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ORIGINAL ARTICLE

An Effective Criterion for Finite Monodromy of ℓ-Adic Sheaves

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Abstract
We provide an effective version of Katz' criterion for finiteness of the monodromy group of a lisse, pure of weight zero, ℓ-adic sheaf on a normal variety over a finite field, depending on the numerical complexity of the sheaf.

Keywords l-adic cohomology · Monodromy

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1 Introduction

Let X be a smooth, geometrically irreducible variety over a field $k = \mathbb{F}_q$ of characteristic p. Fix a prime $\ell \neq p$, and consider the category $\mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ of lisse $\bar{\mathbb{Q}}_\ell$ sheaves on X. A sheaf will be said to be pure (of a certain weight) if it is so for every embedding $\bar{\mathbb{Q}}_\ell \to \mathbb{C}$.

Fix a geometric generic point $\bar{\eta}$ of X. Every sheaf $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ of rank r corresponds to a continuous representation $\pi_1(X, \bar{\eta}) \to \mathrm{GL}(r, \bar{\mathbb{Q}}_\ell)$. The Zariski closure of its image (respectively of the image of the subgroup $\pi_1(\overline{X}, \bar{\eta})$, where $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$) is called the *arithmetic monodromy group* (resp. the *geometric monodromy group*) of $\mathcal{F}$. It is well known that, under certain conditions, these groups govern the distribution of the Frobenius traces of the sheaf $\mathcal{F}$ on the set of rational points of X over larger and larger extensions of $\mathbb{F}_q$.

Several methods have been used in the literature to determine these groups, such as using local monodromies to deduce enough properties of it so that there is only one possibility [3, Chapter 11], or computing their moments and applying Larsen's alternative [6, Chapter 2]. In some cases, one can determine that there are only two options: either the group is finite of it is one specific group [11]. In any case, the question of determining whether the monodromy groups are finite is an interesting one.

In general, we have the following criterion, proven in [4, 8.14] for curves and in [7, Proposition 2.1] for higher dimensional varieties:

Dedicated to Professor Pham Huu Tiep on the occasion of his 60th birthday.

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Proposition 1 *Suppose that $\mathcal{F}$ is geometrically irreducible and pure of weight 0 and its determinant is arithmetically of finite order. Then the following conditions are equivalent:*

1. *The geometric monodromy group G_{geom} of $\mathcal{F}$ is finite.*
2. *The arithmetic monodromy group G_{arith} of $\mathcal{F}$ is finite.*
3. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius trace of $\mathcal{F}$ at t is an algebraic integer.*
4. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius eigenvalues of $\mathcal{F}$ at t are roots of unity.*

However, this criterion does not provide an effective algorithm, since one needs to check that condition (3) or (4) holds for *every* finite extension of $\mathbb{F}_q$. In practice, one usually needs to resort to ad-hoc methods to check that these conditions hold for a particular $\mathcal{F}$ (see e.g. [7, Theorem 3.1]). In this article, we prove the following effective version of the criterion, relying on the complexity of $\mathcal{F}$ as defined by Sawin [10, Definition 6.4].

Theorem 2 *Suppose that X is given with a projective embedding $u : X \hookrightarrow \mathbb{P}^n_k$. Then there exist explicit constants $N_1 = N_1(u, d, r, C)$ and $N_2 = N_2(u, d, r, C)$ such that for every geometrically irreducible $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ of rank r, pure of weight 0, whose determinant is arithmetically of finite order, with complexity $c_u(\mathcal{F}) \leq C$ and E-defined, where $\mathbb{Q} \subseteq E$ is a finite extension of degree $\leq d$, the following conditions are equivalent:*

1. *The geometric monodromy group G_{geom} of $\mathcal{F}$ is finite.*
2. *The arithmetic monodromy group G_{arith} of $\mathcal{F}$ is finite.*
3. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ of degree $\leq N_1$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius trace of $\mathcal{F}$ at t is an algebraic integer.*
4. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ of degree $\leq N_2$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius eigenvalues of $\mathcal{F}$ at t are roots of unity.*

If X is a smooth curve, we have a more explicit version which does not make use of the general concept of complexity:

Theorem 3 *Suppose that X is a smooth curve, Y its smooth projective closure and $D = Y \backslash X$. Then there exist explicit constants $N_1 = N_1(X, d, r, e)$ and $N_2 = N_2(X, d, r, e)$ such that for every geometrically irreducible $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ of rank r, pure of weight 0, whose determinant is arithmetically of finite order, all whose breaks at every $t \in D(\overline{\mathbb{F}}_q)$ are $\leq e$, and E-defined, where $\mathbb{Q} \subseteq E$ is a finite extension of degree $\leq d$, the following conditions are equivalent:*

1. *The geometric monodromy group G_{geom} of $\mathcal{F}$ is finite.*
2. *The arithmetic monodromy group G_{arith} of $\mathcal{F}$ is finite.*
3. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ of degree $\leq N_1$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius trace of $\mathcal{F}$ at t is an algebraic integer.*
4. *For every finite extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^r}$ of degree $\leq N_2$ and every $t \in X(\mathbb{F}_{q^r})$, the Frobenius eigenvalues of $\mathcal{F}$ at t are roots of unity.*

Unfortunately, the explicit constants N_1 and N_2 we obtain in these theorems are still too large to be useful in practice, so at the moment these results are mainly of theoretical interest. See Section 4 for some numerical examples and some potential ways to optimize them.

2 ℓ-Adic Sheaves and Trace Functions

Let X be as in the previous section. Every sheaf $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ defines a trace function $\Phi_\mathcal{F} : \coprod_{m\geq 1} X(k_m) \to \bar{\mathbb{Q}}_\ell$ (where k_m denotes the degree m extension of k) given by

$$\Phi_\mathcal{F}(m, t) = \mathrm{Tr}\left(Frob_{k_m,t}|\mathcal{F}_{\bar{t}}\right),$$

where $t \in X(k_m)$ and $\bar{t}$ is a geometric point over t. By Chevotarev's density theorem, two semisimple sheaves $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ are isomorphic if and only if $\Phi_\mathcal{F} = \Phi_\mathcal{G}$.

For every $m \geq 1$, denote by $\Phi_{\mathcal{F},m} : X(k_m) \to \bar{\mathbb{Q}}_\ell$ the restriction of $\Phi_\mathcal{F}$ to $X(k_m)$. For a smooth curve X, Deligne proved [1] the following bounded version of the previous statement. Let Y be a smooth compactification of X, and $D = Y \backslash X$. For every $s \in D(\bar{k})$ and $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$, let $\alpha_s(\mathcal{F})$ be the largest break of $\mathcal{F}$ at s. Then we have

Theorem 4 [1, Proposition 2.5] *Let X be a smooth curve and $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ two lisse semisimple sheaves of rank r. Then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if $\Phi_{\mathcal{F},m} = \Phi_{\mathcal{G},m}$ for every $m \leq N$, where*

$$N = 2r + \left\lfloor 2\log_q^+\left(2r^2\left(b_1(X) + \max_{s\in D(\bar{k})} \sup\{\alpha_s(\mathcal{F}), \alpha_s(\mathcal{G})\}\right)\right)\right\rfloor,$$

$\log_q^+ = \max\{0, \log_q\}$ *and $b_1(X) = \dim \mathrm{H}_c^1(X, \bar{\mathbb{Q}}_\ell)$ is the first Betti number with compact supports of X.*

We will generalize this to higher dimensional varieties, using Sawin's complexity theory [10] as a replacement for the ramification data. Assume that X is geometrically irreducible, smooth quasi-projective of dimension d over k, given with an embedding $u : X \hookrightarrow \mathbb{P}_k^n$. See [10, Definitions 3.2 and 6.4] for the definition of the complexity $c_u(\mathcal{F}) \in \mathbb{N}$ of a lisse sheaf $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ (or, more generally, an object of $D_c^b(X, \bar{\mathbb{Q}}_\ell)$). By [10, Theorem 5.2] there is an explicit constant A_n such that $c_u(\mathcal{F} \otimes \mathcal{G}) \leq A_n c_u(\mathcal{F}) c_u(\mathcal{G})$ for any $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$. In fact, by [10, Theorem 8.1], one may take

$$A_n = \frac{2^{17}}{3^4} e^{4/3} 13^n (n+2)!$$

We then have the following generalization of Theorem 4:

Theorem 5 *Let $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ be two lisse semisimple sheaves of rank r of complexity $\leq C$. Then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if $\Phi_{\mathcal{F},m} = \Phi_{\mathcal{G},m}$ for every $m \leq N$, where*

$$N = 2r + \lfloor 2\log_q^+(2A_n C^2) \rfloor.$$

The proof follows closely that of [1, Proposition 2.5]. As there, we can decompose $\mathcal{F}$ and $\mathcal{G}$ as direct sums

$$\mathcal{F} \cong \bigoplus_{i\in I} p_{i*}(\mathcal{H}_i \otimes \mathcal{W}_i),$$
$$\mathcal{G} \cong \bigoplus_{i\in I} p_{i*}(\mathcal{H}_i \otimes \mathcal{W}_i')$$

where, for every i, there is some $n_i \geq 1$ such that $\mathcal{H}_i$ is a geometrically irreducible lisse sheaf on $X_{n_i} := X \otimes_k k_{n_i}$ with determinant of finite order, $\mathcal{W}_i$ and $\mathcal{W}_i'$ are geometrically constant on X_{n_i} (and at least one of them is non-zero), $p_i : X_{n_i} \to X$ is the natural projection, and the $\mathcal{H}_i$ and their Galois conjugates are pairwise geometrically non-isomorphic.

Let $\mathcal{H}_{i,j}$ for $0 \leq j < n_i$ denote the Frobenius conjugates of $\mathcal{H}_i$. For every $m \geq 1$, let I_m be the set $\{i \in I : n_i | m\}$. As in [1, Lemme 2.6], we can show:

Lemma 6 *Let* $C = \max\{c_u(\mathcal{F}), c_u(\mathcal{G})\}$ *and* $N_0 = 2\log_q^+(2A_n C^2)$*. For* $m > N_0$*, the functions* $\Phi_{\mathcal{H}_{i,j},m} : X(k_m) \to \bar{\mathbb{Q}}_\ell$ *(*$i \in I_m$*,* $0 \leq j < n_i$*) are linearly independent.*

Let $\sum_{i,j} \lambda_{i,j} \Phi_{\mathcal{H}_{i,j},m} = 0$ be a non-trivial linear combination, and assume without loss of generality that $\lambda_{i_0,j_0} = 1$ and $|\lambda_{i,j}| \leq 1$ for every i, j. We may also assume that $\mathcal{H}_{i_0,j_0}$ is a direct summand of $\mathcal{F}$, by interchanging $\mathcal{F}$ and $\mathcal{G}$ if necessary. Then

$$
\begin{aligned}
0 &= \sum_{t \in X(k_m)} \left(\sum_{i,j} \lambda_{i,j} \Phi_{\mathcal{H}_{i,j},m}(t) \right) \overline{\Phi_{\mathcal{H}_{i_0,j_0},m}(t)} \\
&= \sum_{i,j} \lambda_{i,j} \sum_{t \in X(k_m)} \Phi_{\mathcal{H}_{i,j},m}(t) \overline{\Phi_{\mathcal{H}_{i_0,j_0},m}(t)} = \sum_{i,j} \lambda_{i,j} \sum_{t \in X(k_m)} \Phi_{\mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}},m}(t) \\
&= \sum_{i,j} \lambda_{i,j} \sum_{a=0}^{2d} (-1)^a \mathrm{Tr}\left(F_{k_m} | \mathrm{H}^a_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) \right).
\end{aligned}
$$

Since the $\mathcal{H}_{i,j}$ are geometrically irreducible and pairwise non-isomorphic, we have $\mathrm{H}^{2d}_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) = 0$ if $(i,j) \neq (i_0, j_0)$ and $\bar{\mathbb{Q}}_\ell(-d)$ if $(i,j) = (i_0, j_0)$, so

$$
0 = q^{md} + \sum_{i,j} \lambda_{i,j} \sum_{a=0}^{2d-1} (-1)^a \mathrm{Tr}\left(F_{k_m} | \mathrm{H}^a_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) \right)
$$

and

$$
\begin{aligned}
q^{md} &= \left| \sum_{i,j} \lambda_{i,j} \sum_{a=0}^{2d-1} (-1)^a \mathrm{Tr}\left(F_{k_m} | \mathrm{H}^a_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) \right) \right| \\
&\leq \sum_{i,j} \sum_{a=0}^{2d-1} \left| \mathrm{Tr}\left(F_{k_m} | \mathrm{H}^a_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) \right) \right| \\
&\leq q^{md - \frac{m}{2}} \sum_{i,j} \sum_{a=0}^{2d-1} \dim \mathrm{H}^a_c(X \otimes \bar{k}, \mathcal{H}_{i,j} \otimes \widehat{\mathcal{H}_{i_0,j_0}}) \\
&\leq q^{md - \frac{m}{2}} \sum_{a=0}^{2d-1} \dim \mathrm{H}^a_c(X \otimes \bar{k}, (\mathcal{F} \oplus \mathcal{G}) \otimes \widehat{\mathcal{F}})
\end{aligned}
\tag{1}
$$

since $\oplus_{i,j} \mathcal{H}_{i,j}$ and $\mathcal{H}_{i_0,j_0}$ are direct summands of $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F}$ respectively.

By the properties of the complexity [10, Theorem 5.1, Lemma 6.12, Theorem 8.1], we have

$$\begin{aligned}&\sum_{a=0}^{2d-1} \dim \mathrm{H}_c^a(X \otimes \bar{k}, (\mathcal{F} \oplus \mathcal{G}) \otimes \widehat{\mathcal{F}})\\ &\leq \sum_{a=0}^{2d} \dim \mathrm{H}_c^a(X \otimes \bar{k}, \mathcal{F} \otimes \widehat{\mathcal{F}}) + \sum_{a=0}^{2d} \dim \mathrm{H}_c^a(X \otimes \bar{k}, \mathcal{G} \otimes \widehat{\mathcal{F}})\\ &\leq A_n(c_u(\mathcal{F})^2 + c_u(\mathcal{F})c_u(\mathcal{G})) \leq 2A_nC^2.\end{aligned}$$

So, from (1), we deduce

$$q^{m/2} \leq 2A_nC^2$$

or

$$m \leq 2\log_q(2A_nC^2).$$

The proof of Theorem 5 now concludes exactly as in [1, 2.8]: for every $m > N_0$, we have

$$\Phi_{\mathcal{F},m} = \sum_{i \in I_m} \sum_{j=1}^{n_i} \mathrm{Tr}(F_{k_m}|\mathcal{W}_i)\Phi_{\mathcal{H}_{i,j},m}$$

and

$$\Phi_{\mathcal{G},m} = \sum_{i \in I_m} \sum_{j=1}^{n_i} \mathrm{Tr}(F_{k_m}|\mathcal{W}_i')\Phi_{\mathcal{H}_{i,j},m}$$

so, by the linear independence of the $\Phi_{\mathcal{H}_{i,j},m}$, $\Phi_{\mathcal{F},m} = \Phi_{\mathcal{G},m}$ if and only if $\mathrm{Tr}(F_{k_m}|\mathcal{W}_i) = \mathrm{Tr}(F_{k_m}|\mathcal{W}_i')$ for every $i \in I_m$. For every $i \in I$, $\mathcal{W}_i$ and $\mathcal{W}_i'$ have dimension at most $\lfloor r/n_i \rfloor$ so, in order to show they are isomorphic (that is, that they have the same eigenvalues for the action of $F_{k_{n_i}}$), it suffices to show that the traces of the action of $F_{k_{n_i}}^l = F_{k_{ln_i}}$ on them coincide for $\lfloor 2r/n_i \rfloor$ consecutive values of l [1, Lemme 2.9]. But, under the hypotheses of Theorem 5, these traces coincide for $N_0 < ln_i \leq N_0 + 2r$, in which there are at least $\lfloor 2r/n_i \rfloor$ possible values of l.

3 An Effective Criterion for Finite Monodromy

For a finite extension E of $\mathbb{Q}$ and a positive integer r, let $M(E, r)$ denote the least common multiple of the $n \geq 1$ such that $[E(\zeta_n) : E] \leq r$, where ζ_n is a primitive n-th root of unity. For instance, we have

$$M(\mathbb{Q}, r) = \prod_{\lambda \text{ prime } \leq r+1} \lambda^{\lfloor 1+\log_\lambda \frac{r}{\lambda-1} \rfloor}$$

and, in general, $M(E, r) \leq M(\mathbb{Q}, r \cdot [E : \mathbb{Q}])$, since

$$[E(\zeta_n) : E] \leq r \Rightarrow [\mathbb{Q}(\zeta_n) : \mathbb{Q}] \leq [E(\zeta_n) : \mathbb{Q}] \leq r \cdot [E : \mathbb{Q}].$$

Given an $\mathcal{H} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$, we say that $\mathcal{H}$ is E-valued if, for every $m \geq 1$ and $x \in U(\mathbb{F}_{q^m})$, the characteristic polynomial of the Frobenius action on $\mathcal{H}$ at x has coefficients in E. By [8, Théorème VII.6], if $\mathcal{H}$ is irreducible and its determinant is arithmetically of finite order then it is E-valued for some finite extension $\mathbb{Q} \subseteq E$.

Proposition 7 *Let X be a smooth variety given with a projective embedding $u : X \hookrightarrow \mathbb{P}_k^n$, $\mathcal{H}$ a geometrically irreducible lisse ℓ-adic sheaf on X of rank r, pure of weight 0, of* *complexity $c_u(\mathcal{H}) = C$, and whose determinant is arithmetically of finite order, let $\mathbb{Q} \subseteq E$ be a finite extension such that $\mathcal{H}$ is E-valued, and $M = M(E, r)$. Let*

$$N := 2R + \lfloor 2\log_q^+(2A_n(A_n^{M-1}C^M + r \cdot c_u(X))^2)\rfloor,$$

where

$$R = \sum_{\substack{i=0 \\ i \text{ even}}}^{r-1} \binom{r+M-i-1}{M}\binom{M-1}{i}.$$

Then $\mathcal{H}$ has finite (arithmetic and geometric) monodromy if and only if all Frobenius eigenvalues of $\mathcal{H}$ at x are roots of unity for every $m \le N$ and every $x \in X(\mathbb{F}_{q^m})$.

Proof Suppose that the Frobenius eigenvalues of $\mathcal{H}$ at x are roots of unity for every $m \le N$ and every $x \in X(\mathbb{F}_{q^m})$. Since these eigenvalues are roots of a polynomial of degree r over E, their order divides M by definition. That is, the M-th power of Frobenius acts trivially on $\mathcal{H}_{\bar{x}}$ for every $x \in X(\mathbb{F}_{q^m})$, $m \le N$.

Let $\mathcal{H}^{[M]} := \sum_{i=1}^{M}(-1)^{i-1}i[\mathrm{Sym}^{M-i}\mathcal{H}\otimes\wedge^i\mathcal{H}]$ be the M-th Adams power of $\mathcal{H}$. It is an element of the Grothendieck group of the category of constructible sheaves on X and, by [2, 1], its trace function is given by $\Phi_{\mathcal{H}^{[M]}}(m, x) = \Phi_{\mathcal{H}}(Mm, x)$. By [9, Proposition 3.4], we have the optimized expression $\mathcal{H}^{[M]} = \sum_{i=0}^{M-1}(-1)^i[\mathcal{H}_i]$, where $\mathcal{H}_i \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ denotes the sheaf defined by $\mathcal{H}om_{\mathfrak{S}_M}(\wedge^i V, \mathcal{H}^{\otimes M})$, where V is a constant sheaf of rank $M-1$ on X, $\mathfrak{S}_M$ acts on V via its standard $(M-1)$-dimensional representation, and on $\mathcal{H}^{\otimes M}$ by permutation of the factors. The relationship between both expressions for the Adams power is given by $\mathrm{Sym}^{M-i}\mathcal{H} \otimes \wedge^i\mathcal{H} \cong \mathcal{H}_i \oplus \mathcal{H}_{i-1}$. By the previous paragraph, we have

$$\Phi_{\mathcal{H}^{[M]}}(m, x) = r = \Phi_{\bar{\mathbb{Q}}_\ell^r}(m, x)$$

for every $m \le N$ and $x \in X(\mathbb{F}_{q^m})$. This can be rewritten in terms of "real" sheaves by splitting the positive and negative components of $\mathcal{H}^{[M]}$: let $\mathcal{F} = \oplus_{i \text{ even}}\mathcal{H}_i$ and $\mathcal{G} = \oplus_{i \text{ odd}}\mathcal{H}_i$, then

$$\Phi_{\mathcal{F}}(m, x) = \Phi_{\mathcal{G}\oplus\bar{\mathbb{Q}}_\ell^r}(m, x)$$

for every $m \le N$ and $x \in X(\mathbb{F}_{q^m})$.

Since $\mathcal{F}$ and $\mathcal{G}$ are subsheaves of $\mathcal{H}^{\otimes M}$, their complexity is bounded by that of $\mathcal{H}^{\otimes M}$, which in turn, applying [10, Theorem 5.2] repeatedly, is bounded by $A_n^{M-1}C^M$. Then the complexity of $\mathcal{G} \oplus \bar{\mathbb{Q}}_\ell^r$ is bounded by $A_n^{M-1}C^M + r \cdot c_u(X)$.

Since $\mathcal{F}$ and $\mathcal{G} \oplus \bar{\mathbb{Q}}_\ell^r$ have rank $\sum_{i \text{ even}} \binom{r+M-i-1}{M}\binom{M-1}{i}$ [9, Remark 3.6] and are pure of weight 0, by Theorem 5 we conclude that $\mathcal{F} \cong \mathcal{G}\oplus\bar{\mathbb{Q}}_\ell^n$ or, equivalently, $\mathcal{H}^{[M]} = [\mathcal{F}]-[\mathcal{G}] = [\bar{\mathbb{Q}}_\ell^n]$ in the Grothendieck group. That is, the M-th power of Frobenius acts trivially on $\mathcal{H}_{\bar{x}}$ for every $x \in X(\mathbb{F}_{q^m})$ and every $m \ge 1$. By Proposition 1, we conclude that $\mathcal{H}$ has finite monodromy. $\square$

If X is a smooth curve, then we can improve the bound by using Theorem 4 instead of Theorem 5. Let Y be the smooth projective closure of X and $D := Y\backslash X$. We then get

Proposition 8 *Let $\mathcal{H}$ be a geometrically irreducible lisse ℓ-adic sheaf on X of rank r, pure of weight 0 and whose determinant is arithmetically of finite order, let $\mathbb{Q} \subseteq E$ be a finite extension such that $\mathcal{H}$ is E-valued, and $M = M(E, r)$. For every $x \in D(\overline{\mathbb{F}_q})$, assume that the breaks of $\mathcal{H}$ at x are $\le e_x$, and let $e := \sum_{x\in D(\overline{\mathbb{F}_q})} e_x$ and*

$$N = 2R + \lfloor 2\log_q^+\left(2R^2\,(b_1(X) + e)\right)\rfloor,$$

where

$$R=\sum_{\substack{i=0\\ i\ even}}^{r-1}\binom{r+M-i-1}{M}\binom{M-1}{i}.$$

Then $\mathcal{H}$ has finite (arithmetic and geometric) monodromy if and only if all Frobenius eigenvalues of $\mathcal{H}$ at t are roots of unity for every $m\le N$ and every $x\in X(\mathbb{F}_{q^m})$.

Proof The proof goes exactly as in Proposition 7, using Theorem 4 and the fact that all breaks of $\mathcal{H}^{\otimes r}$ at $x\in D(\overline{\mathbb{F}_q})$ are $\le e_x$. □

Next, we give similar results based on the integrality of the Frobenius traces instead of its Frobenius eigenvalues, which are generally easier to compute. We start by showing

Lemma 9 *Let $\mathbb{Q}_p\subseteq E_\pi$ be a finite extension with ramification index e, and let $\alpha_1,\dots,\alpha_r\in E_\pi$. Let $a=\lfloor\log_p r\rfloor$ and*

$$N:=r\left(1+\left\lfloor\frac{e}{p-1}\left(1-\frac{1}{p^a}\right)\right\rfloor\right).$$

Then $\alpha_1,\dots,\alpha_r$ are integral if and only if $\sum_{i=1}^r\alpha_i^k$ is integral for every $1\le k\le N$.

Proof Let $M=1+\left\lfloor\frac{e}{p-1}\left(1-\frac{1}{p^a}\right)\right\rfloor$; $p_k:=\sum_{i=1}^n\alpha_i^{Mk}$ for $k\ge 1$ and s_k be the k-th elementary symmetric function on $\alpha_1^M,\dots,\alpha_r^M$ for $k=0,\dots,r$. By Newton's identities, $ks_k=\sum_{i=1}^k(-1)^{i-1}s_{k-i}p_i$, so $k!s_k$ is integral for every $k=0,\dots,r$.

If ν denotes the valuation on E_π, normalized so that $\nu(p)=1$, we get

$$\nu(s_k)\ge-\nu(k!)=-\sum_{i=1}^\infty\left\lfloor\frac{k}{p^i}\right\rfloor,$$

so the Newton polygon of the polynomial $\prod_{i=1}^r(1-\alpha_i^MT)$ is bounded below by the polygon with vertices

$$\left\{\left(k,-\sum_{i=1}^\infty\left\lfloor\frac{k}{p^i}\right\rfloor\right);k=0,1,\dots,r\right\}$$

and, in particular, its largest slope (in absolute value) is bounded by

$$\frac{1+p+\cdots+p^{a-1}}{p^a}=\frac{1}{p-1}\left(1-\frac{1}{p^a}\right)$$

so $\nu(\alpha_i^M)\ge-\frac{1}{p-1}\left(1-\frac{1}{p^a}\right)$ for every i, and

$$\nu(\alpha_i)\ge-\frac{1}{M(p-1)}\left(1-\frac{1}{p^a}\right)>-\frac{1}{e}.$$

But $\nu(E_\pi)=\frac{1}{e}\mathbb{Z}$, so we conclude that $\nu(\alpha_i)\ge 0$ for every $i=1,\dots,n$. □

Theorem 10 *Let $\mathcal{H}$ be a geometrically irreducible lisse ℓ-adic sheaf on X of rank r, pure of weight 0, of complexity $c_u(\mathcal{H})=C$, and whose determinant is arithmetically of finite order, let $\mathbb{Q}\subseteq E$ be finite extension such that $\mathcal{H}$ is E-valued, f be the maximum among the ramification indices of the primes of E above p, and $a=\lfloor\log_p n\rfloor$. Let*

$$N:=r\left(1+\left\lfloor\frac{rf}{p-1}\left(1-\frac{1}{p^a}\right)\right\rfloor\right)(2R+\lfloor2\log_q^+(2A_n(A_n^{M-1}C^M+r\cdot c_u(X))^2)\rfloor)$$

where

$$R = \sum_{\substack{i=0 \\ i \text{ even}}}^{r-1} \binom{r+M-i-1}{M}\binom{M-1}{i}.$$

Then $\mathcal{H}$ has finite (arithmetic and geometric) monodromy if and only if $\Phi_{\mathcal{H}}(m, x)$ is integral at all places of E over p for every $m \le N$ and every $x \in X(\mathbb{F}_{q^m})$.

Proof Assume that all Frobenius traces of $\mathcal{H}$ at x are integral at all places of E over p for every $m \le N$ and every $x \in X(\mathbb{F}_{q^m})$. For every $m \ge 1$ and every $x \in X(\mathbb{F}_{q^m})$, the Frobenius eigenvalues of $\mathcal{H}$ at x are contained in some extension of E of degree $\le r$, in which all primes above p have ramification index $\le rf$. By Lemma 9, all Frobenius eigenvalues of $\mathcal{H}$ at x are integral for every $m \le N'$ and every $x \in X(\mathbb{F}_{q^m})$ at all places over p, where

$$N' = 2R + \lfloor 2\log_q(2A_n(A_n^{M-1}C^M + r \cdot c_u(X))^2)\rfloor.$$

By [8, Théorème VII.6], they are also integral at all other non-archimedean places, so they are algebraic integers. Since their product (which is the Frobenius trace at x of $\det(\mathcal{H})$, which is arithmetically of finite order by hypothesis) is a root of unity, they must all be roots of unity, and we conclude by Proposition 7. □

For X a smooth curve we get the following, more optimized, result by using Proposition 8 instead of Proposition 7:

Theorem 11 *Let $\mathcal{H}$ be a geometrically irreducible lisse ℓ-adic sheaf on X of rank r, pure of weight 0 and whose determinant is arithmetically of finite order, let $\mathbb{Q} \subseteq E$ be a finite extension such that $\mathcal{H}$ is E-valued, f be the maximum among the ramification indices of the primes of E above p, and $a = \lfloor \log_p r\rfloor$. For every $x \in D(\overline{\mathbb{F}_q})$, assume that the breaks of $\mathcal{H}$ at x are $\le e_x$, and let $e := \sum_{x \in D(\overline{\mathbb{F}_q})} e_x$ and*

$$N = r\left(1 + \left\lfloor \frac{rf}{p-1}\left(1 - \frac{1}{p^a}\right)\right\rfloor\right)(2R + \lfloor 2\log_q^+\left(2R^2\left(b_1(X) + e\right)\right)\rfloor),$$

where

$$R = \sum_{\substack{i=0 \\ i \text{ even}}}^{r-1} \binom{r+M-i-1}{M}\binom{M-1}{i}.$$

Then $\mathcal{H}$ has finite (arithmetic and geometric) monodromy if and only if $\Phi_{\mathcal{H}}(m, x)$ is integral at all places of E over p for every $m \le N$ and every $x \in X(\mathbb{F}_{q^m})$.

4 Examples

Let $q = p$ be a prime, $X = \mathbb{A}^1_{\mathbb{F}_p}$ the affine line, fix an additive character $\psi : \mathbb{F}_p \to \mathbb{C}$ and an integer $n \ge 2$, and consider the sheaf $\mathcal{F} \in \mathcal{S}(X, \bar{\mathbb{Q}}_\ell)$ whose trace function at $t \in X(\mathbb{F}_{p^r})$ is given by

$$-\frac{1}{G^r}\sum_{x \in \mathbb{F}_{p^r}} \psi\left(\mathrm{Tr}_{\mathbb{F}_{p^r}/\mathbb{F}_p}(x^n + tx)\right),$$

where $G = -\sum_{x \in \mathbb{F}_p} \psi(x^2)$ is the Gauss sum if $p \ne 2$ and $1 + i$ if $p = 2$. That is, the (normalized) Fourier transform of the pull-back of the Artin–Schreier sheaf $\mathcal{L}_\psi$ by the n-th power map. The sheaf $\mathcal{F}$ is lisse of rank $n-1$, pure of weight 0 and $\mathbb{Q}(\zeta_p)$-valued, where ζ_p is a p-th root of unity (respectively $\mathbb{Q}(i)$-valued) if $p \neq 2$ (resp. $p = 2$). We have

$$M = M(\mathbb{Q}(\zeta_p), n-1) = p^{\lfloor 1+\log_p(n-1)\rfloor} \prod_{\substack{\lambda \leq n \text{prime} \\ \lambda \neq p}} \lambda^{\lfloor 1+\log_\lambda \frac{n-1}{\lambda-1}\rfloor}$$

if $p \neq 2$ and

$$M = M(\mathbb{Q}(i), n-1) = 2^{\lfloor 2+\log_2(n-1)\rfloor} \prod_{\substack{\lambda \leq n \text{prime} \\ \lambda \neq 2}} \lambda^{\lfloor 1+\log_\lambda \frac{n-1}{\lambda-1}\rfloor}$$

if $p = 2$, $b_1(X) = 0$, $e = \frac{1}{n-1}$ and $f = \max\{2, p-1\}$ (as p is totally ramified in $\mathbb{Q}(\zeta_p)$ if $p \neq 2$), so in this case we get finite monodromy iff all Frobenius eigenvalues are roots of unity at all points defined over extensions of $\mathbb{F}_p$ of degree up to

$$2R + \lfloor 2\log_p^+ \left(2R^2/(n-1)\right)\rfloor,$$

or if all Frobenius traces are integral at all points over extensions of $\mathbb{F}_p$ of degree up to

$$(n-1)\left(1 + \left\lfloor (n-1)\left(1 - \frac{1}{p^{\lfloor \log_p(n-1)\rfloor}}\right)\right\rfloor\right)\left(2R + \left\lfloor 2\log_p^+ \left(2R^2/(n-1)\right)\right\rfloor\right)$$

if $p \neq 2$ and

$$(n-1)\left(1 + \left\lfloor 2(n-1)\left(1 - \frac{1}{p^{\lfloor \log_2(n-1)\rfloor}}\right)\right\rfloor\right)\left(2R + \left\lfloor 2\log_2^+ \left(2R^2/(n-1)\right)\right\rfloor\right)$$

if $p = 2$. Even for $p = 2$, this gives degrees up to 61, 1178 and 18432701 respectively for $n = 3, 4, 5$ for the eigenvalues criterion, far beyond what we can compute in practice.

Let now $X = \mathbb{G}_{m,\mathbb{F}_q}$ be the one-dimensional torus over $\mathbb{F}_q$ where $q = p^r$, fix an additive character $\psi : \mathbb{F}_p \to \mathbb{C}$ and two disjoint sets of multiplicative characters $\chi = \{\chi_1, \ldots, \chi_a\}$ and $\rho = \{\rho_1, \ldots, \rho_b\}$, and consider the (normalized) hypergeometric sheaf

$$\mathcal{H} := \left(\frac{1}{G^{r(a+b-1)}}\right)^{deg} \mathcal{H}_1(!; \psi \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}; \chi; \rho)$$

[4, Chapter 8]. Assume $a > b$ and, if $p = 2$, also $a + b$ odd, then $\mathcal{H}$ is lisse of rank a on X, pure of weight 0 [4, Theorem 8.4.2], and $\mathbb{Q}(\zeta_{pm})$-valued, where $m = \mathrm{lcm}(\mathrm{ord}(\chi_i), \mathrm{ord}(\rho_j))$. Here

$$M = M(\mathbb{Q}(\zeta_{mp}), a) = p^{\lfloor 1+\log_p a\rfloor} \prod_{\lambda | m \text{prime}} \lambda^{\lfloor \mathrm{ord}_\lambda a + \log_\lambda a\rfloor} \prod_{\substack{\lambda \leq a+1 \text{prime} \\ \lambda \nmid mp}} \lambda^{\lfloor 1+\log_\lambda \frac{a}{\lambda-1}\rfloor},$$

$b_1(X) = 1$, $e = \frac{1}{a-b}$ and $f = p - 1$ (as p is totally ramified in $\mathbb{Q}(\zeta_p)$ and unramified in $\mathbb{Q}(\zeta_m)$), so in this case we get finite monodromy iff all Frobenius eigenvalues are roots of unity at all points defined over extensions of $\mathbb{F}_q$ of degree up to

$$2R + \left\lfloor 2\log_q^+ \left(2R^2\left(1 + \frac{1}{a-b}\right)\right)\right\rfloor \le 2R + \lfloor 2\log_q^+ \left(4R^2\right)\rfloor,$$

or if all Frobenius traces are integral at all points over extensions of $\mathbb{F}_p$ of degree up to

$$a\left(1 + \left\lfloor a\left(1 - \frac{1}{p^{\lfloor \log_p a\rfloor}}\right)\right\rfloor\right)(2R + \lfloor 2\log_q^+ \left(2R^2(1 + 1/(a-b))\right)\rfloor)$$
$$\le a\left(1 + \left\lfloor a\left(1 - \frac{1}{p^{\lfloor \log_p a\rfloor}}\right)\right\rfloor\right)(2R + \lfloor 2\log_q^+(4R^2)\rfloor).$$

For $p = 2$ and $a = 2$, this gives degrees up to 54 and 124 respectively for $m = 3, 5$ for the eigenvalues criterion, which is again beyond what we can compute in practice.

Proposition 7 can be improved in particular cases if we can get better estimates for the dimensions of the cohomology groups of the tensor powers of $\mathcal{F}$ (for instance, if the trace functions of these tensor powers can be written in terms of sums of additive or multiplicative characters, one may use the estimates for the sum of the Betti numbers given in the last section of [5]).

The main obstacle that makes the bounds for N so large in these theorems is the big rank of the components of $\mathcal{F}^{[M]}$. It is easy to see that the N in Theorem 5 can not be made smaller than $O(r)$ - even in dimension 0, one could take for instance the push-forward of the trivial sheaf from $\operatorname{Spec}\mathbb{F}_{q^r}$ to $\operatorname{Spec}\mathbb{F}_q$, which has rank r and trace 0 over $\mathbb{F}_{q^i}$ for $i < r$. However, in the proof of Proposition 7 we have equality not only of the Frobenius traces of $\mathcal{F}$ and $\mathcal{G} \oplus \bar{\mathbb{Q}}_\ell^r$, but also of their Frobenius characteristic polynomials. Any improvement of Theorem 5 in the case where we have equality of Frobenius characteristic polynomials, and not just of traces, would lead to similar improvements in the size of the N's in the theorems, which could hopefully make them usable in practice.

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