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Mixed Variational Inequality Interval-valued Problem: Theorems of Existence of Solutions

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Abstract. In this article, our efforts focus on finding the conditions for the existence of solutions of Mixed Stampacchia Variational Inequality Interval-valued Problem on Hadamard manifolds with monotonicity assumption by using KKM mappings. Conditions that allow us to prove the existence of equilibrium points in a market of perfect competition. We will identify solutions of Stampacchia variational problem and optimization problem with the interval-valued convex objective function, improving on previous results in the literature. We will illustrate the main results obtained with some examples and numerical results.

1. Introduction

The best strategy for solving an optimization problem may involve solving other related problems. The variational inequalities problems carry out this intermediate work.

Within the variational inequalities, in this article, we will study the mixed problems of variational inequality that have applications in circuits in electronics and energy control problems (see [1, 13]). Also, the general economic equilibrium problem and oligopolistic equilibrium problem can be formulated as mixed variational inequality problem (see Konnov and Volotskaya [24]).

Specifically, this article addresses these mixed problems of variational inequality in a novel context, such as Hadamard's manifolds with interval-valued functions.

The classical *mixed variational inequality problem* (MV) consists of finding a point $\overline{x} \in K$ such that

$$\langle G(\overline{x}), x - \overline{x} \rangle + h(x) - h(\overline{x}) \geq 0, \quad \forall\, x \in K,$$

where K is usually assumed to be a nonempty convex set in the real Euclidean space $\mathbb{R}^n$, $G\colon K \to \mathbb{R}^n$ and $h\colon K \to \mathbb{R}$ (see [20]).

(a) If $h = 0$ then the MV problem reduces the classical single-valued variational inequality.

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(b) If $G = 0$ then the MV problem is the usual optimization problem.

(c) If $\langle G(\overline{x}), x - \overline{x}\rangle = F(\overline{x}, x)$, then we are in the presence of the mixed equilibrium problem where F be a bifunction satisfying the property $F(x, x) = 0$ (see [20]).

The name is because they are a mixture of two problems, a variational one and an optimization one. The problem is composed of an operator G and a function h such that one can impose stronger conditions on the operator and weaken those of the function and vice versa. In the formulation of an oligopolistic equilibrium model G represents demand and h represents supply.

In this paper, we give new results of existence of solutions for mixed variational inequality problems under more general assumptions. We will use the Knaster–Kuratowski and Mazurkiewicz theory and served Nash to prove the existence of his equilibrium points.

Taking measurements leads to inaccuracies. One way of catching inaccuracy is with Interval Analysis. The Interval Analysis was introduced by Ramon Moore [28] in 1966 as a tool for automatic error control. Intervals allow the manipulation of truncation errors.

For example, during the Gulf War, a US Patriot missile failed to intercept another due to errors generated by inadequate numerical approximations implemented in the Patriot software (http://www.gao.gov/products/IMTEC-92-26) and the same occurred in the launch of the Ariane rocket, see the file http://esamultimedia.esa.int/docs/esa-x-1819eng.pdf.

In this article we will move within the environment of Hadamard's manifolds. In non-linear spaces, such as the Hadamard manifolds, we extend concepts such as convex sets where geodesic arcs connect two points instead of linear segments. Unsuspectedly, these spaces present some advantage over the linear ones such as that sets that are not convex in the usual sense are convex within these manifolds. We can transform non-convex problems with Euclidean metrics into convex problems with related metrics with all the advantages that this entails (see Colao et al. [9]) and non-monotone vector fields can be reduced into monotone by choosing an appropriate Riemannian metric.

With applications in expected fields such as in optimization problems related to engineering [26], stereo vision processing [27], machine learning and computer vision [31], and in others not so expected, for example, in Economics, within Game Theory, specifically in the achievement of Nash equilibrium points where strategy sets and payoff functions are geodesically convex, see [25], or for Stackelberg's equilibrium points on Hadamard manifolds [36].

The state of the art is as follows. Firstly, in 1980, Giannessi [15] introduced the vector variational inequalities of Stampacchia type, and secondly, in 1998, Giannessi [16] proposed Minty-type inequalities. One fundamental problem for variational inequality problem is the existence issue of solutions. Németh [29] was the first to consider the variational inequality on Hadamard manifolds, and he proved the existence of solutions in this space.

Our article starts on an initial one from 2003 given by Ruiz-Garzón et al. [33], where we established the existence of solutions for the Variational-like Inequality problem under conditions of pseudomonotonicity in n-dimensional Euclidean spaces.

In 2010, Jiang, Pang and Shen [22] studied the existence of solutions of generalized vector variational-type inequalities without the assumption of monotony in Banach spaces by using Brouwer's fixed point theorem.

In 2013, Zhou and Huang [41] introduced the KKM mapping on the Hadamard manifolds. They obtained an existence of weak minimum for a constrained vector optimization problem via a vector variational inequality with real valued functions. New optimality conditions for the mathematical programming problem were given on Hadamard manifolds using generalized convexity in [2, 42].

In 2016, Jana and Nahak [20] proved the existence of solutions of mixed equilibrium problems on Hadamard manifolds and algorithms that converge to the solution of such problems but not with interval-valued functions.

In 2018, Ruiz-Garzón et al. [35] obtained optimality conditions and duality results for constrained multiobjective programming in the Riemannian manifolds context.

In 2020, Wang et al. [40] related the mixed variational inequality with the Nash equilibrium problem on Riemannian manifolds.

Recently, there are two important articles for this work that we present. These are two articles from the year 2020 by S.-I. Chen [6, 7]. On one hand, in [6] Chen studies single existence results for vector variational problems with $f \colon M \to Y$ assuming that M is a finite dimensional Hadamard manifold and Y a Banach space. On the other hand, in [7] Chen studies the KKT conditions of optimality for optimization problem with interval-valued functions, but he does not study existence theorems.

In 2021, Grad and Lara [17] implement an algorithm for solving convex mixed variational inequalities on Euclidean spaces.

Our interest is to generalize all these results by studying the solubility of mixed variational problems with interval-valued functions assessed on Hadamard manifolds. This study is new and non-existent in the current literature.

Contributions. We have organized the contents of this paper as follows. In Section 2, we will recall those elements typical of manifolds, the arithmetic of intervals, and the differentiability of interval-valued functions. We will introduce the Stampacchia and Minty version of the Mixed Variational Inequality Interval-valued Problem. In Section 3, we will prove the solution of the Mixed Stampacchia Variational Inequality Interval-valued Problem (MSVIP) with monotonicity hypothesis via Fan's lemma. In Section 4, we will relate the Interval-valued Optimization Problem (IVOP) and Variational Inequality Interval-valued problems of the Stampacchia and Minty type under convex environments and we will finish by looking at the Walras' economic equilibrium point existence conditions. Finally, Section 5 presents the conclusions to this study. Let us face our goal.

2. Tools

This section will review the concepts and techniques of manifolds, intervals, and interval-valued functions that we will use.

2.1. Tools for manifolds

Let's show some definitions related to manifolds. Let M be a Riemannian manifold endowed with a Riemannian metric on a tangent space T_xM.

With the metric, we can define the corresponding norm denoted by $\|\cdot\|_x$. The length of a piecewise C^1 curve $\gamma\colon [a,b] \to M$ is defined by

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

The distance d that induces the original topology on M, defined as

$$d(x,y) = \inf\{L(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve joining } x \text{ and } y, \forall\, x,y \in M\}.$$

This definition allows us to define the concept of minimal geodesic as any path γ joining x and y in M such that $L(\gamma) = d(x,y)$. If M is complete, then any points in M can be joined by a minimal geodesic.

In this paper, we will work with Hadamard manifolds, a particular case of Riemannian manifolds.

Definition 2.1. We will say that a Hadamard's manifold is a simply connected complete Riemannian manifold of nonpositive sectional curvature.

Example 2.2. (a) The Hyperbolic plane is defined as $\mathcal{H}^2 = \{(x,y) \in \mathbb{R}^2; y > 0\}$ with the Riemannian metric defined as

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

The Hyperbolic plane is a Hadamard manifold with sectional curvature $\kappa = -1$.

(b) The space of positive-definite matrices S^n_{++} is an example of Hadamard manifold with the Riemannian metric $\langle U, V\rangle = \langle X^{-1}UX^{-1}, V\rangle$.

(c) The space $M = \mathbb{R} \times \mathcal{H}^1$ is a Hadamard manifold as they are a Cartesian product of two Hadamard's manifolds.

In the differentiable case, the derivatives of the curves at a point x on the manifold lie in a vector space T_xM. We denote by T_xM the n-dimensional tangent space of M at x, and denote by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M.

Just as T_xM is a linear space, but the same is not true for M. Generally, properties on manifolds are usually carried to the tangent plane and vice versa and that is carried out by two functions: the Riemannian exponential function and its inverse, exp and $\exp^{-1}$, respectively.

Let $\overline{T}M$ be an open neighborhood of M such that $\exp_x\colon \overline{T}M \to M$ is defined as $\exp_x(tv) = \gamma(t) = xe^{(v/x)t}$ for every $v \in \overline{T}M$, where γ is the geodesic starting at x with velocity v (i.e., $\gamma(0) = x$, $\gamma'(0) = v$).

Remark 2.3. (a) If $M = \mathbb{R}^p_+$ then $\exp_x^{-1} y = y - x$, where we denote by $\mathbb{R}^p_+$ the nonnegative orthant of $\mathbb{R}^p$, and $\mathbb{R}_+ := \mathbb{R}^1_+$.

(b) When M is a Hadamard manifold, then $\exp_x$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique minimal geodesic joining x to y. M is diffeomorphic to the Euclidean space $\mathbb{R}^n$, thus is, Hadamard manifolds and Euclidean spaces have similar differential structure and geometrical properties (see [9, Proposition 2.1]).

As in n-dimensional spaces, we will use a concept similar to the convex set.

Definition 2.4. [41] A subset K of M is said to be a geodesic convex if, for any two points $x, y \in K$, the geodesic γ of M has endpoints x and y are belonging to K; that is, if $\gamma\colon [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, then $\gamma(t) = \exp_x(t \exp_x^{-1} y) \in K$ for all $t \in [0,1]$.

Example 2.5. The set $X = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 4 \leq (x_1 - 1)^2 + x_2^2\}$ is not convex in the usual sense with $X \subset \mathbb{R}^2$, but X is a geodesic convex on the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, as it is the image of a geodesic segment.

Definition 2.6. [23] The geodesic convex hull of a set $K \subset M$ is the smallest geodesic convex subset of M containing K. It is denoted by $\operatorname{co}(K)$.

Definition 2.7. Let x_1, x_2 be any points in a Hadamard manifold M. The geodesic convex combination of x_1 and x_2 is the geodesic joining x_1 and x_2, and it is denoted by

$$\operatorname{com}_{(x_1,x_2)}(t_2) = \gamma_{x_2,x_1}(t_2) = \exp_{x_2}(t_2 \exp_{x_2}^{-1} x_1), \quad \forall\, t_2 \in [0,1].$$

Zhou et al. [43] gave the following theorems that we will use in this article.

Theorem 2.8. *Let M be a Hadamard manifold. A set $K \subset M$ is a geodesic convex if and only if it contains all geodesic combinations of elements of K.*

Theorem 2.9. *Let $K \subset M$ be any set in a Hadamard manifold M. Then* $\operatorname{co}(K)$ *consists of all the geodesic combinations of elements of K.*

Theorem 2.10. *For any two subsets K_1, K_2 of a Hadamard manifold M such that $K_1 \subseteq K_2$, then* $\operatorname{co}(K_1) \subseteq \operatorname{co}(K_2)$.

We will also extend the concept of convex function.

Definition 2.11. [38] Let M be a Hadamard manifold and let $K \subseteq M$ be a geodesic convex set. A function $\theta\colon K \to \mathbb{R}$ is said to be geodesic convex (GCX) if and only if for any geodesic γ of K, the composition function $\theta \circ \gamma\colon [0,1] \to \mathbb{R}$ is convex, i.e.,

$$(\theta \circ \gamma)(ta + (1-t)b) \le t(\theta \circ \gamma)(a) + (1-t)(\theta \circ \gamma)(b)$$

for every $a, b \in \mathbb{R}$, and $t \in [0,1]$.

In [10], the authors give an example of a nonconvex function that is nevertheless convex on a Riemannian manifold endowed with an adequate metric.

Example 2.12. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = e^{x_1}(\cosh(x_2) - 1)$ is not convex but f is geodesic convex on Riemannian manifold M_g with the metric $g(x) = \operatorname{diag}(1, e^{2x_1})$.

2.2. Tools for intervals

In this section, we will show the arithmetic related to intervals. We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_C = \big\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \le \overline{a}\big\}.$$

Let $A = [\underline{a}, \overline{a}] = \{a : \underline{a} \le a \le \overline{a}, a \in \mathbb{R}\}$ be a closed interval, where $\underline{a}$ and $\overline{a}$ mean lower and upper bounds of A. If $\underline{a} = \overline{a}$, then $A = [a, a] = a$ is a real number. We have

(a) $A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$ where $B = [\underline{b}, \overline{b}]$.

(b) $\lambda A = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}], & \lambda \ge 0, \\ [\lambda\overline{a}, \lambda\underline{a}], & \lambda < 0 \end{cases}$ where λ is a real number.

The space $\mathcal{K}_C$ is not a linear space since it does not posses an additive inverse and therefore subtraction is not well defined (see Diamond and Kloeden [11]), i.e., if we define

$$A - B = A + (-1)B$$

then, in general, $A - A \neq \{0\}$. For example,

$$[1,2] - [1,2] = [1,2] + [-2,-1] = [-1,1] \neq [0,0].$$

Hukuhara [19] made one of the first attempts. It is said that the Hukuhara difference exists, $A -_H B$, and it is equal to C (H-difference) if $A = B + C$.

The H-difference of two intervals only exists if the widths are such that $\operatorname{len}(A) \geq \operatorname{len}(B)$ where for $A = [\underline{a}, \overline{a}]$, $\operatorname{len}(A) = \overline{a} - \underline{a}$. For example, $[1,2] -_H [4,8]$ does not exist because $\operatorname{len}([1,2]) < \operatorname{len}([4,8])$.

To overcome these drawbacks, Stefanini and Bede [37] introduced the concept of *generalized Hukuhara difference* of two intervals $A, B \in \mathcal{K}_C$ (gH-difference for short), and it is defined as follows:

$$A \ominus_{gH} B = C \quad \Longleftrightarrow \quad A = B + C \quad \text{or} \quad B = A + (-1)C.$$

With this definition, this difference $[1,2] \ominus_{gH} [4,8] = [-6,-3]$ exists because $[4,8] = [1,2] + (-1)[-6,-3]$.

Thus, the gH-difference is a generalization of the H-difference. Chalco-Cano et al. [5] have shown that $A \ominus_{gH} B = \big[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\big]$.

We also need to establish an order to decide when one interval is greater than another. As we knew, the usual ordering "$\leq$" is a total ordering in $\mathbb{R}$; that is, for any two real numbers in $\mathbb{R}$, we can determine their order without difficulty. However, for any two closed intervals in $\mathbb{R}$, there is no natural ordering among the set of all closed intervals in $\mathbb{R}$, and we have to define it (see [34, Definition 2.2]).

Definition 2.13. Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two closed intervals in $\mathbb{R}$. We write

(a) $A \underline{\preceq} B \iff \underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.

(b) $A \preceq B \iff A \underline{\preceq} B$ and $A \neq B$, i.e., $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, with a strict inequality.

(c) $A \prec B \iff \underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

2.3. Tools for interval-valued functions on Hadamard manifolds

In this section, we deal with interval-valued functions on Hadamard manifolds. We will establish the concepts of differentiability, and convexity that we need.

Let D be a subset of a Hadamard manifold M endowed with the Riemannian metric. The function $f\colon D \to \mathcal{K}_C$ is called an *interval-valued function*, i.e., $f(x)$ is a closed interval in $\mathbb{R}$ for each $x \in M$. We will denote $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are real-valued functions and satisfy $f^L(x) \leq f^U(x)$ for every $x \in M$. The functions $f^L, f^U\colon D \to \mathbb{R}$ are called endpoint functions of f. For interval-valued functions on Hadamard manifolds we can define

Definition 2.14. [7] Let M be a Hadamard manifold, let $D \subseteq M$ be a nonempty open geodesic convex set. An interval-valued function $f\colon D \to \mathcal{K}_C$ is called directionally gH-differentiable map along the geodesic γ at $x \in D$, if and only if the limit

$$f'_{gH}(x;\gamma) = \lim_{t\to 0^+} \frac{f(\exp_x(t \exp_x^{-1} y)) \ominus_{gH} f(x)}{t}$$

exists.

Remark 2.15. (a) The previous definition is an extension of the one given by Zhou and Huang [41] for real valued functions.

(b) If f is differentiable map along the geodesic $\gamma(t) = \exp_x(t \exp_x^{-1} y)$ at $x \in M$, then

$$f'_{gH}(x; \exp_x^{-1} y) = df_x(\exp_x^{-1} y) = \langle \operatorname{grad} f, \exp_x^{-1} y\rangle.$$

Moreover, we can define the convexity of an interval-valued function that is so important in the study of optimality conditions.

Definition 2.16. [7] Let $D \subseteq M$ be a nonempty open geodesic convex set and $f\colon D \to \mathcal{K}_C$ be an interval-valued function. Then f is said to be interval-valued geodesic convex (IGCX) at $x \in D$, if for each $y \in D$,

$$f(\gamma(t)) \preceq tf(y) + (1-t)f(x), \quad \forall\, t \in [0,1],$$

where $\gamma(t) = \exp_x(t \exp_x^{-1} y)$ for every $t \in [0,1]$.

We will use the concept of convex function given by Chen [7].

Lemma 2.17. *Let M be a Hadamard manifold and let $D \subseteq M$ be a geodesic convex set. A differential function $f\colon K \to \mathcal{K}_C$ along the geodesic γ is said to be interval-valued geodesic convex (IGCX) on K if and only if for any $x, y \in K$,*

$$f(x) \ominus_{gH} f(\overline{x}) \succeq df_{\overline{x}}(\exp_{\overline{x}}^{-1} x) = f'_{gH}(x; \exp_x^{-1} y).$$

Example 2.18. Let $M = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be endowed with the Riemannian metric defined by $\langle \cdot,\cdot\rangle = g(x)uv$ with $g(x) = x^{-2}$, where $g\colon \mathbb{R}_{++} \to (0,+\infty)$. Let $K = \{x \mid x = e^t, t \in [0,1]\}$ be a subset of M.

Let $\gamma(t) = xe^{(v/x)t} = \exp_x(tv)$ be a geodesic and therefore $\exp_x^{-1} y = x\ln(y/x)$. We have $f(x) = [f^L(x), f^U(x)] = [x^3, x^3+1]$ and $h(x) = [x, 2x]$. In terms of g we obtain that

$$\operatorname{grad} f(x) = g(x)^{-1} f'(x) = x^2 f'(x),$$

where f' denotes the first derivatives of f in the Euclidean sense.

We can calculate

$$\begin{aligned} df_x(\exp_x^{-1} y) &= \langle \operatorname{grad} f(x), \exp_x^{-1} y\rangle = \langle [x^2(3x^2), x^2(3x^2)], x\ln(y/x)\rangle \\ &= x^{-2}[3x^4 \cdot x\ln(y/x), 3x^4 \cdot x\ln(y/x)] = [3x^3\ln(y/x), 3x^3\ln(y/x)] \end{aligned}$$

and

$$f(y) \ominus_{gH} f(x) \succeq df_x(\exp_x^{-1} y), \quad \forall\, x, y \in K.$$

Therefore, f is a differential interval-valued convex mapping.

We will continue by generalizing the Stampacchia and Minty versions of these variational problems on Hadamard manifold given by [8] to interval-valued functions. The close relationship of the solutions of these problems means that they can be considered as "dual" and "primal" problems. As a general rule, the Minty type formulation is easier to solve. We can introduce

Definition 2.19. (a) Mixed Stampacchia Variational Inequality Interval-valued Problem (MSVIP): Find a point $x \in K$ such that there exists no $y \in K$ such that

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \prec [0, 0],$$

where $f\colon K \to \mathcal{K}_C$ is a gH-differentiable function along the geodesic γ and $h\colon K \to \mathcal{K}_C$ is an interval-valued function.

(b) Mixed Minty Variational Inequality Interval-valued Problem (MMVIP): Find a point $x \in K$ such that there exists no $y \in K$ such that

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \succ [0, 0].$$

The following problems are the special cases of MSVIP:

(a) If $M = \mathbb{R}^p_+$ with K being a nonempty, closed, and convex subset of M and the image set is $\mathbb{R}^n$, then the MSVIP reduces to the problem considered by Facchinei and Pang [12].

(b) If $M = \mathbb{R}^p_+$ with K being a nonempty, closed, and convex subset of M, the image set is $\mathbb{R}^n$ and $h = 0$, $\forall\, x, y \in K$, then the MSVIP reduces to the following classical variational problem (VIP): Find a point $x \in K$ such that $\langle \nabla f(x), y - x\rangle \geq 0$, $\forall\, y \in K$, which has been considered by Hartman and Stampacchia [18].

Our goal in the next section is to prove the solubility of MSVIP.

3. Existence results

The Knaster–Kuratowski and Mazurkiewicz (KKM) theory is born as a continuation of the fixed point theorems and they have an essential role in the search for solutions of variational problems with the monotonicity hypothesis, as we will see.

Definition 3.1 (KKM mapping)**.** Let K be a nonempty closed convex subset of a Hadamard manifold M and $F\colon K \to 2^K$ be a multi-valued mapping. F is called a KKM mapping if $\operatorname{co}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ for any finite set $\{x_1, x_2, \dots, x_n\}$ of K.

Lemma 3.2 (Fan)**.** *Let K be a nonempty closed geodesic convex subset of a Hadamard manifold M and $F\colon K \to 2^K$ be a multi-valued mapping. Suppose that F is a KKM mapping. If $F(x)$ is closed for each $x \in K$ and compact for some $x \in K$, then $\bigcap_{x\in K} F(x) \neq \emptyset$.*

We need to extend the hemicontinuous concept or continuous over linear segments to geodesics.

Definition 3.3. [20] A function $\operatorname{grad} f\colon K \to \mathcal{K}_C$ is said to be geodesic hemicontinuous if for every geodesic γ, whenever $t \to 0$,

$$\operatorname{grad} f(\gamma(t)) \to \operatorname{grad} f(\gamma(0)).$$

The concept of monotonicity and its generalization has a primary role in investigating existence results for variational inequalities [33]. We will extend the pseudomonotonicity concept between Euclidean spaces given by Ruiz-Garzón et al. [33] to an interval-valued function on Hadamard manifolds.

Definition 3.4. A mapping $\operatorname{grad} f\colon K \to \mathcal{K}_C$ is said to be h-pseudomonotone (PM) if for every $x, y \in K$, such that

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \not< [0,0],$$

then

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \not> [0,0],$$

where $h\colon K \to \mathcal{K}_C$ is an interval-valued function.

Example 3.5. As Example 2.18, we have $f(x) = [f^L(x), f^U(x)] = [x^3, x^3+1]$ and $h(x) = [x, 2x]$. In terms of g we obtain that

$$\operatorname{grad} f(x) = g(x)^{-1} f'(x) = x^2 f'(x),$$

where f' denotes the first derivatives of f in the Euclidean sense. We can calculate

$$\begin{aligned} df_x(\exp_x^{-1} y) &= \langle \operatorname{grad} f(x), \exp_x^{-1} y\rangle = \langle [x^2(3x^2), x^2(3x^2)], x\ln(y/x)\rangle \\ &= x^{-2}[3x^4 \cdot x\ln(y/x), 3x^4 \cdot x\ln(y/x)] = [3x^3\ln(y/x), 3x^3\ln(y/x)]. \end{aligned}$$

The mapping $\operatorname{grad} f$ is h-pseudo monotone (PM) because $\forall\, x, y \in K$, such that

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) = [3x^3 \ln(y/x), 3x^3 \ln(y/x)] + ([y, 2y] \ominus_{gH} [x, 2x]) \not\prec [0, 0],$$

then

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \not\succ [0, 0].$$

In the following theorem, we will prove the equivalence of Stampacchia and Minty type problems through h-pseudomonotonicity.

Lemma 3.6 (Minty)**.** *Let K be a nonempty, compact, and geodesic convex subset of a Hadamard manifold M with constant sectional curvature $\kappa \leq 0$. Let $f\colon K \to \mathcal{K}_C$ be a differentiable function along the geodesic γ. Suppose that*

(a) *Let* $\operatorname{grad} f\colon K \to \mathcal{K}_C$ *be a geodesic hemicontinuous and h-pseudomonotone (PM) mapping.*

(b) *The function $h\colon K \to \mathcal{K}_C$ is interval-valued geodesic convex (IGCX) on K.*

There exists $x \in K$, such that for all $y \in K$,

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \not\prec [0, 0] \tag{3.1}$$

if and only if for all $y \in K$,

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \not\succ [0, 0]. \tag{3.2}$$

Proof. By the h-pseudomonotonicity of $\operatorname{grad} f$ if x satisfies (3.1), then (3.2) holds for $y \in K$.

Conversely, let $\gamma(t) = \exp_x(t \exp_x^{-1} y)$ be a geodesic for each $t \in [0, 1]$ from $\gamma(t)$ to $\gamma(0) = x$, hence by (3.2), $t > 0$. Since K is geodesic convex, then $\gamma(t) \in K$. Suppose $x \in K$ satisfies Minty type inequality (3.2), and we will prove that (3.1) holds, thus is, x is a solution of MSVIP, then

$$df_{\gamma(t)}(\exp_{\gamma(t)}^{-1} x) + (h(x) \ominus_{gH} h(\gamma(t))) \not\succ [0, 0], \quad \forall\, t \in [0, 1].$$

As h is interval-valued convex, then

$$h(\gamma(t)) \ominus_{gH} h(x) \preceq t[h(y) \ominus_{gH} h(x)].$$

We have that

$$\begin{gathered} df_{\gamma(t)}(\exp_{\gamma(t)}^{-1} x) \preceq h(\gamma(t)) \ominus_{gH} h(x) \preceq t(h(y) \ominus_{gH} h(x)), \\ df_{\gamma(t)}(-\exp_{\gamma(t)}^{-1} x) \succeq -t(h(y) \ominus_{gH} h(x)), \\ df_{\exp_x(t \exp_x^{-1} y)}(\exp_{\exp_x(t \exp_x^{-1} y)}^{-1} x) \succeq t(h(x) \ominus_{gH} h(y)), \quad \forall\, y \in K. \end{gathered}$$

Let P_t denote the parallel transport along the geodesic $\gamma(t) = \exp_x(t \exp_x^{-1} y)$. Since the parallel transport is an isometry,

$$tP_t df_{\exp_x(t\exp_x^{-1} y)}(\exp_x^{-1} y) \succeq t(h(x) \ominus_{gH} h(y)), \quad \forall\, y \in K. \tag{3.3}$$

By dividing $t > 0$, the geodesic hemicontinuity of $\operatorname{grad} f$ and tacking $t \to 0$ in (3.3) one has

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \nprec [0,0], \quad \forall\, y \in K.$$

Therefore, x is a solution of MSVIP. This proof is completed. $\square$

We will prove the solvability of the MSVIP problem.

Theorem 3.7. *Let K be a nonempty, compact, and geodesic convex subset of a Hadamard manifold M with constant sectional curvature $\kappa \leq 0$. Let $f\colon K \to \mathcal{K}_C$ be a differentiable function along the geodesic γ.*

(a) *Let $\operatorname{grad} f\colon K \to \mathcal{K}_C$ be a geodesic hemicontinuous and h-pseudomonotone (PM) mapping.*

(b) *The function $h\colon K \to \mathcal{K}_C$ is interval-valued convex (IGCX) on K.*

(c) *The function $y \mapsto df_{x_0}(\exp_{x_0}^{-1} y)$ is interval-valued convex (IGCX), $\forall\, y \in K$.*

Then MSVIP is solvable.

Proof. Define mappings $F, G\colon K \to 2^K$ by

$$F(y) = \{x \in K : df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \nprec [0,0]\},$$
$$G(y) = \{x \in K : df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \nsucc [0,0]\}.$$

To show the existence solution of MSVIP is sufficient to show that

$$\bigcap_{y\in K} F(y) \neq \emptyset.$$

Let us see that we are in a position to apply Lemma 3.2.

Step 1: Firstly, we show that G is a KKM mapping by proving that F is a KKM mapping. So we have to prove that for any choice $x_1, x_2, \ldots, x_n \in K$,

$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

Indeed, assume that F is not a KKM mapping, then there exist $x_0 \in K$, such that $x_0 \in \operatorname{co}\{x_1, x_2, \ldots, x_n\}$ but $x_0 \notin \bigcup_{i=1}^n F(x_i)$. Thus is,

$$df_{x_0}(\exp_{x_0}^{-1} x_i) + (h(x_i) \ominus_{gH} h(x_0)) \prec [0,0], \quad \forall\, i \in \{1, 2, \ldots, n\}.$$

Therefore

$$x_i \in A = \{y \in K : df_{x_0}(\exp_{x_0}^{-1} y) + (h(y) \ominus_{gH} h(x_0)) \prec [0,0]\}.$$

By hypothesis, $y \mapsto df_{x_0}(\exp_{x_0}^{-1} y)$ is interval-valued convex (IGCX) also h is interval-valued convex (IGCX) function on K therefore $y \mapsto df_{x_0}(\exp_{x_0}^{-1} y) + h(y)$ is interval-valued convex (IGCX) function. Hence the set A is a geodesic convex set. Thus is,

$$x_0 \in \mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq A.$$

On the other hand,

$$[0,0] = df_{x_0}(\exp_{x_0}^{-1} x_0) + (h(x_0) \ominus_{gH} h(x_0)) \prec [0,0].$$

Contradiction. Therefore, F is KKM mapping and so is G.

Step 2*:* Secondly, we can show that $F(y) \subset G(y)$. Since grad f is geodesic hemicontinuous and h-pseudomonotone, it follows from Lemma 3.6 that $F(y) \subset G(y)$ for all $y \in K$. $F(y)$ is a KKM-function, then $G(y)$ is a KKM-function and $\bigcap_{y\in K} F(y) = \bigcap_{y \in K} G(y)$.

Step 3*:* Furthermore, since K is bounded, $G(y)$ is bounded. Moreover, it is obvious that $G(y)$ is closed in K, and therefore $G(y)$ is compact. It follows from Lemma 3.2 that $\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \emptyset$, which implies that there exists $x \in K$ such that $\forall\, y \in K$,

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \not\prec [0,0].$$

That is to say, MSVIP is solvable. □

Remark 3.8. (a) Theorem 3.7 generalizes the Existence Theorem 3.1 given by Jiang et al. [22] for solutions of variational inequality problem from Banach spaces and Hadamard manifolds to interval-valued functions, Theorem 2.2 given by Jana and Nahak [20], Theorem 3.2 given by Colao et al. [9], and Theorem 3.8 given by Jayswal et al. [21].

(b) The geodesic convexity of the set A associated to the function $y \mapsto df_{x_0}(\exp_{x_0}^{-1} y)$ is ensured by being geodesic convex the end point functions following the proof that we can find in Chen and Huang [8, Theorem 3.9], as well as in Chen [6, Theorem 3.1], Zhou and Huang [41, Corollary 3.1], Wang et al. [39, Theorem 4.2], and Ferreira et al. [14, Corollary 3.1].

Corollary 3.9. *Let K be a nonempty, compact, and geodesic convex subset of a Hadamard manifold M with constant sectional curvature $\kappa \leq 0$. Let $f\colon K \to \mathcal{K}_C$ be a differentiable function along the geodesic γ.*

(a) *Let* $\operatorname{grad} f\colon K \to \mathcal{K}_C$ *be a geodesic hemicontinuous and* h*-pseudomonotone (PM) mapping.*

(b) *The function* $h\colon K \to \mathcal{K}_C$ *is interval-valued convex (IGCX) on* K*.*

(c) *For all* $x \in K$*, the set* $A = \{y \in K : df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \prec [0,0]\}$ *is geodesic convex.*

Then MSVIP is solvable.

Proof. In the same way as in Theorem 3.7, we obtain the result since $F(y)$ is a KKM-function as a consequence of assumption (c) and Lemma 3.6. □

Remark 3.10. Hypothesis (c) extends to the case of interval-valued functions other similar ones that can be found in Theorem 3.2 given by Colao et al. [9] and in Theorem 3.4 given by Al-Homidan et al. [3].

We can also guarantee the uniqueness of the solution.

Definition 3.11. A mapping $\operatorname{grad} f\colon K \to \mathcal{K}_C$ is said to be stricly h-pseudomonotone (SPM) if for every $x, y \in K$, $x \neq y$, such that

$$df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \not\prec [0,0],$$

then

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \not\succeq [0,0],$$

where $h\colon K \to \mathcal{K}_C$ is an interval-valued function.

Theorem 3.12. *Let* K *be a nonempty, compact, and geodesic convex subset of a Hadamard manifold* M *with constant sectional curvature* $\kappa \leq 0$*. Let* $f\colon K \to \mathcal{K}_C$ *be a differentiable function along the geodesic* γ*.*

(a) *Let* $\operatorname{grad} f\colon K \to \mathcal{K}_C$ *be a strictly* h*-pseudomonotone (SPM) and geodesic hemicontinuous mapping.*

(b) *The function* $h\colon K \to \mathcal{K}_C$ *is interval-valued convex (IGCX) on* K*.*

(c) *For all* $x \in K$*, the set* $A = \{y \in K : df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) \prec [0,0]\}$ *is geodesic convex.*

Then MSVIP has a unique solution.

Proof. Since the SPM implies the PM, by Theorem 3.7, we are guaranteed the existence of a solution. Let's see the uniqueness. Suppose that MSVIP has two distinct solutions, say x_1 and x_2, then for every $x_1, x_2 \in K$, $x_1 \neq x_2$, we have

$$df_{x_1}(\exp_{x_1}^{-1} x_2) + (h(x_2) \ominus_{gH} h(x_1)) \not\prec [0,0] \tag{3.4}$$

and

$$df_{x_2}(\exp_{x_2}^{-1} x_1) + (h(x_1) \ominus_{gH} h(x_2)) \not\prec [0,0]. \tag{3.5}$$

From grad f is (SPM), it follows from (3.4) that

$$df_{x_2}(\exp_{x_2}^{-1} x_1) + (h(x_1) \ominus_{gH} h(x_2)) \not\succeq [0,0].$$

Contradiction with (3.5). □

In Nguyen et al. [30] the authors established the existence a unique solution of VIP under strong pseudomonotonicy assumptions. We demand in our work a more general condition as the strictly pseudomonotonicity. Also this result is a generalization of Corollary 5.1 given by Ruiz-Garzón et al. [33] in finite dimensional spaces.

Example 3.13. Let $M = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be endowed with the Riemannian metric defined by $\langle \cdot, \cdot \rangle = g(x)uv$ with $g(x) = x^{-2}$, where $g\colon \mathbb{R}_{++} \to (0, +\infty)$. Let $K = \{x \mid x = e^t, t \in [0,1]\}$ be a subset of M.

Let $\gamma(t) = xe^{(v/x)t} = \exp_x(tv)$ be a geodesic and therefore $\exp_x^{-1} y = x\ln(y/x)$. We have $f(x) = [f^L(x), f^U(x)] = [\ln^2(x), \ln^2(x) + 1]$ and $h(x) = [x, 2x]$. As grad $f(x) = g(x)^{-1} f'(x) = x^2 f'(x)$, where f' denotes the first derivatives of f in the Euclidean sense, then we can calculate

$$\begin{aligned} df_x(\exp_x^{-1} y) &= \langle \operatorname{grad} f(x), \exp_x^{-1} y\rangle = \left\langle \left[x^2 \frac{2}{x}\ln(x), x^2 \frac{2}{x}\ln(x)\right], x\ln(y/x)\right\rangle \\ &= \langle [2x\ln(x), 2x\ln(x)], x\ln(y/x)\rangle = x^{-2}[2x\ln(x)x\ln(y/x), 2x\ln(x)x\ln(y/x)] \\ &= [2\ln(x)\ln(y/x), 2\ln(x)\ln(y/x)]. \end{aligned}$$

The mapping grad f is strictly h-pseudomonotone (SPM) because for every $x, y \in K$, such that

$$\begin{aligned} df_x(\exp_x^{-1} y) + (h(y) \ominus_{gH} h(x)) &= [2\ln(x)\ln(y/x), 2\ln(x)\ln(y/x)] + ([y, 2y] \ominus_{gH} [x, 2x]) \\ &\not\prec [0,0], \end{aligned}$$

then

$$df_y(\exp_y^{-1} x) + (h(x) \ominus_{gH} h(y)) \not\succeq [0,0].$$

All assumptions of Theorem 3.12 holds and the MSVIP problem has a unique solution $x = 1 \in K$ such that there exists no $y \in K$ such that

$$[2\ln(x)\ln(y/x), 2\ln(x)\ln(y/x)] + ([y, 2y] \ominus_{gH} [x, 2x]) \not\prec [0, 0].$$

In the next section, we will show the relationships of the existence results already achieved with economic problems and mathematical optimization problems.

4. Applications

Economists have always been interested in studying theorems on the existence of equilibrium points in a market. Walrasian equilibrium points are those where the quantity demanded is equal to the quantity supplied. We consider a market structure with perfect competition. Perfect competition in a market means that none of the agents can influence the good or service price, thus is, there are many producers of a very homogeneous good, where the market (or equilibrium) price arises from the law of supply and demand.

We suppose n commodities and a price vector $p \in \mathbb{R}^n_+ = M$. And as a novelty, we can define the value $E(p)$ of the excess demand mapping as interval-valued function $E\colon \mathbb{R}^n_+ \to \mathcal{K}_C$.

In Konnov [24], we can see that a vector $\overline{p} \in M$ is said to be an equilibrium price vector if we can solve the following variational inequality: find $\overline{p} \geq 0$ and $\overline{q} \in E(\overline{p})$ such that there exists no $p \in \mathbb{R}^n_+$ such that

$$\langle -\overline{q}, \exp_{\overline{p}}^{-1} p\rangle = \langle -\overline{q}, p - \overline{p}\rangle \prec [0, 0].$$

If we suppose that each price of a commodity involved in the market structure has a lower positive bound and may have an upper bound and we denote the excess of demand mapping as $E(p) = D(p) - S(p)$, where D and S are the demand and supply mappings and set $G = -D$, each producer supplies a single commodity, then the *problem of finding an equilibrium price* (EPP) consists of: find $\overline{p} \in K \subseteq M$ and $\overline{s}_i \in S_i(\overline{p})$, $i = 1, 2, \dots, n$, such that there exists no $p \in K$ such that

$$\langle G(\overline{p}), p - \overline{p}\rangle + \sum_{i=1}^{n} \overline{s}_i(p_i - \overline{p}_i) \prec [0, 0].$$

This problem is nothing but MSVIP. By Theorem 3.7, we would have the conditions under which there is an equilibrium price. These conditions include the pseudomonotonicity of the demand and the convexity of $\overline{s}_i$.

Also, we use variational inequalities problems to obtain solutions to optimization problems. Let's see.

We can consider the following *Interval-valued Optimization Problem* on Hadamard manifolds (IVOP) defined as,

$$\min\ f(x) = [f^L(x), f^U(x)] \quad \text{s.t.} \quad x \in K,$$

where $f\colon K \to \mathcal{K}_C$ and K is a subset of a Hadamard manifold M.

This section considers $h = 0$, and we identify solutions of SVIP and IVOP problems.

Theorem 4.1. *Let K be a nonempty, compact, and geodesic convex subset of a Hadamard manifold M with constant sectional curvature $\kappa \leq 0$. Suppose $f\colon K \to \mathcal{K}_C$ is a differential interval-valued convex mapping along the geodesic γ. The point x is SVIP solution if and only if x is solution of IVOP.*

Proof. By reductio ad absurdum. Suppose that x is not a solution of IVOP. Then, there exists $y \in K$ such that $f(x) \ominus_{gH} f(y) \succ [0, 0]$. By the interval-valued convexity of f, we have

$$[0, 0] \succ f(y) \ominus_{gH} f(x) \succeq df_x(\exp_x^{-1} y).$$

Hence, we find a point $x \in K$ such that there exists $y \in K$ such that

$$df_x(\exp_x^{-1} y) \prec [0, 0].$$

Contradiction with x is solution of SVIP.

Conversely, suppose that x is not a solution of SVIP. Then, we find a point $x \in K$ such that $df_y(\exp_y^{-1} x) \succ [0, 0]$. By the interval-valued convexity

$$f(x) \ominus_{gH} f(y) \succeq df_y(\exp_y^{-1} x) \succ [0, 0].$$

Contradiction with x is a solution of IVOP. □

As a consequence of Lemma 3.6 and Theorem 4.1, we have

Corollary 4.2. *Let K be a nonempty, compact, and geodesic convex subset of a Hadamard manifold M with constant sectional curvature $\kappa \leq 0$. Suppose*

(a) *Let $f\colon K \to \mathcal{K}_C$ be a differential and interval-valued convex mapping.*

(b) *Let* $\operatorname{grad} f\colon K \to \mathcal{K}_C$ *be a geodesic hemicontinuous and pseudomonotone mapping.*

There exists $x \in K$, such that for all $y \in K$ the following problems SVIP, MVIP, and IVOP are equivalents.

To sum up,

$$\text{MVIP} \iff \text{SVIP} \iff \text{IVOP}.$$

Therefore, we can reach the Interval-valued Optimization Problem on Hadamard manifolds IVOP through the solutions of SVIP Problem.

Example 4.3. As Example 3.13, let $M = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be endowed with the Riemannian metric defined by $\langle \cdot, \cdot \rangle = g(x)uv$ with $g(x) = x^{-2}$, where $g\colon \mathbb{R}_{++} \to (0, +\infty)$. Let $K = \{x \mid x = e^t, t \in [0, 1]\}$ be a subset of M.

Let $\gamma(t) = xe^{(v/x)t} = \exp_x(tv)$ be a geodesic and therefore $\exp_x^{-1} y = x \ln(y/x)$. We have $f(x) = [f^L(x), f^U(x)] = [\ln^2(x), \ln^2(x) + 1]$. As $\operatorname{grad} f(x) = g(x)^{-1} f'(x) = x^2 f'(x)$, where f' denotes the first derivatives of f in the Euclidean sense, then we can calculate

$$\begin{aligned} df_x(\exp_x^{-1} y) &= \langle \operatorname{grad} f(x), \exp_x^{-1} y \rangle = \left\langle \left[x^2 \frac{2}{x} \ln(x), x^2 \frac{2}{x} \ln(x) \right], x \ln(y/x) \right\rangle \\ &= \langle [2x \ln(x), 2x \ln(x)], x \ln(y/x) \rangle = x^{-2} [2x \ln(x) x \ln(y/x), 2x \ln(x) x \ln(y/x)] \\ &= [2 \ln(x) \ln(y/x), 2 \ln(x) \ln(y/x)]. \end{aligned}$$

The mapping $\operatorname{grad} f$ is pseudomonotone (PM) because for every $x, y \in K$, such that

$$df_x(\exp_x^{-1} y) = [2 \ln(x) \ln(y/x), 2 \ln(x) \ln(y/x)] \not\prec [0, 0],$$

then $df_y(\exp_y^{-1} x) \not\succ [0, 0]$.

Also, the function f is interval-valued geodesic convex (IGCX) on K because $\forall\, x, y \in K$,

$$[\ln^2(y), \ln^2(y) + 1] \ominus_{gH} [\ln^2(x), \ln^2(x) + 1] \succeq [2 \ln(x) \ln(y/x), 2 \ln(x) \ln(y/x)].$$

All assumptions of Corollary 4.2 holds and the SVIP problem has a solution $x = 1 \in K$ such that there exists no $y \in K$ such that

$$df_x(\exp_x^{-1} y) = \langle \operatorname{grad} f(x), \exp_x^{-1} y \rangle = [2 \ln(x) \ln(y/x), 2 \ln(x) \ln(y/x)] \prec [0, 0].$$

The solution to this SVIP problem coincides with MVIP and IVOP problems.

Remark 4.4. In this example, as $\operatorname{grad} \overline{f}(x) = \operatorname{grad} \underline{f}(x)$ and taking $\lambda_n = 1/2 - 1/(n + 3)$, then we can applied the Proximal Point Algorithm 3 given by Ansari and Babu [4] for finding the solution of variational inequality problem, where the algorithm becomes

$$0 = 2x_n \ln(x_n) - \frac{1}{\lambda_n} x_{n+1} \ln \left(\frac{x_n}{x_{n+1}} \right).$$

With the initial guess $x_0 = 2.5$, the iterative points are given in Table 4.1 and Figure 4.1.

n	x_n with initial guest $x_0 = 2.5$	n	x_n with initial guest $x_0 = 2.5$
0	2.5	17	1.00016182
1	2.08094536	18	1.00008899
2	1.65212955	19	1.00004873
3	1.37942868	20	1.00002658
4	1.22220021	21	1.00001444
5	1.13113516	22	1.00000782
6	1.07757916	23	1.00000423
7	1.04579493	24	1.00000228
8	1.0268919	25	1.00000122
9	1.01568265	26	1.00000065
10	1.00907571	27	1.00000035
11	1.0052112	28	1.00000019
12	1.00296952	29	1.0000001
13	1.00167999	30	1.00000005
14	1.00094411	31	1.00000003
15	1.00052731	32	1.00000001
16	1.00029286	33	1

Table 4.1: Finite convergence for Proximal Point Algorithm.

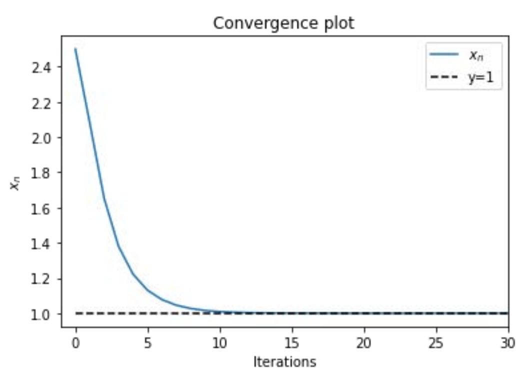

Figure 4.1: Converge plot of iterative process.

5. Conclusions

This paper has introduced the Stampacchia and Minty version of the Mixed Variational Inequality Interval-valued Problem on Hadamard manifolds and proved the existence of solutions with monotonicity hypothesis of the gradient of the function. The results proved in this article allow us to draw the following conclusions:

– We have used the gH-differentiability for interval-valued functions on Hadamard manifolds and we extended the Stampacchia and Minty versions of these variational problems on Hadamard manifold given by Chen and Huang [8] to interval-valued functions.

– We generalized the Existence Theorems for solutions of vector variational inequality problem from Borel and Euclidean spaces given by Jiang et al. [22], Jana and Nahak [20], and Jayswal et al. [21] to Hadamard manifolds via Fan's lemma. We have illustrated the main results obtained with some examples and numerical results.

– We proved the existence of walrasian equilibrium points in a market of perfect competition with interval-valued functions.

– We identified the solutions of Interval-valued Optimization Problem (IVOP) and Variational Inequality Interval-valued problems of the Stampacchia and Minty type under convex environments.

In our opinion, in the future, iterative methods should be proposed to effectively reach solutions to these problems involving interval-valued functions in nonlinear spaces in a similar way as proposed by Noor et al. [32], for example.

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