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Approximation and entropy numbers of composition operators

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Abstract: We give a survey on approximation numbers of composition operators on the Hardy space, on the disk and on the polydisk, and add corresponding new results on their entropy numbers, revealing how they are different.

Keywords: approximation numbers; composition operator; entropy numbers; polydisk; pluricapacity

MSC: primary: 47B33; secondary: 30H10; 32A35; 46B06

1 Introduction

This paper surveys results on approximation numbers of composition operator on the Hardy space, and gives new results, on their entropy numbers, in one or several dimensions. In various papers (see [3, 19–22]), pretty sharp estimates are obtained for them, either for classes of examples like the lens maps or the cusp maps, or in the general case. In particular, a few properties are investigated, related with the so-called "spectral radius type formula", obtained, in dimension one through a result of Widom in [21], and, partially in dimension $N \geq 2$ [22, 23], through a result of Nivoche [26] and Zakharyuta [33]. One of our main results (quoted in dimension one) was the implication (using Green capacity considerations):

$$\|\varphi\|_\infty = 1 \quad \Rightarrow \quad \lim_{n\to\infty} [a_n(C_\varphi)]^{1/n} = 1\,, \tag{1.1}$$

where $a_n(C_\varphi)$ is the n-th approximation number of C_φ. Note that it is straightforward that if $\|\varphi\|_\infty = r < 1$, then $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} \leq r < 1$.

Another way of measuring the compactness of operators is using the entropy numbers instead of the approximation numbers. Those numbers stand a little apart in the jungle of "s-numbers", even though they seem to be the most natural for the study of compactness, since their membership in c_0 characterizes compactness, even in the general framework of arbitrary Banach spaces.

Given a compact operator $T\colon H_1 \to H_2$ between Hilbert spaces, the relation between its entropy numbers $e_n(T)$ and its approximation (if one prefers singular) numbers $a_n(T)$ is theoretically known, through a general result on diagonal operators on ℓ^2, recalled in Theorem 2.2 to follow, and through the Schmidt decomposition of T. This comparison can be thought useless, since in principle we do not know better the numbers $a_n(T)$ than the numbers $e_n(T)$. But in our case, with $T = C_\varphi$, namely $T(f) = f \circ \varphi$ where φ is an analytic self-map of the polydisk $\mathbb{D}^N$, the situation is slightly different. Answering a question of J. Wengenroth [30] about the

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behavior of the entropy numbers of composition operators, we give in this paper estimates for these numbers, analog to that on the approximation numbers. In particular, we have:

$$\|\varphi\|_\infty = 1 \quad \Rightarrow \quad \lim_{n\to\infty} [e_n(C_\varphi)]^{1/\sqrt{n}} = 1\,. \tag{1.2}$$

The proofs are not difficult, but as indicated for example by the comparison between (1.1) and (1.2), the statements feature a very different behavior of those entropy numbers, which deserves attention. This difference is actually more transparent in the polydisk $\mathbb{D}^N$, and the main interest of this paper is to point out how the dependence of the entropy numbers with respect to the dimension N differs from that of the approximation numbers.

The paper is organized as follows. Section 1 is this introduction. In Section 2 we recall the necessary background. In Section 3, we survey results on approximation numbers and give the corresponding new results on entropy numbers. We first begin with general facts, and then give specific results, with a particular interest to the examples of the lens maps and the cusp map. For any non-constant analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, we have:

$$\lim_{n\to\infty} [a_n(C_\varphi)]^{1/n} = \exp\big(-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]\big)\,, \tag{1.3}$$

where $\mathrm{Cap}\,[\varphi(\mathbb{D})]$ is the Green capacity of $\varphi(\mathbb{D})$, from which it follows that $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} = 1$ if and only if $\|\varphi\|_\infty = 1$ (Theorem 3.1). We moreover prove that:

$$\lim_{n\to\infty} [e_n(C_\varphi)]^{1/\sqrt{n}} = \exp\big(-\sqrt{\log 2/\mathrm{Cap}\,[\varphi(\mathbb{D})]}\big)\,, \tag{1.4}$$

from which it follows that $\lim_{n\to\infty}[e_n(C_\varphi)]^{1/\sqrt{n}} = 1$ if and only if $\|\varphi\|_\infty = 1$ (Theorem 3.2).

For the lens map λ_θ with parameter θ, we have (the constants depending only on θ):

$$\alpha\, \mathrm{e}^{-C\, n^{1/2}} \le a_n(C_{\lambda_\theta}) \le \beta\, \mathrm{e}^{-c\, n^{1/2}}\,, \tag{1.5}$$

whereas:

$$\alpha'\, \mathrm{e}^{-C'\, n^{1/3}} \le e_n(C_{\lambda_\theta}) \le \beta'\, \mathrm{e}^{-c'\, n^{1/3}}\,. \tag{1.6}$$

For the cusp map χ, we have, for absolute constants:

$$\alpha\, \mathrm{e}^{-Cn/\log n} \le a_n(C_\chi) \le \beta\, \mathrm{e}^{-cn/\log n}\,, \tag{1.7}$$

and

$$\alpha'\, \mathrm{e}^{-C'\sqrt{n/\log n}} \le e_n(C_\chi) \le \beta'\, \mathrm{e}^{-c'\sqrt{n/\log n}}\,. \tag{1.8}$$

Section 4 is concerned by the multivariate case, again first in the general case, then on specific cases. We are in particular interested by the multi-lens map Λ_θ, defined by:

$$\Lambda_\theta(z_1, \dots, z_N) = \big(\lambda_\theta(z_1), \lambda_\theta(z_2), \dots, \lambda_\theta(z_N)\big)\,,$$

and the multi-cusp map Ξ defined by:

$$\Xi\,(z_1, \dots, z_N) = \big(\chi(z_1), \chi(z_2), \dots, \chi(z_N)\big)\,.$$

We prove that:

$$a\, \exp(-C\, n^{1/(2N)}) \le a_n(C_{\Lambda_\theta}) \le b\, \exp(-c\, n^{1/(2N)}) \tag{1.9}$$

$$a'\, \exp(-C'\, n^{1/(2N+1)}) \le e_n(C_{\Lambda_\theta}) \le b'\, \exp(-c'\, n^{1/(2N+1)}) \tag{1.10}$$

and

$$a\, \exp[-C\, n^{1/N}/\log n] \le a_n(C_\Xi) \le b\, \exp[-c\, n^{1/N}/\log n] \tag{1.11}$$

$$\begin{aligned} a'\, \exp\big[-C'\, n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big] &\le e_n(C_\Xi) \\ &\le b'\, \exp\big[-c'\, n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big]\,. \end{aligned} \tag{1.12}$$

Section 5 is more specifically devoted to the multidimensional case, in connection with the notion of Monge-Ampère (or Bedford-Taylor) pluricapacity, which recently turned out to play an important role in connection with composition operators [23].

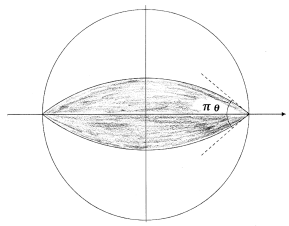

Figure 1: *Lens map domain*

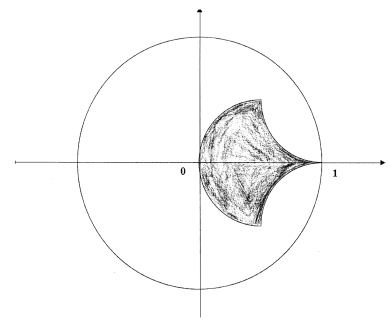

Figure 2: *Cusp map domain*

2 Background and notation

We denote by $\mathbb{D}$ the open unit disk and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle; m is the normalized Lebesgue measure on $\mathbb{T}$: $dm(t) = dt/2\pi$, and H^2 is the usual Hardy space on $\mathbb{D}$. By Littlewood's subordination principle, every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ (also called *Schur function*) defines a bounded operator $C_\varphi\colon H^2 \to H^2$ by $C_\varphi(f) = f \circ \varphi$, called the *composition operator of symbol* φ.

2.1 Two types of Schur functions

Lens maps

For $0 < \theta < 1$, the *lens map* λ_θ with parameter θ is obtained by sending conformally the unit disk $\mathbb{D}$ onto the right half-plane $\Pi = \{z \in \mathbb{C}\,;\ \Re e\, z > 0\}$; then making $u \mapsto u^\theta$, and coming back to $\mathbb{D}$ (see [29, page 27]). Namely:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}\,. \tag{2.1}$$

It is a conformal map from $\mathbb{D}$ onto the domain represented on Figure 1.

Cusp map

The *cusp map* is a conformal mapping χ sending the unit disk $\mathbb{D}$ onto the domain represented on Figure 2. This map was first introduced in [14].

To obtain it, we first map $\mathbb{D}$ onto the half-disk $\mathbb{D}^+ = \{z \in \mathbb{D}\,;\ \Re e\, z > 0\}$. To do that, map $\mathbb{D}$ onto itself by $z \mapsto iz$; then map $\mathbb{D}$ onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}\,;\ \Im m\, z > 0\}$ by

$$T(u) = i\,\frac{1+u}{1-u}\,.$$

Take the square root to map $\mathbb{H}$ in the first quadrant $Q_1 = \{z \in \mathbb{H}\,;\ \Re e\, z > 0\}$, and go back to the half-disk $\{z \in \mathbb{D}\,;\ \Im m\, z < 0\}$ by T^{-1}: $T^{-1}(s) = \frac{1+is}{is-1}$; finally, make a rotation by i to go onto $\mathbb{D}^+$. We get

$$\chi_0(z) = \frac{\left(\dfrac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\dfrac{z-i}{iz-1}\right)^{1/2} + 1}\,. \tag{2.2}$$

One has $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$ and $\chi_0(-i) = i$. The half-circle $\{z \in \mathbb{T}\,;\ \Re e\, z \geq 0\}$ is mapped onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively,

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi}\chi_1(z) + 1, \quad \chi_3(z) = \frac{1}{\chi_2(z)}\,, \tag{2.3}$$

and finally:

$$\chi(z) = 1 - \chi_3(z)\,. \tag{2.4}$$

Hence:

$$1 - \chi(z) = \frac{1}{1 + \frac{2}{\pi}\,\log\big(1/|\chi_0(z)|\big) - i\frac{2}{\pi}\arg\chi_0(z)}\,. \tag{2.5}$$

Note that χ_2 maps $\mathbb{D}$ onto the half-strip $\{z \in \mathbb{C}\,;\ \Re\mathrm{e}\, z > 1 \text{ and } |\Im\mathrm{m}\, z| < 1\}$. One has $\varphi(1) = 1$, $\varphi(-1) = 0$, $\varphi(i) = (1+i)/2$ and $\varphi(-i) = (1-i)/2$.

The domain $\varphi(\mathbb{D})$ is edged by three circular arcs of radii $1/2$ and of respective centers $1/2$, $1 + i/2$ and $1 - i/2$. The real interval $]-1, 1[$ is mapped onto the real interval $]0, 1[$ and the half-circle $\{\mathrm{e}^{i\theta}\,;\ |\theta| \le \pi/2\}$ is sent onto the two circular arcs tangent at 1 to the real axis.

2.2 Approximation and entropy numbers

Given an operator $T\colon X \to Y$ between Banach spaces, recall (see [6]) that we can attach to this operator five non-increasing sequences (a_n), (b_n), (c_n), (d_n), (e_n) of non-negative numbers (depending on T), respectively the sequences of *approximation*, *Bernstein*, *Gelfand*, *Kolmogorov*, and *entropy* numbers of T. We only define here the first one and the last one. The approximation numbers are defined as:

$$a_n(T) = \inf\{\|T - R\|\,;\ \operatorname{rank}(R) < n\}\,, \qquad n \ge 1\,. \tag{2.6}$$

The entropy numbers are defined for $n \ge 1$ as:

$$e_n(T) = \inf\{\varepsilon > 0\,;\ N\big(T(B_X), \varepsilon B_Y\big) \le 2^{n-1}\}\,, \quad n \ge 1\,, \tag{2.7}$$

where B_X and B_Y are the respective closed unit balls of X and Y, and where, for $A, B \subseteq Y$, $N(A, B)$ denotes the smallest number of translates of B needed to cover A (see [6, Chapter 1], or [28, Chapter 5]).

All those sequences (a_n), (b_n), (c_n), (d_n), (e_n), say (u_n), share the ideal property:

$$u_n(ATB) \le \|A\|\, u_n(T)\, \|B\|\,. \tag{2.8}$$

For Hilbert spaces, it turns out that

$$a_n = b_n = c_n = d_n = s_n\,, \tag{2.9}$$

where (s_n) designates the sequence of singular numbers; but entropy numbers stay a little apart.

For Banach spaces X and Y and $T\colon X \to Y$, we have, in general, for $\alpha > 0$:

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le C_\alpha \sup_{1\le k\le n} k^\alpha a_k(T)$$

([5, Theorem 1], see also [28, Theorem 5.2]), and, if X and $Y^\star$ are of type 2:

$$a_n(T) \le K\, e_n(T)\,, \qquad \text{for all } n \ge 1$$

([12, Corollary 1.6]), where $K = \kappa\,[T_2(X)T_2(Y^\star)]^2$; in particular, if T acts between Hilbert spaces:

$$a_n(T) \le 4\, e_n(T)\,, \qquad \text{for all } n \ge 1$$

(see [28, Theorem 5.3]).

Recall (see [1, Definition 6.2.10, p. 137], [17, Définition III.3, p. 162], or [18, Definition IV.3, p. 180]) that a Banach space X has type 2 if there is a constant C such that for every finite sequence $x_1, x_2, \dots, x_n$ in X we have:

$$\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2\right)^{1/2} \le C\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{1/2},$$

the smallest such constant C is denoted $T_2(X)$. Every Hilbert space has type 2, thanks to the parallelogram identity.

Those inequalities indicate that entropy numbers are always bigger than singular numbers, up to a constant, and that, as far as the scale of powers n^α is implied, they are dominated by approximation numbers in a weak sense. But it turns out that, individually, they can be much bigger than the latter for composition operators, as we shall see.

We will rely on the following estimate ([6, Proposition 1.3.2, p. 17]), in which ℓ^2 denotes the space of square-summable sequences $x = (x_k)_{k\ge 1}$ of complex numbers. This estimate is given for the sequence (ε_n), but $e_n = \varepsilon_{2^{n-1}}$, by definition.

Theorem 2.1. (see [6, p. 17]) *There exist absolute constants $0 < a < b$ such that, for any diagonal compact operator $\Delta\colon \ell^2 \to \ell^2$ with positive and non-increasing eigenvalues $(\sigma_k)_{k\ge 1}$, namely $\Delta\big((x_k)_k\big) = (\sigma_k x_k)_k$, we have, for all $n \ge 1$:*

$$a \sup_{k\ge 1}\left[2^{-n/2k}\left(\prod_{j=1}^{k}\sigma_j\right)^{1/k}\right] \le e_n(\Delta) \le b \sup_{k\ge 1}\left[2^{-n/2k}\left(\prod_{j=1}^{k}\sigma_j\right)^{1/k}\right]. \tag{2.10}$$

A useful corollary of Theorem 2.1 is the following.

Theorem 2.2. *Let $T\colon H_1 \to H_2$ be a compact operator between the complex Hilbert spaces H_1 and H_2, and let $(a_n)_{n\ge 1}$ be its sequence of approximation numbers. Then, for all $n \ge 1$:*

$$\alpha \sup_{k\ge 1}\left[2^{-n/2k}\left(\prod_{j=1}^{k}a_j\right)^{1/k}\right] \le e_n(T) \le \beta \sup_{k\ge 1}\left[2^{-n/2k}\left(\prod_{j=1}^{k}a_j\right)^{1/k}\right], \tag{2.11}$$

where α and β are positive numerical constants.

Proof. Let $Tx = \sum_{n=1}^{\infty} s_n(x \mid u_n)\, v_n$ the Schmidt decomposition of T, where $(u_n)_n$ and $(v_n)_n$ are orthonormal sequences of H_1 and H_2, respectively, and $(s_n)_n$ is the sequence of singular numbers of T. Let $\Delta\colon \ell_2 \to \ell_2$ the diagonal operator with diagonal values s_n, $n \ge 1$. Then $T = V_1 \Delta\, U_1$ and $\Delta = V_2 T U_2$, with $U_1 x = \big((x \mid u_n)\big)_n$, $V_1\big((t_n)_n\big) = \sum_n t_n v_n$, $U_2\big((t_n)_n\big) = \sum_n t_n u_n$ and $V_2 x = \big((x \mid v_n)\big)_n$. We have $\|U_1\|, \|V_1\|, \|U_2\|, \|V_2\| \le 1$; hence the result follows from Theorem 2.1 and the ideal property (2.8). □

This theorem might be thought useless, because we do not know better the numbers a_n than the numbers e_n! In our situation, this is not the case, since we made a more or less systematic study of the approximation numbers a_n for composition operators in [3, 19–21] for example.

3 The 1-dimensional case

3.1 General results

In [21], we coined the parameter

$$\beta_1(T) = \lim_{n\to\infty}\big[a_n(T)\big]^{1/n} \tag{3.1}$$

and its versions $\beta_1^+(T)$, $\beta_1^-(T)$ with an upper limit and a lower limit respectively. The following result, proved in [21, Theorem 3.1] for $\|\varphi\|_\infty < 1$ and in [21, Theorem 3.14] for $\|\varphi\|_\infty = 1$, shows in particular that no lower or

upper limit is needed for $\beta = \beta_1$, and provides a simpler proof of the second item in Theorem 3.1 below than in our initial proof of [19, Theorem 3.4].

For the definition of the Green capacity $\mathrm{Cap}\,(A)$ of a Borel subset A of $\mathbb{D}$, $0 \le \mathrm{Cap}\,(A) \le \infty$, we refer to [21, Section 2.3].

Theorem 3.1. *Let $\Omega = \varphi(\mathbb{D})$, with $\varphi : \mathbb{D} \to \mathbb{D}$ a non-constant analytic map. Then:*

1) *One always has $\beta_1^-(C_\varphi) = \beta_1^+(C_\varphi) =: \beta_1(C_\varphi)$ and*

$$\beta_1(C_\varphi) = \exp[-1/\mathrm{Cap}\,(\Omega)] > 0\,. \tag{3.2}$$

2) *In particular, one has the equivalence*

$$\beta_1(C_\varphi) = 1 \quad \Leftrightarrow \quad \|\varphi\|_\infty = 1\,. \tag{3.3}$$

The first item says in particular that we always have:

$$a_n(C_\varphi) \gtrsim r^n \tag{3.4}$$

for some positive constant $r < 1$ [19, Theorem 3.1]. This was actually first pointed out by Parfenov [27].

The second item says that the behavior $a_n(C_\varphi) \approx r^n$ is only obtained when $\|\varphi\|_\infty < 1$ ([19, Theorem 3.4] or [21, Theorem 3.14]).

For entropy numbers, another parameter emerges:

$$\gamma_1(T) = \lim_{n\to\infty} \big[e_n(T)\big]^{1/\sqrt{n}} \tag{3.5}$$

and its $\gamma_1^+(T)$ and $\gamma_1^-(T)$ versions.

Theorem 3.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant symbol and $\Omega = \varphi(\mathbb{D})$. Then*

1) *$\gamma_1^-(C_\varphi) = \gamma_1^+(C_\varphi) =: \gamma_1(C_\varphi)$ and*

$$\gamma_1(C_\varphi) = \exp\big[-\sqrt{\log 2/\mathrm{Cap}\,(\Omega)}\big] > 0\,. \tag{3.6}$$

2) *In particular, one has the equivalence*

$$\gamma_1(C_\varphi) = 1 \quad \Leftrightarrow \quad \|\varphi\|_\infty = 1\,. \tag{3.7}$$

Proof. Set $\rho = 1/\mathrm{Cap}\,(\Omega)$ for simplicity of notations. Let $\varepsilon > 0$, and C_ε a positive constant which depends only on ε and can vary from a formula to another. Theorem 3.1 implies $a_k \le C_\varepsilon\, \mathrm{e}^{\varepsilon k}\mathrm{e}^{-k\rho}$, whence

$$(a_1 \cdots a_k)^{1/k} \le C_\varepsilon\, \mathrm{e}^{\varepsilon k/2}\, \mathrm{e}^{-\rho k/2}\,.$$

Theorem 2.2 gives

$$e_n(C_\varphi) \le C_\varepsilon \sup_{k\ge 1} \big[\,\mathrm{e}^{\varepsilon k/2}\, \mathrm{e}^{-(n/k)(\log 2/2)+(\rho k/2)}\big]\,.$$

This supremum is essentially attained for $k = \lceil \sqrt{(n\,\log 2)/\rho}\,\rceil$ where $\lceil\,.\,\rceil$ stands for the integer part, and gives

$$e_n(C_\varphi) \le C_\varepsilon\, \mathrm{e}^{\frac{\varepsilon}{2}\sqrt{n\log 2/\rho}}\, \mathrm{e}^{-\sqrt{n\rho\,\log 2}}\,.$$

This implies $\gamma_1^+(C_\varphi) \le \mathrm{e}^{\frac{\varepsilon}{2}\sqrt{\log 2/\rho}}\mathrm{e}^{-\sqrt{\rho\log 2}}$, and finally

$$\gamma_1^+(C_\varphi) \le \mathrm{e}^{-\sqrt{\rho\log 2}}\,.$$

The lower bound $\gamma_1^-(C_\varphi) \ge \mathrm{e}^{-\sqrt{\rho\log 2}}$ is proved similarly.

This clearly ends the proof, since we know that $\mathrm{Cap}\,(\Omega) = \infty$ if and only if $\|\varphi\|_\infty = 1$ (see [21, Theorem 3.13]). □

3.2 Estimates for approximation numbers

Estimates of approximation numbers of composition operators can be obtained by using the boundary behavior of the symbol and Blaschke products for the upper estimates, and the radial behavior of this symbol, reproducing kernels, and interpolating sequences, for the lower estimates.

In order to treat simultaneously the two cases of the lens maps and of the cusp map, we will put ourselves in a more general situation.

We say that a continuous function $\omega\colon [0,1] \to \mathbb{R}_+$ is a *modulus of continuity* if it is increasing, subadditive, i.e. $\omega(s+t) \le \omega(s) + \omega(t)$, and vanishes at 0.

Upper estimates

Let φ be a symbol in the disk algebra, i.e. $\varphi\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, and such that $\varphi(\partial\mathbb{D}) \cap \partial\mathbb{D} = \{\xi_1, \dots, \xi_p\}$. We say that this symbol is *boundary regular* if, writing $\xi_j = \mathrm{e}^{it_j}$, we have:

1) for some positive constant C, we have $|\varphi(\mathrm{e}^{it}) - \varphi(\mathrm{e}^{it_j})| \le C\left(1 - |\varphi(\mathrm{e}^{it})|\right)$, for t in a neighborhood of t_j and for $j = 1, \dots, p$;

2) for some modulus of continuity ω and for some positive constant c, we have $c\,\omega(|t - t_j|) \le |\varphi(\mathrm{e}^{it}) - \varphi(\mathrm{e}^{it_j})|$, for t in a neighborhood of t_j and for $j = 1, \dots, p$.

The following result is proved in [20, Theorem 2.3].

Theorem 3.3. *Let φ be a symbol in the disk algebra whose image touches $\partial\mathbb{D}$ at the points $\xi_1, \dots, \xi_p$, and nowhere else, and such that φ is boundary regular. Then, there are constants $\kappa, K, L > 0$, depending only on φ, such that, for every $q \ge 1$:*

$$a_q(C_\varphi) \le K \sqrt{\frac{\omega^{-1}(\kappa\, 2^{-N_q})}{\kappa\, 2^{-N_q}}}\,, \tag{3.8}$$

where N_q is the largest integer such that $p\,Nd_N < q$, with d_N the integer part of $\sigma \log \frac{\kappa\, 2^{-N}}{\omega^{-1}(\kappa\, 2^{-N})} + 1$.

This theorem allows to give an upper estimate for all approximation numbers $a_n(C_\varphi)$, $n \ge 1$ when we can interpolate between the integers Nd_N and $(N+1)\,d_{N+1}$, but this is not the case in general. Nevertheless, this is the case in the examples below.

Theorem 3.4.

1) *For the lens map λ_θ with parameter θ, we have, for some positive constants α and β, depending on θ:*

$$a_n(C_{\lambda_\theta}) \le \alpha\, \mathrm{e}^{-\beta\sqrt{n}}\,. \tag{3.9}$$

2) *For the cusp map χ, we have, for some positive constants α and β:*

$$a_n(C_\chi) \le \alpha\, \mathrm{e}^{-\beta n/\log n}\,. \tag{3.10}$$

Proof. 1) The map λ_θ satisfies the conditions of Theorem 3.3 with $\omega(h) = h^\theta$, with $p = 2$. We have $\omega^{-1}(h) = h^{1/\theta}$. Hence $d_N \approx N$, $N_q \approx \sqrt{q}$, and we then get from (3.8) that $a_q(C_{\lambda_\theta}) \le \alpha\, 2^{-\delta N}$ for $q \gtrsim N^2$, with $\delta > 0$, which gives (3.9).

2) The map χ satisfies the conditions of Theorem 3.3 with $p = 1$ and $\omega(h) = 1/(\log 1/h)$. Then, $\omega^{-1}(h) = \mathrm{e}^{-1/h}$ and $d_N \approx 2^N$, so that $N_q \approx \log q$ and $2^{N_q} \approx q/\log q$, and (3.10) follows. □

The proof of Theorem 3.3 is based on the following Lemma 3.5, with a suitable choice of the Blaschke product of length pNd (d a positive integer to be specified):

$$B(z) = \prod_{j=1}^{p} \prod_{k=1}^{N} \left[\frac{z - p_{j,k}}{1 - \overline{p_{j,k}}\, z}\right]^d,$$

where $p_{j,k} = (1 - 2^{-k})\xi_j$, for $j = 1, \dots, p$ and $k = 1, 2, \dots$. We do not give the details here and refer to [20, Proof of Theorem 2.3].

Lemma 3.5 ([15, Lemma 2.4]). *For every Blaschke product B with less than N zeros (each of them being counted with its multiplicity), one has:*

$$\left[a_N(C_\varphi)\right]^2 \lesssim \sup_{0<h<1,|\xi|=1} \frac{1}{h} \int_{S(\xi,h)} |B(z)|^2 \, dm_\varphi(z)\,, \tag{3.11}$$

where $S(\xi, h) = \{z \in \overline{\mathbb{D}}\,;\ |z-\xi| \le h\}$ and m_φ is the pull-back measure by φ of the normalized Lebesgue measure m on $\mathbb{T}$.

The proof of Lemma 3.5 comes from the estimate of the Carleson norm in the Carleson's theorem and from the fact that the subspace BH^2 is of codimension $\le N-1$, leading to a majorization of the Gelfand number $c_N(C_\varphi)$, and then using (2.9). For the convenience of the reader, we reproduce the proof.

Proof of Lemma 3.5. The subspace BH^2 is of codimension $\le N-1$. Therefore, $a_N = c_N(C_\varphi) \le \left\|C_{\varphi|BH^2}\right\|$, where c_N is the n-th Gelfand number and where we used the equality $a_N = c_N$ occurring in the Hilbertian case, as recalled in the introduction. Now, since $\|Bf\|_{H^2} = \|f\|_{H^2}$ for any $f \in H^2$, we have:

$$\begin{aligned}\left\|C_{\varphi|BH^2}\right\|^2 &= \sup_{\|f\|_{H^2}\le 1} \int_{\mathbb{T}} |B\circ\varphi|^2\,|f\circ\varphi|^2\,dm \\ &= \sup_{\|f\|_{H^2}\le 1} \int_{\overline{\mathbb{D}}} |B|^2|f|^2\,dm_\varphi = \|R_\mu\|^2,\end{aligned}$$

where $\mu = |B|^2 m_\varphi$ and where $R_\mu\colon H^2 \to L^2(\mu)$ is the restriction map. Of course, μ is a Carleson measure for H^2 since $\mu \le m_\varphi$. Now, Carleson's embedding theorem says us that $\|R_\mu\|^2 \le \kappa^2 \sup_{0<h<1,\xi\in\mathbb{T}} \frac{\mu[\overline{S(\xi,h)}]}{h}$ (see [9], Remark after the proof of Theorem 9.3, at the top of page 163; actually, in that book, Carleson's windows $W(\xi, h)$ are used instead of pseudo-Carleson's windows $S(\xi, h)$, but that does not matter, since $W(\xi, h) \subseteq S(\xi, 2h)$: if $r \ge 1-h$ and $|t-t_0| \le h$, then $|re^{it} - e^{it_0}| \le |re^{it} - e^{it}| + |e^{it} - e^{it_0}| \le 2h$). □

Lower estimates

We consider symbols φ taking real values in the real axis (*i.e.* its Taylor series has real coefficients) and such that $\lim_{r\to 1^-} \varphi(r) = 1$, with a given speed.

We say that φ is *radially regular* if it takes real values on $]-1, 1[$ and there exists a modulus of continuity $\omega\colon [0,1] \to [0,2]$ such that $1-\varphi(r) \le \omega(1-r)$ for $0 \le r < 1$. Then we have the following result ([20, Theorem 3.2]).

Theorem 3.6. *Let φ be a radially regular symbol. Then, for the approximation numbers $a_n(C_\varphi)$ of the composition operator C_φ of symbol φ, one has the following lower bound.*

$$a_n(C_\varphi) \ge c \sup_{0<\sigma<1} \sqrt{\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}} \exp\left[-\frac{20}{1-\sigma}\right], \tag{3.12}$$

where $a = 1-\varphi(0) > 0$ and c is another constant depending only on φ.

For our examples, we get:

Theorem 3.7.

1) *For the lens map λ_θ of parameter θ, we have, for some positive constants c and C, depending on θ:*

$$a_n(C_{\lambda_\theta}) \ge c\,\exp\left(-C\sqrt{n}\right). \tag{3.13}$$

2) *For the cusp map χ, we have, for some positive constants c and C:*

$$a_n(C_\varphi) \ge c\exp(-C\,n/\log n). \tag{3.14}$$

Proof. 1) The lens map λ_θ satisfies the conditions of Theorem 3.6 with $\omega^{-1}(h) \approx h^{1/\theta}$. We get the result by adjusting $\sigma = 1 - 1/\sqrt{n}$.

2) The cusp map χ satisfies the conditions of Theorem 3.6 with $\omega^{-1}(h) \approx \mathrm{e}^{-C'/h}$, and by taking $\sigma = \exp(-\log n/2n)$, we get the result. □

The tools for proving Theorem 3.6 are the following.

Recall (see [13, pages 194–195], or [25, pages 302–303]) that if (z_j) is a Blaschke sequence, its Carleson constant δ is defined as

$$\delta = \inf_{j\geq 1}(1 - |z_j|^2)\,|B'(z_j)|\,,$$

where B is the Blaschke product whose zeros are the points z_j. Recall also (see [7], [13, pages 194–195], or [25, pages 302–303]) that an *interpolation sequence* (z_n) with (best) interpolation constant C is a sequence (z_n) (necessarily Blaschke, i.e. $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$) in the unit disk such that, for any bounded sequence (w_n) of scalars, there exists a bounded analytic function f (i.e. $f \in H^\infty$) such that:

$$f(z_n) = w_n\,,\ \forall\, n \geq 1, \quad \text{and} \quad \|f\|_\infty \leq C \sup_{n\geq 1} |w_n|\,.$$

Now (see [11, Chapter VII, Theorem 1.1]), every H^∞-interpolation sequence (z_j) is a Blaschke sequence and its Carleson constant δ is connected to its interpolation constant C by the inequalities

$$1/\delta \leq C \leq \kappa/\delta^2 \tag{3.15}$$

where κ is an absolute constant (actually $C \leq \kappa_1(1/\delta)(1+\log 1/\delta)$). Now, if (z_j) is a H^∞-interpolation sequence with constant C, the sequence of the normalized reproducing kernels $f_j = K_{z_j}/\|K_{z_j}\|$, viz $f_j(z) = \frac{1-|z_j|^2}{1-\bar{z}_j z}$, satisfies

$$C^{-1}\left(\sum_j |\lambda_j|^2\right)^{1/2} \leq \left\|\sum_j \lambda_j f_j\right\|_{H^2} \leq C\left(\sum_j |\lambda_j|^2\right)^{1/2} \tag{3.16}$$

(see [19, Lemma 2.2]).

We then use the following lemma [20, Lemma 3.3], with $u_j \in [0, 1)$ defined inductively by $u_0 = 0$ and the relation:

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)] \quad \text{with } 1 > u_{j+1} > u_j\,,$$

using the intermediate value theorem, where $0 < \sigma < 1$ is a fixed positive number. Note that, when setting $v_j = \varphi(u_j)$, we have $-1 < v_j < 1$,

$$\frac{1 - v_{j+1}}{1 - v_j} = \sigma, \tag{3.17}$$

and

$$1 - v_n = a\,\sigma^n\,, \tag{3.18}$$

with $a = 1 - \varphi(0)$.

Lemma 3.8. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Let $u = (u_1, \dots, u_n)$ be a finite sequence in $\mathbb{D}$ and set $v_j = \varphi(u_j)$, $v = (v_1, \dots, v_n)$. Denote by δ_v the Carleson constant of the finite sequence v and set*

$$\mu_n^2 = \inf_{1\leq j\leq n} \frac{1 - |u_j|^2}{1 - |\varphi(u_j)|^2}\,.$$

Then, for some constant $c' > 0$, we have the lower bound:

$$a_n(C_\varphi) \geq c'\,\delta_v^4\,\mu_n. \tag{3.19}$$

For the details of the proof, we refer to [20, Proof of Theorem 3.2, pages 556-557].

3.3 Entropy numbers

In this section, we give the behavior of the entropy numbers of the lens maps and the cusp map, seeing that they significantly differ from that of the approximation numbers.

Theorem 3.9. *Let λ_θ be the lens map with parameter θ. Then, with positive constants a, b, a', b' depending only on θ:*

$$a' \, \mathrm{e}^{-b' n^{1/3}} \le e_n(C_{\lambda_\theta}) \le a \, \mathrm{e}^{-b n^{1/3}} . \tag{3.20}$$

Proof. We use Theorem 3.9. It follows, using Theorem 2.2, writing $a_k = a_k(C_{\lambda_\theta})$, that $(a_1 \cdots a_k)^{1/k} \le a \, \mathrm{e}^{-b\sqrt{k}}$ and that, for some positive constant C:

$$e_n(C_{\lambda_\theta}) \le C \exp\left[- \left((n/2k) \log 2 + b k^{1/2} \right) \right] .$$

Taking $k = \lceil n^{2/3} \rceil$ gives the claimed upper bound, with a different value of b.

The lower bound is proved similarly, using Theorem 3.13 and the left inequality in Theorem 2.2. □

Theorem 3.10. *Let χ be the cusp map. Then, with positive constants a, b, a', b':*

$$a' \, \mathrm{e}^{-b' \sqrt{n/\log n}} \le e_n(C_\chi) \le a \, \mathrm{e}^{-b \sqrt{n/\log n}} . \tag{3.21}$$

Proof. The proof follows the same lines as in Theorem 3.9. Using Theorem 3.10, we get, with another b:

$$\prod_{j=1}^{k} [a_j(C_\chi)] \le \mathrm{e}^{-b \, k^2/\log k} ,$$

and Theorem 2.2, with the choice $k = \lceil \sqrt{n \log n} \rceil$, gives the upper bound, with a different value of b.

The lower bound is obtained similarly, using now Theorem 3.14 and Theorem 2.2. □

4 The multidimensional case

4.1 General results

Let $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$ be an analytic map. We will say that φ is non-degenerate if $\varphi(\mathbb{D}^N)$ has non-empty interior, equivalently if $\det \varphi'(z) \neq 0$ for at least one point $z \in \mathbb{D}^N$.

Let now $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$ be a non-degenerate analytic map inducing a bounded composition operator $C_\varphi \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ (this is not always the case as soon as $N > 1$, even if φ is injective and hence non-degenerate, see for example [8, p. 246], when the polydisk is replaced by the ball; but similar examples exist for the polydisk). Assume moreover that C_φ is a compact operator.

Theorem 4.1. *Let $C_\varphi \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ be a compact composition operator, with φ non-degenerate. We have:*

1) $e_n(C_\varphi) \ge c \exp\left(- C \, n^{\frac{1}{N+1}} \right)$, *for some constants $C > c > 0$, depending on φ;*

2) *if $\|\varphi\|_\infty < 1$, then $e_n(C_\varphi) \le C \exp\left(- c \, n^{\frac{1}{N+1}} \right)$, with $C > c > 0$ depending on φ.*

Proof. 1) It is proved in [3, Theorem 3.1] that, for a non-degenerate map φ, it holds:

$$a_k(C_\varphi) \ge a' \, \mathrm{e}^{-b' k^{1/N}} .$$

As in the previous section, it follows from Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \ge \mathrm{e}^{-b'' k^{1/N}}$, and then, taking $k = \lceil n^{N/(N+1)} \rceil$, that:

$$e_n(C_\varphi) \ge c \, \mathrm{e}^{-C n^{1/(N+1)}} .$$

2) Similarly, for $\|\varphi\|_\infty < 1$, it is proved in [3, Theorem 5.2] that

$$a_k(C_\varphi) \le C\,\mathrm{e}^{-ck^{1/N}}\,,$$

and we get the result from Theorem 2.2. □

Those estimates motivate the introduction of the parameter

$$\gamma_N(C_\varphi) = \lim_{n\to\infty} \left[e_n(C_\varphi)\right]^{\frac{1}{n^{1/(N+1)}}}\,. \tag{4.1}$$

We define similarly $\gamma_N^{\pm}(C_\varphi)$, and will say more on it in next section.

4.2 Specific results

When $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ is a non degenerate analytic map inducing a compact composition operator $C_\Phi\colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$, a version of Theorem 3.6 is proved in [3, Theorem 4.2]. For upper bounds, a result was obtained in [3, Theorem 5.5] when $\Phi = \varphi_1 \otimes \cdots \otimes \varphi_N$, where $\varphi_1, \dots, \varphi_N\colon \mathbb{D} \to \mathbb{D}$ are symbols inducing compact composition operators on $H^2(\mathbb{D})$. We do not give here the precise statements and refer to the paper [3]. However, we recall the result that can be obtained from that for the multi-lens maps and the multi-cusp map ([3, Theorem 6.1 and Theorem 6.2]). First, we define the multi-lens map Λ_θ and the multi-cusp map.

Let λ_θ be lens maps with parameter θ. The multi-lens map Λ_θ with parameter θ the multi-cusp map on the polydisk $\mathbb{D}^N$ are defined respectively as:

$$\Lambda_\theta(z_1, \dots, z_N) = \big(\lambda_\theta(z_1), \lambda_\theta(z_2), \dots, \lambda_\theta(z_N)\big)\,, \tag{4.2}$$

and:

$$\Xi\,(z_1, \dots, z_N) = \big(\chi(z_1), \chi(z_2), \dots, \chi(z_N)\big)\,. \tag{4.3}$$

for $(z_1, \dots, z_N) \in \mathbb{D}^N$.

Theorem 4.2.

1) *For positive constants a, b, a', b' depending only on θ and N, one has*

$$a'\,\mathrm{e}^{-b' n^{1/(2N)}} \le a_n(C_{\Lambda_\theta}) \le a\,\mathrm{e}^{-b\,n^{1/(2N)}}\,. \tag{4.4}$$

2) *For positive constants a, b, a', b' depending only on N, one has*

$$a'\,\mathrm{e}^{-b'\,n^{1/N}/\log n} \le a_n(C_\Xi) \le a\,\mathrm{e}^{-b\,n^{1/N}/\log n}\,, \tag{4.5}$$

With the same proof as in Theorem 3.9 and Theorem 3.10, we obtain estimates for the entropy numbers.

Theorem 4.3.

1) *Let Λ_θ be the multi-lens map with parameter θ. Then:*

$$a'\,\exp\left(-b' n^{1/(2N+1)}\right) \le e_n(C_{\Lambda_\theta}) \le a\,\exp\left(-b\,n^{1/(2N+1)}\right)\,. \tag{4.6}$$

2) *Let $\Xi\colon \mathbb{D}^N \to \mathbb{D}^N$ be the multi-cusp map. Then:*

$$\begin{aligned} a'\,\exp\big[-b'\,n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big] \\ \le e_n(C_\Xi) \le a\,\exp\big[-b\,n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big]\,. \end{aligned} \tag{4.7}$$

5 Links with pluricapacity and Zakharyuta's results

Here, in dimension $N \geq 2$, the situation is satisfactory for upper bounds (see [22]); for lower bounds, see [23]. The notion involved is now that of pluricapacity, or Monge-Ampère capacity, coined by Bedford and Taylor in [4]. More precisely, if A is a Borel subset of $\mathbb{D}^N$, we refer to [22] or [23] for the definition of its pluricapacity $\mathrm{Cap}_N(A)$, belonging to $[0, +\infty]$, and set:

$$\tau_N(A) = \frac{1}{(2\pi)^N} \mathrm{Cap}_N(A) \tag{5.1}$$

$$\Gamma_N(A) = \exp\left[- \left(\frac{N!}{\tau_N(A)} \right)^{1/N} \right] \tag{5.2}$$

$$\beta_N^+(T) = \limsup_{n\to\infty} \left[a_n(T) \right]^{1/n^{1/N}} . \tag{5.3}$$

We temporarily assume that $\|\varphi\|_\infty < 1$ so that $K = \overline{\varphi(\mathbb{D}^N)}$ is a compact subset of $\mathbb{D}^N$. We proved in [22, Theorem 6.4], relying on positive results of Nivoche ([26]) and Zaharyuta ([33, Proposition 6.1])[1] on the so-called Kolmogorov conjecture, that:

Theorem 5.1. *It holds:*

$$\beta_N^+(C_\varphi) \leq \Gamma_N(K) . \tag{5.4}$$

We have the following result, which extends the previous result in dimension 1.

Theorem 5.2. *The following upper bound holds:*

$$\gamma_N^+(C_\varphi) \leq \exp\left(- \beta_N \rho^{N/(N+1)} \right) , \tag{5.5}$$

where

$$\rho = \left(\frac{N!}{\tau_N(K)} \right)^{1/N} = 2\pi \left(\frac{N!}{\mathrm{Cap}_N(K)} \right)^{1/N} , \tag{5.6}$$

and

$$\beta_N = (\log 2)^{1/(N+1)} \left(\frac{2N}{N+1} \right)^{N/(N+1)} . \tag{5.7}$$

Proof. Abbreviate $a_n(C_\varphi)$ and $e_n(C_\varphi)$ to a_n and e_n, and set $\alpha = N/(N+1)$. Let $\varepsilon > 0$. Theorem 5.1 implies:

$$a_k \leq C_\varepsilon \, \mathrm{e}^{\varepsilon k^{1/N}} \mathrm{e}^{-\rho k^{1/N}} ,$$

so:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon \, \mathrm{e}^{\varepsilon k^{1/N}} \mathrm{e}^{-\rho \alpha k^{1/N}} .$$

Apply once more Theorem 2.2 to obtain:

$$e_n \leq C_\varepsilon \sup_{k\geq 1} \mathrm{e}^{\varepsilon k^{1/N}} \exp\left[- \left(\frac{n}{2k} \log 2 + \rho \, \alpha \, k^{1/N} \right) \right] .$$

The supremum is essentially attained for k the integral part of $(\log 2/2\rho\alpha)^\alpha n^\alpha$ and then, in view of (5.7) and $\alpha/N = 1 - \alpha$, as n goes to infinity:

$$\frac{n}{k} \log 2 + \rho \, \alpha \, k^{1/N} \sim (\log 2)^{1-\alpha} (2\rho \, \alpha)^\alpha \, n^{1-\alpha} .$$

1 A more general result has recently been proved by Bandtlow and Nivoche in [2].

Finally, with $c_\alpha = (\log 2/2\rho\,\alpha)^{1-\alpha}$, we have:

$$\begin{aligned} e_n &\le C_\varepsilon\, \mathrm{e}^{\varepsilon\, c_\alpha n^{1-\alpha}} \exp\left(-\beta_N \rho^\alpha n^{1-\alpha}\right) \\ &= C_\varepsilon\, \mathrm{e}^{\varepsilon\, c_\alpha n^{1/(N+1)}} \exp\left(-\beta_N \rho^{N/(N+1)} n^{1/(N+1)}\right). \end{aligned}$$

This clearly ends the proof of Theorem 5.2. □

Remark. We have so far no sharp lower bound for entropy numbers, at least when $\|\varphi\|_\infty = 1$, since we already fail to have one in general for approximation numbers (see however [23]). In dimension 1, we used the fact that every *connected* Borel subset E of $\mathbb{D}$ such that $\bar{E} \subseteq \mathbb{D}$, the Green capacity of E is equal to that of $\bar{E}$ ([21, Theorem 2.3]).

Besides, let $J\colon H^\infty(\mathbb{D}^N) \to \mathcal{C}(K)$ be the canonical embedding, when $K \subseteq \mathbb{D}^N$ is a "condenser", namely a compact subset of $\mathbb{D}^N$ such that any bounded analytic function on $\mathbb{D}^N$ which vanishes on K vanishes identically, which is moreover "regular". The positive solution to the Kolmogorov conjecture can be expressed in terms of the Kolmogorov numbers $d_n(J)$ of J or equivalently, in terms of the entropy numbers $e_n(J)$ of J ([32, Theorem 5], generalizing Erokhin's result in dimension 1 appearing in his posthumous paper [10] and methods due to Mityagin [24] and Levin and Tikhomirov [16]; see also [33, Lemma 2.2]). The result is that, taking $K = \overline{\varphi(\mathbb{D}^N)}$, one has, with sharp constants c_K, c'_K depending on the pluricapacity of K in $\mathbb{D}^N$:

$$d_n(J) \approx \mathrm{e}^{-c_K\, n^{1/N}} \quad \text{and} \quad e_n(J) \approx \mathrm{e}^{-c'_K\, n^{1/(N+1)}}. \tag{5.8}$$

This jump from the exponent $1/N$ to the exponent $1/(N+1)$ is reflected in our Theorem 5.2, through the new parameter γ_N^+.

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