particles that are close to the trapped-circulating boundary. Here, ρ_p is the poloidal gyro-radius, and L is the equilibrium radial scale length. Of course, for high n modes, e.g. in rippled tokamaks, the resonance conditions can be satisfied for deeply trapped and well circulating particles as well, as long as there is only one class of trapped particles in the equilibrium.

As it has been demonstrated, the poloidal mode coupling can be treated by adopting the Eulerian approach [17–19]. In that approach, the explicit poloidal mode coupling is annihilated by choosing a new set of the poloidal and toroidal angle like variables. Thus, the solution is relatively succinct in terms of the new variables.

We, however, do not use the Krook model for the collision operator, even though that model can be used to produce the correct transport coefficients. We instead adopt the test particle operator [24] and employ the appropriate approximation valid for the localized distribution resulting from the resonance [19]. We take this opportunity to demonstrate the collisional resonance broadening in the distribution function that cannot be obtained using a Krook model. The width of the resonance layer depends on the collisional operator used.

We also want to demonstrate the relation between the nonlinear trapping regime (or superbanana regime) and the quasilinear plateau regime. We will show that when the mirroring frequency (i.e. nonlinear bounce frequency) is smaller than the effective collision frequency, quasilinear plateau regime prevails. In the opposite limit, the nonlinear trapping regime dominates. The 'mirror force' is the force that is responsible for the nonlinear trapping. Of course, it is not the same as the well-known mirror force of the $\mu\vec{\nabla}B$. Here, μ is the magnetic moment, and B is the magnetic field strength. We formulate the calculation of the quasilinear plateau accordingly and, thus, it differs from the conventional approach. The formulation offers physics insight on the transition between the nonlinear trapping and quasilinear plateau regimes. As a result, we provide an approximate analytic formula that connects the asymptotic limits of these two regimes, which is not yet in any existing numerical codes, to facilitate plasma modeling.

The rest of the paper is organized as follows. The magnetic coordinates, field representations, and relevant physical quantities are presented in section 2. In section 3, we develop the equation that governs the nonlinear trapping and quasilinear plateau regimes. We solve the drift kinetic equation in section 4. The previously published results in the nonlinear trapping regime are summarized in section 4.1. In section 4.2, we take the limit of the quasilinear plateau regime to neglect the mirror force-like term. In addition, we simplify the test particle collision operator utilizing the localization property of the perturbed distribution function. Thus, not only the transport coefficients but also the distribution function can be accurately described. Approximate analytic formulas that yield the proper asymptotic limits of the two regimes discussed here are presented in section 5. The concluding remarks are given in section 6.

2. Magnetic coordinates and field representations

We adopt Hamada coordinates (V, θ, ζ) for the equilibrium magnetic field $\vec{B}$ such that

$$\overrightarrow{B} = \psi'\vec{\nabla}V \times \vec{\nabla}\theta - \chi'\vec{\nabla}V \times \vec{\nabla}\zeta, \tag{3}$$

where V is the volume enclosed inside the flux surface divided by $4\pi^2$, θ is the poloidal angle, ζ is the toroidal angle, $\psi' = \vec{B}\cdot\vec{\nabla}\zeta$, and $\chi' = \vec{B}\cdot\vec{\nabla}\theta$ [25]. The corresponding covariant representation is

$$\overrightarrow{B} = F\vec{\nabla}\zeta + G\vec{\nabla}\theta + \vec{\nabla}\varphi, \tag{4}$$

where F is the poloidal current outside the flux surface multiplied by $2/c$, G is the toroidal current inside the flux surface multiplying by $2/c$, c is the speed of light, φ is the solution of the equation $\vec{B}\cdot\vec{\nabla}\varphi = B^2 - \langle B^2\rangle$ and angular brackets denote flux surface average [26].

The Eulerian approach can be consistently employed to solve the drift kinetic equation for the distribution in all the sub-cyclotron frequency range [17–19, 27, 28]. In that approach a particular set of angle-like variables is chosen so that the angle dependences in the equilibrium bounce-transit and drift frequencies are appropriately averaged along the magnetic field line [17–19]. As a result, the frequency coefficients of the drift kinetic equation are not secular and depend only on the radial coordinate. This dramatically simplifies the analytic solution procedure. A poloidal angle-like variable η is needed to replace the angle θ. For trapped particles, η is defined as

$$\eta = \frac{\pi}{\int_{\theta_{t_2}}^{\theta_{t_1}} d\theta \frac{B}{B_m}\frac{\upsilon}{|\upsilon_\parallel|}} \int_{\theta_m}^{\theta} d\theta \frac{B}{B_m}\frac{\upsilon}{\upsilon_\parallel}, \tag{5}$$

where θ_{t_1} and θ_{t_2} are the turning points of the trapped particles, B_m is the minimum value of $B = \left|\vec{B}\right|$, occurring at the poloidal angle θ_m, υ is the particle speed, and $\upsilon_\parallel$ is the particle speed that is parallel to $\vec{B}$. The angle η has a period of 2π for all trapped particles. For circulating particles, the angle η is defined as

$$\eta = \frac{2\pi}{\int_{-\pi}^{\pi} d\theta \frac{B}{B_m}\frac{\upsilon}{|\upsilon_\parallel|}} \int_{\theta_m}^{\theta} d\theta \frac{B}{B_m}\frac{\upsilon}{|\upsilon_\parallel|}. \tag{6}$$

The corresponding bounce ω_b and transit ω_t frequencies are, respectively,

$$\omega_b = \frac{\upsilon\chi'}{B_m}\frac{\pi}{\int_{\theta_{t_2}}^{\theta_{t_1}} d\theta \frac{B}{B_m}\frac{\upsilon}{|\upsilon_\parallel|}}, \tag{7}$$

and

$$\omega_t = \frac{\upsilon\chi'}{B_m}\frac{2\pi}{\int_{-\pi}^{\pi} d\theta \frac{B}{B_m}\frac{\upsilon}{|\upsilon_\parallel|}}. \tag{8}$$

A toroidal angle-like variable $\overline{\zeta_0}$ is defined to replace the field line label $\zeta_0 = q\theta - \zeta$. Explicitly,

$$\overline{\zeta_0} = \zeta_0 - \frac{(F - \partial\varphi/\partial\zeta_0)\upsilon_\parallel}{\Omega\chi'}q'\theta - \int^{\theta} \frac{Bd\theta}{\upsilon_\parallel\chi'} \times \left[\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0 - \frac{\upsilon_\parallel}{B}\frac{\partial}{\partial\upsilon}\left(\frac{(F-\partial\varphi/\partial\zeta_0)\upsilon_\parallel}{\Omega}q'\theta\right) - \left\langle\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0\right\rangle_{b,t}\right], \tag{9}$$

where prime denotes $\partial/\partial\upsilon$, Ω is the gyro-frequency, $\vec{\upsilon_d}$ is the drift velocity and $\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0 = (\upsilon_\parallel/B)\vec{\nabla}\cdot\left[(\upsilon_\parallel/\Omega)\vec{B}\times\vec{\nabla}\zeta_0\right]$. The θ integral is an integral along the magnetic field line. The average $\langle\cdot\rangle_{b,t}$ denotes the average over the bounce and transit orbits respectively [17–19]. Explicitly expressions for $\langle\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0\rangle_{b,t}$ can be found in [5, 17]. Because the integrand is periodic, the indefinite integral is well-defined.

The Eulerian approach can accommodate the effects of finite banana width in the radial direction V as demonstrated in [17]. It can also include the effects of the finite orbit width in the toroidal direction that are imbedded in the θ-integral. All these effects can be reflected in Fourier amplitudes in the new angle variables.

We also note in passing that the exact expression of $\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0$ can be used without the need of the low β and large aspect ratio approximations. Here, β is the ratio of the plasma pressure to the magnetic field pressure.

In terms of $(P_\zeta, \eta, \overline{\zeta_0})$, the perturbed magnetic field that breaks the toroidal symmetry can be expressed as

$$B = B_S - B_0\sum_{l,n}\{b_{lnc}cos\left[(l-nq)\eta + n\overline{\zeta_0}\right] + b_{lns}\sin\left[(l-nq)\eta + n\overline{\zeta_0}\right]\}, \tag{10}$$

where B_S is the toroidally symmetric equilibrium magnetic strength, B_0 is the magnetic field strength on the magnetic axis, and b_{lnc} and b_{lns} are the Fourier coefficients. The finite radial mode width, although can be included as shown in [17], is neglected in equation (10) for simplicity. However, finite orbit width effects in the toroidal direction can be easily included in equation (10) because in general b_{lnc} and b_{lns} are functions of (υ, λ) where $\lambda = \mu B_m/E$, $\mu = \upsilon_\perp^2/(2B)$, $E = \upsilon^2/2$, and $\upsilon_\perp$ is the particle speed perpendicular to the equilibrium magnetic field.

3. Drift kinetic equation

The drift kinetic equation in $(P_\zeta, \theta, \zeta_0, E, \mu)$ coordinates is

$$\left(\upsilon_\parallel\hat{n} + \vec{\upsilon_d}\right)\cdot\vec{\nabla}\theta\frac{\partial f}{\partial\theta} + \upsilon_d.\vec{\nabla}\zeta_0\frac{\partial f}{\partial\zeta_0} + \dot{P}_\zeta\frac{\partial f}{\partial P_\zeta} = C(f), \tag{11}$$

where $P_\zeta = \chi - (F - \partial\varphi/\partial\zeta_0)\upsilon_\parallel/\Omega, \hat{n} = \vec{B}/B, C(f)$ denotes collision operator, and

$$\dot{P}_\zeta = \frac{\upsilon_\parallel}{B}\left[\frac{\partial}{\partial\zeta_0}\left(\frac{\upsilon_\parallel B^2}{\Omega}\right)\right]. \tag{12}$$

The $\partial\varphi/\partial\zeta_0$ term in P_ζ can be neglected for practical applications.

The drift kinetic equation in equation (11) differs from most of the treatments on the theory of neoclassical toroidal plasma viscosity in that the radial variable is P_ζ not χ. Consequently, it is $\dot{P}_\zeta$ not $\vec{\upsilon_d}\cdot\vec{\nabla}\chi$ that is the drive of the equation. The physics reason for the difference is that the equilibrium radial drift motion is not included in the conventual approach. When the radial motion is included, there is also an additional term in the toroidal drift that depends on dq/dV as shown in [15–17]. This additional drift can be viewed either as a modification in the toroidal drift or as a modification on the safety factor q that a particle experiences. The latter view adopted here simplifies the solution procedure dramatically as shown in [18, 19]. Physically, this implies that both trapped and circulating particles follow the constant $q = q(P_\zeta)$ line when the radial drift motion is included. We should remark that even though $q = q(P_\zeta)$ does not appear in the resonant condition, the $q = q(P_\zeta)$ is actually imbedded in the Fourier coefficients b_{lnc} and b_{lns} for the equilibrium trapped particles.

In terms of $(P_\zeta, \overline{\zeta_0}, \eta, E, \mu)$ the drift kinetic equation can be cast as

$$\sigma_t\omega_{b,t}\frac{\partial f}{\partial\eta} + \langle\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0\rangle_{b,t}\frac{\partial f}{\partial\overline{\zeta_0}} + \dot{P}_\zeta\frac{\partial f}{\partial P_\zeta} = C(f), \tag{13}$$

after neglecting $\vec{\upsilon_d}\cdot\vec{\nabla}\theta$, which is $\rho_p/L < 1$ and corresponds physically to the neglect of the potato orbits [29]. The physics of the potato orbits can be included by defining a more complicated angle η. Here, $\sigma_t = \pm 1$ denotes the sign of $\upsilon_\parallel$ for the equilibrium circulating particles and $\sigma_b = 1$ for the equilibrium trapped particles and is not displayed for simplicity.

We simplify the drift kinetic equation by assuming that broken toroidal symmetry induced $\dot{P}_\zeta$ is smaller than $\omega_{b,t}$ and $\langle\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0\rangle_{b,t}$ such that the leading order equation in this ordering is

$$\sigma_t\omega_{b,t}\frac{\partial f_0}{\partial\eta} + \langle\vec{\upsilon_d}\cdot\vec{\nabla}\zeta_0\rangle_{b,t}\frac{\partial f_0}{\partial\overline{\zeta_0}} + \dot{P}_\zeta\frac{\partial f_0}{\partial P_\zeta} = C(f_0), \tag{14}$$

and the corresponding solution is

$$f_0 = f_0(P_\zeta, E, \lambda). \tag{15}$$

The anisotropic dependence, i.e. λ dependence has to be at most no more than the first order in the small parameter of the gyro-radius ordering, i.e. $\rho_p/L < 1$. If we had included a source term in equation (14), f_0 can be a slowing down distribution; a detailed discussion on this issue is presented in [19] and will not be repeated here.

The next order equation is

$$\sigma_t\omega_{b,t}\frac{\partial f_1}{\partial\eta}+\langle\vec{v_d}\cdot\vec{\nabla}\zeta_0\rangle_{b,t}\frac{\partial f_1}{\partial\overline{\zeta_0}}+\dot{P}_\zeta\frac{\partial f_1}{\partial P_\zeta}+\dot{P}_\zeta\frac{\partial f_0}{\partial P_\zeta}=C(f_1). \tag{16}$$

The reason that $\partial f_1/\partial P_\zeta$ term is included in equation (16), is that in the nonlinear trapping regime, $\partial f_1/\partial P_\zeta\sim\partial f_0/\partial P_\zeta$ in the vicinity of the resonance as demonstrated in [19]. We will summarize the results in the nonlinear trapping regime and solve equation (16) in the quasilinear plateau regime here.

4. Solution to drift kinetic equation

We approach the quasilinear plateau regime from the collisional end of the nonlinear trapping regime by increasing collision frequency. Thus, we focus on the transition between these two regimes to discuss the physics that has not been addressed in the theory of neoclassical toroidal plasma viscosity. We will demonstrate the physics using a single mode, e.g. the perturbed magnetic field strength, δB,

$$\delta B=-b_{lnc}cos\left[(l-nq)\eta+n\overline{\zeta_0}\right]. \tag{17}$$

For the quasilinear plateau regime, Fourier modes are independent from each other, and the results are additive. Thus, the result of a single mode is adequate. In the nonlinear trapping regime, resonances must be well separated from each other. Thus, as long as the resonances are far from overlapping, the result of a single mode is relevant to the modeling of tokamak physics.

4.1. Summary on results of the nonlinear trapping regime

In the nonlinear trapping regime, drift kinetic equation is solved by a subsidiary expansion [18]. The small parameter is $\nu_{\rm eff}<\omega_{\rm b,nl}$, where the effective collision frequency $\nu_{\rm eff}=\nu/f_{\rm t}^2$, $\omega_{\rm b,nl}$ is the bounce frequency of the nonlinearly trapped particles, and ν is the typical collision frequency. Explicitly [18],

$$f_t=\frac{nq}{|l-nq|}\sqrt{\delta B/B}\left(\frac{\rho_p}{r}s\right)^{1/2}, \tag{18}$$

and

$$\omega_{b,nl}=\frac{v}{Rq}\sqrt{\delta B/B}\left(n^2q^2\frac{\rho_p}{r}s\right)^{1/2}, \tag{19}$$

where r is the minor radius, $s=d\ln q/d\ln r$ is the shear parameter, and R is the major radius. In this regime, the nonlinear particle orbit can be calculated from a constant of motion Z, which is [18]

$$Z=\frac{1}{2}\frac{\partial\omega}{\partial P_{\zeta_0}}\left(P_\zeta-P_{\zeta_0}\right)^2+\omega_0\left(P_\zeta-P_{\zeta_0}\right)-\frac{v^2}{2}\left(1-3\frac{{v_{\parallel 0}}^2}{v^2}\right)\frac{B}{\Omega}nb_{lnc}cos\,y, \tag{20}$$

where $\omega=\sigma_t\left[l-\delta_tnq\left(P_\zeta\right)\right]\omega_{b,t}+n\omega_d$, $y=\left[l-\delta_tnq\left(P_\zeta\right)\right]\eta+n\overline{\zeta_0}$, $\delta_{\rm t}=1$ for equilibrium circulating particles, otherwise $\delta_{\rm t}=0$, P_{ζ_0} is P_ζ in the vicinity of the resonance that satisfies the resonance conditions, and $v_{\parallel 0}^2$ is $l=0$ component of $v_\parallel^2$. Using Z, the nonlinear particle orbit can be expressed in a pendulum from

$$\omega=\sigma_\omega\widehat{\omega}\sqrt{\widehat{k}^2-sin^2\left(y/2\right)}, \tag{21}$$

where $\sigma_\omega=\pm1$ is the sign of ω, the magnitude of ω is

$$\widehat{\omega}=2\sqrt{\left|\frac{\partial\omega}{\partial P_{\zeta_0}}\frac{v^2}{2}\left(1-3\frac{{v_{\parallel_0}}^2}{v^2}\right)\frac{B}{\Omega}nb_{lnc}\right|} \tag{22}$$

and $\hat{k}^2$ is

$$\widehat{k}^2=\frac{{\omega_0}^2}{4\left|\frac{\partial\omega}{\partial P_{\zeta_0}}\frac{v^2}{2}\left(1-3\frac{{v_{\parallel_0}}^2}{v^2}\right)\frac{B}{\Omega}nb_{lnc}\right|}. \tag{23}$$

Here, we remark that $\hat{k}^2=0$ corresponds to $\omega_0=0$, i.e. the position in the phase space where the linear resonance occurs. Thus, a finite value of $\hat{k}^2<1$ corresponds to a $\omega_0\neq0$ in the linear phase but is trapped when the nonlinear trapping occurs. This is the consequence of the choice of evaluating Z [18].

Utilizing y and Z as independent variables, the drift kinetic equation can be cast as

$$\omega\frac{\partial f_1}{\partial y}+\dot{P}_\zeta\frac{\partial f_0}{\partial P_\zeta}=C(f_1). \tag{24}$$

Equation (24) has been solved in the nonlinear trapping regime in [18]. The transport fluxes, i.e. neoclassical toroidal plasma viscosity in that regime are

$$\begin{pmatrix}\vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\\ \vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}\end{pmatrix}=-\frac{e\chi'}{c}{v_t}^3\int dxx^2\begin{pmatrix}1\\ x^2-\frac{5}{2}\end{pmatrix}\nu_DI_k\times\left[\frac{\pi v\widehat{\omega}{\omega_\nu}^2}{B_m\omega_{b,t}\left|\frac{\partial\omega}{\partial\lambda}\right|\left(\partial\omega/\partial P_{\zeta_0}\right)^2}\frac{\partial f_0}{\partial P_{\zeta 0}}\right]_r, \tag{25}$$

where $x=v/v_t$, $\vec{e_\zeta}=\vec{\nabla}V\times\vec{\nabla}\theta$, $\overleftrightarrow{\pi}$ is the viscous tensor, $\overleftrightarrow{\Theta}$ is the heat viscous tensor, e is the electric change, $v_t=\sqrt{2T/M}$ is the thermal speed, T is the temperature, and M is the mass. The subscript r indicates that the quantities inside the square brackets are evaluated at the resonance pitch angle for a given E. The symbol ${\omega_\nu}^2$ is defined as [18]

$$ {\omega_\nu}^2 = 2\left(\frac{\upsilon_\parallel}{\upsilon}\right)^2 \frac{B_m}{B}\lambda\left(\frac{\partial\omega}{\partial\lambda}\right)^2 + 2\upsilon^2 \frac{\sum_b {\nu_\parallel}^{ab}}{\sum_b \nu_D^{ab}}\left(\frac{\partial\omega}{\partial\upsilon}\right)^2, \quad (26) $$

where a and b denote plasma species, $\nu_\parallel^{ab}$ is the parallel diffusion frequency, ν_D^{ab} is the pitch angle scattering frequency; the latter two frequencies are defined in [24]. We remark here that a is the test particle species and b is the field particle species. The viscous forces we show here are for the species a; the subscript a is omitted for simplicity. The notation I_k is a number resulting from the pitch angle integral $I_k = \int_0^\infty d\hat{k}^2 \left(\left\langle \frac{\hat{\omega}}{|\omega|}\right\rangle_y - \frac{H}{\langle |\omega|/\hat{\omega}\rangle_y}\right)$, where H is a step function that vanishes for nonlinearly trapped particles, $\langle\cdot\rangle_y = (2\pi)^{-1}\int_{-\pi}^{\pi} dy(\cdot)$ for nonlinearly circulating particles, $\langle\cdot\rangle = (2\pi)^{-1}\int_{-y_t}^{y_t} dy(\cdot)$ with turning points at $\pm y_t$ for nonlinearly trapped particles.

The approximate test particle collision operator used in deriving viscous forces is

$$ C(f) = \nu_D {\omega_\nu}^2 \frac{\partial^2 f}{\partial\omega^2}, \quad (27) $$

which is not a model collision operator. As a matter of fact, the collision frequency is enhanced for the localized distribution in ω. The enhanced effective collision frequency would not be obtained if a Krook model had been employed. The same is the case in the quasilinear plateau regime to be demonstrated in the next subsection.

4.2. Quasilinear plateau regime

Our approach in treating bounce-transit and drift resonance in the quasilinear plateau regime differs from all other approaches by including the mirror force like physics. The governing equation is equation (24) which has the independent variables (Z, y, E, λ). In the quasilinear plateau regime, we change the independent variables from (Z, y, E, λ) to (Z, y, E, ω), and recast equation (24) into

$$ \omega\frac{\partial f_1}{\partial y} + \omega\frac{\partial\omega}{\partial y}\frac{\partial f_1}{\partial\omega} + \dot{P}_\zeta\frac{\partial f_0}{\partial P_\zeta} = C(f_1), \quad (28) $$

where the second term on the left side is the mirror-force-like term that controls the nonlinear trapping. In the nonlinear trapping regime, the mirror-force like term is indispensable. However, in the quasilinear plateau regime, the frequency implied by this term is smaller than the collision frequency and can be neglected; this criterion sets the low collision frequency bound of the quasilinear plateau regime, i.e.

$$ \nu_D {\omega_\nu}^2 > \hat{\omega}^3, \quad (29) $$

a bound that has not been obtained previously in other treatments. Thus, our treatment of the quasilinear plateau and nonlinear trapping regimes is similar to that of the banana and plateau regimes in standard neoclassical theory [2, 30, 31]. Here, we note that if we had used a Krook model for $C(f)$, we could not obtain equation (29) properly.

Neglecting the mirror-force like term as is appropriate, the governing equation in quasilinear plateau regime is

$$ \omega\frac{\partial f_1}{\partial y} + \dot{P}_\zeta\frac{\partial f_0}{\partial P_\zeta} = C(f_1), \quad (30) $$

where ω is an independent variable. The resonance at $\omega = 0$ is now resolved by collisions. For a single mode, equation (30) can be expressed explicitly as

$$ \omega\frac{\partial f_1}{\partial y} - \frac{\upsilon^2}{2}\left(1 - 3\frac{{\upsilon_{\parallel 0}}^2}{\upsilon^2}\right)\frac{mc}{e} nb_{lnc}\sin y \frac{\partial f_0}{\partial P_{\zeta_0}} = \nu_D{\omega_\nu}^2\frac{\partial^2 f_1}{\partial\omega^2}, \quad (31) $$

where $\upsilon_{\parallel 0}^2$ is $l = 0$ component of $\upsilon_\parallel^2$. The width of the resonance layer $\Delta\omega$ can be estimated to be, using equation (31),

$$ |\Delta\omega| \sim \left(\nu_D{\omega_\nu}^2\right)^{1/3}. \quad (32) $$

Here, we note that the layer width in equation (32) cannot be obtained if a Krook model is used for $C(f)$. Equation (32) can be solved utilizing the function $Hi(z)$ that satisfies [32]

$$ \frac{d^2 w}{dz} - zw = 1, \quad (33) $$

where

$$ w = \pi Hi = \int_0^\infty dt\, e^{zt - t^3/3}; $$

and the solution is

$$ f_1 = -i\frac{\pi}{2}\frac{\upsilon^2}{2}\left(1 - 3\frac{{\upsilon_{\parallel 0}}^2}{\upsilon^2}\right)\frac{mc}{e} nb_{lnc}\frac{\partial f_0}{\partial P_{\zeta_0}}\left(\nu_D{\omega_\nu}^2\right)^{-1/3} \times \left[Hi(z)e^{iy} - Hi(-z)e^{-iy}\right], \quad (34) $$

where $z = -i\omega/\left(\nu_D{\omega_\nu}^2\right)^{1/3}$. The distribution f_1 satisfies the boundary condition that $f_1 \to 0$ as $|z| \to \infty$.

The neoclassical toroidal plasma viscosity can be calculated straightforwardly using f_1 in equation (34). The definition for neoclassical toroidal viscosity is

$$ \langle \vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\rangle = -\left\langle\frac{M}{2}\int d\vec{\upsilon}\left(\upsilon^2 - 3{\upsilon_\parallel}^2\right) f\frac{1}{B}\frac{\partial B}{\partial\zeta_0}\right\rangle, \quad (35) $$

where the angular brackets denote the flux surface average in the quasilinear plateau regime. Substituting f_1 in equation (34) into equation (35) yields

$$ \begin{pmatrix} \langle\vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\rangle \\ \langle\vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}\rangle \end{pmatrix} = -\chi'\frac{M^2 cn^2}{eB_m}\frac{\pi^2}{8}{\upsilon_t}^8 \times \int dx\, x^7\begin{pmatrix}1 \\ x^2 - \frac{5}{2}\end{pmatrix} \times \left(1 - 3\frac{{\upsilon_{\parallel 0}}^2}{\upsilon^2}\right)^2 \times \frac{{b_{lnc}}^2}{\omega_{b,t}\left|\frac{\partial\omega}{\partial\lambda}\right|}\frac{\partial f_0}{\partial P_{\zeta_0}}, \quad (36) $$

where the energy integral is performed over the region that satisfies the resonance condition.

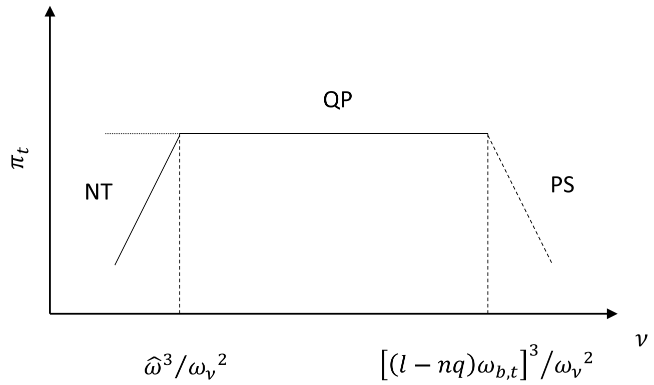

Figure 1. Schematic plot of neoclassical toroidal viscosity $\vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\equiv\pi_t$ versus collision frequency ν in log–log scale. The transitions between nonlinear trapping (NT), quasilinear plateau (QP) and Pfirsch–Schlüter (PS) regimes are noted. When the resonances overlap in the nonlinear trapping regime, the result of the quasilinear plateau regime persists.

5. Analytic expression for nonlinear trapping and quasilinear plateau regimes

The asymptotic results for neoclassical toroidal plasma viscosity in the nonlinear trapping and quasilinear plateau regimes can be reproduced from the following analytic expressions

$$\begin{pmatrix} \langle \vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\rangle \\ \langle \vec{e_\zeta}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}\rangle \end{pmatrix} = -\frac{e\chi'}{cB_m}v_t{}^4\int dx\,x^3\begin{pmatrix}1\\ x^2-\frac{5}{2}\end{pmatrix}\hat{\omega}\left(\frac{\partial\omega}{\partial P_{\zeta_0}}\right)^{-2} \times\frac{\partial f_0/\partial P_{\zeta_0}}{\omega_{b,t}\left|\partial\omega/\partial\lambda\right|}\times\frac{\kappa_{NT}\kappa_{QP}}{\kappa_{NT}+\kappa_{QP}}, \quad (37)$$

where $\kappa_{\mathrm{NT}}=\pi\nu_{\mathrm{D}}I_{\mathrm{k}}\omega_\nu{}^2$, and $\kappa_{\mathrm{QP}}=\left(\pi^2/32\right)\hat{\omega}^3$. To obtain equation (37), we have used $\hat{\omega}^2$ to represent the perturbed field $b_{\mathrm{lnc}}{}^2$. Some of the results in equation (37) are not in any existing numerical codes. The formulas can be useful in modeling tokamak experiments and energetic alpha particle transport losses. A schematic plot of the neoclassical toroidal plasma viscosity versus collision frequency ν is shown in figure 1. Here, we remark that when the resonances overlap in nonlinear trapping regime the result of the quasilinear plateau regime persists.

In the collisional regime, the viscous forces connect to those in the Pfirsch–Schlüter regime shown in [2].

6. Discussions and concluding remarks

We have presented an equation for treating the bounce-transit and drift resonances that can accommodate both the nonlinear trapping and quasilinear plateau physics. When the effective collision frequency is less than the nonlinear trapping frequency, the mirror-force like term cannot be neglected and the nonlinear trapping physics is dominant. In the opposite limit, the physics of the quasilinear plateau prevails.

The physics of the nonlinear trapping and quasilinear plateau regimes discussed here is similar to that of the electrostatic Landau damping. The nonlinear trapping physics is analogous to the electrostatic trapping in the Landau theory [33] and the quasilinear plateau regime to the linear Landau damping, in which there is no collision frequency dependence in the damping rate.

We calculate the neoclassical toroidal plasma viscosity in the quasilinear plateau regime by including the effects of the equilibrium radial drift in the safety factor q experienced by the particles instead of in the toroidal drift. This is consistent with the choice of P_ζ as an independent radial variable. Even though we have only used a single mode in the theory, the results can be easily generated to multiple modes because they are additive in the quasilinear plateau regime.

We note that because toroidal drift speed is smaller than the nominal bounce and transit speeds by a factor of ρ_{p}/L, the bounce-transit and drift resonances most likely occur in the vicinity of the trapped-circulating boundary where the real values of the bounce and transit frequencies are small. Of course, for high n mode, the resonance can be satisfied for deeply trapped and well circulating particles.

We have also constructed analytic formulas that can reproduce the respective asymptotic limits of the nonlinear trapping and quasilinear plateau regimes. The results of these formulas are not in existing numerical codes. The formulas are useful in modeling tokamak physics, including the transport losses of energetic alpha particles. As noted in [19], the energetic alpha particle transport losses can be comparable to those of neoclassical theory for $\delta B/B\sim 10^{-4}$. Because plasma parameters are not always in the asymptotic limits, the detailed value can only be calculated numerically. The formulas presented here can serve that purpose.

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