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Groups, Geometry, and Dynamics

On localizations of quasi-simple groups with given countable center

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Dedicated to the memory of Rüdiger Göbel

Abstract. A group homomorphism $i: H \to G$ is a *localization* of H, if for every homomorphism $\varphi: H \to G$ there exists a unique endomorphism $\psi: G \to G$ such that $i\psi = \varphi$ (maps are acting on the right). Göbel and Trlifaj asked in [18, Problem 30.4(4), p. 831] which abelian groups are centers of localizations of simple groups. Approaching this question we show that every countable abelian group is indeed the center of some localization of a quasi-simple group, i.e., a central extension of a simple group. The proof uses Obraztsov and Ol′shanskii's construction of infinite simple groups with a special subgroup lattice and also extensions of results on localizations of finite simple groups by the second author and Scherer, Thévenaz and Viruel.

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1. Introduction

Simple groups are a good source of many kinds of realization theorems. For example, we know from Droste, Giraudet, and Göbel [8] that every group can be expressed as the outer automorphism of some simple group. Using Ol′shanskii [25], Obraztsov [24] proved that every abelian group is the center of some infinite quasi-simple group. In [25] we can find Burnside groups of large prime exponent with many extra properties. In this paper we apply Obraztsov [24] to find localizations of (quasi)-simple groups with given countable center, see Theorem 2.3.

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Before going into details of our results, let us recall the importance of localization, especially in Group Theory and Topology. In general, the goal of defining a localization is simplifying an object or category, sometimes by passing from global to local, sometimes by inverting morphisms. Since the seventies, these constructions have entered with force in the categories of groups and spaces, showing their potential. Localization of spaces was introduced by Sullivan [33] in his proof of the Adams conjecture, Hilton, Mislin, and Roitberg [20] introduced the group version while generalizing abelianization to nilpotent groups, Bousfield [6] provided the bridge between these two types of functors, and Farjoun [10] gave an unified treatment with the notion of f-localization with respect to a morphism f. Note that values of group f-localization functors $\eta_H\colon H \to LH$ are localizations in the sense defined above, and conversely, given a localization $f\colon H \to G$, we have that $G = L_f H$ where L_f is the localization functor with respect to f (see e.g., [7, Lemma 2.1]).

We recall now several subjects where these localization tools have been of great importance: concerning (co)homology of groups, the p-primary analysis based on p-localization is crucial in the modern theory of fusion systems of Broto, Levi, and Oliver [5], which led in particular to the surprising fact that the $\mathbb{F}_p$-homology of a group G determines the homotopy structure (mod p) of its classifying space. Also, the exotic non-perfect group localization of a perfect group discovered in [31] uses as input the fundamental group of a universal acyclic space [3] which in turn determines the plus-construction, the key tool used by Quillen [29] to define algebraic K-theory out of the general linear groups. In a more geometric context, localization plays a key role in the celebrated proof that all finite loop spaces are manifolds [2], and very recently [1] localizations have been defined which are compatible with actions of groups.

More related to our approach in this paper, (co)localizations of Burnside groups (which in turn are themselves localizations of free groups) have appeared at least in two different contexts: in relation with amenability phenomena [13], and from the point of view of combinatorial group theory, for example in [18] and [19]. In particular, the existence of $2^{\aleph_0}$ varieties of groups not closed under cellular covers (another name for colocalizations) is shown in the last references, complementing the fact that there are $2^{\aleph_0}$ varieties closed under taking cellular covers [14]. On the other hand, cellular covers of simple groups were described in [4], see also [9].

Our first attempt to deal with the problem of Göbel and Trlifaj was to use Libman's work and subsequent constructions of large arbitrary single-group localizations. Libman [22] showed that the natural inclusion $A_n \hookrightarrow A_{n+1}$ of alternating groups (for $n \geq 7$) is a localization. Indeed, this motivated the study of localizations between simple groups in [30, 32, 28]. First examples of infinite localizations of A_n (for $n \geq 10$) were also given in [22]; in this case $A_n \hookrightarrow SO(n)$ is a localization. In fact, for every non-abelian simple group S and cardinal κ, there exists a simple localization of cardinality $\geq \kappa$. This was proved under the (GCH) in [15] and extended to any universe of set theory in [17] jointly with Shelah. Recall that GCH (Generalized Continuum Hypothesis) states that given any infinite cardinal λ, there is no other cardinal λ' such that $\lambda < \lambda' < 2^{\lambda}$.

In this paper we will need simple groups with large Schur multipliers and thus must discard the use of finite simple groups; see Karpilovsky [21] for the restricted size of Schur multipliers of finite simple groups. Simple groups from [15, 17] are here of no use, and we take advantage of the constructions in [25, 24]. Relying on these Burnside groups, our main result Theorem 5.9 gives a partial positive answer to Problem 4 in [18, Chapter 30] for countable abelian groups.

Let F be a free abelian group of countable rank and A be any epimorphic image. Moreover, let H be a complete, co-hopfian simple group without involutions, and suppose that $m > 10^{75}$ or $m = \infty$, such that all proper subgroups of H are cyclic of an order dividing m if $m < \infty$, and of infinite order otherwise. The existence of these groups H follows from [24, 25]. Then we will show the following claims.

(a) There exists a localization $H \subset G$ with G complete, simple, co-hopfian G and Schur multiplier $\mathrm{Mult}(G) \cong F$.

(b) This localization lifts to a localization $\widetilde{H} \to \widetilde{G}$ between the corresponding universal central extensions, where $\mathfrak{z}(\widetilde{G}) \cong F$.

(c) This localization also lifts to another localization $\bar{H} \subset \bar{G}$, between the corresponding central extensions, where $\mathfrak{z}\bar{G} \cong A$.

By (b) all free, abelian groups of rank $\leq 2^{\aleph_0}$ can arise as centers of localizations of simple groups. More generally, we will represent countable abelian groups as centers of localizations of (quasi)-simple groups by (c). This also leads to the following open question.

Problem 1.1. Is there a simple, complete co-hopfian group H without involutions and with trivial Schur multiplier?

Corollary 1.2. *Let A be any abelian group and let H be a simple, co-hopfian, complete group without involutions, of exponent $m > 10^{75}$ or $m = \infty$, such that all proper subgroups of H are cyclic of an order dividing m if $m < \infty$, and of infinite order otherwise, and with Schur multiplier* $\mathrm{Mult}(H) = 1$. *Then there exists a localization $H \subseteq L$ with $\mathfrak{z}L = A$, and L/A is a complete, simple and co-hopfian group of exponent m.*

The simple groups in Theorem 5.9 can be investigated more thoroughly if results on finite simple groups obtained in [30] and [32] are first extended to infinite groups. It is important to observe that localizations $H \to G$ between two simple groups often induce localizations $\mathrm{Aut}(H) \to \mathrm{Aut}(G)$, as will be discussed next. We will also consider the induced homomorphisms $\widetilde{H} \to \widetilde{G}$ between their corresponding universal and intermediate central extensions.

2. Localization of infinite simple groups

First we extend some results from [30] to infinite simple groups. If G is a group and $g, h \in G$, then $hg^* = h^g = g^{-1}hg$ is conjugation and $c\colon G \to \operatorname{Aut} G$, $g \mapsto g^*$, is the conjugation map with kernel $\mathfrak{z}G$, the center of G. As usual, let $G^* = \{g^* \mid g \in G\} = \operatorname{Inn} G \subseteq \operatorname{Aut} G$. It is immediate for $\alpha \in \operatorname{Aut} G$ and $g \in G$ that

$$g^*\alpha^* = (g\alpha)^* \quad \text{or, equivalently,} \quad (gc)\alpha^* = (g\alpha)c, \tag{2.1}$$

because $gc\alpha^* = g^*\alpha^* = \alpha^{-1}g^*\alpha$, $(g\alpha)c = (g\alpha)^*$ and for any $h\alpha \in G$ we have

$$(h\alpha)\alpha^{-1}g^*\alpha = hg^*\alpha = (g^{-1}hg)\alpha = (g\alpha)^{-1}(h\alpha)(g\alpha) = (h\alpha)^{g\alpha} = (h\alpha)(g\alpha)^*.$$

Now suppose that G and H are groups with trivial center.

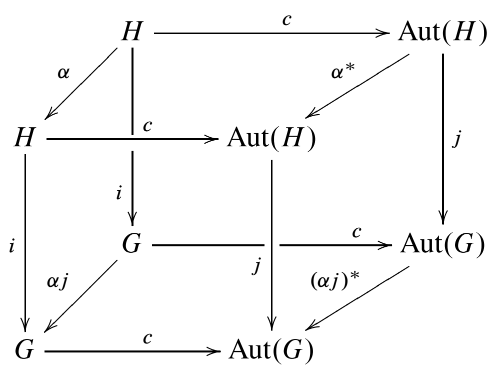
Let $H \subseteq G$, and $h \in H$. In order to emphasize *conjugation by h extended to G*, we also write $h^{*G} \in G^*$, thus $h^* = h^{*H} \subseteq h^{*G}$.

Definition 2.1. If $j\colon \operatorname{Aut} H \to \operatorname{Aut} G$, then we say that j *extends* $H^* \subseteq G^*$ if $h^* j = h^{*G}$ for all $h \in H$.

An easy argument from [30, Lemma 1.2] can be used for a more general

Lemma 2.2. *Suppose that G and H are groups with trivial center. If $j\colon \operatorname{Aut} H \to \operatorname{Aut} G$ extends $H^* \subseteq G^*$, then also any $\alpha \in \operatorname{Aut} H$ extends to $\alpha j \in \operatorname{Aut} G$, i.e., $(\alpha j) \restriction H = \alpha$.*

Proof. If $h \in H$, then we must show that $h\alpha = h(\alpha j)$, that is, the left square in the following cube commutes.



First we apply (2.1) twice to H and G, respectively, and get $(h\alpha)^* = h^*\alpha^*$ as well as $(h(\alpha j))^{*G} = h^{*G}(\alpha j)^*$ (i.e. top and bottom squares commute). Moreover, since j is a homomorphism, it follows $(h^*\alpha^*)j = h^{*G}(\alpha j)^*$ (i.e. right square commutes). Recall now that $\alpha \in \operatorname{Aut} H$. From these three equalities and the assumption $(h\alpha)^* j = (h\alpha)^{*G}$ on j we get $(h(\alpha j))^{*G} = h^{*G}(\alpha j)^* = (h^*\alpha^*)j = (h\alpha)^* j = (h\alpha)^{*G}$, hence $(h\alpha)^{*G} = (h(\alpha j))^{*G}$. Since G has trivial center, it follows $h\alpha = h(\alpha j)$ and $(\alpha j) \restriction H = \alpha$. □

The lemma will be used to characterize certain localizations $H \subseteq G$. When G and H are finite simple groups, then the next Theorem 2.3 is due to [30, Theorem 1.4] (and a variation of [15, Corollary 4]), because finite simple groups are obviously co-hopfian. Recall that a group G is *co-hopfian* if every monomorphism $G \hookrightarrow G$ is an automorphism of G. In one direction of our extension we will need that G is co-hopfian.

Theorem 2.3. *Let $H \subseteq G$ be an extension of two simple groups, and assume that G is co-hopfian. Then, $H \subseteq G$ is a localization if and only if the following conditions hold*:

(1) *there is a monomorphism $j\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ which extends $H^* \subseteq G^*$*;

(2) *any subgroup of G isomorphic to H is conjugate to H in $\operatorname{Aut} G$*;

(3) *the centralizer $\mathfrak{c}_{\operatorname{Aut} G} H = 1$ is trivial.*

Proof. Assume that $H \subseteq G$ is a localization. Denote the inclusion by $i\colon H \hookrightarrow G$.

(1) If $\alpha \in \operatorname{Aut} H$, then for $\alpha i\colon H \to G$ there is a unique $\beta \in \operatorname{End} G$ such that $\alpha = \beta \restriction H$. Since G is simple, the map β is injective. The map $\alpha^{-1} i\colon H \to G$ also has a unique extension $\gamma \in \operatorname{End} G$ such that $\alpha^{-1} = \gamma \restriction H$, and from $\alpha^{-1}\alpha = \mathrm{id}_H$ and uniqueness of localizations it follows $\gamma\beta = \mathrm{id}_G$. Hence β is also surjective and $\beta \in \operatorname{Aut} G$. It is clear by the uniqueness of localizations that the map

$$j\colon \operatorname{Aut} H \longrightarrow \operatorname{Aut} G, \quad \alpha \longmapsto \alpha j = \beta,$$

is a monomorphism. Note that $h \in H$ by definition of j is mapped to $h^* j = h^{*G}$, hence j extends $H^* \subseteq G^*$ and (1) holds.

(2) If K is a subgroup of G and isomorphic to H, then there is some monomorphism $\alpha\colon H \to K \subseteq G$. By the localization there is $\beta \in \operatorname{End} G$ such that $\beta \restriction H = \alpha$, and since G is co-hopfian, it follows that $\beta \in \operatorname{Aut} G$. Hence $\beta^{-1} K \beta = K^{\beta^*} = H$, as required.

(3) If $\beta \in \operatorname{Aut} G$, and $\alpha = \beta \restriction H = \mathrm{id}_H$, then by the uniqueness of extensions it follows $\beta = \mathrm{id}_G$, hence (3) holds.

Conversely, we assume that H, G are simple, G is also co-hopfian and we want to show that $H \subseteq G$ is a localization, if (1), (2), (3) hold. If $\varphi \in \operatorname{Hom}(H, G)$, then we must show that there is a unique $\beta \in \operatorname{End} G$ such that $\varphi = \beta \restriction H$. If φ is the trivial homomorphism, then we choose β also trivial and note that β is unique, because G is simple and $0 \neq H \subseteq \operatorname{Ker}\beta$ forces $G\beta$ to be trivial too. We may assume that $\varphi \neq 0$, and φ is injective, because also H is simple. If $K = H\varphi$, then there is an isomorphism $\alpha\colon H \to K$, and by (1) and (2) we find $\beta \in \operatorname{Aut} G$ such that $K^{\beta^*} = H$. The automorphism $\alpha\beta \in \operatorname{Aut} H$ extends to an automorphism $\psi \in \operatorname{Aut} G$ by (1) (using Lemma 2.2). Then $\psi\beta^{-1}$ is the desired extension of α. Its uniqueness follows from (3) and the assumption that G is co-hopfian. □

We will apply the following immediate consequence of Theorem 2.3.

Corollary 2.4. *Let $H \subseteq G$ be an extension of simple, complete groups. If $H \subseteq G$ is a localization, then the following two conditions hold*:

(1) *any subgroup of G which is isomorphic to H is conjugate to H*;

(2) $\mathfrak{c}_G(H) = 1$.

Conversely, if G is co-hopfian and (1) *and* (2) *hold, then $H \subseteq G$ is a localization.*

3. Automorphism groups and localizations

We next extend some results from Section 2 of [32] to simple groups of arbitrary size. Recall that $\operatorname{Out} G$ denotes the outer automorphism group $\operatorname{Aut} G/G$ of a simple group G (where we identify G and G^*).

Lemma 2.1 of [32] holds for infinite groups, too.

Lemma 3.1. *Let G be a non-abelian simple group. Then any proper normal subgroup of* Aut G *contains G. In particular, any endomorphism of* Aut G *is either a monomorphism or contains G in its kernel.*

Lemma 2.2 of [32] states that any non-abelian finite simple subgroup of Aut G is contained in G. The easy argument is based on the solution of the Schreier conjecture, which ensures that Out G is solvable for every finite non-abelian simple group G. But for infinite groups this is no longer true, because from [8] we know that all groups are outer automorphism groups of simple groups. In order to proceed to infinite groups we will assume that the outer automorphism group is hyperabelian, which extends solvability. Recall that a group A is called *hyperabelian* if every non-trivial epimorphic image of A has a non-trivial abelian normal subgroup. In particular, A can be filtered as an ascending (possibly transfinite) union $\bigcup K_i$ of normal subgroups with abelian factors K_{i+1}/K_i. Hence, $\operatorname{Hom}(H, A) = 0$ for all non-abelian simple groups H.

Lemma 3.2. *If G is a simple group and* Out G *is hyperabelian, then any non-abelian simple subgroup of* Aut G *is contained in G.*

Proof. We identify G and G^*, which is normal in $\operatorname{Aut} G$. If $H \neq 1$ is a simple non-abelian subgroup of $\operatorname{Aut} G$, then $H \cap G$ is normal in H. Hence it is either 1 or H by simplicity of H. If $H \cap G = 1$ then $H \cong GH/G$ is a subgroup of $\operatorname{Out} G$ which, by assumption, is hyperabelian, hence H is cyclic, a contradiction. Otherwise, $H \cap G = H$, hence $H \subseteq G$ as required. □

Next we want to extend [32, Theorem 2.3]. We will follow the arguments in [32, p. 770], but must adjust two of the conditions. Since $\operatorname{Aut} G$ need not be complete, we must replace $\operatorname{Aut} G$ by $\operatorname{Aut}(\operatorname{Aut} G)$. Moreover (similar to the above), we assume that $\operatorname{Aut} G$ is co-hopfian and will apply Lemma 3.2.

Theorem 3.3. *Let $j\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ be an inclusion of the automorphism groups of two non-abelian simple groups H and G. Assume that $\operatorname{Aut} G$ is co-hopfian and $\operatorname{Out} G$ is hyperabelian. Then $j\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ is a localization, if and only if the following four conditions are satisfied*:

(a) *if $\Omega = \{k\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G\}$ denotes the set of all monomorphisms of $\operatorname{Aut} H$ in $\operatorname{Aut} G$, then $\operatorname{Aut}(\operatorname{Aut} G)$ acts transitively on Ω*;

(b) $\mathfrak{c}_{\operatorname{Aut}(\operatorname{Aut} G)} \operatorname{Aut} H = 1$;

(c) *any homomorphism $\varphi'\colon \operatorname{Aut} H/(\operatorname{Aut} H \cap G) \to \operatorname{Aut} G$ extends uniquely to a $\psi'\colon \operatorname{Out} G \to \operatorname{Aut} G$ such that $k\psi' = \pi\varphi'$, where $\pi\colon \operatorname{Out} H \twoheadrightarrow \operatorname{Aut} G/\operatorname{Aut} H \cap G$ is the canonical projection* (*see Figure 1*);

(d) *if a homomorphism $\varphi\colon \operatorname{Aut} H \to \operatorname{Aut} G$ contains H in its kernel, then also $\operatorname{Aut} H \cap G \subseteq \operatorname{Ker} \varphi$.*

Proof. Suppose that $j\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ is a localization and let $\varphi\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ be any monomorphism. Then φ extends to a homomorphism $\psi\colon \operatorname{Aut} G \to \operatorname{Aut} G$ such that $j\psi = \varphi$. We must show that $\psi \in \operatorname{Aut} G$.

If $\operatorname{Ker} \psi \neq 0$, then $G \subseteq \operatorname{Ker} \psi$ by Lemma 3.1, and thus $H \subseteq \operatorname{Ker} \psi$, which is impossible since φ is monomorphism. Thus ψ is injective and also surjective, because $\operatorname{Aut} G$ is co-hopfian. It follows that ψ is an automorphism. Hence, both (a) and (b) follow clearly.

To show (c), let $\varphi'\colon \operatorname{Aut} H/\operatorname{Aut} H \cap G \to \operatorname{Aut} G$ be any homomorphism (see Figure 1). Let $\varphi = p\varphi'\colon \operatorname{Aut} H \twoheadrightarrow \operatorname{Aut} H/\operatorname{Aut} H \cap G \to \operatorname{Aut} G$. Since j is a localization, there exists a unique $\psi\colon \operatorname{Aut} G \to \operatorname{Aut} G$ such that $j\psi = \varphi$. And since φ is not injective, also ψ is not injective, and $G \subseteq \operatorname{Ker} \psi$ by Lemma 3.1. Hence ψ factors through $\operatorname{Out} G$ and there is a unique homomorphism $\psi'\colon \operatorname{Out} G \to \operatorname{Aut} G$ such that $p\psi' = \psi$.

We claim $\pi\varphi' = k\psi'$. Composing the left-hand side with p we have $p\pi\varphi' = p\varphi' = \varphi$ and on the right-hand side by the above $pk\psi' = jp\psi' = j\psi = \varphi$, hence $p\pi\varphi' = pk\psi'$. Since p is surjective, it follows $\pi\varphi' = k\psi'$.

Suppose that there exists another $\psi'': \operatorname{Out} G \to \operatorname{Aut} G$ such that $k\psi'' = \pi\varphi'$. Then $pk\psi' = pk\psi''$, and therefore $jp\psi' = jp\psi''$. Since j is a localization, it follows $p\psi' = p\psi''$, and therefore $\psi' = \psi''$, because p is an epimorphism. Hence (c) holds.

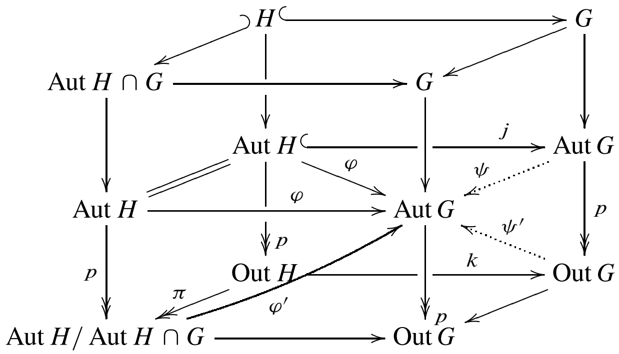


Figure 1. Diagram for (c) and (d).

For (d) suppose that $\varphi: \operatorname{Aut} H \to \operatorname{Aut} G$ is a homomorphism with $H \subseteq \operatorname{Ker} \varphi$ and (by the localization j) choose $\psi: \operatorname{Aut} G \to \operatorname{Aut} G$ such that $j\psi = \varphi$. Then, $\operatorname{Ker} \psi \supseteq (\operatorname{Ker} \varphi) j \neq 0$ and, by Lemma 3.1, $G \subseteq \operatorname{Ker} \psi$, and therefore, $\operatorname{Aut}(H) \cap G \subseteq \operatorname{Ker} \varphi$.

Suppose conversely that $(a) - (d)$ hold, and let $\varphi: \operatorname{Aut} H \to \operatorname{Aut} G$ be any group homomorphism. If $\operatorname{Ker}(\varphi) = 0$, then $\varphi \in \Omega$ and by (a) there exists an automorphism $\psi: \operatorname{Aut} G \to \operatorname{Aut} G$ such that $j\psi = \varphi$. Uniqueness is guaranteed by (b).

If $\operatorname{Ker}(\varphi) \neq 0$, then $H \subseteq \operatorname{Ker}(\varphi)$ by Lemma 3.1. And by (d) also $\operatorname{Aut} H \cap G \subseteq \operatorname{Ker}(\varphi)$, and therefore, φ factors through some homomorphism $\varphi': \operatorname{Aut} H/ \operatorname{Aut} H \cap G \to \operatorname{Aut} G$ such that $p\varphi' = p\pi\varphi' = \varphi$. By (c) this homomorphism extends uniquely to some $\psi': \operatorname{Out} G \to \operatorname{Aut} G$ such that $k\psi' = \pi\varphi'$.

We next show that the composition $\psi := p\psi'$ satisfies $j\psi = \varphi$. Indeed,

$$jp\psi' = pk\psi' = p\pi\varphi' = \varphi.$$

For the uniqueness suppose that another homomorphism $\hat{\psi}: \operatorname{Aut} G \to \operatorname{Aut} G$ satisfies $j\hat{\psi} = \varphi$. Then $\operatorname{Ker} \hat{\psi} \neq 0$ and $G \subseteq \operatorname{Ker}(\hat{\psi})$ by Lemma 3.1, hence $\hat{\psi}$ factors through some $\hat{\psi}': \operatorname{Out} G \to \operatorname{Aut} G$ such that $p\hat{\psi}' = \hat{\psi}$. As $p: \operatorname{Aut} H \twoheadrightarrow \operatorname{Out} H$ is an epimorphism, and

$$pk\hat{\psi}' = jp\hat{\psi}' = j\hat{\psi} = \varphi = p\pi\varphi'$$

we have $k\hat{\psi}' = \pi\varphi'$. By uniqueness of (c) we get $\hat{\psi}' = \psi'$, and therefore $\hat{\psi} = \psi$, as desired. $\square$

Observation 3.4. If $H \subseteq G$ is a localization of non-abelian simple groups, and G is co-hopfian, then every monomorphism $\varphi\colon H \hookrightarrow G$ extends to a unique $\psi \in \operatorname{Aut} G$ such that $\psi \restriction H = \varphi$.

If the assumption of co-hopfian is removed, localizations $\varphi\colon H \hookrightarrow G$ for which the previous observation does not hold can be found, as it is shown in the following (abelian) examples: the localization $\mathbb{Z} \hookrightarrow \mathbb{Z}[1/2]$, where $\mathbb{Z} \overset{3}{\hookrightarrow} \mathbb{Z}[1/2]$, does not extend to any automorphism, and neither does the isomorphism $\mathbb{Z}[1/2] \overset{2}{\to} \mathbb{Z}[1/2]$ restrict to an isomorphism on $\mathbb{Z}$.

A version of [32, Theorem 2.4] also extends to infinite groups (by adding co-hopfian).

Theorem 3.5. *Let $H \subseteq G$ be a localization of non-abelian simple groups, suppose that G and $\operatorname{Aut} G$ are co-hopfian, $\operatorname{Aut} H$ is complete and $\operatorname{Out} G$ is hyper-abelian. Then any extension $j\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ is a localization, if and only if conditions (b), (c) and (d) as in Theorem 3.3 are satisfied. If also $\operatorname{Aut} G$ is complete, then j is a localization, if and only if (c) and (d) hold.*

Proof. We apply the previous Theorem 3.3 for the converse implication. Let $\varphi\colon \operatorname{Aut} H \hookrightarrow \operatorname{Aut} G$ be any monomorphism decomposed as

$$\operatorname{Aut} H \xrightarrow{\varphi} \varphi(\operatorname{Aut} H) \overset{k}{\hookrightarrow} \operatorname{Aut} G,$$

and let $\varphi'\colon H \overset{\cong}{\to} H' \overset{k}{\hookrightarrow} G$ be its restriction. If the localization $H \subseteq G$ is denoted by i, then by Observation 3.4, there exists a unique $\alpha \in \operatorname{Aut} G$ such that $\varphi' k\alpha = i$. And clearly, conjugation $\alpha^*\colon \operatorname{Aut} G \to \operatorname{Aut} G$ restricts to an automorphism $\alpha^*\colon \operatorname{Aut}(H') \to \operatorname{Aut} H$ such that $k\alpha^* = \alpha^* j$. By completeness the composite $\varphi\alpha^*\colon \operatorname{Aut} H \to \operatorname{Aut} H$ is conjugation β^* by some automorphism $\beta \in \operatorname{Aut} H$. Now $\beta^* j = ((\beta)j)^* j$, by uniqueness of localization, and then

$$j((\beta)j)^*(\alpha^*)^{-1} = \beta^* j(\alpha^*)^{-1} = \beta^*(\alpha^*)^{-1}k = \varphi,$$

therefore $\psi = ((\beta)j)^*(\alpha^*)^{-1}$ extends φ, i.e., $j\psi = \varphi$, as desired.

Now suppose that $\operatorname{Aut} G$ is complete. If $\beta^*\colon \operatorname{Aut} G \cong \operatorname{Aut} G$ is an inner automorphism that centralizes $\operatorname{Aut} H$, then it centralizes H, and therefore the map $\beta\colon G \to G$ is the identity homomorphism, by uniqueness of the localization $H \subseteq G$. □

4. Universal central extensions and localizations

We next extend results from Section 1 of [32] to non-abelian simple groups of arbitrary size.

Corollary 4.7 below was stated in [32] only for finite groups and was used to find many examples of non-simple localizations of finite non-abelian simple groups, thus showing that simplicity is not preserved under localization functors in general. For infinite groups this corollary could be useful to realize any abelian group as the center of some simple localization (see Corollary 4.7 and Question 5.11).

Let G be a non-abelian simple group and denote by $\mathrm{Mult}(G) = H_2(G;\mathbb{Z})$ its Schur multiplier. Consider

$$0 \longrightarrow \mathrm{Mult}(G) \hookrightarrow \widetilde{G} \twoheadrightarrow G \longrightarrow 1$$

the universal central extension of G. Recall that the group $\mathrm{Mult}(G)$ coincides with the center $\mathfrak{z}\widetilde{G}$ of $\widetilde{G}$.

If G is a non-abelian simple group, then any proper normal subgroup of $\widetilde{G}$ is contained in $\mathrm{Mult}(G)$. In particular, any non-trivial endomorphism of $\widetilde{G}$ is either a monomorphism, or its kernel is contained in $\mathrm{Mult}(G)$.

Proposition 4.1. *Let G be any group and let $H \subset G$ be a maximal subgroup. Suppose that H is non-abelian simple and G has some element of order p, where p is the index of H in G. If the inclusion $H \hookrightarrow G$ is a localization, then G is simple.*

Proof. For the convenience of the reader we repeat the proof of [32, Proposition 1.1]. Suppose that H is normal in G. Since H is maximal, $G/H \cong C_p$. By hypothesis, there exists an element $x \in G$ of order p, yielding a non-trivial endomorphism $\psi: G \to G/H \cong C_p \hookrightarrow G$. But $H\psi = 0$, hence $\psi = 0$, because $H \hookrightarrow G$ is a localization, which is a contradiction. So H is not normal in G and suppose, by contradiction, that G has a proper normal subgroup N. Then $N \cap H = 1$, otherwise $N \cap H = H$ would imply $N = G$ by the maximality of H and the fact that H is not normal. Therefore, $G/N \cong H$. But then the identity homomorphism $\mathrm{id}: G \to G$ and the projection $G \twoheadrightarrow G/N$ extend an inclusion $H \hookrightarrow G$, which contradicts the uniqueness of localizations. □

The following will be applied in our main Theorem 5.9.

Proposition 4.2. *Let $i: H \hookrightarrow G$ be an inclusion of two non-abelian simple groups, and $j: \widetilde{H} \to \widetilde{G}$ be the induced homomorphism between the universal central extensions. If every homomorphism $\widetilde{H} \to \widetilde{G}$ sends* $\mathrm{Mult}(H)$ *into* $\mathrm{Mult}(G)$*, then i is a localization if and only if j is a localization.*

Proof. As in [32, Proposition 1.2], we show first that $p: \widetilde{G} \twoheadrightarrow G$ and $q: \widetilde{H} \twoheadrightarrow H$ induce a bijection $F: \mathrm{Hom}(\widetilde{H}, \widetilde{G}) \longrightarrow \mathrm{Hom}(H, G)$ characterized by the property that $F(\tilde{\varphi})$ (with $\tilde{\varphi} \in \mathrm{Hom}(\widetilde{H}, \widetilde{G})$) is the unique morphism $\varphi: H \to G$ such that

$\tilde{\varphi}p = q\varphi$. If $\tilde{\varphi}: \widetilde{G} \to \widetilde{H}$ is any homomorphism, then by hypothesis $(\mathrm{Mult}(H))\tilde{\varphi} \subseteq \mathrm{Mult}(G)$, hence it induces a unique homomorphism $\varphi: H \to G$ such that $\tilde{\varphi}p = q\varphi$. To show that F is surjective, we use that the universal central extension is a cellular cover (i.e. a co-localization functor), hence any homomorphism $\varphi: H \to G$ induces a unique homomorphism $\tilde{\varphi}: \widetilde{H} \to \widetilde{G}$ such that $\tilde{\varphi}p = q\varphi$. □

The next results are parallel to [32, Corollary 1.7, Proposition 1.4].

Corollary 4.3. *Let $i: H \hookrightarrow G$ be an inclusion of two non-abelian simple groups. Moreover, let H be superperfect, $j: H = \widetilde{H} \hookrightarrow G$ be the induced homomorphism on the universal central extensions. Then $i: H \hookrightarrow G$ is a localization if and only if $j: H \hookrightarrow \widetilde{G}$ is a localization.*

The universal property of the universal central extension yields the following well-known fact:

Corollary 4.4. *If G is a non-abelian simple group and $p: \widetilde{G} \twoheadrightarrow G$ is its universal central extension, then* $\mathrm{Aut}(\widetilde{G}) \cong \mathrm{Aut}\, G$.

Proposition 4.5. *Let G be a non-abelian simple group with non-trivial Schur multiplier. Then, $p: \widetilde{G} \twoheadrightarrow G$ is a localization if and only if there are no monomorphisms $\widetilde{G}/N \hookrightarrow G$ for any normal subgroup $N \subsetneq \mathrm{Mult}(G)$ of $\widetilde{G}$.*

Proof. Suppose that $\widetilde{G} \twoheadrightarrow G$ is a localization and suppose that there exists a monomorphism $\varphi: \widetilde{G}/N \hookrightarrow G$ for some $N \neq \mathrm{Mult}(G)$. By the localization property there is some endomorphism $\varphi': G \to G$ such that $p\varphi' = q\varphi$, where $q: \widetilde{G} \twoheadrightarrow \widetilde{G}/N$ is the canonical projection. But then, for $x \in \mathrm{Mult}(G) \setminus N$, we get $1 \neq xq\varphi = xp\varphi' = 1$, which is a contradiction.

Conversely, let $\varphi: \widetilde{G} \to G$ be any homomorphism. If $\varphi = 0$, then obviously φ extends to $\varphi' = 0$ on G. If $\varphi \neq 0$, then $\mathrm{Ker}\,\varphi \subseteq \mathrm{Mult}(G)$, and by hypothesis $\mathrm{Ker}\,\varphi = \mathrm{Mult}(G)$ and hence φ factors through a homomorphism $\varphi': G \to G$. Uniqueness of φ' is guaranteed in any case because $p: \widetilde{G} \twoheadrightarrow G$ is an epimorphism.

Finally [32, Theorem 1.5] extends to infinite simple groups. □

Theorem 4.6. *Let $i: H \hookrightarrow G$ be an inclusion of two non-abelian simple groups and $j: \widetilde{H} \to \widetilde{G}$ be the induced homomorphism on the universal central extensions. Assume that G does not contain any non-split central extension of H as a subgroup. Then $i: H \hookrightarrow G$ is a localization if and only if $j: \widetilde{H} \to \widetilde{G}$ is a localization.*

Proof. Let $\varphi: \widetilde{H} \to \widetilde{G}$, then $\mathrm{Mult}(\widetilde{H})/\mathrm{Ker}\,\varphi p \to \widetilde{H}\varphi p \twoheadrightarrow H$ is a non-split central extension of H such that $\widetilde{H}\varphi p \subseteq G$. The hypothesis on G implies $\widetilde{H}\varphi p = H$, hence $\mathrm{Ker}\,\varphi p = \mathrm{Mult}(H)$ and $\mathrm{Mult}(H)\varphi \subseteq \mathrm{Mult}(G)$. We can then apply Proposition 4.2 and the theorem follows. □

As in [32], if we assume that H has no non-trivial central extensions then the following holds.

Corollary 4.7. *Let $i\colon H \hookrightarrow G$ be an inclusion of two non-abelian simple groups and assume that H is superperfect* (*i.e.,* $\mathrm{Mult}(H) = 1$). *Let also $j\colon H = \widetilde{H} \hookrightarrow \widetilde{G}$ denote the induced homomorphism on the universal central extensions. Then $i\colon H \hookrightarrow G$ is a localization if and only if $j\colon H \hookrightarrow \widetilde{G}$ is a localization.*

5. Centers and localizations of quasi-simple groups

Using ideas from Ol′shanskii [25] in [23, Theorem B] (or [24]) Obraztsov showed the existence of simple complete groups with a very special lattice of subgroups. We will say that a group G satisfies the *Ol′shanskii's subgroup property,* (with respect to some family $\{G_i \mid i \in I\}$), if every proper subgroup of G is either cyclic or contained in a conjugate of some G_i. These strong results depend heavily on the assumption that G has no involutions. For finite groups this is equivalent to saying that G has odd order and thus must be solvable by the Feit–Thompson Theorem [12], so the simple groups considered in this context must be infinite. We start with two important definitions (see [25], Section 35):

Definition 5.1. Given a family $\{G_i \mid i \in I\}$ of groups, the *free amalgam* of the groups G_i is the set $\cup_{i \in I} G_i$, with $G_i \cap G_j = \{1\}$ for every $i \neq j$.

Observe that no group structure is imposed, a priori, in a free amalgam.

Definition 5.2. Given a group G, a family $\{G_i \mid i \in I\}$ and the corresponding free amalgam $\mathcal{A}$, a map $\mathcal{A} \to G$ is an *embedding* if it is injective and the restriction to every G_i is a group homomorphism.

In the sequel we will build on the mentioned results of Ol′shanskii and Obraztsov in order to generalize this notion of embedding to free products. With the notation of the previous definition, consider an embedding $f\colon \mathcal{A} \hookrightarrow G$, and the unique map $g\colon \mathcal{A} \to *_{i \in I} G_i$ that restricts to the inclusion $G_i \hookrightarrow *_{\in I} G_i$ for every $i \in I$. By the universal property of the free product of groups, there is a unique homomorphism $h\colon *_{\in I} G_i \to G$ such that $gh = f$. This fact motivates the following definition:

Definition 5.3. Let $F = *_{i \in I} G_i$ be the free product of the given family of groups. We call a homomorphism $\sigma\colon F \to G$ a $*$*-embedding* if $\sigma \restriction G_i\colon G_i \to G$ is a monomorphism for each $i \in I$ and $G_i\sigma \cap G_j\sigma = 1$ for all $i \neq j \in I$. This is equivalent to saying that

$$\sigma \restriction \big(\dot{\bigcup}_{i \in I} (G_i \setminus \{1\}) \cup \{1\}\big) \text{ is an injection.}$$

By the previous observation, every embedding gives rise to a unique $*$-embedding. Observe however that a $*$-embedding is not necessarily injective, and in fact they can have very large kernels.

Before we present our first result concerning $*$-embeddings, we need one more definition. Recall that a group G is *aspherical* (or *diagrammatically aspherical*) if there exists no reduced spherical diagram over G consisting only of 0-cells; a thorough introduction to the theory of diagrams is Chapter 4 of [25].

Let us introduce now our first embedding result:

Theorem 5.4. *Let $\{G_i \mid i \in I\}$, $|I| \geq 2$, be a countable family of countable groups without involutions, of exponent $m > 10^{75}$ or $m = \infty$. Then there is a $*$-embedding $F = \star_{i \in I} G_i \to G$ such that the following holds*:

(1) *G is simple, complete, aspherical and has no involutions*;

(2) *G satisfies Ol′shanskii's subgroup property for $\{G_i \mid i \in I\}$*;

(3) *the Schur multiplier* $\mathrm{Mult}(G)$ *is a free abelian group of infinite* (*countable*) *rank*;

(4) *if $m = \infty$, then G is torsion-free and if $m < \infty$, then G has exponent m.*

Proof. Let $\mathcal{A}$ be the free amalgam associated to the family $\{G_i \mid i \in I\}$, $|I| \geq 2$. According to above, to find a $*$-embedding $F = \star_{i \in I} G_i \to G$ it is enough to find an embedding $\mathcal{A} = \star_{i \in I} G_i \to G$ and then to extend it by linearity.

Now items (1), (2), and (4) are a direct consequence of [23, Theorem B] (see also [25, p. 386, Theorem 35.1]). For (3) note that the group G of the theorem is aspherical, and hence it has free abelian Schur multiplier of rank the size of the group G, see [25, p. 334, Theorem 31.1]. □

We derive the following immediate result.

Corollary 5.5. *Assuming the statement of Theorem 5.4, with $\{G_i \mid i \in I\}$ for $|I| \geq 2$, such that there are no embeddings $G \hookrightarrow G_i$ for any $i \in I$, then G is co-hopfian. The additional hypothesis is satisfied if G_i is hyperabelian or co-hopfian for $i \in I$.*

Proof. Suppose by contradiction that $\varphi\colon G \hookrightarrow G$ is a proper monomorphism, then $G\varphi \subsetneq G$, and by Ol′shanskii's subgroup property for $\{G_i \mid i \in I\}$ there are $i \in I$ and $g \in G$ such that $G\varphi \subseteq G_i^g$. Hence $\varphi(g^{-1})^*$ is an embedding of G into G_i, which contradicts our assumption. Hence G must be co-hopfian.

Assuming the additional hypothesis, we must show that an embedding $\psi\colon G \hookrightarrow G_i$ does not exist. If G_i is hyperabelian, then also G must be hyperabelian and cannot be simple, as shown in Theorem 5.4(1). Otherwise G_i must be co-hopfian and $G_i \subset G$ together with ψ leads to a proper embedding $G_i \subset G \hookrightarrow G\psi \subseteq G_i$, so G_i cannot be co-hopfian. Hence ψ is not possible. □

We will also need a version of the above for $|I| \leq 1$. If $I = \emptyset$, we will replace I by $I' = \{0, 1\}$ and let $G_0 = G_1 = \mathbb{Z}_m$ (with m as in the theorem and $\mathbb{Z}_m = \mathbb{Z}$, if $m = \infty$). If $I = \{0\}$, then choose $G_0 = H$ co-hopfian and $G_1 = \mathbb{Z}_m$. Thus we can apply Theorem 5.4 and Corollary 5.5 for I' (because $\mathbb{Z}_m$ is hyperabelian and H is co-hopfian) and get for any I (also for $|I| = 0$ and $|I| = 1$) the following immediate

Corollary 5.6. *Let $H = 1$ or H be a countable co-hopfian group without involutions of any exponent $m > 10^{75}$ or $m = \infty$. Then there is a group G such that the following holds*:

1. *$H \subseteq G$, with G countable, simple, complete, aspherical and without involutions*;
2. *if $H \neq 1$, then G satisfies Ol′shanskii's subgroup property for H. If $H = 1$, then G satisfies Ol′shanskii's subgroup property for the empty set of groups*;
3. *the Schur multiplier* $\mathrm{Mult}(G)$ *is a free abelian group of infinite* (*countable*) *rank*;
4. *if $m = \infty$, then G is torsion-free, and if $m < \infty$, then G has exponent m*;
5. *G is co-hopfian.*

Theorem 5.7. *Let H be any simple, co-hopfian and complete group, which embeds properly into a simple, complete group G satisfying Ol′shanskii's subgroup property with respect to H. Then G is co-hopfian, and $H \subseteq G$ is a localization.*

Proof. The group G is co-hopfian by Corollary 5.5. We can now use Corollary 2.4 to show that $H \subseteq G$ is a localization.

Let H' be any subgroup of G isomorphic to H. If $G \cong H$, then $H \subset G \cong H$ contradicts co-hopfian. Hence $G \not\cong H'$ and $H' \subsetneq G$ is not cyclic. By Ol′shanskii's subgroup property there is $g \in G$ such that $H' \subseteq H^g$. Hence there is the embedding $\varphi(g^{-1})^*$ of $H' \cong H$ into H, which forces $H' = H^g$, because H is co-hopfian. Thus (1) of Corollary 2.4 follows.

Suppose that the centralizer $\mathfrak{c}_G H \neq 1$. If $H \mathfrak{c}_G H = G$, then H is normal in G, but G is simple, which is impossible. Thus $H \mathfrak{c}_G H \neq G$ is not cyclic, and by Ol′shanskii's subgroup property, $H \mathfrak{c}_G H \subseteq H^g$ for some $g \in G$. Since H has trivial center, it follows from $\mathfrak{c}_G H \neq 1$ that $H \subsetneq H \mathfrak{c}_G H$ leads to a proper embedding of H into itself (induced by g), which contradicts our assumption that H is co-hopfian. Thus $\mathfrak{c}_G H = 1$, part (2) of Corollary 2.4 holds, and the corollary can be applied. Thus Theorem 5.7 follows. □

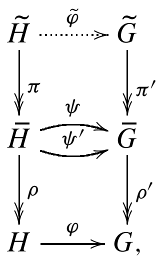
In the main theorem we shall need the following basic fact on central extensions. (Note that this holds for arbitrary cellular covers, where π is assumed to be only an epimorphism.)

Lemma 5.8. *Suppose that* $\widetilde{H} \overset{\pi}{\twoheadrightarrow} \overline{H} \overset{\rho}{\twoheadrightarrow} H$ *and* $\widetilde{G} \overset{\pi'}{\twoheadrightarrow} \overline{G} \overset{\rho'}{\twoheadrightarrow} G$ *are sequences of central extensions, where* $\widetilde{H}$ *and* $\widetilde{G}$ *are the universal central extensions. Let* $\psi, \psi' \colon \overline{H} \to \overline{G}$ *be two homomorphisms.*

(*a*) *If* $\pi\psi = \tilde{\varphi}\pi' = \pi\psi'$ *for some* $\tilde{\varphi}\colon \widetilde{H} \to \widetilde{G}$, *then* $\psi = \psi'$.

(*b*) *If* ψ *and* ψ' *extend* $\varphi\colon H \to G$, *then* $\psi = \psi'$.

Proof. Part (a) follows, since π is epimorphism. For (b) let $\tilde{\varphi}\colon \widetilde{H} \to \widetilde{G}$ be the unique extension of φ. Then also $\tilde{\varphi}$ is the extension of both ψ and ψ', so the upper square in the following diagram commutes:



Hence (a) applies, and we get $\psi = \psi'$. □

The next main result was already discussed in the introduction.

Theorem 5.9. *Let* F *be a free abelian group of countable rank and* A *be any epimorphic image. Moreover, let* H *be a countable, complete, co-hopfian simple group without involutions, and suppose that* $m > 10^{75}$ *or* $m = \infty$ *is such that all proper subgroups of* H *are cyclic of an order dividing* m *if* $m < \infty$, *and of infinite order otherwise.*

(a) *Then there exists a localization* $H \subset G$, *with* G *complete, simple, co-hopfian and Schur multiplier* $\mathrm{Mult}(G) \cong F$.

(b) *This localization lifts to a localization* $\widetilde{H} \to \widetilde{G}$ *between the corresponding universal central extensions, where* $\mathfrak{z}(\widetilde{G}) \cong F$.

(c) *This localization lifts to another localization* $\overline{H} \subset \overline{G}$, *between the corresponding central extensions, where* $\mathfrak{z}\overline{G} \cong A$.

By Theorem 5.4 there is an inclusion $H \subset G$ into a complete simple group G of exponent m with $\mathrm{Mult}(G)$ isomorphic to F. By Theorem 5.7 the group G is co-hopfian, and this embedding is a localization, hence (a) holds.

Part (b) follows directly from Theorem 4.6. Suppose that G contains a non-split central extension $\overline{H}$ of H. Then $\overline{H}$ would be a non-cyclic proper subgroup of G, which by the Ol′shanskii's subgroup property satisfies $\overline{H} \subseteq H^g \cong H$ for some $g \in G$. Since $\overline{H}$ is not simple, it must be a proper subgroup of H, hence cyclic by the assumption on H, which is a contradiction. Therefore, the additional hypothesis of Theorem 4.6 holds, and we get that $\widetilde{H} \to \widetilde{G}$ is a localization.

For part (c) we first define $\overline{G}$ and $\overline{H}$. The epimorphism $\eta\colon F = \operatorname{Mult}(G) \twoheadrightarrow A$ induces a push-out diagram of central extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & \overline{G} & \xrightarrow{p} & G & \longrightarrow & 1. \end{array}$$

First note that $\pi^{-1}\mathfrak{z}\overline{G}$ is a normal subgroup of $\widetilde{G}$ which contains F and also $\pi^{-1}\mathfrak{z}\overline{G} \subseteq \operatorname{Mult}(G) = F$, hence $\mathfrak{z}\overline{G} = F\pi = A$.

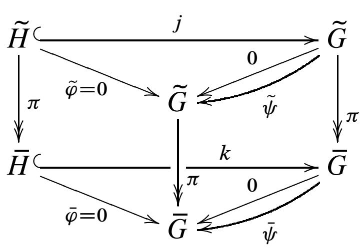
Let $\overline{H} := (\widetilde{H})j\pi$, then we have a diagram of central extensions

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{j} & \widetilde{G} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \overline{H} & \xhookrightarrow{k} & \overline{G} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ H & \xhookrightarrow{i} & G, \end{array}$$

where $\widetilde{H} \twoheadrightarrow \overline{H}$ is the universal central extension, and $k\colon \overline{H} \subset \overline{G}$ is the inclusion extending $i\colon H \subset G$. Note that k is a strict inclusion $\overline{H} \subsetneq \overline{G}$, otherwise we would have $\overline{H} \cong \overline{G}$, and thus $H \cong \overline{H}/\mathfrak{z}(\overline{H}) \cong \overline{G}/\mathfrak{z}(\overline{G}) \cong G$. This would yield a strict monomorphism $H \hookrightarrow G \cong H$, contradicting that H is co-hopfian.

For (c) it remains to show that $\overline{H} \subset \overline{G}$ is the desired localization. Thus let $\bar{\varphi}\colon \overline{H} \to \overline{G}$ be any homomorphism.

If $\bar{\varphi} = 0$, then $0 \in \operatorname{End}\overline{G}$ extends $\bar{\varphi}$, and also $\tilde{\varphi} = 0\colon \widetilde{H} \to \widetilde{G}$ extends uniquely $\bar{\varphi} = 0$. If $\bar{\psi}\colon \overline{G} \to \overline{G}$ is a homomorphism which extends $\bar{\varphi} = 0$, and $\tilde{\psi}\colon \widetilde{G} \to \widetilde{G}$ is the unique extension of $\bar{\psi}$, then $j\tilde{\psi} = \tilde{\varphi} = 0$ by the uniqueness property of universal central extensions.



Then by the uniqueness property for the localization j we have $\tilde{\psi} = 0$, and therefore $\pi\bar{\psi} = \pi 0$. Since π is an epimorphism we obtain $\bar{\psi} = 0$ as desired (see Lemma 5.8 (a)).

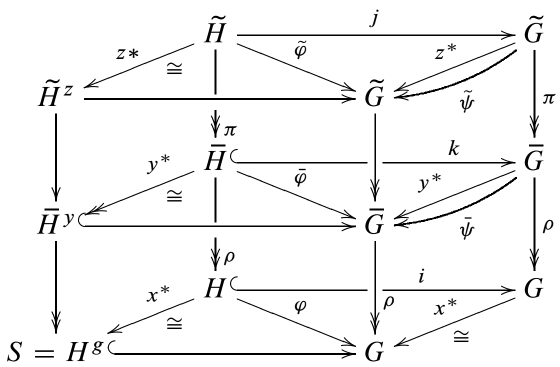
Suppose finally that $\bar{\varphi} \neq 0$. Hence $\operatorname{Ker}\bar{\varphi} \subseteq \mathfrak{z}(\bar{H})$, because $\bar{H}/\mathfrak{z}(\bar{H}) \cong H$ is simple. Then $\operatorname{Ker}\bar{\varphi}\rho \neq \bar{H}$, otherwise $\bar{\varphi}\rho = 0\rho$ and this would imply that both $\bar{\varphi}$ and 0 extend the trivial map $0\colon H \to G$, and then $\bar{\varphi} = 0$, by Lemma 5.8 (b). We have that $\operatorname{Ker}\bar{\varphi}\rho \subseteq \mathfrak{z}(\bar{H})$, because H is simple.

Let $S = \bar{H}\bar{\varphi}\rho \cong \bar{H}/\operatorname{Ker}\bar{\varphi}\rho \neq 0$ and consider the canonical epimorphism $S \twoheadrightarrow H$, induced by the inclusion $\operatorname{Ker}\bar{\varphi}\rho \subseteq \mathfrak{z}(\bar{H})$. In particular, S cannot be cyclic. By Ol′shanskii's subgroup property it follows that either $S = G$ or $S \subseteq H^g \subset G$ for some $g \in G$.

In the first case we have $S = G$ and thus an epimorphism $G = S \twoheadrightarrow H$, which must be an isomorphism, as G is simple. This yields a proper inclusion $H \subsetneq G \cong H$ of H, which contradicts the assumption that H is co-hopfian.

In the second case we have a monomorphism $S \subseteq H^g \cong H$ and the epimorphism $S \twoheadrightarrow H$, as before. But, by hypothesis, proper subgroups of H are cyclic, thus $S = H^g$.

If $\alpha\colon H \to S$ is the given isomorphism, then $H\alpha = H^g$ and $\alpha(g^{-1})^* \in \operatorname{Aut} H$. Since H is complete ($\operatorname{Aut} H = H^*$), we find $h \in H$ such that $\alpha(g^{-1})^* = h^*$, hence $\alpha = (hg)^*$. Denote $x = hg$ and choose preimages $y \in \bar{H}$, and $z \in \widetilde{H}$ such that $y\rho = x$ and $z\pi = y$.



Now we can apply Lemma 5.8(b) to infer that $\bar{\varphi} = y^* \restriction \bar{H}$ and then clearly $y^*\colon \bar{G} \to \bar{G}$ extends $\bar{\varphi}$, i.e., $ky^* = \bar{\varphi}$.

We show next that this extension is unique. If $\bar{\psi}\colon \bar{G} \to \bar{G}$ is another homomorphism such that $k\bar{\psi} = \bar{\varphi}$, then $\bar{\psi}$ lifts to a unique $\widetilde{\psi}$ such that $\widetilde{\psi}\pi = \pi\bar{\psi}$. Now $jz^* = \widetilde{\varphi}$ and also $j\widetilde{\psi} = \widetilde{\varphi}$. For the second equality use that $j\widetilde{\psi} = j\pi\bar{\psi} = k\pi\bar{\psi} = \pi\varphi$, and the uniqueness of liftings to universal central extensions. Hence, the uniqueness of extensions of the localization j yields $\widetilde{\psi} = z^*$. Thus we have $\pi\bar{\psi} = \pi y^*$, and since π is an epimorphism, we get $\bar{\psi} = y^*$. Thus $\bar{H} \subset \bar{G}$ is also a localization, as desired.

Assuming that there are such groups H with trivial Schur multiplier, we get the following corollary.

Corollary 5.10. *Let A be any countable abelian group and let H be any countable, non-abelian simple group without involutions of exponent $m > 10^{75}$ or $m = \infty$. Assume that H is co-hopfian, complete, with Schur multiplier* $\mathrm{Mult}(H) = 1$. *Then there exists a localization $H \subset \overline{G}$ with $\mathfrak{z}\overline{G} = A$, and G is a complete, simple and co-hopfian group of exponent m.*

Proof. From Corollary 5.6 we obtain an inclusion $H \subset G$ which is a localization by Theorem 5.7 (or Theorem 5.9(b)). Corollary 4.3 says that $H \subset \widetilde{G}$ is also a localization. By the proof of part (c) of Theorem 5.9 (when $\widetilde{H} = \overline{H} = H$), we notice that it is not necessary to assume that all proper subgroups of H are cyclic of exponent m. Indeed, following the argument of the proof, the epimorphism $S \twoheadrightarrow H$ is an isomorphism, and S cannot be a proper subgroup of H^g, because otherwise we would have a strict monomorphism $H \cong S \subsetneq H^g \cong H$, which would contradict that H is co-hopfian. The proof can then continue without this additional assumption. Hence we get a localization $H = \overline{H} \subset \overline{G}$, with center $\mathfrak{z}\overline{G} = A$, satisfying the desired properties. □

In view of the last corollary it would be interesting to answer the following question.

Problem 5.11. Is there a simple, complete co-hopfian group H without involutions and trivial Schur multiplier?

Remark 5.12. In Theorem 5.9 we can replace the property that all proper subgroups of H are cyclic by the more general property that H does not contain any non-split central extension of itself as a proper subgroup.

According to Proposition 4.5 the simple groups satisfying this property are exactly those groups H whose universal central extension $\widetilde{H} \twoheadrightarrow H$ is a localization.

In the theory of the localization of groups (and other categories) we sometimes find morphisms that are localizations and cellular covers (or co-localizations) at the same time; see e.g., [11] and [16].

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