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ORIGINAL PAPER

Total and non-total suborbits for hypercyclic operators

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Abstract

In this note, it is proved that if X is a separable infinite dimensional Fréchet space that admits a continuous norm then, given a closed infinite dimensional subspace of X, there exists a hypercyclic operator admitting a dense orbit which in turn admits a suborbit all of whose sub-suborbits are total in the prescribed subspace. This is related to a recently published result asserting that every supercyclic vector for an operator on a Hilbert space supports a non-total suborbit. Here we also extend this result to normed spaces.

Keywords Orbit under an operator · Suborbit · Hypercyclic operator · Supercyclic operator · Total set

Mathematics Subject Classification 46A99 · 47A16

1 Introduction

If X is a topological vector space (TVS) over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), then a set $A \subset X$ is said to be *total* provided that its linear span $\operatorname{span}(A)$ is dense in X (in such a case, we say that A spans X). More generally, if Y is a closed subspace of X, then we say that A is *total in* Y if $\overline{\operatorname{span}}(A) = Y$. As usual, we denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Assume that $x_0 \in X$ and that $T : X \to X$ is a selfmap of X. Then the *orbit* and the *projective orbit* of x_0 under T are respectively defined as the sets $\mathrm{O}(x_0, T) = \{T^n x_0 : n \in \mathbb{N}_0\}$ and $\mathbb{K} \cdot \mathrm{O}(x_0, T) = \{\lambda T^n x_0 : \lambda \in \mathbb{K}, n \in \mathbb{N}_0\}$. To

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every infinite subset $J = \{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}_0$ we can associate a *suborbit* $\mathrm{O}(x_0, T, J) = \{T^n x_0 : n \in J\} = \{T^{n_j} x_0 : j \in \mathbb{N}\}$ (so $\mathrm{O}(x_0, T) = \mathrm{O}(x_0, T, \mathbb{N}_0)$).

If T is an operator (that is, linear and continuous), then T is said to be *hypercyclic* (*supercyclic*, *cyclic*, resp.) provided that there is a vector $x_0 \in X$ –called *hypercyclic* (*supercyclic*, *cyclic*, resp.) for T– such that $\mathrm{O}(x_0, T)$ ($\mathbb{K} \cdot \mathrm{O}(x_0, T)$, $\mathrm{span}(\mathrm{O}(x_0, T))$, resp.) is dense in X. The reader is referred to, for instance, [12] or [14] for background on TVSs, while the fundamentals on hypercyclicity theory can be found in [4, 9].

Observe that hypercyclic implies supercyclicity, and in turn, supercyclicity implies cyclicity, and that these three kinds of operators can only live in separable TVSs. Moreover, only infinite dimensional TVSs can support hypercyclic operators (see [9]). If X supports a supercyclic operator then either X is infinite dimensional or $\dim(X) \in \{0, 1, 2\}$ if $\mathbb{K} = \mathbb{R}$ (or $\dim(X) \in \{0, 1\}$ if $\mathbb{K} = \mathbb{C}$, resp.) (see [10]). The intermediate notion of supercyclic operators was introduced in 1974 by Hilden and Wallen in [11]. The existence of hypercyclic operators on separable infinite dimensional Banach (Fréchet, resp.) was proved in [5] (in [2, 6], resp.).

Recently, and inspired by the fact that –for metrizable TVSs– a hypercyclic vector must possess a suborbit tending to zero, Faghih and Hedayatian [7] have established the following interesting Theorem 1 below, where we have used the terminology given in the first paragraph.

Theorem 1 *Let X be an infinite dimensional Hilbert space and T be an operator on X. If x_0 is a supercyclic vector for T, then there is a suborbit of x_0 under T that is non-total in X.*

This statement opens a door to research the dynamics of suborbits of linear operators enjoying some cyclicity property.

The aim of this short note is to contribute to this research in the setting of a certain class of Fréchet spaces including Banach spaces. Our findings are connected to the conclusion of Theorem 1. In fact, we shall establish the existence of hypercyclic (hence supercyclic) operators admitting dense orbits enjoying the property that every infinite subset of some suborbit spans a prescribed closed infinite dimensional subspace. The precise statement will be provided in Theorem 2.4 of the next section. The result is preceded by an assertion on existence of total subsets being infinitely compressible in some sense. Finally, in Sect. 3, Theorem 1 will be extended to normed spaces, so giving in this setting an affirmative answer to Question 1 posed in [7].

2 Total suborbits

We begin with an assertion (Theorem 2.2) about existence of infinite total sets that cannot be "trivially minimalized" we respect to totality. Prior to this, let us recall the following important statement, that is due to Bonet and Peris [6, Lemma 2] in the case that the space is not isomorphic to $\mathbb{K}^{\mathbb{N}}$. It will be used in our proof. By X^* we shall denote, as usual, the topological dual space of X. And ω will stand for the space $\mathbb{K}^{\mathbb{N}}$ of all scalar sequences, that becomes a Fréchet space when endowed with the product topology.

Theorem 2.1 *If X is an infinite dimensional separable Fréchet space, then there is a sequence $\{e_n\}_{n \geq 1} \subset X$ as well as a sequence $\{\varphi_m\}_{m \geq 1} \subset X^*$ with the following properties:*

(a) $\varphi_n(e_n) \in (0, 1]$ *for all* $n \in \mathbb{N}$ *and* $\varphi_m(e_n) = 0$ *if* $m \neq n$.
(b) *The set* $\{e_n : n \in \mathbb{N}\}$ *is total in* X.
(c) $e_n \to 0$ *as* $n \to \infty$.

Proof As said before, the conclusion was provided by [6, Lemma 2] in the case that X is not (isomorphic to) ω. If $X = \omega$, then simply take $e_n := (0, 0, \dots, 0, 1, 0, 0, \dots)$ (with the nonzero term at the nth place), and $\varphi_m := \pi_m$, the mth projection, that is, $\pi_m(x_1, x_2, x_3, \dots) = x_m$. □

Theorem 2.2 *Assume that X is a separable Fréchet space. Then there exists an infinite total subset A of X enjoying the property that any infinite subset of it is also total.*

Proof We first deal with the finite dimensional case, which will later inspire the infinite dimensional one. Then X is isomorphic to $\mathbb{K}^N$ for some $N \in \mathbb{N}$, so that we can assume $X = \mathbb{K}^N$. Consider the infinite set $A := \{u_1, u_2, u_3, \dots\}$ of $\mathbb{K}^N$ given by

$$u_k := (k, k^2, \dots, k^N) \quad (k \in \mathbb{N}).$$

Note that the conclusion is trivial if $N = 1$, so that we can suppose $N \geq 2$. If B is an infinite subset of A, then $B = \{u_k : k \in M\}$, where M is an infinite subset of $\mathbb{N}$. Take N pairwise different vectors $v_j = u_{k_j} = (k_j, k_j^2, \dots, k_j^N) \in B$, so that $k_j \in M$ for $j = 1, \dots, N$. Now the matrix of the v_j's with respect to the canonical basis of $\mathbb{K}^N$ is $\left(k_j^l\right)_{(j,l) \in \{1,\dots,N\}}$, a Vandermonde matrix, whose determinant is

$$k_1 \cdots k_N \cdot \prod_{1 \leq i < j \leq N} (k_j - k_i),$$

which is nonzero because the k_i's are pairwise different. Then the v_j's are linearly independent. Hence

$$\mathbb{K}^N = \operatorname{span}\{v_j\}_{1 \leq j \leq N} \subset \operatorname{span}(B) \subset \overline{\operatorname{span}}(B) \subset \mathbb{K}^N,$$

and so $\overline{\operatorname{span}}(B) = \mathbb{K}^N$, that is, B is total.

Next, we face the infinite dimensional case. According to Theorem 2.1, we can select a pair of sequences $\{e_p\}_{p \geq 1} \subset X$, $\{\varphi_q\}_{q \geq 1} \subset X^*$ satisfying properties (a)–(b)–(c). Since X is locally convex and metrizable, its topology can be defined by an increasing sequence of seminorms

$$\| \cdot \|_1 \leq \| \cdot \|_2 \leq \| \cdot \|_3 \leq \cdots .$$

For each $a > 1$, consider the sequence of vectors

$$S_\nu := \sum_{n=1}^{\nu} a^{-n} e_n \quad (\nu = 1, 2, 3, \dots).$$

Fix a neighborhood U of 0. Then there are $m \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$U \supset B_{m,\varepsilon} := \{x \in X : \|x\|_m < \varepsilon\}.$$

It follows from (c) that $\|e_n\|_m \to 0$ as $n \to \infty$, and also $a^{-n} \to 0$, so that there is $n_0 \in \mathbb{N}$ such that

$$\|e_n\|_m \leq 1 \text{ and } a^{-n} < \frac{\varepsilon(a-1)}{a} \text{ for all } n \geq n_0. \tag{2.1}$$

If $\nu > \mu \geq n_0$ then we get from (2.1) that

$$\begin{aligned} \|S_\nu - S_\mu\|_m &= \| \sum_{n=\mu+1}^{\nu} a^{-n} e_n \|_m \leq \sum_{n=\mu+1}^{\nu} a^{-n} \|e_n\|_m \leq \sum_{n=\mu+1}^{\nu} a^{-n} \\ &< a^{-\mu}(a^{-1} + a^{-2} + a^{-3} + \cdots) = a^{-\mu} \cdot \frac{a}{a-1} < \varepsilon. \end{aligned}$$

Therefore $S_\nu - S_\mu \in B_{m,\varepsilon}$, and so $S_\nu - S_\mu \in U$ for all $\nu > \mu \geq n_0$. In other words, $\{S_\nu\}_{n\geq 1}$ is a Cauchy sequence in X. The completeness of X implies the existence of a limit $u_a \in X$ for this sequence. Thus, we obtain a family of vectors

$$u_a = \sum_{n=1}^{\infty} a^{-n} e_n \quad (a > 1). \tag{2.2}$$

These vectors are linearly independent. Indeed, assume, by way of contradiction, the existence of m (≥ 2) nonzero scalars $\lambda_1, \ldots, \lambda_m$ and of pairwise different reals $a_1, \ldots, a_m \in (1, +\infty)$ with $\lambda_1 u_{a_1} + \cdots + \lambda_m u_{a_m} = 0$. By using the convergence of the series defining the u_{a_i}'s, we derive that

$$\sum_{n=1}^{\infty} \left(\lambda_1 a_1^{-n} + \cdots \lambda_m a_m^{-n}\right) e_n = 0. \tag{2.3}$$

Since the φ_q's are linear and continuous, an application of (a) for $q = 1, \ldots, m$ to (2.3) yields (after dividing by $\varphi_q(e_q)$, that is nonzero) the squared homogeneous linear system

$$a_1^{-n}\lambda_1 + \cdots a_m^{-n}\lambda_m = 0 \quad (n = 1, 2, \ldots, m)$$

with unknowns $\lambda_1, \ldots, \lambda_m$. Again, its associated matrix is a Vandermonde one with nonzero determinant, because the numbers $a_1^{-1}, \ldots, a_m^{-1}$ are nonzero and pairwise different. Thus, the unique solution of such a system is

$$\lambda_1 = \cdots = \lambda_m = 0,$$

which is a contradiction. This shows the linear independence of the vectors u_a and, in passing, we obtain that they are mutually different. In particular, the set

$$A := \{u_{1+k} : k \in \mathbb{N}\} \tag{2.4}$$

is infinite.

Finally, assume that B is an infinite subset of A. Then there is a strictly increasing sequence $k_1 < k_2 < k_3 < \cdots$ of natural numbers such that $B = \{u_{1+k_j} : j \in \mathbb{N}\}$. Our goal is to prove that $\overline{\text{span}}(B) = X$. By a well known consequence of the Hahn–Banach theorem, it is enough to show that, if a functional $\varphi \in X^*$ vanishes on B, then $\varphi = 0$. Take $\varphi \in X^*$ such that $\varphi(x) = 0$ for all $x \in B$. Then $\varphi(u_{1+k_j}) = 0$ for all $j \in \mathbb{N}$. According to (2.2), we get

$$\sum_{n=1}^{\infty} (1 + k_j)^{-n} \varphi(e_n) = 0 \text{ for all } j \in \mathbb{N}. \tag{2.5}$$

Since φ is continuous and the seminorms $\|\cdot\|_n$ define the topology of X, there are $m \in \mathbb{N}$ and a constant $C \in (0, +\infty)$ such that

$$|\varphi(x)| \leq C \|x\|_m \text{ for all } x \in X.$$

But the sequence $\{\|e_n\|_m\}_{n\geq 1}$ is bounded as it converges (to 0). In particular, $|\varphi(e_n)| \leq K$ for all $n \in \mathbb{N}$ and some absolute constant K. It follows that the radius of convergence of the power series $\sum_{n=1}^{\infty} \varphi(e_n)\lambda^n$ $(\lambda \in \mathbb{K})$ is at least one. Consequently, the function $F : \{|\lambda| < 1\} \to \mathbb{K}$ given by

$$F(\lambda) := \sum_{n=1}^{\infty} \varphi(e_n)\lambda^n$$

is well defined and analytic. It follows from (2.5) that

$$F\Big(\frac{1}{1+k_j}\Big) = 0 \text{ for all } j \in \mathbb{N}.$$

Since $\frac{1}{1+k_j} \to 0$ as $j \to \infty$ and $0 \in \{|\lambda| < 1\}$, we get that F vanishes at all points of a subset of $\{|\lambda| < 1\}$ having an accumulation point in $\{|\lambda| < 1\}$. Now, the Analytic Continuation Principle tells us that $F \equiv 0$. Therefore all of its Taylor coefficients at the origin are zero, that is, $\varphi(e_n) = 0$ for all $n \in \mathbb{N}$. By linearity, we get $\varphi(x) = 0$ for all $x \in \operatorname{span}(\{e_n\}_{n\geq 1})$. To conclude, the continuity of φ together with property (b) of Theorem 2.1 yields $\varphi = 0$, as required. $\square$

The following crucial assertion about existence of hypercyclic operators with prescribed orbits will be needed. It is due to Albanese [1], who extended a corresponding result in the setting of Banach spaces given by Grivaux [8].

Theorem 2.3 *Let X be a separable infinite dimensional Fréchet space admitting a continuous norm. Suppose that $\{v_n : n = 0, 1, 2, \dots\}$ is a dense set of linearly independent vectors of X. Then there is a hypercyclic operator T on X such that* $\operatorname{orb}(v_0, T) = \{v_n : n \in \mathbb{N}_0\}$.

We are now ready to establish our main result.

Theorem 2.4 *Suppose that X is a separable infinite dimensional Fréchet space that admits a continuous norm. Let $u_0 \in X \setminus \{0\}$ and Y be a closed infinite dimensional subspace of X. Then there exists a hypercyclic operator T on X satisfying the following properties:*

(a) *u_0 is hypercyclic for T.*
(b) *There exists an infinite subset $J \subset \mathbb{N}_0$ such that* $\operatorname{orb}(u_0, T, L)$ *is total in Y for any infinite $L \subset J$.*

Proof Let us fix $u_0 \in X$ and $Y \subset X$ as in the assumptions of the theorem. Since Y is also a separable infinite dimensional Fréchet space, we infer from Theorem 2.2 (and its proof) the existence of a countably infinite linearly independent set

$$A = \{x_1, x_2, x_3, \dots, x_n, \dots\} \subset Y$$

all of whose infinite subsets are total in Y.

Two cases are possible: either $u_0 \in \operatorname{span}(A)$ or $u_0 \notin \operatorname{span}(A)$. In the second case we keep A as it stands now. In the first one, there is a unique, finite set $F \subset \mathbb{N}$ such that u_0 can be expanded as a linear combination of the x_i's $(i \in F)$ with nonzero coefficients. Then we would replace A by $A \setminus F$, which is also infinite, so sharing the same property of A with respect to its infinite subsets. Therefore we are allowed to start with the fact that $u_0, x_1, x_2, x_3, \dots$ are linearly independent.

Since X is separable and metrizable, there is an open basis $\{G_n : n \in \mathbb{N}\}$ for the topology of X. As an easy consequence of the Baire category theorem, the dimension of X cannot be countable. Then $\operatorname{span}(A \cup \{u_0\})$ is a proper subspace of X, and so it has empty interior. Hence we can choose a vector

$$y_1 \in G_1 \setminus \operatorname{span}(A \cup \{u_0\}).$$

By a similar reason, we can pick a vector

$$y_2 \in G_2 \setminus \operatorname{span}(A \cup \{u_0, y_1\}).$$

By following this procedure, we get a sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ satisfying $y_n \in G_n$ (so $\{y_n\}_{n\in\mathbb{N}}$ is dense) and y_{n+1} is not in the linear span of $\{u_0, x_1, x_2, \dots\} \cup \{y_1, \dots, y_n\}$. This implies that the set

$$S := \{u_0, x_1, y_1, x_2, y_2, x_3, y_3, \dots\}$$

is dense (because $\{y_n\}_{n\in\mathbb{N}}$ is) and linearly independent. Then an application of Theorem 2.3 with

$$v_0 := u_0 \text{ and } \{v_n : n \in \mathbb{N}_0\} := S$$

provides a hypercyclic operator T such that $\mathrm{orb}(u_0, T) = S$. Since S is dense, we have (a). Finally, (b) is fulfilled if we select

$$J := \{n_1 < n_2 < n_3 < \cdots\},$$

where n_k is defined as the unique $m \in \mathbb{N}$ for which $T^m u_0 = x_k$. □

3 Non-total suborbits of supercyclic vectors

This short section is devoted to extend Theorem 1 from Hilbert spaces to the wider setting of normed spaces. Our approach will be necessarily different because the inner product of the space played a central role in the proof of [7, Theorem 1]. We shall make use of the following auxiliary result that is a weak version of Lemma 2.3 in [3] due to Bamerni, Kadets and Kiliçman. We remark that the completeness of X is not needed in the proof of such lemma.

Lemma 3.1 *Let $\mathcal{A}$ be a dense subset of a normed space X and $e \in X$ be a fixed vector with $\|e\| > 1$. Then for every finite dimensional subspace $Y \subset X$ with $\mathrm{dist}(e, Y) > 1$ there is a vector $a \in \mathcal{A}$ such that*

$$\mathrm{dist}(e, \mathrm{span}(Y \cup \{a\})) > 1.$$

We are now ready to prove the promised extension of Theorem 1.

Theorem 3.2 *Let X be an infinite dimensional normed space and T be an operator on X. If x_0 is a supercyclic vector for T, then there is a suborbit of x_0 under T that is non-total in X.*

Proof Let $e \in X$ be a fixed vector with $\|e\| > 1$, and let $Y_0 := \{0\}$. Then

$$\mathrm{dist}(e, Y_0) = \|e\| > 1.$$

It follows from Lemma 3.1 and the denseness of $\mathcal{A}_1 := \mathbb{K} \cdot \mathrm{O}(x_0, T) \setminus Y_0$ the existence of a vector $y_1 \in \mathcal{A}_1$ such that

$$\mathrm{dist}(e, \mathrm{span}(Y_0 \cup \{y_1\})) > 1.$$

Hence $\mathrm{dist}(e, \mathrm{span}(Y_1)) > 1$, where we have set $Y_1 := \mathrm{span}(\{y_1\})$. There exist $n_1 \in \mathbb{N}_0$ and $\lambda_1 \in \mathbb{K} \setminus \{0\}$ for which $y_1 = \lambda_1 T^{n_1} x_0$. Let

$$Z_1 := \mathrm{span}(\{x_0, T x_0, T^2 x_0, \dots, T^{n_1} x_0\}.$$

Now, Y_1 (and also Z_1) is a finite dimensional subspace of X the set $\mathcal{A}_2 := \mathbb{K} \cdot \mathrm{O}(x_0, T) \setminus Z_1$ is dense, because Z_1 is closed and has empty interior. A further application of Lemma 3.1 yields the existence of a vector $y_2 = \lambda_2 T^{n_2} x_0 \in \mathcal{A}_2$ such that $\mathrm{dist}(e, Y_2) > 1$, where

$$Y_2 := \mathrm{span}(Y_1 \cup \{y_2\}) = \mathrm{span}(\{y_1, y_2\}).$$

Again, there exist $n_2 \in \mathbb{N}_0$ and $\lambda_2 \in \mathbb{K} \setminus \{0\}$ for which $y_2 = \lambda_2 T^{n_2} x_0$. Observe that, since $y_2 \notin Z_1$, we have $n_2 > n_1$. Define

$$Z_2 := \mathrm{span}(\{x_0, T x_0, T^2 x_0, \dots, T^{n_2} x_0\}$$

and consider $\mathcal{A}_3 := \mathbb{K} \cdot \mathrm{O}(x_0, T) \setminus Z_2$. By following this procedure we can construct inductively a sequence of vectors $y_k = \lambda_n T^{n_k} x_0$ such that

$$n_1 < n_2 < n_3 < \dots, \ \lambda_k \neq 0 \text{ and } \mathrm{dist}(e, Y_k) > 1 \ \text{ for all } \ k \in \mathbb{N},$$

where

$$Y_k := \mathrm{span}\{y_1, \dots, y_k\} = \mathrm{span}(\{T^{n_1} x_0, T^{n_2} x_0, \dots, T^{n_k} x_0\}).$$

Finally, let $J := \{n_1, n_2, \dots\}$, consider the suborbit $O(x_0, T, J)$ and note that its linear span equals $\bigcup_{k=1}^{\infty} Y_k$. Therefore

$$\mathrm{dist}(e, \overline{\mathrm{span}}(O(x_0, T, J))) = \mathrm{dist}(e, \mathrm{span}(O(x_0, T, J))) \geq 1.$$

Thus, $\overline{\mathrm{span}}(O(x_0, T, J)) \neq X$, as required. □

4 Final remarks

1. Concerning Theorem 2.1, it should be said that Lemma 2 in [6] actually gives more information about the system $((e_n), (\varphi_n))$. Namely, the family (φ_n) can be chosen to be equicontinuous in X^* (not possible if $X = \omega$: see [6, p. 589]). In fact, the result is a generalization of a theorem due to Ovsepian and Pelczynski [13] about existence of complete total biorthogonal systems in separable infinite dimensional Banach spaces.
2. From the proof of Theorem 2.2 it is easily derived the following strengthening: Assume that X is a Fréchet space. Then X is infinite dimensional and separable if and only if it contains an infinite linearly independent subset A all of whose infinite subsets are total. Furthermore, according to (2.4), we can choose A to be a sequence; namely, $\{u_{1+n}\}_{n \geq 1}$. Moreover, if $\{V_n\}_{n \geq 1}$ is a nondecreasing of 0-neighborhoods in X, then there exist $c_n \in (0, +\infty)$ such that $c_n u_{1+n} \in V_n$ for all $n \in \mathbb{N}$, so $c_n u_{1+n} \to 0$. Now, for every $J \subset \mathbb{N}$, it is clear that $\{c_n u_{1+n}\}_{n \in J}$ spans the same subspace as $\{u_{1+n}\}_{n \in J}$. Consequently, the set A can be selected to be, in addition, relatively compact.
3. Theorem 2.4 holds in the special case $Y = X$, so providing for every nonzero vector u_0 an operator T such that u_0 is hypercyclic for T and all infinite subsets of some suborbit $\mathrm{orb}(u_0, T, J)$ are total. Note that this does not contradict Theorem 1 in [7] (see Sect. 1) nor Theorem 3.2. All that can be inferred is that J must be non-cofinite (as constructed, in fact, in the proof of Theorem 2.4).
4. The space ω is the most emblematic separable Fréchet space lacking a continuous norm. It would be interesting to know whether or not the conclusion of Theorem 2.4 holds for ω.

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