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Evolution algebras whose evolution operator is a homomorphism

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Abstract
This article deals with the evolution operator of evolution algebras. We give a theorem that allows to characterize these algebras when this operator is a homomorphism of algebras of rank $n-2$ and this result in turn allows us to extend the classification of this type of algebras, given in a previous result by ourselves in 2021, up to the case of dimension 4. For this purpose, we analyze and make use of an algorithm for the degenerate case. A computational study of the procedure is also made.

KEYWORDS
algorithm, classification, evolution algebras, evolution operator, homomorphism

1 | INTRODUCTION

On the one hand, evolution algebras, which were firstly introduced by Tian in his Ph.D. thesis[1] in 2004, have many connections with different mathematical theories, which makes them especially interesting for finding links between concepts from different branches of Mathematics, such as graph theory, group theory, dynamical systems, stochastic processes and mathematical physics.[2-6] On the other hand, they have also many applications to different scientific disciplines, such as physics, engineering and, above all, biology.[7-10] Both aspects explain the growing interest of researchers in them today. However, only the classification of these algebras up to dimension 3 has been obtained.[11-13] This article is related to this topic, being the main objective to classify a special family of evolution algebras in dimension 4.

All the information of these algebras is collected in the so-called structure matrix, from which the main operator, that reveals the dynamics of the algebra, is defined. In this article we will focus on the study of this operator, specifically in the case where it is a homomorphism of algebras. This fact has been previously analyzed in Reference 14, hence our purpose is to obtain new results in this subject and develop the methods outlined in that paper for higher-dimensional evolution algebras.

The structure of this article is as follows. In Section 2 we briefly introduce the main definitions and properties of evolution algebras. Section 3 begins with the review of the main results obtained in Reference 14, to later present a new result that allows to extend the classification of these algebras up to dimension 4. For this purpose, we analyze and make use of an algorithm for the degenerate case, doing a computational study of the procedure. At the end of this section, we explicitly classify these algebras in dimension 4. Finally, the article ends with some conclusions in Section 4.

2 | PRELIMINARIES ON EVOLUTION ALGEBRAS

Let $E \equiv (E, +, \cdot)$ be an algebra over a field $\mathbb{K}$. It is said that E is an *evolution algebra* if there exists a basis $\mathcal{B} = \{e_i : i \in \Lambda\}$ of E, where Λ is an index set, such that $e_i \cdot e_j = 0$, if $i \neq j$. The basis $\mathcal{B}$ is called a natural basis. Since $\mathcal{B}$ is a basis, the product $e_j \cdot e_j = e_j^2$ can be written as $\sum_{i\in\Lambda} a_{ij} e_i$, with $a_{ij} \in \mathbb{K}$, where only a finite quantity of a_{ij}, called *structure constants*, are nonzero for each $j \in \Lambda$ fixed. So, the product on E is determined by the *structure matrix* $A = (a_{ij})$.

Tian defined in Reference 15 the *evolution operator* associated with $\mathcal{B}$ as the endomorphism $L : E \to E$ which maps each generator into its square, that is, $L(e_j) = e_j^2 = \sum_{i\in\Lambda} a_{ij} e_i$, for all $j \in \Lambda$. The matrix representation of the evolution operator with respect to the basis $\mathcal{B}$ is the structure matrix A.

Throughout this article we will differentiate between two types of algebras, the *nondegenerate* ones, when $e_j^2 \neq 0$, for all $j \in \Lambda$, and the *degenerate* ones, when there exists j such that $e_j^2 = 0$.

In general, evolution algebras are neither associative nor power-associative, but they are commutative and hence flexible.

3 | EVOLUTION ALGEBRAS WHOSE EVOLUTION OPERATOR IS A HOMOMORPHISM OF ALGEBRAS

An endomorphism $g : E \to E$ is called a homomorphism of algebras if $g(x \cdot y) = g(x) \cdot g(y)$, for all $x, y \in E$. In this section, we study those evolution algebras over $\mathbb{C}$ such that $L \in Der(E)$, that is, the evolution operator is a derivation. This condition, as shown in Reference 14, is equivalent to

$$A\left(c_j \odot (c_j - \vec{1})\right) = 0, \quad \forall\, j, \tag{1}$$

$$A\left(c_i \odot c_j\right) = 0, \quad \forall\, i \neq j, \tag{2}$$

where c_j is the jth column of A and $\odot$ denotes the Hadamard product, that is, the element-by-element multiplication.

A study is carried out in Reference 14 to characterize this type of algebras, up to ordering of the basis elements, reaching the following results in the nondegenerate case.

Proposition 1. *Let E be an evolution algebra of dimension n such that $L \in End(E)$. Let c_k be a nonnull column that can be written as a linear combination of others columns. Then, $Ac_k = Ac_k^{(2)} = 0$. In particular, if E is nondegenerate and A has rank $n-k$, then there are at least $k+1$ of these columns.*

Theorem 1. *Let E be an evolution algebra with structure matrix $A = (a_{ij})$ such that L is an automorphism. Then, $L \in End(E)$ if and only if A is obtained by permuting the columns of the identity matrix.*

Theorem 2. *Let $n \geq 2$. Let E be a nondegenerate evolution algebra, with natural basis $\mathcal{B} = \{e_i : i = 1, \dots, n\}$ and structure matrix $A = (a_{ij})$ with rank $n-1$. Then, $L \in End(E)$ if and only if there exists a rearrangement of the basis $\mathcal{B}$ such that the structure matrix has the form*

$$\begin{pmatrix} A_1 & A_2 \\ O & P \end{pmatrix},$$

where

- $A_1 = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}$, *for some $a \neq 0$.*
- *A_2 is a $2 \times (n-2)$ matrix whose two rows are equal.*
- *O is the $(n-2) \times 2$ zero matrix.*
- *P is a $(n-2) \times (n-2)$ matrix obtained by permuting the columns of the identity matrix of size $n-2$.*

Theorem 3. *Let E be a nondegenerate evolution algebra of dimension n. Suppose the structure matrix A has range 1. Then, $L \in End(E)$ if and only if $\vec{\lambda} = (1, \lambda_2, \dots, \lambda_n)^t$ is orthogonal to c_1 and $c_1^{(2)}$, where $c_j = \lambda_j c_1$, for all $j \neq 1$.*

If E is degenerate, the following algorithm to obtain all the structure matrices A of these algebras in dimension n is given.

Step 1 Obtain all matrices A' of dimension $n-1$, both degenerate and nondegenerate.
Step 2 For each A' obtained, find the vector space orthogonal to $\{c'_j \odot (c'_j - 1), c'_i \odot c'_j; i \neq j\}$. Then, the searched matrices are those of the form $A = \begin{pmatrix} A' & 0 \\ v^t & 0 \end{pmatrix}$, with v belonging to this orthogonal space.

Additionally, the complete classification for dimensions less than four is given in Reference 14.

3.1 | Characterization for rank $n-2$

We give next a characterization for rank $n-2$, which will be used later to obtain the complete classification of these algebras in dimension 4.

Theorem 4. *Let $n \geq 4$. Let E be a nondegenerate evolution algebra with natural basis $\mathcal{B} = \{e_i : i = 1, \ldots, n\}$ and structure matrix $A = (a_{ij})$ of rank $n-2$. Then, $L \in End(E)$ if and only if there exists a rearrangement of the elements of the basis $\mathcal{B}$ such that one of the following cases is obtained*

1. *The structure matrix has the form*

$$\begin{pmatrix} A_1 & A_2 \\ O & P \end{pmatrix},$$

where

- *O is the $(n-r) \times r$ zero matrix.*
- *P is a $(n-r) \times (n-r)$ matrix obtained by permuting the columns of the identity matrix of size $n-r$.*
- *A_1 and A_2 are matrices of dimension $r \times r$ and $r \times (n-r)$, respectively, and match one of these descriptions*

(a) *$r = 3$ and there exist $a, b \neq 0$, with $a \neq -b$, such that*

$$A_1 = \begin{pmatrix} a & b & -a-b \\ a & b & -a-b \\ a & b & -a-b \end{pmatrix}.$$

A_2 has all its rows the same.

(b) *$r = 3$ and there exist $a, b \neq 0$, such that*

$$A_1 = \begin{pmatrix} a & -a & b \\ a & -a & b \\ 0 & 0 & 0 \end{pmatrix}.$$

A_2 has the first two rows the same. In the third row there is at most a 1.

(c) *$r = 3$ and there exist $a, b, c \neq 0$ and different from each other, such that*

$$\begin{pmatrix} a & a\lambda & a\mu \\ b & b\lambda & b\mu \\ c & c\lambda & c\mu \end{pmatrix},$$

with $\lambda = -\frac{a(c-a)}{b(c-b)}$ and $\mu = -\frac{a(b-a)}{c(b-c)}$.
Column j of A_2 is of the form $(\alpha_j + \beta_j a, \alpha_j + \beta_j b, \alpha_j + \beta_j c)^t$, where there is at most a nonnull α_j and equal to 1.

(d) *$r = 4$ and there exist a, b, c, d, with $ad - bc \neq 0$, such that*

$$A_1 = \begin{pmatrix} a & -a & c & -c \\ a & -a & c & -c \\ b & -b & d & -d \\ b & -b & d & -d \end{pmatrix}.$$

A_2 has its first two rows the same and its last two rows the same.

2. *Same assumptions as (1b), with a 1 in the third row of A_2 and canceling in P the column that corresponds to this element.*
3. *Same assumptions as (1c), with an $\alpha_j = 1$ and canceling in P the column that corresponds to this element.*

Proof. Let $V = null(A)$ and $\Omega = \{i : \exists\, v = (v_1, \dots, v_n)^t \in V, v_i \neq 0\}$. If $i \in \Omega$, then there exists $v = (v_1, \dots, v_n)^t \in \mathbb{C}^n$ such that $Av = 0$ and $v_i \neq 0$. From $Av = 0$, we get that $\sum_{i=1}^{n} v_i c_i = 0$. Since $v_i \neq 0$, Proposition (1) implies $c_i \in V$. That is, if $i \in \Omega$, then $c_i \in V$.

Let us see that $|\Omega| \leq 4$. By contradiction, suppose $|\Omega| > 4$. Then, there are more than four columns that belong to V. Since the dimension of V is 2, the rank of the matrix formed by these columns is less than or equal to 2. This implies that the rank of A is less than $2 + (n-4) = n-2$, which is a contradiction.

In addition, $|\Omega| \neq 1$, since otherwise any vector of V would have all its coordinates null, except one, from which it follows that V would be contained in a one-dimensional space, which is not possible since its dimension is 2.

Similarly, $|\Omega| \neq 2$, since if $\Omega = \{i, j\}$, then V is the space generated by e_i and e_j. From $Ae_j = 0$, it would be deduced that $c_j = 0$, which is not possible.

Therefore, $|\Omega|$ is 3 or 4. We analyze these two cases

- If $|\Omega| = 3$, we can assume by a rearrangement of the basis that $\Omega = \{1, 2, 3\}$. Then, $c_1, c_2, c_3, c_1^{(2)}, c_2^{(2)}, c_3^{(2)} \in V$, and therefore these columns can only have their first three entries nonnull. We split the matrix A as follows

$$A = \begin{pmatrix} A_1 & A_2 \\ O & P \end{pmatrix},$$

with A_1 of dimension 3×3. Clearly O is the zero matrix. Note that if $v = (v_1, v_2, v_3, 0, \dots, 0)^t \in V$, then the product $Av = 0$ is equivalent to the product $A_1(v_1, v_2, v_3)^t = 0$. Since the only nonnull coordinates of the vectors in V are the first three ones, we decompose $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-3}$, and we identify the subspace V with its projection in $\mathbb{R}^3$.
Let $j \geq 4$. From the condition (1), all the coordinates of $c_j \odot (c_j - \vec{1})$, except the first three ones, are null. That is, a_{ij} is 1 or 0, for all $i \geq 4$.
Considering the constraints of the conditions (1) and (2) to A_1, we deduce that this matrix must be the structure matrix of a nondegenerate evolution algebra of dimension 3 whose evolution operator is a homomorphism of algebras. Moreover, the rank of this matrix must be less than or equal to 2, since $c_1, c_2, c_3 \in V$ and $dim(V) = 2$. We know all these matrices (up to rearrangement of the basis vectors) so we can distinguish the following cases
  - If A_1 is of the form

$$\begin{pmatrix} a & -a & \lambda \\ a & -a & \lambda \\ a & -a & 1 \end{pmatrix},$$

    for some $a \neq 0$. In this case $c_3 \notin V$, which is a contradiction.
  - If A_1 is of the form

$$\begin{pmatrix} a & b & -a-b \\ a & b & -a-b \\ a & b & -a-b \end{pmatrix},$$

    for some $a, b \neq 0$, with $a \neq -b$.
    Let $j \geq 4$. Since $c_1 \odot c_j \in V$, then $(a_{1j}, a_{2j}, a_{3j}) \in V$. From this, along with $c_j \odot (c_j - \vec{1}) \in V$, it follows that $(a_{1j}^2, a_{2j}^2, a_{3j}^2) \in V$. That is

$$\begin{cases} a(a_{1j} - a_{3j}) + b(a_{2j} - a_{3j}) = 0, \\ a(a_{1j}^2 - a_{3j}^2) + b(a_{2j}^2 - a_{3j}^2) = 0. \end{cases}$$

Let us see that the only solution to this system is $a_{1j} = a_{2j} = a_{3j}$, so all the rows in A_2 are equal. The second equation is equivalent to

$$a(a_{1j} - a_{3j})(a_{1j} + a_{3j}) + b(a_{2j} - a_{3j})(a_{2j} + a_{3j}) = 0.$$

From the first equation, we obtain that $b(a_{2j} - a_{3j}) = -a(a_{1j} - a_{3j})$, so the previous equation remains

$$a(a_{1j} - a_{3j})(a_{1j} - a_{2j}) = 0,$$

from which it follows that $a_{1j} = a_{2j} = a_{3j}$.
Let us see the structure of the submatrix P. From the condition (1), it follows that all the entries in this matrix are zeros and ones. From the condition (2), it follows that there are not more than a 1 per row, since otherwise the product of the corresponding columns would have one of its last $n-3$ coordinates equal to 1, which cannot be true because $\Omega = \{1, 2, 3\}$. In addition, any column c_j, with $j \geq 4$, must have a 1, since otherwise that column would be a multiple of c_1 and the rank of A would be less than $n-2$. Thus, P is a matrix obtained by permuting the columns of the identity matrix of size $n-4$.

– If A_1 is of the form

$$\begin{pmatrix} a & -a & b \\ a & -a & b \\ 0 & 0 & 0 \end{pmatrix},$$

for some $a, b \neq 0$.
Let $j \geq 4$. Since $c_1 \odot c_j \in V$, then $(a_{1j}, a_{2j}, 0) \in V$, from which we get $a_{1j} = a_{2j}$. Since $c_j \odot (c_j - \vec{1}) \in V$, then

$$u := \begin{pmatrix} 0 \\ 0 \\ a_{3j}(a_{3j}-1) \end{pmatrix} = \begin{pmatrix} a_{1j}(a_{1j}-1) \\ a_{1j}(a_{1j}-1) \\ a_{3j}(a_{3j}-1) \end{pmatrix} - \begin{pmatrix} a_{1j}(a_{1j}-1) \\ a_{1j}(a_{1j}-1) \\ 0 \end{pmatrix} \in V.$$

From $Au = 0$, we get that $a_{3j}(a_{3j} - 1)c_3 = 0$, so $a_{3j} \in \{0, 1\}$.
Therefore, the first and second rows of A_2 are the same, and in the third one a 1, at most, can appear, since otherwise the condition (2) would not be fulfilled. Only in the case in which this 1 exists (say, in position $(3, j_0)$), it could happen that the matrix P is not a permutation by columns of the identity matrix, since the 1 located in the j_0th column could be removed from P, leaving the corresponding column null.

– If A_1 is of the form

$$\begin{pmatrix} a & a\lambda & a\mu \\ b & b\lambda & b\mu \\ c & c\lambda & c\mu \end{pmatrix},$$

for some $b, c \neq 0$, with $b \neq c$, $\lambda = -\frac{a(c-a)}{b(c-b)}$ and $\mu = -\frac{a(b-a)}{c(b-c)}$. Since λ and μ cannot be null, also $b \neq a \neq c$.
Then, $\{c_1, c_1^{(2)}\}$ is a basis of V, since a vector and its square are linearly independent if and only if the vector has at least two different nonnull coordinates. In addition, we have that $a \neq 0$ by the definition of Ω.
Let $j \geq 4$. Since $c_1 \odot c_j \in V$, then there exist α_j and β_j such that

$$\begin{pmatrix} aa_{1j} \\ ba_{2j} \\ ca_{3j} \end{pmatrix} = \alpha_j \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \beta_j \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}.$$

Therefore

$$\begin{cases} a_{1j} = \alpha_j + \beta_j a, \\ a_{2j} = \alpha_j + \beta_j b, \\ a_{3j} = \alpha_j + \beta_j c. \end{cases}$$

Since

$$\underbrace{c_j \odot (c_j - 1)}_{\in V} = (\vec{\alpha}_j + \beta_j c_1) \odot (\vec{\alpha}_j + \beta_j c_1 - \vec{1})$$
$$= \underbrace{\beta_j^2 c_1^{(2)}}_{\in V} + \underbrace{\beta_j(2\alpha_j - 1)c_1}_{\in V} + \vec{\alpha}_j^{(2)} - \vec{\alpha}_j,$$

then $\vec{\alpha}_j^{(2)} - \vec{\alpha}_j \in V$. Since

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(a-c)(b-c) \neq 0,$$

then there are no nonzero constant vectors in V, so $\alpha_j \in \{0, 1\}$.
Since

$$\underbrace{c_i \odot c_j}_{\in V} = (\vec{\alpha}_i + \beta_i c_1) \odot (\vec{\alpha}_j + \beta_j c_1)$$
$$= \underbrace{\beta_i \beta_j c_1^{(2)}}_{\in V} + \underbrace{(\alpha_i \beta_j + \alpha_j \beta_i) c_1}_{\in V} + \vec{\alpha}_i \odot \vec{\alpha}_j,$$

then $\vec{\alpha}_i \odot \vec{\alpha}_j \in V$, so there can only exist, at most, one nonnull α_j.
Only if there is a $\alpha_{j_0} = 1$, it could happen that the matrix P is not a permutation by columns of the identity matrix, since the 1 located in the j_0th column could be removed from P, leaving this column null.

- If $|\Omega| = 4$, we can assume by a rearrangement of the basis that $\Omega = \{1, 2, 3, 4\}$. Then, $c_j, c_j^{(2)} \in V$, for $j = 1, 2, 3, 4$. Analogously to the previous case, we split the matrix A as

$$A = \begin{pmatrix} A_1 & A_2 \\ O & P \end{pmatrix},$$

with A_1 of dimension 4×4 and O the zero matrix, and we identify the subspace V with its projection in $\mathbb{R}^4$.
Rows of A_1 provide implicit equations of V, so the rank of A_1 must be 2. Suppose, by reordering the basis, that the first two rows of A_1 are linearly independent.
We distinguish two cases

  – All the nonnull entries of c_j are equal, for every $j \in \{1, 2, 3, 4\}$. Let us set $i_0 \in \{3, 4\}$. Suppose that the row i_0 is not null. Since it is a linear combination of the first two rows and the nonnull elements in a column are equal, the only possibility is that row i_0 is equal to row 1 or row 2.
  Let us rearrange the first four elements of the basis in such a way that c_1 is the column with the highest number of zeros, regardless of the fact that maybe the first two rows are not linearly independent.

    * If there are no zeros in c_1, then there are no zeros in A_1 and all rows are equal, which is not possible due to the rank of A_1 is 2.

* If there is a 0 in c_1 at position i_0, from $Ac_1 = 0$ we obtain that $c_1 + c_2 + c_3 + c_4 - c_{i_0} = 0$, that is to say, there are three columns that add up to 0. If there is a column without zeros, we obtain that $c_1 + c_2 + c_3 + c_4 = 0$, so $c_{i_0} = 0$, which cannot be possible. Therefore there is a zero in all columns. Since $c_1 \odot c_j \in V$, if these zeros are in a position $i_1 \neq i_0$, we obtain that $c_1 + c_2 + c_3 + c_4 - c_{i_0} - c_{i_1} = 0$, from which we get $c_{i_1} = 0$. Since this is not possible, all zeros are in the same row, and the other three rows of A_1 would be the same, which is a contradiction due to the rank of A.
* If there are three zeros in c_1, from $Ac_1 = 0$ we get that there is a null column.
* There cannot be four zeros in c_1 because $c_1 \neq 0$.

Therefore there are two zeros in c_1. A similar reasoning to the one done above shows us that, if there are zeros in one column, there must be two of them and in the same or opposite positions as in c_1.
Reordering the elements 2, 3, 4 of the basis (this leaves the first column with the same number of zeros), we can assume that $c_1 = (a, a, 0, 0)^t$ or $c_1 = (0, 0, a, a)^t$, with $a \neq 0$. Since $c_1 \in V$, it is necessary that $c_2 = -c_1$. In these cases A_1 can be

$$\begin{pmatrix} a & -a & c & -c \\ a & -a & c & -c \\ 0 & 0 & c & -c \\ 0 & 0 & c & -c \end{pmatrix}, \quad \begin{pmatrix} a & -a & 0 & 0 \\ a & -a & 0 & 0 \\ 0 & 0 & c & -c \\ 0 & 0 & c & -c \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & c & -c \\ 0 & 0 & c & -c \\ a & -a & c & -c \\ a & -a & c & -c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & c & -c \\ 0 & 0 & c & -c \\ a & -a & 0 & 0 \\ a & -a & 0 & 0 \end{pmatrix}.$$

Recall that the rows of these matrices determine the implicit equations of V. From condition (2) we get that the first two rows of A_2 are equal, and the last two as well. The matrix P is the expected one.

– There exists $j \in \{1, 2, 3, 4\}$ such that c_j has at least two distinct nonnull values. By reordering the basis, we assume that this column is the first one, $c_1 = (a, b, c, d)^t$, and that $a \neq b$. Then, $\{c_1, c_1^{(2)}\}$ is a basis of V. Also $a, b, c, d \neq 0$, by the definition of Ω. Let λ_j, μ_j be such that $c_j = \lambda_j c_1 + \mu_j c_1^{(2)}$. Let j_0 be such that $\mu_{j_0} \neq 0$ (there must exist at least one or the rank of A_1 would be 1). Then

$$\underbrace{c_1 \odot c_{j_0}}_{\in V} = \underbrace{\lambda_{j_0} c_1^{(2)}}_{\in V} + \mu_{j_0} c_1^{(3)},$$

that is to say, $c_1^{(3)} \in V$. Then

$$0 = \begin{vmatrix} a^3 & a^2 & a \\ b^3 & b^2 & b \\ c^3 & c^2 & c \end{vmatrix} = \underbrace{abc(a-b)}_{\neq 0}(a-c)(b-c),$$

$$0 = \begin{vmatrix} a^3 & a^2 & a \\ b^3 & b^2 & b \\ d^3 & d^2 & d \end{vmatrix} = \underbrace{abd(a-b)}_{\neq 0}(a-d)(b-d).$$

Hence $c, d \in \{a, b\}$. Let us see the different possibilities

* If $c = d = a$, then $c_1 = (a, b, a, a)$, and another basis of V is $\{(1, 0, 1, 1)^t, (0, 1, 0, 0)^t\}$. Since $A(0, 1, 0, 0)^t = 0$, then $c_2 = 0$, which is a contradiction.
* If $c = d = b$, then $c_1 = (a, b, b, b)$, and another basis of V is $\{(1, 0, 0, 0)^t, (0, 1, 1, 1)^t\}$. Since $A(1, 0, 0, 0)^t = 0$, then $c_1 = 0$, which is a contradiction.

TABLE 1 Computing time

Input	Computing time (s)	Input	Computing time (s)
$n = 2$	0.063	$n = 80$	26.703
$n = 6$	0.219	$n = 100$	44.469
$n = 10$	0.406	$n = 120$	69.309
$n = 20$	1.375	$n = 140$	101.391
$n = 30$	3.078	$n = 160$	143.172
$n = 40$	5.531	$n = 180$	195.469
$n = 60$	13.656	$n = 200$	258.422

Therefore $\{c, d\} = \{a, b\}$. By a rearrangement of the basis we can assume that $c_1 = (a, a, b, b)$. Them $\{(1, 1, 0, 0)^t, (0, 0, 1, 1)^t\}$ is a basis of V, from which it follows that $c_2 = -c_1$ and $c_4 = -c_3$. Hence A_1 is of the form

$$\begin{pmatrix} a & -a & c & -c \\ a & -a & c & -c \\ b & -b & d & -d \\ b & -b & d & -d \end{pmatrix},$$

with $ad - bc \neq 0$ so that the rank of A_1 is 2. The first two rows of A_2 are equal, and the last two as well. The matrix P is the expected one. ■

3.2 | The algorithm for the degenerate case

In this subsection we implement the algorithm for the degenerate case in Python and we make a computational study of its complexity.

The code to execute the second step of the algorithm for a given matrix is as follows

```
from sympy import Matrix,matrix_multiply_elementwise,ones

def algorithm(A):
    n = A.shape[0]
    M1 = Matrix([[matrix_multiply_elementwise(A.col(i),A.col(i)-ones(n,1)).T]
                 for i in range(n)])
    M2 = Matrix([[matrix_multiply_elementwise(A.col(i),A.col(j)).T]
                 for i in range(n) for j in range(i+1,n)])
    M = Matrix([[M1],[M2]])
    return M.nullspace()
```

To do the complexity study, we have considered the family of evolution algebras whose structure matrix is a block matrix with blocks of the form

$$\begin{pmatrix} a & -a \\ a & -a \end{pmatrix}.$$

It is easy to check that all these matrices correspond to evolution algebras whose evolution operator is a homomorphism of algebras. The algorithm has been implemented with an Intel Core i5-1135G7 processor (frequency of 2.4 GHz) and 16 GB of RAM. Table 1 shows the computing time used to return the output of the whole procedure according to the dimension n of the algebra.

TABLE 2 Real computing time and expected time by the cubic fit in higher dimensions

Input	Computing time (s)	Expected time (s)	Relative error
$n = 220$	334.894	333.528	0.0041
$n = 240$	424.325	422.120	0.0052
$n = 260$	529.316	525.170	0.0078
$n = 280$	649.094	643.775	0.0082
$n = 300$	785.872	779.031	0.0087

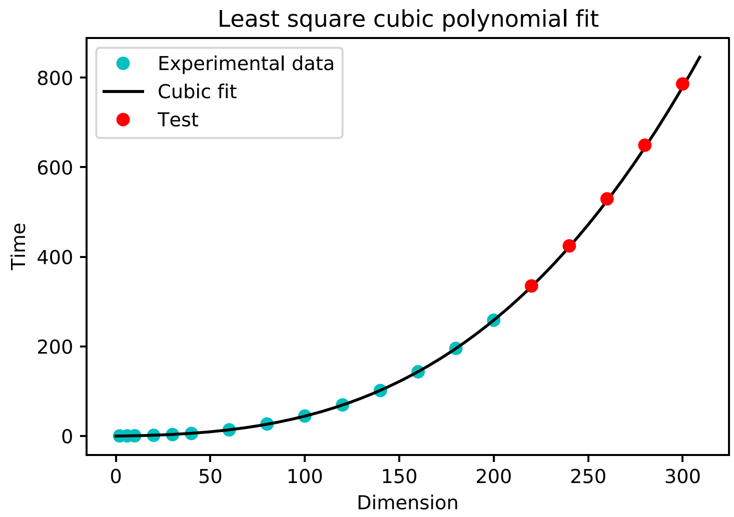

FIGURE 1 Least square cubic polynomial fit to the experimental data and test data for predictions

Let us compute the complexity of the algorithm taking into account the number of operations carried out in the worst case. For this purpose, we use the big O notation: Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) = O(g(x))$ if there exist $M \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$, for all $x \geq x_0$.

In our algorithm, the first loop is executed n times, and the second loop is executed $\frac{n(n-1)}{2} = O(n^2)$ times. Functions *matrix_multiply_elementwise*, *col*, T has complexity $O(n)$, so the complexity of the loops are $O(n^2)$ and $O(n^3)$, respectively. If function *nullspace* runs in $O(n^3)$, then the complexity of the algorithm is $O(n^3)$. To check if this assumption is correct, we calculate the least square cubic polynomial fit to our experimental data in Table 1, and we make predictions for larger dimensions ($n = 220, 240, 260, 280, 300$). As can be seen in Table 2 and Figure 1, our data fits this model, so our algorithm has complexity $O(n^3)$.

3.3 | Four-dimensional evolution algebras

In this subsection we give the complete classification of these algebras in dimension 4, up to reordering of the basis elements.

3.3.1 | Nondegenerate evolution algebras

By Theorem 1, those of maximum rank are

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \text{ and } \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}.$$

From Theorem 2, those of rank $n-1=3$ are the following, with $a \neq 0$,

$$\begin{pmatrix} a & -a & \lambda & \mu \\ a & -a & \lambda & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -a & \lambda & \mu \\ a & -a & \lambda & \mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For rank $n-2=2$, we get the following matrices from Theorem 4

- For all $a, b \neq 0$, with $a \neq -b$,

$$\begin{pmatrix} a & b & -a-b & \lambda \\ a & b & -a-b & \lambda \\ a & b & -a-b & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- For all $a, b \neq 0$

$$\begin{pmatrix} a & -a & b & \lambda \\ a & -a & b & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & -a & b & \lambda \\ a & -a & b & \lambda \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & -a & b & \lambda \\ a & -a & b & \lambda \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- For all $a, b, c \neq 0$ and different from each other

$$\begin{pmatrix} a & a\lambda & a\mu & a\beta \\ b & b\lambda & b\mu & b\beta \\ c & c\lambda & c\mu & c\beta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & a\lambda & a\mu & 1+a\beta \\ b & b\lambda & b\mu & 1+b\beta \\ c & c\lambda & c\mu & 1+c\beta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & a\lambda & a\mu & 1+a\beta \\ b & b\lambda & b\mu & 1+b\beta \\ c & c\lambda & c\mu & 1+c\beta \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

  where $\lambda = -\frac{a(c-a)}{b(c-b)}$ and $\mu = -\frac{a(b-a)}{c(b-c)}$.
- For all a, b, c, d, with $ad - bc \neq 0$,

$$\begin{pmatrix} a & -a & c & -c \\ a & -a & c & -c \\ b & -b & d & -d \\ b & -b & d & -d \end{pmatrix}.$$

Finally, from Theorem 3, the matrices of rank 1 are of the form

$$\begin{pmatrix} a & \lambda_1 a & \lambda_2 a & \lambda_3 a \\ b & \lambda_1 b & \lambda_2 b & \lambda_3 b \\ c & \lambda_1 c & \lambda_2 c & \lambda_3 c \\ d & \lambda_1 d & \lambda_2 d & \lambda_3 d \end{pmatrix},$$

with $(1, \lambda_1, \lambda_2, \lambda_3)^t$ orthogonal to $\langle (a,b,c,d)^t, (a^2,b^2,c^2,d^2)^t \rangle$, $\lambda_1, \lambda_2, \lambda_3 \neq 0$ and $(a,b,c,d) \neq (0,0,0,0)$. We distinguish between the following cases

- If a, b, c or d is null, we can assume that $d = 0$, by reordering the basis. Then, $(1, \lambda_1, \lambda_2)^t$ is orthogonal to $\langle (a, b, c)^t, (a^2, b^2, c^2)^t \rangle$, that is, the problem has been reduced in one dimension. In this case, the solutions are (see Reference 14)
  - For all $a, b, c \neq 0$ and different from each other

$$\begin{pmatrix} a & \lambda_1 a & \lambda_2 a & \lambda_3 a \\ b & \lambda_1 b & \lambda_2 b & \lambda_3 b \\ c & \lambda_1 c & \lambda_2 c & \lambda_3 c \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

  with $\lambda_1 = -\frac{a(c-a)}{b(c-b)}$, $\lambda_2 = -\frac{a(b-a)}{c(b-c)}$ and $\lambda_3 \neq 0$.
  - For all $a, \lambda_3 \neq 0$ and $\lambda_1 \neq 0, -1$

$$\begin{pmatrix} a & \lambda_1 a & -(\lambda_1 + 1)a & \lambda_3 a \\ a & \lambda_1 a & -(\lambda_1 + 1)a & \lambda_3 a \\ a & \lambda_1 a & -(\lambda_1 + 1)a & \lambda_3 a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

  - For all $a, \lambda_2, \lambda_3 \neq 0$

$$\begin{pmatrix} a & -a & \lambda_2 a & \lambda_3 a \\ a & -a & \lambda_2 a & \lambda_3 a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- If $a, b, c, d \neq 0$, then the following system must be fulfilled

$$\begin{cases} a + b\lambda_1 + c\lambda_2 + d\lambda_3 = 0, \\ a^2 + b^2\lambda_1 + c^2\lambda_2 + d^2\lambda_3 = 0. \end{cases}$$

  From the first equation, we obtain $\lambda_3 = -\frac{a+b\lambda_1+c\lambda_2}{d}$. Substituting in the second equation, we get

$$b(b-d)\lambda_1 + c(c-d)\lambda_2 + a(a-d) = 0.$$

  We distinguish the following cases
  - If $c = d$, then $b(b-d)\lambda_1 + a(a-d) = 0$. If $b = d$, we get $a = b = c = d$ and we obtain the matrices

$$\begin{pmatrix} a & \lambda_1 a & \lambda_2 a & -(1 + \lambda_1 + \lambda_2)a \\ a & \lambda_1 a & \lambda_2 a & -(1 + \lambda_1 + \lambda_2)a \\ a & \lambda_1 a & \lambda_2 a & -(1 + \lambda_1 + \lambda_2)a \\ a & \lambda_1 a & \lambda_2 a & -(1 + \lambda_1 + \lambda_2)a \end{pmatrix}.$$

    If $b \neq d$, we get

$$\lambda_1 = -\frac{a(a-d)}{b(b-d)}, \quad \lambda_3 = -\frac{a(a-b) + c(d-b)\lambda_2}{d(d-b)} = -\frac{a(a-b)}{d(d-b)} - \lambda_2.$$

  - If $c \neq d$, then we obtain

$$\lambda_2 = -\frac{a(a-d) + b(b-d)\lambda_1}{c(c-d)}, \quad \lambda_3 = -\frac{a(a-c) + b(b-c)\lambda_1}{d(d-c)}.$$

3.3.2 | Degenerate evolution algebras

Let us make use of the algorithm for all the matrices A' of dimension 3. All these matrices are (see Reference 14)

- For $\delta, \delta' \in \{0,1\}$ and $a, b \in \mathbb{C}$

$$\begin{pmatrix} \delta & 0 & 0 \\ \delta' & 0 & 0 \\ a & b & 0 \end{pmatrix}.$$

If $a = b = 0$ or $\{a, b\} = \{0,1\}$, then there are no conditions on v. In another case, $v_3 = 0$.
- For $\delta \in \{0,1\}$, $a \in \mathbb{C} \setminus \{0,1\}$ and $b \in \mathbb{C}$

$$\begin{pmatrix} \delta_1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}.$$

If $b \in \{0,1\}$, then $v_2 = 0$. In another case, $v_3 = -\frac{a(a-1)}{b(b-1)} v_2$.
- For $a, b \in \mathbb{C}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ a & b & 0 \end{pmatrix}.$$

If $a = b = 0$ or $\{a, b\} = \{0,1\}$, then there are no conditions on v. In another case, $v_3 = 0$.
- For $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$

$$\begin{pmatrix} a & -a & 0 \\ a & -a & 0 \\ b & -b & 0 \end{pmatrix}.$$

If $b = 0$, then $v_2 = -v_1$. If $b = a$, then $v_3 = -v_1 - v_2$. Otherwise it must be fulfilled that $a(v_1 + v_2) + bv_3 = a^2(v_1 + v_2) + b^2 v_3 = 0$. It implies that $v_3 = 0$ and $v_2 = -v_1$.
- For

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

there are no conditions on v.
- For $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$

$$\begin{pmatrix} a & -a & b \\ a & -a & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We have the condition $v_2 = -v_1$.

- For $a, b, c \in \mathbb{C} \setminus \{0\}$ and different from each other

$$\begin{pmatrix} a & \lambda a & \mu a \\ b & \lambda b & \mu b \\ c & \lambda c & \mu c \end{pmatrix},$$

where $\lambda = -\frac{a(c-a)}{b(c-b)}$ and $\mu = -\frac{a(b-a)}{c(b-c)}$. It must be fulfilled that $av_1 + bv_2 + cv_3 = a^2v_1 + b^2v_2 + c^2v_3 = 0$, and solving this system, we obtain that $v_2 = \lambda v_1$ and $v_3 = \mu v_1$.
- For $a, b \in \mathbb{C} \setminus \{0\}$ and $a \neq -b$

$$\begin{pmatrix} a & b & -a-b \\ a & b & -a-b \\ a & b & -a-b \end{pmatrix}.$$

We get the condition $v_3 = -v_1 - v_2$.
- For $a, b \in \mathbb{C} \setminus \{0\}$

$$\begin{pmatrix} a & -a & b \\ a & -a & b \\ 0 & 0 & 0 \end{pmatrix}.$$

It must be fulfilled the condition $v_2 = -v_1$.

4 | CONCLUSIONS

Classifications in evolution algebras by means of the behavior of the evolution operator are scant, above all for high dimensions. The results given in this article allow to obtain the complete classification up to dimension 4 in the case of homomorphism of algebras, which supposes, in our opinion, a step forward in the study of this topic. However, some related problems remain open. We show here some of them.

- Transfer the results obtained to other branches of Mathematics that have direct connections with evolution algebras. For example, each evolution algebra has associated a directed graph such that the edge starting from e_i to e_j has weight a_{ij}. Starting from this, how is the fact that the evolution operator is an endomorphism of algebras expressed in the graph? What properties do these graphs have?
- Study when the evolution operator is a homomorphism of algebras of rank r, with $2 \leq r \leq n-3$, in order to obtain a characterization of these algebras that allows a complete classification for dimensions greater than 4.
- Classify evolution algebras whose evolution operator satisfies other equalities.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

The authors claim to have contributed equally and significantly in this paper.

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