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Comparison of singular numbers of composition operators on different Hilbert spaces of analytic functions

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ABSTRACT

We compare the rate of decay of singular numbers of a given composition operator acting on various Hilbert spaces of analytic functions on the unit disk $\mathbb{D}$. We show that for the Hardy and Bergman spaces, our results are sharp. We also give lower and upper estimates of the singular numbers of the composition operator with symbol the "cusp map" and the lens maps, acting on weighted Dirichlet spaces.

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1. Introduction

Composition operators are mainly studied on Hilbert spaces of analytic functions, and more specifically on the Hardy space H^2, the Bergman space $\mathfrak{B}^2$, and the Dirichlet space $\mathcal{D} = \mathcal{D}^2$. It is well known, thanks to the Littlewood subordination principle, that every analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator C_φ on H^2 and on $\mathfrak{B}^2$, but not necessarily on $\mathcal{D}^2$ ([36, Chapter 1 and Exercises]; see also [11, Section 6.2]). There exist even composition operators which are not bounded on $\mathcal{D}^2$ but which are in all Schatten classes $S_p(H^2)$ and $S_p(\mathfrak{B}^2)$ with $p > 0$, of both the Hardy space and the Bergman space ([24, Theorem 2.10]). Nevertheless, for compact composition operators, the following results hold: 1) every composition operator which is compact on H^2 is compact on $\mathfrak{B}^2$ (see [36, Theorem 3.5] and [32, Theorem 3.5]); 2) every composition operator that is compact on $\mathcal{D}^2$ is in all the Schatten classes $S_p(H^2)$ for $p > 0$ ([24, Theorem 2.9]); for every $p > 0$, every composition operator that is in $S_p(H^2)$ is in $S_p(\mathfrak{B}^2)$. Since the membership in a Schatten class S_p of an operator on a Hilbert space means that its approximation numbers are ℓ_p-summable, that suggests that there is a strong link between the approximation numbers $a_n^{\mathcal{D}^2}(C_\varphi)$, $a_n^{H^2}(C_\varphi)$ and $a_n^{\mathfrak{B}^2}(C_\varphi)$ of the composition operator C_φ on $\mathcal{D}^2$, H^2 and $\mathfrak{B}^2$ respectively.

The aim of this paper is to prove that, indeed, in some sense $a_n^{\mathcal{D}^2}(C_\varphi)$ is "greater" than $a_n^{H^2}(C_\varphi)$, which is "greater" than $a_n^{\mathfrak{B}^2}(C_\varphi)$. We recover then that $C_\varphi \in S_p(H^2)$ implies that $C_\varphi \in S_p(\mathfrak{B}^2)$ (Section 3). In Section 3.6, we also give some results about conditional multipliers.

In Section 4 we give an example with C_φ compact on H^2 but not in any Schatten class $S_p(\mathfrak{B}^2)$ for $p < \infty$. We prove that $C_\varphi \in S_p(H^2)$ implies that $C_\varphi \in S_{p/2}(\mathfrak{B}^2)$ and give an example with $C_\varphi \in S_p(H^2)$ but $C_\varphi \notin S_q(\mathfrak{B}^2)$ for any $q < p/2$.

However, our result is not sufficient to explain why the compactness of C_φ on $\mathcal{D}^2$ implies that $C_\varphi \in S_p(H^2)$ for all $p > 0$. A more subtle relationship should exist between $a_n^{\mathcal{D}^2}(C_\varphi)$ and $a_n^{H^2}(C_\varphi)$. In fact, for every composition operator C_φ that is compact on $\mathcal{D}^2$, we have $\lim_{n\to\infty} \big[a_n^{\mathcal{D}^2}(C_\varphi)\big]^{1/n} = \lim_{n\to\infty} \big[a_n^{H^2}(C_\varphi)\big]^{1/n}$ ([28, Theorem 3.1 and Theorem 3.14]); in particular, for symbols φ such that $\|\varphi\|_\infty < 1$, the numbers $a_n^{\mathcal{D}^2}(C_\varphi)$ and $a_n^{H^2}(C_\varphi)$ behave like r^n, with $r = \exp(-1/\mathrm{Cap}[\varphi(\mathbb{D})])$, and where $\mathrm{Cap}[\varphi(\mathbb{D})]$ is the Green capacity of $\varphi(\mathbb{D})$. On the other hand, for the so-called cusp map χ, we have, for some constants $c_1 > c_1' > 0$ ([27, Theorem 4.3]):

$$\mathrm{e}^{-c_1 n/\log n} \lesssim a_n^{H^2}(C_\chi) \lesssim \mathrm{e}^{-c_1' n/\log n} \tag{1.1}$$

and, for some constants $c_2 > c_2' > 0$ ([25, Theorem 3.1]):

$$\mathrm{e}^{-c_2\sqrt{n}} \lesssim a_n^{\mathcal{D}^2}(C_\chi) \lesssim \mathrm{e}^{-c_2'\sqrt{n}}\,, \tag{1.2}$$

which is much greater. In Section 5.2, we show that the behavior of $a_n(C_\chi)$ in (1.1) holds in all weighted Dirichlet spaces $\mathcal{D}_\alpha^2$ for $\alpha > 0$ (with other constants), and hence (1.2) shows that a jump happens for $\alpha = 0$. We also look at the lens maps.

2. Notation and background

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. We denote $dA = dx\,dy/\pi$ the normalized area measure on $\mathbb{D}$. The normalized Lebesgue measure $dt/2\pi$ on $\mathbb{T} = \partial\mathbb{D}$ is denoted dm.

2.1. Hilbert spaces of analytic functions

Recall that the Hardy space H^2 is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 \, dm(\xi) < \infty \, .$$

If $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have $\|f\|_{H^2}^2 = \sum_{k=0}^{\infty} |c_k|^2$.

The Bergman space $\mathfrak{B}^2$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}^2}^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \, .$$

If $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have $\|f\|_{\mathfrak{B}^2}^2 = \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1}$.

More generally, for $\gamma > -1$, the *weighted Bergman space* $\mathfrak{B}_\gamma^2$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}_\gamma^2}^2 = (\gamma + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\gamma \, dA(z) < \infty \, ,$$

and if $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have:

$$\|f\|_{\mathfrak{B}_\gamma^2}^2 = \sum_{k=0}^{\infty} \beta_k |c_k|^2 \, ,$$

with:

$$\beta_k = \frac{k!\,\Gamma(\gamma+2)}{\Gamma(k+\gamma+2)} \approx \frac{1}{(k+1)^{\gamma+1}}$$

(the equivalence depends on γ).

Hence $\mathfrak{B}^2 = \mathfrak{B}_0^2$ and H^2 corresponds to the degenerate case $\gamma = -1$.

The Dirichlet space $\mathcal{D}^2$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathcal{D}^2}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty \, .$$

If $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have $\|f\|^2_{\mathcal{D}^2} = |c_0|^2 + \sum_{k=0}^{\infty} k\,|c_k|^2$.

With the equivalent norm $|||f|||^2_{\mathcal{D}^2} = \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2\, dA(z)$, we have the more pleasant form $|||f|||^2_{\mathcal{D}^2} = \sum_{k=0}^{\infty} (k+1)\,|c_k|^2$.

More generally, for $\alpha > -1$, the *weighted Dirichlet space* $\mathcal{D}^2_\alpha$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|^2_{\mathcal{D}^2_\alpha} = |f(0)|^2 + (\alpha+1) \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \, dA(z) < \infty \,,$$

and if $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have:

$$\|f\|^2_{\mathcal{D}^2_\alpha} = \sum_{k=0}^{\infty} \beta_k |c_k|^2 \,,$$

with $\beta_0 = 1$ and for $k \geq 1$:

$$\beta_k = \frac{k\,.\,k!\,\Gamma(\alpha+2)}{\Gamma(k+\alpha+1)} \approx \frac{1}{(k+1)^{\alpha-1}}$$

(the equivalence depending on α). Another equivalent expression is:

$$\widetilde{\beta}_k = \frac{(k+1)!\,\Gamma(\alpha+2)}{\Gamma(k+\alpha+1)} \,;$$

we have $\beta_k \leq \widetilde{\beta}_k \leq 2\beta_k$.

In particular, for $\gamma > -1$:

$$\mathcal{D}^2 = \mathcal{D}^2_0 \,, \quad H^2 = \mathcal{D}^2_1 \quad \text{and} \quad \mathfrak{B}^2_\gamma = \mathcal{D}^2_{\gamma+2} \,. \tag{2.1}$$

2.2. Composition operators

Any analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ defines a bounded composition operator C_φ on the Hardy space H^2 (see [32, Section 2.2]) and on every weighted Bergman space $\mathfrak{B}^2_\gamma$ for $\gamma > -1$ ([32, Proposition 3.4]), hence on every weighted Dirichlet space $\mathcal{D}^2_\alpha$ with $\alpha \geq 1$. However, this is not always the case on the weighted Dirichlet spaces $\mathcal{D}^2_\alpha$ for $\alpha < 1$ ([32, Proposition 3.12]).

For convenience, we assume that φ is not constant and we say that φ is a *symbol*. We denote φ^* the boundary values function of φ.

The Carleson window of size h centered at $\xi \in \mathbb{T}$ is:

$$W(\xi, h) = \{z \in \mathbb{D}\,;\ |z| \geq 1-h \quad \text{and} \quad -\pi h \leq \arg(\bar{\xi} z) < \pi h\} \,. \tag{2.2}$$

For every integer $n \geq 1$ and for $j = 0, \ldots, 2^n - 1$, we set:

$$W_{n,j} = W(\mathrm{e}^{2ji\pi/2^n}, 2^{-n}) \, . \tag{2.3}$$

We also use the Carleson boxes

$$S(\xi, h) = \{z \in \mathbb{D} \, ; \ |\xi - z| < h\} \, , \tag{2.4}$$

which satisfy $S(\xi, h) \subseteq W(\xi, h) \subseteq S(\xi, 2\pi h)$.

The Hastings-Luecking boxes are defined, for every integer $n \geq 1$ and for $0 \leq j \leq 2^n - 1$, as:

$$R_{n,j} = \Big\{ z \in \mathbb{D} \, ; \ 1 - \frac{1}{2^{n-1}} \leq |z| < 1 - \frac{1}{2^n} \text{ and } \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \Big\} \tag{2.5}$$

A measure μ on $\overline{\mathbb{D}}$ is a *Carleson measure* if $\sup_{\xi \in \mathbb{T}} \mu\big[\overline{W(\xi, h)}\big] = \mathrm{O}\,(h)$. By the Carleson embedding theorem, μ is a Carleson measure if and only if the inclusion map $J_\mu \colon H^2 \to L^2(\mu)$ is bounded. The automatic boundedness of C_φ on H^2 implies that the pull-back measure m_φ, defined as $m_\varphi(B) = m[{\varphi^*}^{-1}(B)]$ for all Borel sets $B \subseteq \overline{\mathbb{D}}$, is a Carleson measure. This composition operator is compact on H^2 if and only if m_φ is supported by $\mathbb{D}$ and $\sup_{\xi \in \mathbb{T}} m_\varphi[W(\xi, h)] = \mathrm{o}\,(h)$ ([31]).

Similar results hold for composition operators on the weighted Bergman spaces ([32, Theorem 4.3]).

2.3. Singular numbers, approximation numbers and Schatten classes

Let H be a separable complex Hilbert space and $T \colon H \to H$ be a compact operator. There exist two orthonormal sequences (u_n) and (v_n) and a non-increasing sequence (s_n) of non-negative numbers with $s_n \underset{n \to \infty}{\longrightarrow} 0$ such that, for all $x \in H$:

$$T(x) = \sum_{n=1}^{\infty} s_n \, \langle x \mid v_n \rangle \, u_n \, . \tag{2.6}$$

This representation $T = \sum_{n=1}^{\infty} s_n \, u_n \otimes v_n$ is called the Schmidt decomposition of T and the numbers $s_n = s_n(T)$ the *singular numbers* of T. They are actually the eigenvalues of $|T| = \sqrt{T^* T}$ rearranged in non-increasing order. In particular $s_1 = \|T\|$.

These numbers have the important "ideal property":

$$s_n(ATB) \leq \|A\| \, s_n(T) \, \|B\| \, .$$

The nth *approximation number* of T defined as:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\| \, . \tag{2.7}$$

It is known that, for all $n \geq 1$, we have $s_n(T) = a_n(T)$ (see [5, p. 155]).

For $p > 0$, the Schatten class $S_p(H)$ is the set of all compact operators $T\colon H \to H$ for which $\|T\|_p^p := \sum_{n=1}^{\infty} [a_n(T)]^p < \infty$.

We have $S_q(H) \subseteq S_p(H)$ for $0 < q \leq p$.

For composition operators C_φ, D. Luecking ([29, Corollary 2]) characterized their membership in the Schatten classes.

For $\gamma > -1$, let $dA_\gamma(z) = (\gamma+1)(1-|z|^2)^\gamma\, dA(z)$. For $\gamma = -1$, we set $H^2 = \mathfrak{B}^2_{-1}$ and $dm = dA_{-1}$. Then, for $\gamma \geq -1$, the composition operator C_φ belongs to $S_p(\mathfrak{B}^2_\gamma)$ if and only if:

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \big(2^{n(\gamma+2)} A_{\gamma,\varphi}(R_{n,j})\big)^{p/2} < \infty\,, \tag{2.8}$$

where $A_{\gamma,\varphi}$ is the pull-back measure of A_γ by φ (by φ^* for $\gamma = -1$).

As usual, the notation $A \lesssim B$ means that $A \leq C\,B$ for some positive constant C, which may depend on some parameters, and $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

3. Comparison of approximation numbers

3.1. Main result

In the introduction, we said that, in some sense, $a_n^{\mathcal{D}^2}(C_\varphi)$ is "greater" than $a_n^{H^2}(C_\varphi)$, which is "greater" than $a_n^{\mathfrak{B}^2}(C_\varphi)$. This vague statement is made more precise in the following result.

Theorem 3.1. *For any symbol φ, we have, for every $n \geq 1$:*

$$\prod_{j=1}^{n} a_j^{\mathfrak{B}^2}(C_\varphi) \leq \prod_{j=1}^{n} a_j^{H^2}(C_\varphi) \leq \prod_{j=1}^{n} a_j^{\mathcal{D}^2}(C_\varphi)\,. \tag{3.1}$$

It is understood that if C_φ is not bounded on $\mathcal{D}^2$, then $a_j^{\mathcal{D}^2}(C_\varphi) = +\infty$.

As a consequence, we recover a previous result ([24, Corollary 3.2]; see also [7, Theorem 2.5]).

Corollary 3.2. *For any symbol φ, we have:*

1) *a) if C_φ is compact on $\mathcal{D}^2$, then C_φ is compact on H^2;*
   b) if C_φ is compact on H^2, then C_φ is compact on $\mathfrak{B}^2$.

Moreover, for every $p > 0$, we have:

2) *a) if $C_\varphi \in S_p(\mathcal{D}^2)$, then $C_\varphi \in S_p(H^2)$;*

b) if $C_\varphi \in S_p(H^2)$, then $C_\varphi \in S_p(\mathfrak{B}^2)$.

Items 1) a) and 2) a) are not sharp, since we proved in [24, Theorem 2.9] that if C_φ is compact on $\mathcal{D}^2$, then C_φ belongs to all Schatten classes $S_p(H^2)$, with $p > 0$. However, we will see in Section 4 that the item 1) b) is sharp, but 2) b) is not.

We will prove these results in a more general setting in Theorem 3.12.

3.2. Subordination of sequences

Let $\mathcal{S}$ be the set of non-increasing sequences $u = (u_j)_{j\geq 1}$ of real numbers. If $u, v \in \mathcal{S}$, the sequence u is said to be *subordinate* to the sequence v, and we write $u \prec v$, if:

$$\sum_{j=1}^{n} u_j \leq \sum_{j=1}^{n} v_j \qquad \text{for all } n \geq 1\,. \tag{3.2}$$

For example, if $u = (1, 1, 0, 0, \ldots)$ and $v = (2, 0, 0, \ldots)$, we have $u \prec v$.

We have a basic stability property of this notion (see [38, Theorem 1.16, p. 13]).

Proposition 3.3. *Let I be an interval of $\mathbb{R}$ and $h\colon I \to \mathbb{R}$ be increasing and convex. Then, if $u, v \in \mathcal{S}$ are sequences of numbers in I, we have:*

$$u \prec v \quad \Longrightarrow \quad h(u) \prec h(v)\,.$$

Proof. We may assume that h is $\mathcal{C}^2$. We fix $n \geq 1$ and set $a = \min\{u_n, v_n\}$. Then, for $x \in I$ and $x > a$:

$$h(x) = h(a) + (x-a)h'(a) + \int_a^{+\infty} (x-t)^+ h''(t)\,dt\,.$$

One easily checks, using (3.2), that $\sum_{j=1}^{n}(u_j - t)^+ \leq \sum_{j=1}^{n}(v_j - t)^+$ for all $t \geq a$. Hence, thanks to the positivity of $h'(a)$ and h'':

$$\begin{aligned}
\sum_{j=1}^{n} h(u_j) &= nh(a) + h'(a)\sum_{j=1}^{n}(u_j - a) + \int_a^{+\infty} \sum_{j=1}^{n}(u_j - t)^+ h''(t)\,dt \\
&\leq nh(a) + h'(a)\sum_{j=1}^{n}(v_j - a) + \int_a^{+\infty} \sum_{j=1}^{n}(v_j - t)^+ h''(t)\,dt \\
&= \sum_{j=1}^{n} h(v_j)\,. \quad \square
\end{aligned}$$

A stronger notion is that of log-subordination.

Definition 3.4. We say that the sequence $u \in \mathcal{S}$ of positive numbers is *log-subordinate* to the sequence $v \in \mathcal{S}$ of positive numbers if $\log u \prec \log v$. In other terms, if:

$$\prod_{j=1}^{n} u_j \leq \prod_{j=1}^{n} v_j \qquad \text{for all } n \geq 1 .$$

The following result will be useful.

Proposition 3.5. *For sequences of positive numbers $u, v \in \mathcal{S}$, the following two conditions are equivalent:*

1) $\log u \prec \log v$*;*
2) $u^p \prec v^p$ *for all* $p > 0$*.*

Proof. If $\log u \prec \log v$, it suffices to apply Proposition 3.3 to the sequences $\log u$ and $\log v$ and to the function $h(x) = \mathrm{e}^{px}$ to get $u^p \prec v^p$.

Conversely, if $u^p \prec v^p$ for all $p > 0$, we have:

$$\Big(\frac{1}{n} \sum_{j=1}^{n} u_j^p \Big)^{1/p} \leq \Big(\frac{1}{n} \sum_{j=1}^{n} v_j^p \Big)^{1/p} ,$$

and letting p going to 0, we get:

$$\Big(\prod_{j=1}^{n} u_j \Big)^{1/n} \leq \Big(\prod_{j=1}^{n} v_j \Big)^{1/n} ,$$

i.e. $\log u \prec \log v$. $\square$

Corollary 3.6. *Let $u, v \in \mathcal{S}$ be two sequences of positive numbers such that u is log-subordinate to v. Then for $N \geq n$:*

$$u_N \leq v_1^{n/N} v_n^{1-n/N} . \tag{3.3}$$

In particular, for any $n \geq 1$:

$$u_{2n} \leq \sqrt{v_1 v_n} . \tag{3.4}$$

Proof. We have:

$$u_N^N \leq \prod_{j=1}^{N} u_j \leq \prod_{j=1}^{N} v_j = \prod_{j=1}^{n} v_j \prod_{j=n+1}^{N} v_j \leq v_1^n v_n^{N-n} ,$$

and (3.3) follows. Now, the choice $N = 2n$ gives (3.4). $\square$

Note that the choice $N = [n \log n]$, the integer part of $n \log n$, can be useful (see [4]).

3.3. Singular numbers

The following Weyl type result is crucial for the proof of our main result. It is certainly known by specialists, but we have not found any reference.

Proposition 3.7. *Let T be a compact operator on a separable complex Hilbert space H and $T = \sum_{j=1}^{\infty} s_j \, u_j \otimes v_j$ its Schmidt decomposition. Then, for every integer $n \geq 1$:*

$$s_1 \cdots s_n = \max \big| \det \big(\langle T f_j \mid g_i \rangle \big)_{i,j} \big| ,$$

where the supremum is taken over all pairs $(f_j)_{1 \leq j \leq n}$ and $(g_i)_{1 \leq i \leq n}$ of orthonormal systems of length n in H.

Proof. First, assume that H is n-dimensional. We may assume that $H = \ell_2^n$ and we denote $(e_i)_{1 \leq i \leq n}$ its canonical basis.

Since $T(v_j) = s_j u_j$, we have $\det \big(\langle T v_j \mid u_i \rangle \big)_{i,j} = s_1 \cdots s_n$.

Now, if $(f_j)_{1 \leq j \leq n}$ and $(g_i)_{1 \leq i \leq n}$ are two orthonormal systems, we consider the following diagram:

$$\ell_2^n \xrightarrow{U} \ell_2^n \xrightarrow{T} \ell_2^n \xrightarrow{V} \ell_2^n$$

where U, V are the unitary operators defined by:

$$U\Big(\sum_{j=1}^{n} t_j e_j \Big) = \sum_{j=1}^{n} t_j f_j \quad \text{and} \quad Vx = \sum_{j=1}^{n} \langle x \mid g_j \rangle \, e_j \, .$$

We observe that

$$\langle VTU e_j \mid e_i \rangle = \langle TU e_j \mid V^* e_i \rangle = \langle T f_j \mid g_i \rangle \, ,$$

so that:

$$\big| \det \big(\langle T f_j \mid g_i \rangle \big)_{i,j} \big| = |\det V| \, |\det T| \, |\det U| = |\det T| = s_1 \cdots s_n \, .$$

In the general case, denote by P_n and Q_n the orthogonal projections onto $F_n := [f_1, \ldots, f_n]$ and $G_n := [g_1, \ldots, g_n]$ respectively. We can see F_n and G_n as isometric copies of ℓ_2^n. Observe that $\langle T f_j \mid g_i \rangle = \langle Q_n T P_n f_j \mid g_i \rangle$. By the above special case, we get, using the ideal property of singular numbers:

$$\big| \det \big(\langle T f_j \mid g_i \rangle \big)_{i,j} \big| = \prod_{j=1}^{n} s_j (Q_n T P_n) \leq \prod_{j=1}^{n} s_j (T) \, . \quad \square$$

3.4. Comparison principle for operators

For convenience, we say that an operator $U: H \to K$ between Hilbert spaces is *unitary* if it is a surjective isometry, even if $H \neq K$.

V. È Kacnel'son ([15]) proved the following result.

Theorem 3.8 *(V. È Kacnel'son). Let H be a separable complex Hilbert space and $(e_i)_{i\geq 0}$ a fixed orthonormal basis of H. Let $A: H \to H$ be a bounded linear operator. We assume that the matrix of A with respect to this basis is lower-triangular: $\langle Ae_j \mid e_i\rangle = 0$ for $i < j$.*

Let $(d_j)_{j\geq 0}$ be an increasing sequence of positive real numbers and D the (possibly unbounded) diagonal operator such that $D(e_j) = d_j e_j$, $j \geq 0$. Then the operator $D^{-1}AD: H \to H$ is bounded and moreover:

$$\|D^{-1}AD\| \leq \|A\| \,. \tag{3.5}$$

In [6], this theorem was extended in the framework of Banach spaces with 1-unconditional basis and used for the study of composition operators, and in [7] to compare the Schatten-class norms of weighted Hilbert spaces of analytic functions.

We have the following generalization (the case $n = 1$ giving $\|D^{-1}AD\| \leq \|A\|$).

Theorem 3.9. *With the notation of Theorem 3.8, and assuming moreover that A is compact, we have, for every $n \geq 1$:*

$$\prod_{j=1}^{n} s_j(D^{-1}AD) \leq \prod_{j=1}^{n} s_j(A)\,. \tag{3.6}$$

In other words, the sequence $\big(s_j(D^{-1}AD)\big)_j$ is log-subordinate to $\big(s_j(A)\big)_j$.

Proof. Let $\mathbb{C}_0$ be the right half-plane $\mathbb{C}_0 = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$ and $H_N = \operatorname{span}\{e_j\,;\ j \leq N\}$. We set:

$$a_{i,j} = \langle Ae_j \mid e_i\rangle$$

and

$$A_N = P_N A P_N\,,$$

where P_N is the orthogonal projection from H into H with range H_N. We consider, for $z \in \overline{\mathbb{C}_0}$:

$$A_N(z) = D^{-z} A_N D^z : H \to H\,,$$

where $D^z(e_n) = d_n^z e_n$.

If $\big(a_{i,j}^N(z)\big)_{i,j}$ is the matrix of $A_N(z)$ on the basis $\{e_j\,;\ j \geq 0\}$ of H, we clearly have:

$$a_{i,j}^N(z) = \begin{cases} a_{i,j}(d_j/d_i)^z & \text{if } i,j \leq N \\ 0 & \text{otherwise}. \end{cases}$$

In particular, we have, by hypothesis:

$$|a_{i,j}^N(z)| \leq \sup_{k,l} |a_{k,l}| := M\,, \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Since $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{i,j\leq N} |a_{i,j}^N(z)|^2 \leq (N+1)^2 M^2$, we get:

$$\|A_N(z)\| \leq (N+1)M \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Let us consider the function $u\colon \overline{\mathbb{C}_0} \to \overline{\mathbb{C}_0}$ defined by:

$$u(z) = \prod_{j=1}^n s_j\big(A_N(z)\big)\,. \tag{3.7}$$

This function u is continuous on $\overline{\mathbb{C}_0}$.

If α denotes a pair (f_j), (g_i) of orthonormal systems of length n of H, we set, for $z \in \overline{\mathbb{C}_0}$:

$$F_\alpha(z) = \det\big(\langle A_N(z) f_j \mid g_i\rangle\big)_{i,j}\,,$$

the function F_α is analytic in $\mathbb{C}_0$ and continuous on $\overline{\mathbb{C}_0}$. By Proposition 3.7, we have $u = \sup_\alpha |F_\alpha|$, so that u is subharmonic in $\mathbb{C}_0$. Moreover:

$$u(z) \leq \|A_N(z)\|^n \leq [(N+1)M]^n \qquad \text{for } z \in \overline{\mathbb{C}_0}\,,$$

and:

$$u(z) = \prod_{j=1}^n s_j(A_N) \leq \prod_{j=1}^n s_j(A) \qquad \text{for } z \in \partial\mathbb{C}_0\,,$$

since the operator $D^z\colon H \to H$ is then unitary. Hence we can use the following form of the maximum principle.

Theorem 3.10 *(Maximum principle). Let Ω be an arbitrary domain in $\mathbb{C}$, with $\Omega \neq \mathbb{C}$, and $u\colon \overline{\Omega} \to \mathbb{R}$ a function subharmonic in Ω, and continuous and bounded above on $\overline{\Omega}$. Then:*

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u\,.$$

This theorem is proved in [3, Theorem 15.1, p. 190] for $u = |f|$, with $f\colon \overline{\Omega} \to \mathbb{C}$ holomorphic in Ω, and continuous and bounded on $\overline{\Omega}$, and in [14, Theorem 5.16, p. 232]. It follows that:

$$\sup_{\mathfrak{Re}\, z \geq 0} u(z) \leq \prod_{j=1}^{n} s_j(A)\,.$$

In particular $u(1) \leq \prod_{j=1}^{n} s_j(A)$, or else:

$$\prod_{j=1}^{n} s_j(D^{-1}A_N D) \leq \prod_{j=1}^{n} s_j(A)\,.$$

Now, since the matrix of $A - A_N$ is lower-triangular, the inequality (3.5), applied to $A - A_N$, gives $\|D^{-1}(A - A_N)D\| \leq \|A - A_N\| \underset{N\to\infty}{\longrightarrow} 0$. Moreover, for each $j \geq 1$, the map $T \in \mathcal{L}(H) \mapsto s_j(T)$ is continuous, since $|s_j(T_1) - s_j(T_2)| \leq \|T_1 - T_2\|$. Then, letting N tend to infinity, we obtain that

$$s_j(D^{-1}A_N D) \underset{N\to\infty}{\longrightarrow} s_j(D^{-1}AD)\,,$$

and the result follows. $\square$

An alternative proof of Theorem 3.9 can be given using antisymmetric tensor products.

Alternative proof of Theorem 3.9. Let I denote the set of all increasing n-tuples $\alpha = (i_1 < i_2 < \cdots < i_n)$ of non-negative integers. Let $(u_\alpha)_{\alpha\in I}$ be the orthonormal basis of $\Lambda^n(H)$, the n-th exterior power of H, defined by:

$$u_\alpha = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}\,, \quad \alpha \in I\,.$$

We use the general fact that:

$$\prod_{j=1}^{n} s_j(D^{-1}AD) = \|\Lambda^n(D^{-1}AD)\|\,,$$

where Λ^n denotes the n-th skew product.

Since $\Lambda^n(UV) = \Lambda^n(U)\Lambda^n(V)$ ([38, page 10]), we get:

$$\begin{aligned}\Lambda^n(D^{-1}AD) = \Lambda^n(D^{-1})\Lambda^n(A)\Lambda^n(D) &= \big[\Lambda^n(D)\big]^{-1}\Lambda^n(A)\Lambda^n(D)\\ &=: \Delta^{-1}\Lambda^n(A)\Delta\,,\end{aligned}$$

where Δ is the diagonal operator on the basis (u_α) with diagonal elements $\delta_\alpha = d_{i_1}\cdots d_{i_n}$ if $\alpha = (i_1 < i_2 < \cdots < i_n)$.

Now, we claim that $\Lambda^n(A)$ is lower triangular in the following sense. If $\alpha = (i_1 < i_2 < \cdots < i_n)$ and $\beta = (j_1 < j_2 < \cdots < j_n)$ are two elements of I, then:

$$\delta_\alpha < \delta_\beta \Longrightarrow \big\langle \Lambda^n(A)\, u_\beta, u_\alpha \big\rangle = 0\,. \tag{3.8}$$

Indeed, assume that $\big\langle \Lambda^n(A)\, u_\beta, u_\alpha \big\rangle \neq 0$. Since:

$$\begin{aligned}\big\langle \Lambda^n(A)\, u_\beta, u_\alpha \big\rangle &= \big\langle Ae_{j_1} \wedge Ae_{j_1} \cdots \wedge Ae_{j_n}, e_{i_1} \wedge e_{i_1} \cdots \wedge e_{i_n} \big\rangle \\ &= \det \big(\langle Ae_{j_p}, e_{i_q} \rangle \big)_{1 \leq p,q \leq n}\,,\end{aligned}$$

it follows, by definition of determinants, that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that:

$$\prod_{1 \leq k \leq n} \langle Ae_{j_k}, e_{i_{\sigma(k)}} \rangle \neq 0\,,$$

implying that $i_{\sigma(k)} \geq j_k$ for each k. But then, since $l \mapsto d_l$ is nondecreasing:

$$\delta_\alpha = \prod_{1 \leq k \leq n} d_{i_{\sigma(k)}} \geq \prod_{1 \leq k \leq n} d_{j_k} = \delta_\beta\,.$$

Now, (3.8) allows to apply Theorem 3.8 to get the result. □

Remark. We could also remark that the function:

$$u(z) = \prod_{1 \leq j \leq n} s_j(D^{-z} A_N D^z) = \|\Lambda^n(D^{-z} A_N D^z)\|$$

is subharmonic since it is a norm, on $\Lambda^n(H)$, and hence a supremum of moduli of the holomorphic functions $z \mapsto l(D^{-z} A_N D^z)$, for l a linear functional on $\Lambda^n(H)$.

Corollary 3.11. *With the notation of Theorem 3.8, $D^{-1}AD$ is compact if A is. Moreover, for any $p > 0$, if $A \in S_p(H)$, so does $D^{-1}AD$, and:*

$$\|D^{-1}AD\|_p \leq \|A\|_p\,.$$

Proof. Since $\big(s_n(D^{-1}AD)\big)_n$ is log-subordinate to $\big(s_n(A)\big)_n$, Corollary 3.6 gives the first assertion, and Proposition 3.5 gives the second one. □

3.5. Application to composition operators

We consider here general weighted Hilbert spaces of analytic functions on $\mathbb{D}$.

Let $\beta = (\beta_k)_{k \geq 0}$ be a sequence of positive numbers such that:

$$\liminf_{k \to \infty} {\beta_k}^{1/k} \geq 1 \tag{3.9}$$

(as we will see right after, this condition ensures that the evaluation maps are bounded) and let $H^2(\beta)$ be the Hilbert space of functions $f(z) = \sum_{k=0}^{\infty} c_k z^k$ such that:

$$\|f\|^2_{H^2(\beta)} := \sum_{k=0}^{\infty} \beta_k \, |c_k|^2 < \infty \, . \tag{3.10}$$

This is a Hilbert space of analytic functions on $\mathbb{D}$ with a reproducing kernel K_a, namely:

$$f(a) = \langle f, K_a \rangle \quad \text{for all } f \in H^2(\beta) \, , \tag{3.11}$$

because the evaluations $f \in H^2(\beta) \mapsto f(a)$ are continuous:

$$\Big| \sum_{k=0}^{\infty} c_k a^k \Big| \leq \Big(\sum_{k=0}^{\infty} \beta_k \, |c_k|^2 \Big)^{1/2} \Big(\sum_{k=0}^{\infty} \beta_k^{-1} |a|^{2k} \Big)^{1/2} < \infty \, ,$$

thanks to condition (3.9).

The canonical orthonormal basis of $H^2(\beta)$ is formed by the normalized monomials

$$e_k^\beta(z) = \frac{z^k}{\sqrt{\beta_k}} \, , \quad k = 0, 1, \dots \, ; \tag{3.12}$$

so we have, for all $a \in \mathbb{D}$:

$$\|K_a\|^2_{H^2_\omega} = \sum_{n=0}^{\infty} |e_n^\beta(a)|^2 = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \, |a|^{2n} \, . \tag{3.13}$$

We refer to [9] or [42] for more on those spaces. See also [16] for an alternative definition.

For example, the weighted Dirichlet space $\mathcal{D}^2_\alpha$ corresponds to $\beta_k \approx (k+1)^{1-\alpha}$. In particular, the Hardy space H^2 corresponds to $\beta_k = 1$, the Bergman space $\mathfrak{B}^2$ to $\beta_k = 1/(k+1)$, and the Dirichlet space $\mathcal{D}^2$ to $\beta_k = (k+1)$.

For the weights

$$\beta_k = \frac{(k+1)! \, \Gamma(\alpha+2)}{\Gamma(k+\alpha+1)} \, ,$$

we get, using the binomial formula $\sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k! \, \Gamma(\alpha)} \, x^k = (1-x)^{-\alpha}$ for $|x| < 1$, that the reproducing kernels are, for $a \neq 0$:

$$K_a^\alpha(z) = \frac{1}{\alpha(\alpha+1)} \, \frac{(1-\bar{a}z)^{-\alpha} - 1}{\bar{a}z} \, , \qquad \text{for } \alpha > 0 \, ; \tag{3.14}$$

$$K_a^0(z) = \frac{1}{\bar{a}z} \, \log \frac{1}{(1-\bar{a}z)} \, , \tag{3.15}$$

(with $K_0^\alpha(z) = 1/(\alpha+1)$ and $K_0^0(z) = 1$).

Let us point out that $\lim_{\alpha\to 0^+} K_a^\alpha(z) = K_a^0(z)$.

Now let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map. We assume that:

$$\varphi(0) = 0 \,. \tag{3.16}$$

This map φ induces *formally* a lower-triangular composition operator C_φ on $H^2(\beta)$ since:

$$\langle C_\varphi(e_j^\beta), e_i^\beta\rangle = \frac{1}{\sqrt{\beta_i\beta_j}}\,\langle \varphi^j, z^i\rangle = 0 \quad \text{for } i < j \,.$$

Remark. We can often omit condition (3.16). In fact, let us consider the automorphisms $\varphi_a\colon \mathbb{D} \to \mathbb{D}$, $a \in \mathbb{D}$, given by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. When C_{φ_a} is *bounded on* $H^2(\beta)$, then, with $a = \varphi(0)$, the function $\psi = \varphi_a \circ \varphi$ satisfies $\psi(0) = 0$ and $\varphi = \varphi_a \circ \psi$; hence $C_\varphi = C_\psi \circ C_{\varphi_a}$ and $C_\psi = C_\varphi \circ C_{\varphi_a}$, so:

$$\|C_{\varphi_a}\|^{-1} a_n(C_\psi) \leq a_n(C_\varphi) \leq \|C_{\varphi_a}\|\, a_n(C_\psi) \,.$$

A necessary condition for having C_{φ_a} bounded on $H^2(\beta)$ is that $\varphi_a \in H^2(\beta)$. Since $\varphi_a(z) = a + \sum_{k=1}^\infty \bar{a}^{k-1}(|a|^2-1)z^k$, we have $\varphi_a \in H^2(\beta)$ for all $a \in \mathbb{D}$ if $\lim_{k\to\infty} \beta_k^{1/k} = 1$.

For weighted Dirichlet spaces $\mathcal{D}_\alpha^2$, with any $\alpha > -1$, the automorphisms φ_a define bounded composition operators on $\mathcal{D}_\alpha^2$. In fact, we have, for $f \in \mathcal{D}_\alpha^2$:

$$\begin{aligned}\|f\circ\varphi_a\|_{\mathcal{D}_\alpha^2}^2 &= |f(a)|^2 + (\alpha+1)\int_{\mathbb{D}} |f'[\varphi_a(z)]|^2 |\varphi_a'(z)|^2 (1-|z|^2)^\alpha \, dA(z) \\ &= |f(a)|^2 + (\alpha+1)\int_{\mathbb{D}} |f'(w)|^2 (1-|\varphi_a(w)|^2)^\alpha \, dA(w) \,.\end{aligned}$$

Since:

$$\frac{1-|\varphi_a(w)|^2}{1-|w|^2} = \frac{1-|a|^2}{|1-\bar{a}w|^2} \,,$$

we have $1 - |\varphi_a(w)|^2 \approx 1 - |w|^2$ and we get:

$$\|f\circ\varphi_a\|_{\mathcal{D}_\alpha^2}^2 \lesssim |f(a)|^2 + (\alpha+1)\int_{\mathbb{D}} |f'(w)|^2 (1-|w|^2)^\alpha \, dA(w) \approx \|f\|_{\mathcal{D}_\alpha^2}^2 \,.$$

For $\alpha \geq 0$, that follows directly from [43, Theorem 1] (see also [35, Section 6.12], [16, Theorem 1.3 and Proposition 3.1] or [33, Theorem 3.1]), since φ_a is univalent.

We will write for short C_φ^β to designate the operator C_φ acting on $H^2(\beta)$.

As an application of the general principles of Section 3.4 we have the following result, whose first items were previously obtained by I. Chalendar and J. Partington in [6] and [7] (actually (3.b) is also proved in [7], but for values $p \geq 1$).

Theorem 3.12. *Let $H^2(\beta)$ and $H^2(\gamma)$ be two weighted Hilbert spaces. Assume that γ is dominated by β in the sense that the sequence (β_k/γ_k) is increasing, so that the continuous inclusion $H^2(\beta) \subseteq H^2(\gamma)$ holds. Then, for $\varphi\colon \mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$:*

1) *if C_φ^β is bounded, C_φ^γ is bounded as well, and $\|C_\varphi^\gamma\| \leq \|C_\varphi^\beta\|$;*
2) *if C_φ^β is compact, so is C_φ^γ;*
3) *the sequence $s^\gamma = \big(s_n(C_\varphi^\gamma)\big)_{n\geq 1}$ is log-subordinate to the sequence $s^\beta = \big(s_n(C_\varphi^\beta)\big)_{n\geq 1}$, so that:*
   a) *$s_{2n}(C_\varphi^\gamma) \leq \sqrt{s_1(C_\varphi^\beta)}\,\sqrt{s_n(C_\varphi^\beta)}$, for all $n \geq 1$;*
   b) *$C_\varphi^\beta \in S_p\big(H^2(\beta)\big) \Longrightarrow C_\varphi^\gamma \in S_p\big(H^2(\gamma)\big)$, for any $p > 0$.*

Remark. Let us mention that we can apply the previous theorem in the framework of weighted Dirichlet spaces. Indeed, let $0 < \beta < \gamma$ and consider the two weights:

$$\beta_k = \frac{k.k!\,\Gamma(\beta+2)}{\Gamma(k+\beta+1)} \qquad \text{and} \qquad \gamma_k = \frac{k.k!\,\Gamma(\gamma+2)}{\Gamma(k+\gamma+1)}$$

associated with the weighted Dirichlet spaces $\mathcal{D}_\beta^2$ and $\mathcal{D}_\gamma^2$ respectively, with $\gamma > \beta$, so that $\mathcal{D}_\beta^2 \subset \mathcal{D}_\gamma^2$. In order to apply our comparison Theorem 3.12, we have to show that the sequence (β_k/γ_k) increases. But

$$\frac{\beta_k}{\gamma_k} = \frac{\Gamma(\beta+2)}{\Gamma(\gamma+2)}\,\frac{\Gamma(\gamma+k+1)}{\Gamma(\beta+k+1)} =: \frac{\Gamma(\beta+2)}{\Gamma(\gamma+2)}\,A_k\,,$$

and, setting $h = \gamma - \beta > 0$ and $x_k = \beta + k + 1$, we see that:

$$A_k = \frac{\Gamma(x_k+h)}{\Gamma(x_k)}\,\cdot$$

Since the function Γ is log-convex, the map $x \mapsto \frac{\Gamma(x+h)}{\Gamma(x)}$ increases on $(0,\infty)$, and we get that the sequence (β_k/γ_k) increases.

Proof of Theorem 3.12. We set $d_k = \sqrt{\beta_k/\gamma_k}$ and $e_k(z) = z^k$.

Let $J\colon H^2(\beta) \to H^2(\gamma)$ the unitary (onto isometry) and diagonal operator defined by $J(e_k) = d_k e_k$, for all $k \geq 0$.

The operator $A = JC_\varphi^\beta J^{-1}$ maps $H^2(\gamma)$ into itself and $s_n(A) = s_n(C_\varphi^\beta)$ for all $n \geq 1$ (in particular $\|A\|_{\mathcal{L}(H^2(\gamma))} = \|C_\varphi^\beta\|_{\mathcal{L}(H^2(\beta))}$). Moreover, A has a lower-triangular matrix.

Now we consider the diagonal operator $D\colon H^2(\gamma) \to H^2(\gamma)$ defined by $D(e_k) = d_k e_k$. In general, it is an unbounded operator. It is plain that $D^{-1}J\colon H^2(\beta) \to H^2(\gamma)$ is the canonical inclusion, since $(D^{-1}J)(e_k) = e_k$ for all $k \geq 0$. Hence $(D^{-1}J)C_\varphi^\beta = C_\varphi^\gamma(D^{-1}J)$, and since $AJ = JC_\varphi^\beta$, we have the following commutative diagram:

$$\begin{tikzcd} & H^2(\beta) \arrow[r, "C_\varphi^\beta"] \arrow[d, "J"'] & H^2(\beta) \arrow[d, "J"] & \\ H^2(\gamma) \arrow[r, "D"'] & H^2(\gamma) \arrow[r, "A"'] & H^2(\gamma) \arrow[r, "D^{-1}"'] & H^2(\gamma) \end{tikzcd}$$

By Theorem 3.9, we get:

$$\log s(D^{-1}AD) \prec \log s(A)$$

(so we have, in particular, $\|D^{-1}AD\|_{\mathcal{L}(H^2(\gamma))} \leq \|A\|_{\mathcal{L}(H^2(\gamma))} = \|C_\varphi^\beta\|_{\mathcal{L}(H^2(\beta))}$). But $D^{-1}AD = C_\varphi^\gamma$, and this proves Theorem 3.12, using Proposition 3.5 and Corollary 3.6. $\square$

Remark. Actually, the same proof gives the following generalization of Theorem 3.12.

Theorem 3.13. *With the hypothesis of Theorem 3.12, let $T\colon \mathcal{H}ol(\mathbb{D}) \to \mathcal{H}ol(\mathbb{D})$ be a linear map such that its restriction T_β to $H^2(\beta)$ is bounded from $H^2(\beta)$ into $H^2(\beta)$ and has a matrix in the canonical basis of $H^2(\beta)$ which is lower-triangular. Then:*

1) *T_γ is bounded as well, and $\|T_\gamma\| \leq \|T_\beta\|$;*
2) *if T_β is compact, so is T_γ;*
3) *the sequence of singular numbers $s(T_\gamma) = \big(s_n(T_\gamma)\big)_{n\geq 1}$ is log-subordinate to the sequence $s(T_\beta) = \big(s_n(T_\beta)\big)_{n\geq 1}$, so that:*
   a) *$s_{2n}(T_\gamma) \leq \sqrt{s_1(T_\beta)}\,\sqrt{s_n(T_\beta)}$, for all $n \geq 1$;*
   b) *$T_\beta \in S_p\big(H^2(\beta)\big) \Longrightarrow T_\gamma \in S_p\big(H^2(\gamma)\big)$, for any $p > 0$.*

3.6. Application to conditional multipliers

We first recall the following well-known proposition (and give a short proof, for sake of completeness). Note that this result does not hold for the Dirichlet spaces $\mathcal{D}_\alpha^2$ when $\alpha \leq 0$ ([40, Theorem 10]; see also [39, Theorem 2.7], [17, Theorem A], and [41, Theorem 4.2]). Recall that it is well-known that the space $\mathcal{M}(H^2)$ of multipliers of H^2 is isometric to H^∞.

Proposition 3.14. *For every $\gamma > -1$, the space $\mathcal{M}(\mathfrak{B}_\gamma^2)$ of multipliers of $\mathfrak{B}_\gamma^2$ is isometric to the space H^∞.*

If H is a Hilbert space of analytic functions on $\mathbb{D}$, containing the constants, and with reproducing kernels K_a, $a \in \mathbb{D}$, then the space $\mathcal{M}(H)$ of multipliers of H is contained contractively into the space H^∞.

Proof. If $hf \in H$ for all $f \in H$, then, taking $f = \mathbb{1}$, we have $h \in H$, so h is analytic. The same proof as in [1, Proposition 3.1] shows that $h \in H^\infty$. For sake of completeness we give a short different proof.

In fact, we have, for all $a \in \mathbb{D}$:

$$M_h^*(K_a) = \overline{h(a)}\, K_a \quad \text{for all } a \in \mathbb{D}\,; \tag{3.17}$$

hence $|h(a)|\,\|K_a\| \leq \|M_h^*\|\,\|K_a\|$, and, since $\|K_a\|$ is not null, that proves that $h \in H^\infty$ and $\|h\|_\infty \leq \|M_h\|$.

Hence $\mathcal{M}(H) \subsetneqq H^\infty$, contractively.

When $H = \mathfrak{B}^2_\gamma$, we have the reverse inclusion. Indeed, for every $h \in H^\infty$, one clearly has $hf \in \mathfrak{B}^2_\gamma$ and $\|hf\|_{\mathfrak{B}^2_\gamma} \leq \|h\|_\infty \|f\|_{\mathfrak{B}^2_\gamma}$ for all $f \in \mathfrak{B}^2_\gamma$, so the multiplication operator $M_h \colon \mathfrak{B}^2_\gamma \to \mathfrak{B}^2_\gamma$ is bounded with norm no greater than $\|h\|_\infty$. $\square$

Now let φ be an analytic self-map of $\mathbb{D}$ and $H = H^2(\beta)$ be a weighted Hilbert space of analytic functions on $\mathbb{D}$, with reproducing kernel K_a, $a \in \mathbb{D}$, on which C_φ acts boundedly. We denote its multiplier set, respectively multiplier set conditionally to φ, by:

$$\mathcal{M}(H) = \{w \in H\,;\; wf \in H \text{ for each } f \in H\} \tag{3.18}$$

and

$$\mathcal{M}(H, \varphi) = \{w \in H\,;\; w\,(f \circ \varphi) \in H \text{ for all } f \in H\}\,. \tag{3.19}$$

We have $\mathcal{M}(H) \subseteq \mathcal{M}(H, \varphi)$.

The set $\mathcal{M}(H, \varphi)$ plays an important role in the study of weighted composition operators.

Definition 3.15. A Hilbert space H of analytic functions on $\mathbb{D}$, containing the constants, and with reproducing kernels K_a, $a \in \mathbb{D}$, is said to be *admissible* if:

- (i) H^2 is continuously embedded in H;
- (ii) $\mathcal{M}(H) = H^\infty$;
- (iii) the automorphisms of $\mathbb{D}$ induce bounded composition operators on H;
- (iv) $\dfrac{\|K_a\|_H}{\|K_b\|_H} \leq h\Big(\dfrac{1 - |b|}{1 - |a|}\Big)$ for $a, b \in \mathbb{D}$, where $h \colon \mathbb{R}^+ \to \mathbb{R}^+$ is an non-decreasing function.

Note that (i) implies that $\|f\|_H \leq C\,\|f\|_{H^2}$ for all $f \in H^2$, for some positive constant C, and so (B_H and B_{H^2} being the unit ball of H and H^2 respectively):

$$\|K_a\|_H = \sup_{f \in B_H} |f(a)| \geq C^{-1} \sup_{f \in B_{H^2}} |f(a)| = C^{-1}(1 - |a|^2)^{-1/2}\,,$$

implying that:

$$\lim_{|a|\to 1^-} \|K_a\|_H = \infty\,.$$

Examples.

1) The weighted Bergman space $\mathfrak{B}^2_\gamma$, with $\gamma > -1$ is admissible.

Indeed, we know that it is continuously embedded in $H^2 = \mathfrak{B}^2_{-1}$; condition (ii) is Proposition 3.14; condition (iii) is satisfied according to the Remark before Theorem 3.12, and $\|K_a\|^2 = \frac{1}{(1-|a|^2)^{\gamma+2}}$, giving (iv).

2) More generally, we have the following result.

Proposition 3.16. *For any decreasing sequence β such that the automorphisms of $\mathbb{D}$ induce bounded composition operators on $H^2(\beta)$, the space $H^2(\beta)$ is admissible.*

Recall that $H^2(\beta)$ is defined in (3.10). A particular case is obtained as follows. Let $\omega\colon (0,1) \to \mathbb{R}_+$ be an integrable function such that, for some positive and locally bounded function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$, we have:

$$\frac{\omega(y)}{\omega(x)} \leq \rho\Big(\frac{y}{x}\Big) \qquad \text{for all } x, y \in (0,1)\,, \tag{3.20}$$

and let H^2_ω be the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|^2_{H^2_\omega} := \int_{\mathbb{D}} |f(z)|^2\, \omega(1-|z|^2)\, dA(z) < \infty\,. \tag{3.21}$$

Such spaces are used in [16] and in [28]. We have $H^2_\omega = H^2(\beta)$ with:

$$\beta_n = 2\int_0^1 r^{2n+1}\omega(1-r^2)\, dr = \int_0^1 t^n \omega(1-t)\, dt\,. \tag{3.22}$$

Indeed, since $\beta_n = \int_0^1 (1-t)^n \omega(t)\, dt$, the sequence $\beta = (\beta_n)_n$ is decreasing. Moreover, the fact that the automorphisms of $\mathbb{D}$ induce bounded composition operators on H^2_ω is proved as in the Remark before Theorem 3.12, namely:

$$\begin{aligned}
\|f\circ\varphi_a\|^2_{H^2_\omega} &= \int_{\mathbb{D}} |f(w)|^2\, |\varphi_a'(w)|^2\, \omega(1-|\varphi_a(w)|^2)\, dA(w) \\
&\leq \Big(\frac{1+|a|}{1-|a|}\Big)^2 \int_{\mathbb{D}} |f(w)|^2\, \rho\Big(\frac{1-|\varphi_a(w)|^2}{1-|w|^2}\Big)\, \omega(1-|w|^2)\, dA(w)
\end{aligned}$$

$$\leq \left(\frac{1+|a|}{1-|a|}\right)^2 c_{\rho,a} \int_{\mathbb{D}} |f(w)|^2 \, \omega(1-|w|^2)\, dA(w)$$
$$=: \kappa_a \, \|f\|^2_{H^2_\omega} \,,$$

where we used that $|\varphi_a'(w)| \leq \frac{1+|a|}{1-|a|}$, that $\frac{1-|\varphi_a(w)|^2}{1-|w|^2} \leq \frac{1+|a|}{1-|a|}$, and that ρ is locally bounded.

Note that $\mathfrak{B}^2_\gamma$, $\gamma > -1$, corresponds to $\omega(t) = (\gamma+1)\, t^\gamma$.

Proof of Proposition 3.16. Condition (i) is satisfied because β is decreasing. Moreover, since β is decreasing, Theorem 3.13, applied to $T = M_w$, with $w \in H^\infty$, ensures that $H^\infty = \mathcal{M}(H^2) \subseteq \mathcal{M}\big(H^2(\beta)\big)$, and, for all $w \in H^\infty$:

$$\|M_w \colon H^2(\beta) \to H^2(\beta)\| \leq \|M_w \colon H^2 \to H^2\| = \|w\|_\infty \,.$$

Now, Proposition 3.14 implies that $H^\infty = \mathcal{M}\big(H^2(\beta)\big)$ and

$$\|M_w \colon H^2(\beta) \to H^2(\beta)\| = \|w\|_\infty$$

for all $w \in H^\infty$.

It remains to show that, for $H = H^2(\beta)$, the condition (iii) implies the condition (iv).

Since $H^2(\beta)$ is isometrically rotation invariant, it is clear that $\|K_a\| = \|K_{|a|}\|$; hence $\|K_x\| \leq \|K_y\|$, for $0 \leq x \leq y < 1$.

Assume now that $0 < y < x < 1$. Let T be the disk automorphism:

$$T(z) = \frac{2z+1}{z+2} \,, \qquad z \in \mathbb{D} \,. \tag{3.23}$$

The fixed points of T are 1 and -1, and $T(0) = 1/2$. We define the sequence $(a_n)_{n\geq 0}$ by induction, with:

$$a_0 = 0 \,, \qquad a_{n+1} = T(a_n) \,.$$

We see that:

$$1 - a_{n+1} = \int_{a_n}^1 T'(x)\, dx = \int_{a_n}^1 \frac{3}{(x+2)^2}\, dx \leq \frac{3}{4}\,(1-a_n) \,; \tag{3.24}$$

so $(a_n)_n$ is increasing and converges to 1. In the same way, we see that:

$$1 - a_{n+1} = \int_{a_n}^1 T'(x)\, dx = \int_{a_n}^1 \frac{3}{(x+2)^2}\, dx \geq \frac{1}{3}\,(1-a_n) \,. \tag{3.25}$$

Since $0 < y < x < 1$, we can find $m \leq n$ such that:

$$a_{m-1} < y < a_m \,, \qquad \text{and} \qquad a_{n-1} < x < a_n \,.$$

We have $\|K_x\| \leq \|K_{a_n}\|$ and $\|K_y\| \geq \|K_{a_{m-1}}\|$. Since $C_T^* K_z = K_{T(z)}$ for all $z \in \mathbb{D}$, we have:

$$\frac{\|K_x\|}{\|K_y\|} \leq \frac{\|K_{a_n}\|}{\|K_{a_{m-1}}\|} \leq \|C_T^*\|^{n-m+1} = \alpha^{n-m+1} \,,$$

with $\alpha = \|C_T\| \geq 1$. Applying (3.24) and (3.25), we get:

$$\frac{1-y}{1-x} \geq \frac{1-a_m}{1-a_{n-1}} \geq \frac{1-a_m}{3(1-a_n)} \geq \frac{1}{3}\Big(\frac{4}{3}\Big)^{n-m} .$$

It suffices now to take $s \geq 0$ such that $(4/3)^s = \alpha$, and $A > 0$ large enough in order that, with the increasing function $h(t) = \max\{At^s, 1\}$, $t > 0$, we have:

$$h\Big(\frac{1-y}{1-x}\Big) \geq \frac{A}{3^s}\,\alpha^{n-m} \geq \frac{A}{3^s \alpha}\,\frac{\|K_x\|}{\|K_y\|} \geq \frac{\|K_x\|}{\|K_y\|} \cdot \qquad \square$$

Let us come back to the conditional multipliers. In general, we obviously have:

$$H^\infty \subseteq \mathcal{M}(H, \varphi) \subseteq H \,. \tag{3.26}$$

The extreme cases were characterized by Attele ([2]) when $H = H^2 = \mathfrak{B}^2_{-1}$ (and Contreras and Hernández-Díaz in [8] for the spaces H^p) as follows.

Theorem 3.17 *(Attele). We have:*

1) $\mathcal{M}(H^2, \varphi) = H^2$ *if and only if* $\|\varphi\|_\infty < 1$.
2) $\mathcal{M}(H^2, \varphi) = H^\infty$ *if and only if* φ *is a finite Blaschke product.*

A key tool for the most delicate second necessary condition is the use of inner and outer functions. We no longer have this tool at our disposal for the admissible spaces $H = H^2(\beta)$, but we can nevertheless state the following analogous result.

Theorem 3.18. *Let* φ *an analytic self-map of* $\mathbb{D}$ *and* H *be an admissible Hilbert space on which* C_φ *acts boundedly. We have:*

1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H, \varphi)$;
2) $\mathcal{M}(H, \varphi) = H$ *if and only if* $\|\varphi\|_\infty < 1$;
3) $\mathcal{M}(H, \varphi) = H^\infty$ *if and only if* φ *is a finite Blaschke product.*

Note that the assumption that C_φ acts boundedly on H is automatically satisfied when $H = H^2(\beta)$ with β decreasing, by Theorem 3.12.

Proof. 1) Suppose first that $\varphi(0) = 0$. Let $w \in \mathcal{M}(H^2, \varphi)$. The weighted composition operator $M_w C_\varphi$ is bounded on H^2, and moreover lower triangular on the canonical basis; applying Theorem 3.13, 1), we get that $M_w C_\varphi$ is bounded on H as well; that is, $w \in \mathcal{M}(H, \varphi)$.

In the general case, let $\varphi(0) = a$, so that $(\varphi_a \circ \varphi)(0) = 0$. Property (iii) implies that

$$\mathcal{M}(H^2, \varphi) = \mathcal{M}(H^2, \varphi_a \circ \varphi) \subseteq \mathcal{M}(H, \varphi_a \circ \varphi) = \mathcal{M}(H, \varphi) \, ,$$

since $f \in H^2$ if and only if $f \circ \varphi_a \in H^2$ and $f \in H$ if and only if $f \circ \varphi_a \in H$.

2) The necessary condition is proved as in [2] for H^2; we recall some details. We start from the (obvious, but useful) mapping equation:

$$(M_w C_\varphi)^* (K_z) = \overline{w(z)} \, K_{\varphi(z)} \, . \tag{3.27}$$

The assumption implies the existence of a constant C such that:

$$\| M_w C_\varphi \|_{\mathcal{L}(H)} \leq C \, \| w \|_H \qquad \text{for all } w \in H \, .$$

As a consequence, for given $z \in \mathbb{D}$:

$$\| (M_w C_\varphi)^* (K_z) \|_H \leq C \, \| w \|_H \| K_z \|_H \, ,$$

that is, in view of (3.27):

$$|w(z)| \, \| K_{\varphi(z)} \|_H \leq C \, \| w \|_H \| K_z \|_H \, . \tag{3.28}$$

Testing this inequality with $w = K_z$ and simplifying by $\| K_z \|_H^2$, we get that $\| K_{\varphi(z)} \|_H \leq C$. Since $\lim_{|a| \to 1^-} \| K_a \|_H = \infty$, as a consequence of (i), this implies that $\| \varphi \|_\infty < 1$, by this same consequence.

For the sufficient condition, observe that if $\| \varphi \|_\infty < 1$, then $f \circ \varphi \in H^\infty$ for all $f \in H$. Since $\mathcal{M}(H) = H^\infty$, according to (iv), we get that $f \circ \varphi \in \mathcal{M}(H)$ and therefore $w (f \circ \varphi) = M_{f \circ \varphi} \, w \in H$ for all $w \in H$. That means that $H \subseteq \mathcal{M}(H, \varphi)$. Therefore, by (3.26), we have $\mathcal{M}(H, \varphi) = H$.

3) The sufficient condition goes as follows: finite Blaschke products φ clearly satisfy (and actually are characterized by: see [2]):

$$R_\varphi := \sup_{z \in \mathbb{D}} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty \, .$$

Now let $w \in \mathcal{M}(H, \varphi)$, so that $C := \| M_w C_\varphi \| < \infty$. We may assume that $\| w \|_H \leq 1$. The mapping equation (3.28) gives, for $z \in \mathbb{D}$:

$$|w(z)|\,\|K_{\varphi(z)}\|_H \leq C\,\|K_z\|_H\,.$$

By (ii), this implies that, for $|z|$ close enough to 1:

$$|w(z)| \leq C\,\frac{\|K_z\|_H}{\|K_{\varphi(z)}\|_H} \leq C\,h\Big(\frac{1-|\varphi(z)|}{1-|z|}\Big) \leq C\,h(R_\varphi)\,.$$

This means that $w \in H^\infty$.

Finally, for the necessary condition, assume that $\mathcal{M}(H,\varphi) = H^\infty$. Then $\mathcal{M}(H^2,\varphi) = H^\infty$, by 1), and then φ is a finite Blaschke product by Attele's theorem (Theorem 3.17). □

Remark. In Proposition 3.16, we assume that the automorphisms of $\mathbb{D}$ induce bounded composition operators on $H^2(\beta)$. It is known ([18, Theorem 1]) that this is not always the case. Let us give a simpler proof, for a particular case. Let $\beta_n = \exp(-\sqrt{n})$, and consider the space $H^2(\beta)$. We then have, for $0 < r = \mathrm{e}^{-\varepsilon} < 1$:

$$\|K_r\|^2 = \sum_{n=0}^{\infty} r^{2n}\exp(\sqrt{n}) = \sum_{n=0}^{\infty}\mathrm{e}^{-2n\varepsilon}\exp(\sqrt{n}) =: S(\varepsilon)\,.$$

We easily see (using e.g. the Euler-MacLaurin formula) that, when $\varepsilon \to 0^+$:

$$S(\varepsilon) \sim I(\varepsilon) := \int_0^\infty \exp(\sqrt{t} - 2\,\varepsilon t)\,dt = (4\varepsilon^2)^{-1}\int_0^\infty \exp\bigg(\frac{\sqrt{x}-x}{2\,\varepsilon}\bigg)\,dx\,.$$

We use the Laplace theorem ([10] p. 125) on the equivalence of integrals:

$$\int_0^\infty \mathrm{e}^{A\varphi(x)}\,dx \sim \sqrt{2\pi(|\varphi''(x_0)|)^{-1}}\,A^{-1/2}\mathrm{e}^{A\varphi(x_0)}\,, \quad \text{as } A\to\infty\,,$$

and apply it to $A = 1/2\varepsilon$ and to the function $\varphi(x) = \sqrt{x} - x$, which takes its maximum at $x_0 = 1/4$, with $\varphi(x_0) = 1/4$. We get that:

$$S(\varepsilon) \approx \varepsilon^{-3/2}\exp\Big(\frac{1}{8\varepsilon}\Big) \approx (1-r)^{-3/2}\exp\Big(\frac{1}{8(1-r)}\Big)\,.$$

Now, consider the automorphism T of $\mathbb{D}$ given by (3.23). For $r < 1$, we have $1 - T(r) \sim (1-r)T'(1) = (1-r)/3$; so:

$$\frac{1}{1-T(r)} - \frac{1}{1-r} = \frac{1}{1-r}\Big(\frac{1-r}{1-T(r)} - 1\Big) \sim \frac{1}{1-r}\Big(\frac{1}{T'(1)} - 1\Big) = \frac{2}{1-r}\,,$$

and that implies that:

$$\frac{\|K_{T(r)}\|^2}{\|K_r\|^2} \approx \exp\left[\frac{1}{8}\left(\frac{1}{1-T(r)} - \frac{1}{1-r}\right)\right] \underset{r\to 1^-}{\longrightarrow} \infty\,.$$

Since $K_{T(r)} = C_T^*(K_r)$, this implies that C_T^*, and hence C_T, is not bounded on $H^2(\beta)$.

4. Schatten classes for Hardy spaces and Bergman spaces

We know that if a composition operator C_φ is compact on the Hardy space H^2, then it is compact on the Bergman space $\mathfrak{B}^2$ (see [32, Proposition 2.7 and Theorem 3.5]). Theorem 4.3 below shows that we cannot expect better.

Let us begin by a preliminary result. Recall that the 2-Carleson function of the analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is:

$$\rho_{\varphi,2}(h) = \sup_{\xi\in\mathbb{T}} A_\varphi\big(W(\xi,h)\big)\,, \tag{4.1}$$

where A is the normalized area measure on $\mathbb{D}$, A_φ is the pull-back measure of A by φ, i.e. $A_\varphi(B) = A[\varphi^{-1}(B)]$ for all Borel sets $B \subseteq \mathbb{D}$. It is well-known (see [13]) that $\rho_{\varphi,2}(h) = \mathrm{O}\,(h^2)$, due to the fact that all composition operators C_φ are bounded on $\mathfrak{B}^2$, and that C_φ is compact on $\mathfrak{B}^2$ if and only if $\rho_{\varphi,2}(h) = \mathrm{o}\,(h^2)$. For Schatten classes, we have the following result.

Proposition 4.1. *If the composition operator $C_\varphi\colon \mathfrak{B}^2 \to \mathfrak{B}^2$ is in the Schatten class $S_p(\mathfrak{B}^2)$ for some $p \in (0,\infty)$, then:*

$$\rho_{\varphi,2}(h) = \mathrm{o}\left(h^2\Big(\log\frac{1}{h}\Big)^{-2/p}\right). \tag{4.2}$$

Proof. We follow the proof of [20, Proposition 3.4]. By [29, Corollary 2] and [20, Proposition 3.3], $C_\varphi \in S_p(\mathfrak{B}^2)$ if and only if:

$$\sum_{n=1}^{\infty}\bigg(\sum_{j=0}^{2^n-1} 4^{np/2}[A_\varphi(W_{n,j})]^{p/2}\bigg) < \infty\,, \tag{4.3}$$

where $W_{n,j} = W(\mathrm{e}^{2j i\pi/2^n}, 2^{-n})$.

Observing that, for $h = 2^{-n}$, we have:

$$\big[\rho_{\varphi,2}(2^{-n})\big]^{p/2} \leq \sum_{j=0}^{2^n-1}\big[A_\varphi(W_{n,j})\big]^{p/2}\,,$$

(4.3) yields:

$$\sum_{n=1}^{\infty}\big[\rho_{\varphi,2}(2^{-n})\big]^{p/2} 4^{np/2} < +\infty\,.$$

By [22, Theorem 3.1], we have a constant $C_0 > 0$ such that:

$$\rho_{\varphi,2}(\varepsilon h) \leq C_0\, \varepsilon^2 \rho_{\varphi,2}(h)$$

for $0 < \varepsilon \leq 1$ and $0 < h < 1$. Hence, if we set:

$$u_n = \left(\frac{\rho_{\varphi,2}(2^{-n})}{4^{-n}}\right)^{p/2},$$

we have, for $n \geq k$:

$$u_n \leq C_0^{p/2}\, u_k\,.$$

The following lemma, whose proof is postponed, then shows that:

$$n \left(\frac{\rho_{\varphi,2}(2^{-n})}{4^{-n}}\right)^{p/2} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{4.4}$$

Lemma 4.2. *Let $\sum u_n$ be a convergent series of positive numbers such that $u_n \leq C\, u_k$ for $n \geq k$, for some positive constant C. Then $u_n = \mathrm{o}\,(1/n)$.*

To finish the proof, it remains to consider, for every $h \in (0, 1/2)$, the integer n such that $2^{-n-1} < h \leq 2^{-n}$; then (4.4) gives:

$$\lim_{h\to 0^+} \left(\frac{\rho_{\varphi,2}(h)}{h^2}\right)^{p/2} \log(1/h) = 0\,,$$

as announced. $\square$

Proof of Lemma 4.2. Let:

$$v_n = \sum_{n/2<k\leq n} u_k\,.$$

Since the series $\sum u_n$ converges, on the one hand, we have $v_n \underset{n\to\infty}{\longrightarrow} 0$, and on the other hand:

$$v_n \geq \frac{n}{2}\, C^{-1} u_n\,. \quad \square$$

Theorem 4.3. *There exists a symbol φ for which the composition operator C_φ is compact on the Hardy space H^2, but is not in any Schatten class $S_p(\mathfrak{B}^2)$ of the Bergman space $\mathfrak{B}^2$ with $p < \infty$.*

Proof. We use a variant of the Shapiro-Taylor map ([37, Section 4]) introduced in [20, Theorem 5.6] for showing that there is a compact composition operator on H^2 which is in no Schatten class $S_p(H^2)$ for $p < \infty$. Let:

$$V_\varepsilon = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0 \text{ and } |z| < \varepsilon\} \tag{4.5}$$

We set:

$$f(z) = z \log(-\log z)\,, \tag{4.6}$$

where log is the principal determination of the logarithm. For $\varepsilon > 0$ small enough, we have $\mathfrak{Re}\, f(z) > 0$ for $z \in V_\varepsilon$. Let $g\colon \mathbb{D} \to V_\varepsilon$ be the conformal map from $\mathbb{D}$ onto V_ε sending $\mathbb{T} = \partial\mathbb{D}$ onto ∂V_ε, and with $g(1) = 0$ and $g'(1) = -\varepsilon/4$. Explicitly, g is the composition of the following maps: a) $\sigma\colon z \mapsto -z$ from $\mathbb{D}$ onto itself; b) $\gamma\colon z \mapsto \frac{z+i}{1+iz}$ from $\mathbb{D}$ onto $P = \{\mathfrak{Im}\, z > 0\}$; c) $s\colon z \mapsto \sqrt{z}$ from P onto $Q = \{\mathfrak{Re}\, z > 0\,, \mathfrak{Im}\, z > 0\}$; d) $\gamma^{-1}\colon z \mapsto \frac{z-i}{1-iz}$ from Q onto $V = \{|z| < 1\,, \mathfrak{Re}\, z > 0\}$, and e) $h_\varepsilon\colon z \mapsto \varepsilon z$ from V onto V_ε.

We then set:

$$\varphi = \exp(-f \circ g)\,. \tag{4.7}$$

This analytic function φ maps $\mathbb{D}$ into itself and we proved in [20, Theorem 5.6] that C_φ is compact on H^2.

For $z = r\mathrm{e}^{i\alpha} \in V_\varepsilon$, we have (see [20, proof of Theorem 5.6]):

$$\mathfrak{Re}\, f(z) = r \log\log\frac{1}{r}\bigg(\cos\alpha + \frac{\alpha\sin\alpha}{\log(1/r)\log\log(1/r)}\bigg) + \mathrm{o}\bigg(\frac{1}{\log(1/r)\log\log(1/r)}\bigg) \tag{4.8}$$

$$\mathfrak{Im}\, f(z) = r \log\log\frac{1}{r}\bigg(\sin\alpha - \frac{\alpha\cos\alpha}{\log(1/r)\log\log(1/r)}\bigg) + \mathrm{o}\bigg(\frac{1}{\log(1/r)\log\log(1/r)}\bigg)\,. \tag{4.9}$$

It follows that:

$$0 < \mathfrak{Re}\, f(z) \lesssim r\log\log\frac{1}{r} \qquad \text{and} \qquad |\mathfrak{Im}\, f(z)| \lesssim r\log\log\frac{1}{r}\,\cdot \tag{4.10}$$

Now, assume that $r \leq h/\log\log(1/h)$. Then $r\log\log(1/r) \lesssim h$. Since $g'(1) \neq 0$, g is bi-Lipschitz in a neighborhood of 1; hence $|\varphi(u)| \approx 1 - \mathfrak{Re}\, f[g(u)]$ and $|\arg\varphi(u)| \approx |\mathfrak{Im}\, f[g(u)]|$, so we have:

$$A_\varphi\big(W(1,h)\big) \gtrsim \big(h/\log\log(1/h)\big)^2.$$

Therefore:

$$\rho_{\varphi,2}(h) \geq A_\varphi\big(W(1,h)\big) \gtrsim \frac{h^2}{\big(\log\log(1/h)\big)^2}\,, \tag{4.11}$$

so (4.2) cannot be satisfied. Hence $C_\varphi \notin S_p(\mathfrak{B}^2)$, whatever $p < \infty$. $\square$

When $C_\varphi \in S_p(H^2)$, we actually have better behavior on the Bergman space.

Theorem 4.4. *For every $p > 0$, we have $C_\varphi \in S_{p/2}(\mathfrak{B}^2)$ when $C_\varphi \in S_p(H^2)$.*

In particular, if C_φ is Hilbert-Schmidt on H^2, then it is nuclear on $\mathfrak{B}^2$ (since on Hilbert spaces the nuclear operators coincide with those in the Schatten class S_1).

Proof. We proved in [22, formula (3.26), page 3963], as a consequence of the main result of [21], that for some positive constants C, C', we have:

$$A_\varphi[W(\xi,h)] \leq C\,\big(m_\varphi[W(\xi,Ch)]\big)^2$$

for all $\xi \in \mathbb{T}$ and $0 < h < 1$ small enough. We may assume, enlarging C' if needed, that $C' = 2^N$ for some positive integer N. Hence if $W_{n,j} = W(\mathrm{e}^{2ji\pi/2^n}, 2^{-n})$ and $W'_{n,j} = W(\mathrm{e}^{2ji\pi/2^n}, 2^N 2^{-n})$, for $n > N$, we have:

$$\sum_{j=0}^{2^n-1} \big(4^n A_\varphi(W_{n,j})\big)^{p/4} \leq C \sum_{j=0}^{2^n-1} \big(2^n m_\varphi(W'_{n,j})\big)^{p/2}.$$

Now, each Carleson window $W'_{n,j}$ of size $2^N 2^{-n}$ is contained in the union of 2^N other Carleson windows $W_{n,j_1}, \ldots, W_{n,j_{2^N}}$ of size 2^{-n} and of less than $N2^{N-1}$ Hastings-Luecking boxes R_{ν,j_l} with $\nu \leq n-1$. Hence:

$$\sum_{j=0}^{2^n-1} \big(4^n A_\varphi(W_{n,j})\big)^{p/4} \leq C_p 2^N \sum_{j=0}^{2^n-1} \big(2^n m_\varphi(W_{n,j})\big)^{p/2} + C_p N 2^{N-1} \sum_{j=0}^{2^n-1} \big(2^n m_\varphi(R_{n,j})\big)^{p/2},$$

(since, for $a, b \geq 0$, we have $(a+b)^r \leq a^r + b^r$ if $0 < r \leq 1$, and $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ if $r \geq 1$).

It follows, thanks to [29, Corollary 2] and [20, Proposition 3.3], that $C_\varphi \in S_p(H^2)$ implies $C_\varphi \in S_{p/2}(\mathfrak{B}^2)$. $\square$

Theorem 4.4 is sharp, as shown by the following result.

Theorem 4.5. *For every p with $0 < p < \infty$, there exists a symbol φ for which the composition operator C_φ is in the Schatten class $S_p(H^2)$ on the Hardy space, but is not in any Schatten class $S_q(\mathfrak{B}^2)$ of the Bergman space with $q < p/2$.*

Before giving the proof, let us mention that this theorem implies (in a strong way) a separation between Schatten classes by composition operators on Bergman spaces. Curiously, we did not find any reference for this result.

Indeed, for every $r > 0$, there exists a symbol φ for which the composition operator C_φ is in the Schatten class $S_{2r}(H^2)$ on the Hardy space, hence in the Schatten class $S_r(\mathfrak{B}^2)$ on the Bergman space by Theorem 4.4, but which is not in any Schatten class $S_q(\mathfrak{B}^2)$ of the Bergman space for $q < r$.

Proof. Again, we use the variant of the Shapiro-Taylor map introduced in [20, Theorem 5.4] in order to have a composition operator in $S_p(H^2)$ but not in $S_q(H^2)$ for $q < p$. For $\varepsilon > 0$ small enough, we set, for $z \in V_\varepsilon$, where V_ε is defined in (4.5):

$$f(z) = z(-\log z)^{2/p}[\log(-\log z)]^s\,, \tag{4.12}$$

with $s > 1/p$.

We set:

$$\varphi = \exp(-f \circ g)\,, \tag{4.13}$$

where g is as in the proof of Theorem 4.3. Then $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic and we proved in [20, Theorem 5.4] that $C_\varphi \in S_p(H^2)$.

For $z = r\mathrm{e}^{i\alpha} \in V_\varepsilon$, we have (see [20, Lemma 5.5]:

$$0 < \mathfrak{Re}\, f(z) \lesssim r\Big(\log\frac{1}{r}\Big)^{2/p}\Big(\log\log\frac{1}{r}\Big)^s \tag{4.14}$$

$$|\mathfrak{Im}\, f(z)| \lesssim r\Big(\log\frac{1}{r}\Big)^{2/p}\Big(\log\log\frac{1}{r}\Big)^s\,. \tag{4.15}$$

As in the proof of Theorem 4.3, that implies that:

$$\rho_{\varphi,2}(h) \geq A_\varphi\big(W(1,h)\big) \gtrsim \frac{h^2}{(\log 1/h)^{4/p}(\log\log 1/h)^{2s}} \tag{4.16}$$

By Proposition 4.1, if C_φ is in $S_q(\mathfrak{B}^2)$, we have:

$$\rho_{\varphi,2}(h) = \mathrm{o}\bigg(\frac{h^2}{(\log 1/h)^{2/q}}\bigg)\,,$$

but, due to (4.16), this is not possible for $q < p/2$. Therefore $C_\varphi \notin S_q$. $\square$

Remark. Actually Theorem 4.4 has a more general form.

Theorem 4.6. *Let $\mathfrak{B}^2_{\gamma_1}$ and $\mathfrak{B}^2_{\gamma_2}$ be two weighted Bergman space of parameter γ_1 and γ_2, with $\gamma_2 > \gamma_1 \geq -1$. Then, for every $p > 0$ and any symbol φ, we have:*

1) *$C_\varphi \in S_p(\mathfrak{B}^2_{\gamma_1})$ implies $C_\varphi \in S_{\widetilde{p}}(\mathfrak{B}^2_{\gamma_2})$, with $\widetilde{p} = \frac{\gamma_1+2}{\gamma_2+2}\, p < p$;*
2) *when φ is finitely valent, the converse holds.*

Note that this theorem gives another, though less explicit, proof of Theorem 4.3 and of Theorem 4.5, as a direct consequence of [20, Theorem 5.4 and Theorem 5.6] since the symbols used in the proof of these theorems (and in that of the above Theorem 4.3 and Theorem 4.5) are univalent. In fact, that the Shapiro-Taylor map, defined in (4.7), is univalent is proved in [37, Lemma 4.1 (a)]. As well, the modified Shapiro-Taylor map, defined in (4.13), is univalent. In fact, its derivative $f'(z)$ is the sum of three terms, the dominant one being $(-\log z)^{2/p} \big(\log(-\log z)\big)^s$; it follows that, for ε small enough and $z = r\,\mathrm{e}^{it}$ with $0 < r < \varepsilon$ and $|t| < \pi/2$, we have:

$$\mathfrak{Re}\, f'(z) \geq \frac{1}{2} (\log 1/r)^{2/p} (\log\log 1/r)^s > 0\,,$$

and it follows that f is univalent in V_ε. In both cases, the symbol $\varphi = \exp(-f \circ g)$ is univalent.

Proof. Recall that D. Luecking and K. H. Zhu proved in [30, Theorem 1 and Theorem 3] that, for $\gamma \geq -1$, we have $C_\varphi \in S_p(\mathfrak{B}^2_\gamma)$ if and only if:

$$\int_{\mathbb{D}} \left(N_{\varphi,\gamma+2}(z) \left(\log \frac{1}{|z|} \right)^{-(\gamma+2)} \right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty\,, \tag{4.17}$$

where $N_{\varphi,\beta}$ ($\beta \geq 1$) is the *weighted Nevanlinna counting function*, defined as:

$$N_{\varphi,\beta}(z) = \sum_{\varphi(w)=z} \left(\log \frac{1}{|w|} \right)^\beta \tag{4.18}$$

if $z \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, and $N_{\varphi,\beta}(z) = 0$ otherwise.

As stated in the introduction, for $\gamma = -1$ we have $\mathfrak{B}^2_{-1} = H^2$.

Now, for $1 \leq \beta_1 < \beta_2$, the ℓ_{β_2}-norm is smaller than the ℓ_{β_1}-norm; so we have:

$$\big[N_{\varphi,\beta_2}\big]^{1/\beta_2} \leq \big[N_{\varphi,\beta_1}\big]^{1/\beta_1}\,. \tag{4.19}$$

It follows that, for $-1 \leq \gamma_1 < \gamma_2$:

$$\begin{aligned}
&\int_{\mathbb{D}} \Big(N_{\varphi,\gamma_2+2}(z) \Big(\log\frac{1}{|z|}\Big)^{-(\gamma_2+2)}\Big)^{\widetilde{p}/2} \frac{dA(z)}{(1-|z|^2)^2} \\
&\qquad \leq \int_{\mathbb{D}} \Big(\big[N_{\varphi,\gamma_1+2}(z)\big]^{\frac{\gamma_2+2}{\gamma_1+2}} \Big(\log\frac{1}{|z|}\Big)^{-(\gamma_2+2)}\Big)^{\widetilde{p}/2} \frac{dA(z)}{(1-|z|^2)^2} \\
&\qquad = \int_{\mathbb{D}} \Big(N_{\varphi,\gamma_1+2}(z) \Big(\log\frac{1}{|z|}\Big)^{-(\gamma_1+2)}\Big)^{p/2} \frac{dA(z)}{(1-|z|^2)^2}\,,
\end{aligned}$$

and that proves that $C_\varphi \in S_{\widetilde{p}}(\mathfrak{B}^2_{\gamma_2})$ if $C_\varphi \in S_p(\mathfrak{B}^2_{\gamma_1})$.

Now, if φ is s-valent, we have:

$$n_\varphi(z) := \sum_{\varphi(w)=z} 1 = \operatorname{card}\{w\in\mathbb{D}\,;\ \varphi(w)=z\} \leq s\,.$$

Using Hölder's inequality, we get, for $1\leq\beta_1<\beta_2$:

$$N_{\varphi,\beta_1}(z) \leq [n_\varphi(z)]^{(\beta_2-\beta_1)/\beta_2}\big[N_{\varphi,\beta_2}(z)\big]^{\beta_1/\beta_2} \leq s^{(\beta_2-\beta_1)/\beta_2}\big[N_{\varphi,\beta_2}(z)\big]^{\beta_1/\beta_2}\,.$$

Therefore:

$$\begin{aligned}
&\int_{\mathbb{D}} \Big(N_{\varphi,\gamma_1+2}(z) \Big(\log\frac{1}{|z|}\Big)^{-(\gamma_1+2)}\Big)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} \\
&\qquad \leq s^{p\,(\beta_2-\beta_1)/2\beta_2} \int_{\mathbb{D}} \Big(N_{\varphi,\gamma_2+2}(z) \Big(\log\frac{1}{|z|}\Big)^{-(\gamma_2+2)}\Big)^{\widetilde{p}/2} \frac{dA(z)}{(1-|z|^2)^2}\,,
\end{aligned}$$

and $C_\varphi \in S_{\widetilde{p}}((\mathfrak{B}^2_{\gamma_2})$ implies that $C_\varphi \in S_p(\mathfrak{B}^2_{\gamma_1})$. $\square$

5. Two examples

5.1. Preliminaries

Theorem 3.12 can be successfully applied to $H^2(\beta)=\mathcal{D}^2_\alpha$ and $H^2(\gamma)=\mathcal{D}^2_{\alpha'}$ with $-1<\alpha<\alpha'$. But as we will see now with the example of the cusp map χ, or of the lens maps, it does not provide as sharp estimates as wished. For example, with $H^2(\beta)=\mathcal{D}^2\subseteq H^2(\gamma)=\mathcal{D}^2_\alpha$ where $\alpha>0$, it gives, using [25, Theorem 3.1]:

$$a_n\big(C_\chi^{\mathcal{D}^2_\alpha}\big) \lesssim \sqrt{a_{n/2}(C_\chi^{\mathcal{D}^2})} \lesssim \exp(-b\sqrt{n})\,,$$

while we will see, using the point of view of weighted composition operators, that actually:

$$a_n(C_\chi^{\mathcal{D}^2_\alpha}) \lesssim \exp\big(-b\,(n/\log n)\big)\,.$$

We now elaborate on this point of view.

Let H be a Hilbert space of analytic functions on $\mathbb{D}$ whose set of multipliers $\mathcal{M}(H)$ is isometrically H^∞. For example, this is the case for $H = \mathfrak{B}^2_\gamma$ for all $\gamma > -1$, as we recall from Proposition 3.14.

Through a standard averaging argument, we easily have the following result (see [26, Lemma 2.2]).

Proposition 5.1. *Let H be a Hilbert space of analytic functions on $\mathbb{D}$ such that $\mathcal{M}(H) = H^\infty$. Let $z = (z_j)$ be a sequence of distinct points of $\mathbb{D}$ which is an interpolation sequence for H^∞ with constant I_z. Then, the sequence (K_{z_j}) is a Riesz sequence for H and moreover, for all $\lambda_1, \ldots, \lambda_n, \ldots \in \mathbb{C}$, we have:*

$$I_z^{-2} \sum_j |\lambda_j|^2 \, \|K_{z_j}\|^2 \leq \Big\| \sum_j \lambda_j K_{z_j} \Big\|^2 \leq I_z^2 \sum_j |\lambda_j|^2 \, \|K_{z_j}\|^2 \, .$$

In [19, Lemma 2.6], we used Proposition 5.1 to prove an estimate from below (the proof was given only for $H = H^2$ and $w \in H^\infty$). We slightly improve this estimate here, with nearly the same proof, as follows.

Theorem 5.2. *Let H be a Hilbert space of analytic functions on $\mathbb{D}$ such that $\mathcal{M}(H) = H^\infty$. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a symbol and $M_w C_\varphi \colon H \to H$ an associated weighted composition operator with weight $w \in H$. We assume that $M_w C_\varphi$ is bounded. Let $u = (u_j)_{1 \leq j \leq n}$ be a sequence of length n of points of $\mathbb{D}$ and $v_j = \varphi(u_j)$, and assume that the points v_j are distinct. Let I_v be the interpolation constant of $v = (v_j)_{1 \leq j \leq n}$. Then, the approximation numbers of $M_w C_\varphi$ satisfy:*

$$a_n(M_w C_\varphi) \geq \inf_{1 \leq j \leq n} \bigg(|w(u_j)| \, \frac{\|K_{v_j}\|}{\|K_{u_j}\|} \bigg) \, I_v^{-2} \, .$$

Proof. Recall that the approximation numbers $a_n(S)$ of an operator S on a Hilbert space coincide with its Bernstein numbers $b_n(S)$. Let E be the span of K_{u_j}, with $1 \leq j \leq n$. Set $\delta = \inf_{1 \leq j \leq n} \frac{\|K_{v_j}\|}{\|K_{u_j}\|} \, |w(u_j)|$. Take $f = \sum_{j=1}^n \lambda_j \, K_{u_j}$ in the unit sphere of E; we hence have $\sum_{j=1}^n |\lambda_j|^2 \|K_{u_j}\|^2 \geq I_u^{-2}$. Setting $T = M_w C_\varphi$, we see that $T^*(f) = \sum_{j=1}^n \lambda_j \overline{w(u_j)} \, K_{v_j}$, so that

$$\begin{aligned} \|T^*(f)\|^2 &\geq I_v^{-2} \sum_{j=1}^n |\lambda_j|^2 |w(u_j)|^2 \|K_{v_j}\|^2 \geq I_v^{-2} \delta^2 \sum_{j=1}^n |\lambda_j|^2 \|K_{u_j}\|^2 \\ &\geq \delta^2 I_v^{-2} I_u^{-2} \geq \delta^2 I_v^{-4} \, . \end{aligned}$$

In the last inequality, we used the obvious inequality $I_u \leq I_v$ (if $f(v_j) = a_j$, $j = 1, \ldots, n$, then $(f \circ \varphi)(u_j) = a_j$ for $j = 1, \ldots, n$, and $\|f \circ \varphi\|_\infty \leq \|f\|_\infty$).

Hence $a_n(T) = a_n(T^*) = b_n(T^*) \geq \delta I_v^{-2}$. $\square$

In order to apply Theorem 5.2 for weighted Dirichlet spaces, we will use the following process.

First, it suffices to prove the lower estimate with $(\mathcal{D}_\alpha^2)^*$ instead of $\mathcal{D}_\alpha^2$, where:

$$(\mathcal{D}_\alpha^2)^* = \{f \in \mathcal{D}_\alpha^2\,;\ f(0) = 0\} \tag{5.1}$$

is the hyperplane of $\mathcal{D}_\alpha^2$ of functions vanishing at 0.

The derivation Δ is by definition a unitary operator from $(\mathcal{D}_\alpha^2)^*$ onto $\mathfrak{B}_\alpha^2$. For any symbol φ vanishing at 0, we have the following diagram, where $w = \varphi'$:

$$(\mathcal{D}_\alpha^2)^* \xrightarrow{\ \Delta\ } \mathfrak{B}_\alpha^2 \xrightarrow{M_w C_\varphi} \mathfrak{B}_\alpha^2 \xrightarrow{\ \Delta^{-1}\ } (\mathcal{D}_\alpha^2)^*\,, \tag{5.2}$$

with obviously (since $\varphi'\,(f' \circ \varphi) = (f \circ \varphi)'$):

$$C_\varphi^{(\mathcal{D}_\alpha^2)^*} = \Delta^{-1}(M_w C_\varphi)\,\Delta\,,$$

which shows that $C_\varphi^{(\mathcal{D}_\alpha^2)^*}$, acting on the delicate space $(\mathcal{D}_\alpha^2)^*$ is unitarily equivalent to the weighted composition operator $M_w C_\varphi$ acting on the more robust space $\mathfrak{B}_\alpha^2$ (in that $\mathcal{M}(\mathfrak{B}_\alpha^2) = H^\infty$). Moreover, $C_\varphi^{(\mathcal{D}_\alpha^2)^*}$ and $M_w C_\varphi \colon \mathfrak{B}_\alpha^2 \to \mathfrak{B}_\alpha^2$ have the same approximation numbers.

5.2. The cusp map on weighted Dirichlet spaces

First, we recall the definition of the cusp map χ. We begin by defining:

$$\chi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1}\, \cdot \tag{5.3}$$

That defines a conformal mapping from $\mathbb{D}$ onto the right half-disk

$$D = \{z \in \mathbb{D}\,;\ \mathfrak{Re}\, z > 0\}$$

such that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2} - 1$. Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi}\,\chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)}\,, \tag{5.4}$$

and finally:

$$\chi(z) = 1 - \chi_3(z)\,, \tag{5.5}$$

where:

$$a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) = 1.56\ldots \in (1, 2) \tag{5.6}$$

is chosen in order that $\chi(0) = 0$. The image Ω of the (univalent) cusp map χ is formed by the intersection of the inside of the disk $D\big(1 - \frac{a}{2}, \frac{a}{2}\big)$ and the outside of the two closed disks $\overline{D\big(1 + \frac{ia}{2}, \frac{a}{2}\big)}$ and $\overline{D\big(1 - \frac{ia}{2}, \frac{a}{2}\big)}$.

Since χ is injective, it follows from Zorboska's characterization in [43] (see also [35, Section 6.12]) that the composition operator C_χ is bounded on $\mathcal{D}^2_\alpha$ for $\alpha \geq 0$. In particular $\chi \in \mathcal{D}^2_\alpha$.

Theorem 5.3. *Let χ be the cusp map acting on the Dirichlet space $\mathcal{D}^2_\alpha$. Then, for some positive constants $b'_\alpha > b_\alpha > 0$, depending only on α, we have, for all $n \geq 1$:*

$$\mathrm{e}^{-b'_\alpha \, n/\log n} \lesssim a_n(C_\chi) \lesssim \mathrm{e}^{-b_\alpha \, n/\log n} \qquad \text{for } \alpha > 0\,, \tag{5.7}$$

and

$$\mathrm{e}^{-b'_0 \sqrt{n}} \lesssim a_n(C_\chi) \lesssim \mathrm{e}^{-b_0 \sqrt{n}} \qquad \text{for } \alpha = 0\,. \tag{5.8}$$

Actually, the proof shows that, for $\alpha > 0$, the constant b_α can be chosen as $c \min(1, \alpha)$, and b'_α can be chosen as $c' \max(1, \alpha)$, where c, c' are absolute positive constants.

Note that the case $\alpha < 0$ is not relevant, since then composition operators C_φ that are compact on $\mathcal{D}^2_\alpha$ must satisfy $\|\varphi\|_\infty < 1$ ([34]).

The estimates (5.8) was first proved in [25, Theorem 3.1], with ad hoc methods. We give here a more transparent proof of the lower bound, based on weighted composition operators acting on $\mathfrak{B}^2_\alpha$, and that works for all $\alpha \geq 0$. Here the weight is χ'. Since $\chi \in \mathcal{D}^2_\alpha$, we have $\chi' \in \mathfrak{B}^2_\alpha$.

We have the following estimates (the first one was given in [27, Lemma 4.2]).

Lemma 5.4. *When $r \to 1^-$, it holds:*

$$1 - \chi(r) \approx \frac{1}{\log[1/(1 - r)]} \qquad \textit{and} \qquad \chi'(r) \approx \frac{1}{(1 - r) \log^2[1/(1 - r)]}\,. \tag{5.9}$$

Proof. For $r \in (0, 1)$, we have ([27, Lemma 4.2]):

$$\chi_0(r) = \tan\left[\frac{1}{2} \arctan\left(\frac{1 - r}{1 + r}\right)\right] = \tan\left(\frac{\pi}{8} - \frac{1}{2} \arctan r\right);$$

hence $\chi_0(r) \approx 1 - r$ and:

$$\chi'_0(r) = -\frac{1 + [\chi_0(r)]^2}{2(1 + r^2)}\,.$$

Using the definitions (5.4) and (5.5), we get:

$$\chi_2' = -\frac{2}{\pi}\,\frac{\chi_0'}{\chi_0} \quad \text{and} \quad \chi_3' = -\frac{a\,\chi_2'}{\chi_2^2}\,\cdot$$

So we have, when $r \to 1^-$:

$$\chi_2(r) \approx \log[1/(1-r)]$$

and:

$$\chi_2'(r) = \Big(\frac{1+[\chi_0(r)]^2}{\pi(1+r^2)}\Big)\,\frac{1}{\chi_0(r)} \approx \frac{1}{\chi_0(r)} \approx \frac{1}{1-r}\,\cdot$$

The result follows. $\square$

Proof of Theorem 5.3.
Proof of the lower bound.
We choose $0 < u_j < 1$ so as to get via (5.9), with $v_j = \chi(u_j)$ and $\varepsilon > 0$ to be adjusted later:

$$1 - v_j = \mathrm{e}^{-j\varepsilon}\,; \tag{5.10}$$

hence:

$$\log\Big(\frac{1}{1-u_j}\Big) \approx \mathrm{e}^{j\varepsilon}\,. \tag{5.11}$$

The interpolation constant I_v of the sequence $v = (v_j)_j$ satisfies $I_v \lesssim 1/\delta_v^2$, where δ_v is its Carleson constant (see [12, Chapter VII, Theorem 1.1]). Since $\frac{1-v_{j+1}}{1-v_j} = \mathrm{e}^{-\varepsilon}$, we have ([26, Lemma 6.4]), for some positive constants c_1, c_2:

$$\delta_v \geq \exp\Big(-\frac{c_1}{1-\mathrm{e}^{-\varepsilon}}\Big) \geq \mathrm{e}^{-c_2/\varepsilon}\,;$$

hence, with $c_0 = 2\,c_2$:

$$I_v \lesssim \mathrm{e}^{\,c_0/\varepsilon}$$

(where the implicit constant does not depend on ε).

The reproducing kernel of the Bergman space $\mathfrak{B}_\alpha^2$ satisfies:

$$\|K_a\| = \frac{1}{(1-|a|^2)^{1+\alpha/2}} \approx \frac{1}{(1-|a|)^{1+\alpha/2}}\,\cdot$$

Using (5.9) and (5.10), we get, for $1 \leq j \leq n$:

$$
\begin{aligned}
|\chi'(u_j)| \, \frac{\|K_{v_j}\|}{\|K_{u_j}\|} & \gtrsim \frac{1}{(1-u_j)\, \log^2[1/(1-u_j)]} \, \frac{(1-u_j)^{1+\alpha/2}}{(1-v_j)^{1+\alpha/2}} \\
& = \frac{1}{\log^2[1/(1-u_j)]} \, \frac{(1-u_j)^{\alpha/2}}{(1-v_j)^{1+\alpha/2}} \\
& \approx \frac{1}{\mathrm{e}^{2j\varepsilon}} \, \frac{\exp(-\frac{\alpha}{2}\, \mathrm{e}^{j\varepsilon})}{\mathrm{e}^{-j\varepsilon(1+\alpha/2)}} = \exp\bigg(- \Big[\frac{\alpha}{2}\, \mathrm{e}^{j\varepsilon} + j\varepsilon \Big(1 - \frac{\alpha}{2}\Big) \Big] \bigg) \\
& \gtrsim \exp\bigg(- \Big[\frac{\alpha}{2}\, \mathrm{e}^{n\varepsilon} + n\varepsilon \Big(1 - \frac{\alpha}{2}\Big) \Big] \bigg) ,
\end{aligned}
$$

since the function $t \mapsto \frac{\alpha}{2}\, \mathrm{e}^t + t\,\big(1 - \frac{\alpha}{2}\big)$ is increasing; in fact, its derivative is positive.

Theorem 5.2 with $w = \chi'$ gives:

$$
a_n^{\mathcal{B}_\alpha^2}(M_{\chi'} C_\chi) \gtrsim \exp\bigg(- \Big[\frac{\alpha}{2}\, \mathrm{e}^{n\varepsilon} + n\varepsilon \Big(1 - \frac{\alpha}{2}\Big) + \frac{2\, c_0}{\varepsilon} \Big] \bigg) . \tag{5.12}
$$

Case $\alpha = 0$. In this case, we have:

$$
a_n^{\mathfrak{B}_0^2}(M_{\chi'} C_\chi) \gtrsim \exp\bigg(- \Big[n\varepsilon + \frac{2\, c_0}{\varepsilon} \Big] \bigg) .
$$

Taking $\varepsilon = 1/\sqrt{n}$, we get:

$$
a_n^{\mathcal{D}_0^2}(C_\chi) = a_n^{\mathfrak{B}_0^2}(M_{\chi'} C_\chi) \gtrsim \mathrm{e}^{-c\sqrt{n}}
$$

for some positive absolute constant c.

Case $\alpha > 0$. We take $\varepsilon = \frac{1}{n} \log\big(\frac{n}{\log n}\big)$ and we get:

$$
a_n^{\mathfrak{B}_\alpha^2}(M_{\chi'} C_\chi) \gtrsim \exp\bigg(- \Big[\frac{\alpha}{2}\, \frac{n}{\log(n)} + \log\Big(\frac{n}{\log(n)}\Big) + 2\, c_0 \frac{n}{\log(n) - \log(\log(n))} \Big] \bigg) .
$$

Since $\log\big(\frac{n}{\log(n)}\big) + 2\, c_0 \frac{n}{\log(n) - \log(\log(n))} = \mathrm{O}\big(\frac{n}{\log(n)}\big)$, we get:

$$
a_n^{\mathcal{D}_\alpha^2}(C_\chi) \gtrsim \exp(-b'_\alpha\, n / \log n) ,
$$

with $b'_\alpha = c' \max(\alpha, 1)$, where c' is some positive absolute constant.

Proof of the upper bound.

For $\alpha = 0$, the upper bound is proved in [25, Theorem 3.1]; for $\alpha > 0$, we will follow that proof, with the same notation, but with the weighted Nevanlinna counting function $N_{\chi,\alpha}$ instead of the counting function n_χ. Recall that the weighted Nevanlinna counting function of the analytic function $\varphi\colon \mathbb{D} \to \mathbb{D}$ is:

$$
N_{\varphi,\alpha}(w) = \sum_{\varphi(z) = w} (1 - |z|^2)^\alpha .
$$

Note that this definition is slightly different from, although equivalent to, that given in (4.18), but it is more convenient here.

Since the cusp map χ is univalent, we have:

$$N_{\chi,\alpha}(w) = \begin{cases} (1-|\chi^{-1}(w)|^2)^\alpha & \text{for } w \in \chi(\mathbb{D})\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

The Schwarz lemma gives $N_{\chi,\alpha}(w) \leq (1-|w|^2)^\alpha$, but the following lemma gives the better estimate:

$$N_{\chi,\alpha}(w) \lesssim \mathrm{e}^{-c_0\alpha/(1-|w|)}\,, \quad \text{for } w \in \chi(\mathbb{D}) \tag{5.14}$$

Lemma 5.5. *We have:*

$$1-|\chi(z)| \gtrsim \frac{1}{\log[1/|1-z|]} \quad \textit{for all } z \in \mathbb{D}\,,$$

so that, for some positive constant c_0:

$$1-|\chi^{-1}(w)| \leq |1-\chi^{-1}(w)| \lesssim \exp\Big(-\frac{c_0}{1-|w|}\Big) \quad \textit{for all } w \in \chi(\mathbb{D})\,.$$

Proof of the lemma. It suffices to look at the neighborhood of 1, since outside the two functions are continuous and do not vanish. Setting $u = 1-z$, an easy computation with Taylor expansions gives that $\frac{z-i}{iz-1} = -1 + iu + \mathrm{o}\,(u)$, as $u \to 0$, so $\big(\frac{z-i}{iz-1}\big)^{1/2} = i[1-iu/2 + \mathrm{o}\,(u)] = i + u/2 + \mathrm{o}\,(u)$, and (recall (5.3) and (5.4)):

$$\chi_0(1-u) = \frac{u}{4} + \mathrm{o}\,(u) \quad \text{as } u \to 0\,;$$

and $|1-\chi(z)| = |\chi_3(z)| \approx \dfrac{1}{|\log(\chi_0(z))|} \gtrsim \dfrac{1}{\log(1/|1-z|)}\,.$

Finally, since the cusp is contained in an angular sector, there exists some $\delta > 0$ such that $1-|\chi(z)| \geq \delta|1-\chi(z)|$ for every $z \in \mathbb{D}$. The result follows. □

Thus we obtain the following estimate.

Lemma 5.6. *We have:*

$$\|\chi^n\|^2_{\mathcal{D}^2_\alpha} \lesssim n^2\,\mathrm{e}^{-\sqrt{2c_0\alpha\,n}}\,.$$

Proof. We have, since $\chi(0) = 0$:

$$\|\chi^n\|^2_{\mathcal{D}^2_\alpha} = \int_{\mathbb{D}} |n\chi^{n-1}(z)\,\chi'(z)|^2\,dA_\alpha(z)$$

$$
\begin{aligned}
&= \int\limits_{\chi(\mathbb{D})} n^2 |w|^{2n-2} N_{\chi,\alpha}(w)\, dA(w) \\
&\leq \int\limits_{\chi(\mathbb{D})} n^2 |w|^{2n-2}\, \mathrm{e}^{-c_0\alpha/(1-|w|)}\, dA(w) \\
&\leq n^2 (1-h)^{2n-2} + n^2 \int\limits_{\chi(\mathbb{D})\cap\{|w|\geq 1-h\}} \mathrm{e}^{-c_0\alpha/(1-|w|)}\, dA(w) \\
&\lesssim n^2\, \mathrm{e}^{-2nh} + n^2\, \mathrm{e}^{-c_0\alpha/h} .
\end{aligned}
$$

Choosing $h = \sqrt{c_0\alpha/2n}$ gives:

$$
\|\chi^n\|^2_{\mathcal{D}^2_\alpha} \lesssim n^2\, \mathrm{e}^{-\sqrt{2c_0\alpha\, n}} . \quad \square
$$

For $f(z) = \sum_{n=0}^\infty c_n z^n$, we define:

$$
(S_N f)(z) = \sum_{n=0}^N c_n z^n .
$$

As a consequence of Lemma 5.6, we have the following majorization.

Lemma 5.7. *We have:*

$$
\|C_\chi - C_\chi S_N\|_{\mathcal{D}^2_\alpha} \lesssim N^{\frac{3+2\alpha}{4}}\, \mathrm{e}^{-\sqrt{2c_0\alpha\, N}} . \tag{5.15}
$$

Proof. It suffices to use the Hilbert-Schmidt norm and Lemma 5.6:

$$
\begin{aligned}
\|C_\chi - C_\chi S_N\|^2_{\mathcal{D}^2_\alpha} \leq \|C_\chi - C_\chi S_N\|^2_{\mathrm{HS}} &\approx \sum_{n>N} \frac{\|\chi^n\|^2_{\mathcal{D}^2_\alpha}}{n^{1-\alpha}} \\
&\lesssim \sum_{n>N} n^{1+\alpha}\, \mathrm{e}^{-\sqrt{2c_0\alpha\, n}} \approx N^{\frac{3}{2}+\alpha}\, \mathrm{e}^{-\sqrt{2c_0\alpha\, N}} . \quad \square
\end{aligned}
$$

Lemma 5.8. *Let J be the canonical injection $J\colon H^2 \to L^2(\mu)$ with $d\mu = N_{\chi,\alpha}\, dA$. Then:*

$$
a_n(C_\chi S_N) \lesssim N^{\frac{1+\alpha}{2}}\, a_n(J) . \tag{5.16}
$$

Proof. Let $f \in (\mathcal{D}^2_\alpha)^*$ and write $f(z) = \sum_{j=1}^\infty c_j z^j$, we have:

$$
\|C_\chi S_N f\|^2_{\mathcal{D}^2_\alpha} = \int\limits_{\mathbb{D}} \Big| \sum_{j=1}^N j c_j w^{j-1} \Big|^2 N_{\chi,\alpha}(w)\, dA(w) = \int\limits_{\mathbb{D}} |(\Delta_N f)(w)|^2\, d\mu(w) ,
$$

where $\Delta_N\colon (\mathcal{D}^2_\alpha)^* \to H^2$ is defined by:

$$(\Delta_N f)(w) = \sum_{j=1}^{N} j c_j w^{j-1} .$$

We hence have $\|C_\chi S_N f\|_{\mathcal{D}^2_\alpha} = \|J \Delta_N f\|_{L^2(\mu)}$ for all $f \in (\mathcal{D}^2_\alpha)^*$. It follows that there exists a contraction $T_N \colon L^2(\mu) \to (\mathcal{D}^2_\alpha)^*$ such that:

$$C_\chi S_N = T_N J \Delta_N .$$

Now:

$$\|\Delta_N f\|^2_{H^2} = \sum_{j=0}^{N} j^{1+\alpha} j^{1-\alpha} |c_j|^2 \leq N^{1+\alpha} \sum_{j=0}^{N} j^{1-\alpha} |c_j|^2 \leq N^{1+\alpha} \|f\|^2_{\mathcal{D}^2_\alpha} ;$$

hence, by the ideal property of approximation numbers:

$$a_n(C_\chi S_N) \approx a_n\big((C_\chi S_N)_{|(\mathcal{D}^2_\alpha)^*}\big) \leq \|T_N\| \, \|\Delta_N\| \, a_n(J) \lesssim N^{\frac{\alpha+1}{2}} \, a_n(J) . \qquad \square$$

Proposition 5.9. *Let $\alpha > 0$ and J be the canonical injection $J \colon H^2 \to L^2(\mu)$ with $d\mu = N_{\chi,\alpha} \, dA$. Then, for some absolute positive constant c:*

$$a_n(J) \lesssim \exp\Big(-c \, \min(\alpha, 1) \, \frac{n}{\log n}\Big) . \tag{5.17}$$

Proof. We use a modification of the Blaschke product of [25, page 168], as follows. Let $r = [\log_2 n]$ be the greatest integer $< \log_2 n$, where $\log_2$ is the binary logarithm, and B_0 be the Blaschke product with simple zeros at the points:

$$z_j = 1 - 2^{-j} , \qquad 1 \leq j \leq r ,$$

and we consider the Blaschke product $B = B_0^n$.

Let $E = BH^2$, which is a subspace of H^2 of codimension $n \, [\log_2 n]$.

We have, by the Carleson embedding theorem for H^2 (see [23, Lemma 2.4]):

$$\|J_{|E}\|^2 \lesssim \sup_{\substack{0<h<1 \\ \xi \in \mathbb{T}}} \frac{1}{h} \int_{S(\xi,h) \cap \Omega} |B|^2 N_{\chi,\alpha} \, dA , \tag{5.18}$$

where $S(\xi, h) = \{z \in \mathbb{D} \, ; \; |z - \xi| < h\}$ and:

$$\Omega = \chi(\mathbb{D}) . \tag{5.19}$$

Note that $A[S(\xi, h) \cap \Omega] \lesssim h^3$ since the area of $\chi(\mathbb{D}) \cap \{|w| \geq 1 - h\}$ is $\approx h^3$; in fact this set is delimited at the cuspidal point 1 by two circular arcs.

Now we majorize the right-hand side of (5.18). For that, we first note that, since Ω is contained in an angular sector, there is an absolute positive constant δ_0 such that $1-|w| \geq \delta_0 |1-w|$ for all $w \in \Omega$. Hence if $w \in S(\xi, h) \cap \Omega$, we have:

$$\delta_0\, |1-w| \leq 1-|w| \leq |\xi - w| < h\,,$$

and $w \in S(1, h/\delta_0)$. Hence $S(\xi, h) \cap \Omega \subseteq S(1, h/\delta_0) \cap \Omega$.

Moreover, we may assume that $h = \delta_0 2^{-l}$ and we separate two cases.

• $l \geq r$.

We simply majorize $|B|$ by 1. Lemma 5.5 and (5.14) lead to:

$$\begin{aligned} \frac{1}{h} \int_{S(\xi,h)\cap\Omega} |B|^2 N_{\chi,\alpha}\, dA &\lesssim \frac{1}{h} \int_{S(\xi,h)\cap\Omega} \exp(-\alpha\, c_0/h)\, dA \\ &\lesssim \frac{1}{h}\, \mathrm{e}^{-\alpha c_0/h}\, A[S(\xi,h)\cap\Omega] \approx h^2\, \mathrm{e}^{-\alpha c_0/h} \\ &\leq \mathrm{e}^{-\alpha\, (c_0/\delta_0)\, 2^l} \leq \mathrm{e}^{-\alpha\, (c_0/\delta_0)\, 2^r} \leq \mathrm{e}^{-\alpha\, (c_0/2\delta_0) n}\,. \end{aligned}$$

• $l < r$.

We write:

$$\begin{aligned} \int_{S(\xi,h)\cap\Omega} |B|^2 N_{\chi,\alpha}\, dA &\leq \int_{S(1,h/\delta_0)\cap\Omega} |B|^2 N_{\chi,\alpha}\, dA \\ &= \int_{S(1,2^{-r})\cap\Omega} |B|^2 N_{\chi,\alpha}\, dA \\ &\qquad + \int_{\{w\in\Omega\,;\ 2^{-r}\leq|w-1|<2^{-l}\}} |B|^2 N_{\chi,\alpha}\, dA\,. \end{aligned}$$

By the previous case, the first integral in the right-hand side, divided by h, is $\lesssim \mathrm{e}^{-\alpha\, (c_0/2\delta_0) n}$.

Now, if $2^{-r} \leq |w-1| < 2^{-l}$, there is some $j \in \{l+1, \ldots, r\}$ such that $2^{-j} \leq |w-1| < 2^{-j+1}$. Then:

$$|w - z_j| \leq |w-1| + |1-z_j| \leq 2^{-j+1} + 2^{-j} = 3\,.\,2^{-j}\,;$$

hence, since $2^{-j} \leq (1/\delta_0)\,(1-|w|)$ and $1-|z_j| = 1-z_j = 2^{-j}$, we have, with $M = \max(3, 1/\delta_0)$:

$$|w - z_j| \leq M\, \min(1-|w|, 1-|z_j|)\,.$$

Therefore [23, Lemma 2.3] shows that the modulus of the j-th factor of $B_0(w)$ is less or equal than $\kappa = M/\sqrt{M^2+1} < 1$. It follows that $|B(w)| = |B_0(w)|^n \leq \kappa^n$. Using $N_{\chi,\alpha}(w) \leq 1$ we obtain:

$$\frac{1}{h} \int\limits_{\{w\in\Omega\,;\ 2^{-r}\leq|w-1|<2^{-l}\}} |B|^2 N_{\chi,\alpha}(w)\, dA \lesssim \frac{1}{h}\, A[S(\xi,h)\cap\Omega]\, \kappa^{2n} \lesssim \kappa^{2n} = \mathrm{e}^{-\varepsilon_0 n}$$

for some absolute constant $\varepsilon_0 > 0$.

We get:

$$\frac{1}{h} \int\limits_{S(\xi,h)\cap\Omega} |B|^2 N_{\chi,\alpha}\, dA \lesssim \mathrm{e}^{-\alpha\,(c_0/2\delta_0) n} + \mathrm{e}^{-\varepsilon_0 n} \lesssim \mathrm{e}^{-c_1 \min(\alpha,1)\, n}\,, \tag{5.20}$$

where $c_1 > 0$ does not depend on α.

In either case, we obtain,

$$\| J_{|E} \| \lesssim \mathrm{e}^{-c_2 \min(\alpha,1)\, n}\,.$$

That means that the Gelfand number $c_{n\,[\log_2 n]}(J)$ is $\lesssim \mathrm{e}^{-c_2 \min(\alpha,1)\, n}$. Since the Gelfand numbers are the same as the approximation numbers on Hilbert spaces, we get:

$$a_{n\,[\log_2 n]}(J) \lesssim \mathrm{e}^{-c_2 \min(\alpha,1)\, n},$$

or, making a change of variables in the indices:

$$a_n(J) \lesssim \exp\Big(-c_3\, \min(\alpha,1)\, \frac{n}{\log n}\Big)\,,$$

as claimed. □

End of the proof of the upper bound. For every operator $R\colon \mathcal{D}^2_\alpha \to \mathcal{D}^2_\alpha$ with rank $< n$, we have:

$$\| C_\chi - R \| \leq \| C_\chi - C_\chi S_N \| + \| C_\chi S_N - R \|\,;$$

so:

$$a_n(C_\chi) \leq \| C_\chi - C_\chi S_N \| + a_n(C_\chi S_N)\,.$$

Using Lemma 5.7, Lemma 5.8 and Proposition 5.9, we obtain:

$$\begin{aligned} a_n(C_\chi) &\lesssim N^{\frac{3+2\alpha}{4}}\, \mathrm{e}^{-\sqrt{2c_0\alpha\, N}} + N^{\frac{\alpha+1}{2}}\, \mathrm{e}^{-c_3\, \min(\alpha,1)\, n/\log n} \\ &\lesssim N^{\frac{3+2\alpha}{4}}\, \big(\mathrm{e}^{-\sqrt{2c_0\alpha\, N}} + \mathrm{e}^{-c_3 \min(\alpha,1)\, n/\log n}\big)\,. \end{aligned}$$

a) Suppose first that $\alpha \leq 1$. Then:

$$a_n(C_\chi) \lesssim N^{\frac{3+2\alpha}{4}}\, \big(\mathrm{e}^{-\sqrt{2c_0\alpha\, N}} + \mathrm{e}^{-c_3\alpha\, n/\log n}\big)\,.$$

Choosing N as the integral part of $\alpha(c_3^2/2c_0)\,(n/\log n)^2$, we get:

$$a_n(C_\chi) \lesssim \Big(\frac{n}{\log n}\Big)^{\frac{3+2\alpha}{2}} \mathrm{e}^{-c_3\alpha n/\log n} \lesssim \Big(\frac{n}{\log n}\Big)^{\frac{5}{2}} \mathrm{e}^{-c_3\alpha n/\log n} \lesssim \mathrm{e}^{-c_3'\alpha n/\log n}\,,$$

for another absolute constant $c_3' < c_3$.

b) For $\alpha > 1$, we have:

$$a_n(C_\chi) \lesssim N^{\frac{3+2\alpha}{4}}\,\big(\mathrm{e}^{-\sqrt{2c_0\alpha\,N}} + \mathrm{e}^{-c_3\,n/\log n}\big)\,.$$

Choosing for N the integral part of $(1/\alpha)(c_3^2/2c_0)(n/\log n)^2$, we get:

$$a_n(C_\chi) \lesssim \Big(\frac{n}{\log n}\Big)^{\frac{3+2\alpha}{2}} \mathrm{e}^{-c_3 n/\log n}\,.$$

However, the term $(n/\log n)^{\frac{3+2\alpha}{2}}$ tends to infinity when α tends to infinity (even if the implicit constants in the inequalities depend on α). In order to have a better estimate, we need a different strategy.

We recall that $\mathcal{D}_1^2 = H^2$. We now use Theorem 3.12, which is licit as indicated in the remarks following the statement of this theorem. Using the previously treated case, we obtain that:

$$a_{2n}^{\mathcal{D}_\alpha^2}(C_\chi) \leq \sqrt{a_1^{H^2}(C_\chi)\, a_n^{H^2}(C_\chi)} \lesssim \exp\Big(-c\,\frac{n}{\log n}\Big)\cdot$$

Hence, rescaling on one hand and using the monotony of the sequence of the approximation numbers on the other hand, we get:

$$a_n^{\mathcal{D}_\alpha^2}(C_\chi) \lesssim \exp\Big(-c'\,\frac{n}{\log n}\Big)\,,$$

where the underlying constants do not depend on α.

That ends the proof of Theorem 5.3. $\square$

Remark 1. Actually, for $\alpha > 0$, the formula (5.12) gives that:

$$a_n^{\mathcal{D}_\alpha^2}(C_\chi) \gtrsim \exp\Big(-\Big[\frac{\alpha}{2}\,\mathrm{e}^{n\varepsilon} + n\varepsilon\Big(1-\frac{\alpha}{2}\Big) + \frac{2\,c_0}{\varepsilon}\Big]\Big)\,. \tag{5.21}$$

Taking $\varepsilon = 1/\sqrt{n}$, we get, for $0 < \alpha < 2$, with $c_1 = 1 + 2\,c_0$, this “bad” estimate:

$$a_n^{\mathcal{D}_\alpha^2}(C_\chi) \gtrsim \exp\Big(-\Big[\frac{\alpha}{2}\,\mathrm{e}^{\sqrt{n}} + c_1\,\sqrt{n}\Big]\Big)\,.$$

Nevertheless, Theorem 3.12, 3) a), gives:

$$a_n^{\mathcal{D}_0^2}(C_\chi) \gtrsim \Big(a_{2n}^{\mathcal{D}_\alpha^2}(C_\chi)\Big)^2 \gtrsim \exp\bigg(-\Big[\alpha\,\mathrm{e}^{\sqrt{2n}} + 2c_1\sqrt{2n}\Big]\bigg).$$

Note that, even though we did not explicitly state them, the implicit constants in these inequalities are $\approx 2^{-\alpha/2}$; so, letting α tend to 0, we obtain, with $c = 2^{\frac{3}{2}}\,c_1$:

$$a_n^{\mathcal{D}_0^2}(C_\chi) \gtrsim \mathrm{e}^{-c\sqrt{n}}\,,$$

explaining the jump between the cases $\alpha > 0$ and $\alpha = 0$.

Remark 2. When $\alpha \to 0^+$, the behavior both of the upper and the lower estimates remains quite far from the one in the case $\alpha = 0$. It would be interesting to get better control on both sides relative to α to understand the breaking point between the case $\alpha > 0$ and the case $\alpha = 0$. Very likely, it would require a different viewpoint and different methods of estimating approximation numbers.

5.3. Lens maps for weighted Dirichlet spaces

In this section, we consider lens maps (see [36, page 27]). Let us recall that for $0 < \theta < 1$, the lens map λ_θ of parameter θ is the map from $\mathbb{D}$ into $\mathbb{D}$ defined by:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}\,. \tag{5.22}$$

It is a conformal map obtained by sending $\mathbb{D}$ onto the right half-plane, then taking the power θ, and going back to $\mathbb{D}$.

Since λ_θ is univalent, it follows from [43, Theorem 1] that the associated composition operator C_{λ_θ} is bounded on $\mathcal{D}_\alpha^2$ for all $\alpha \geq 0$.

Theorem 5.10. *Let $0 < \theta < 1$ and λ_θ be the lens map of parameter θ. Then the composition operator C_{λ_θ} is not compact on $\mathcal{D}^2 = \mathcal{D}_0^2$; but for all $\alpha > 0$, C_{λ_θ} is compact on $\mathcal{D}_\alpha^2$, and moreover there are positive constants $b > b' > 0$, depending only on θ and α, such that, for all $n \geq 1$:*

$$\mathrm{e}^{-b\sqrt{n}} \lesssim a_n(C_{\lambda_\theta}) \lesssim \mathrm{e}^{-b'\sqrt{n}}\,. \tag{5.23}$$

In particular, for $\alpha > 0$, C_{λ_θ} is in all the Schatten classes $S_p(\mathcal{D}_\alpha^2)$ for $p > 0$.

The proof shows that we can take $b = \sqrt{\alpha}\,b_\theta$, where b_θ is a positive constant depending only on θ and that the constant b' can be taken equal to $c\,\frac{2(1-\theta)}{2\alpha+(1-\alpha)\theta}\,\alpha^{3/2}$ for some positive absolute constant c.

Proof. Since λ_θ is univalent, its weighted Nevanlinna counting function is:

$$N_{\lambda_\theta,\alpha}(w) = (1 - |\lambda_\theta^{-1}(w)|)^\alpha \qquad \text{for } w \in \Omega := \lambda_\theta(\mathbb{D})$$

and 0 elsewhere. By [43, Theorem 1], C_{λ_θ} is compact on $\mathcal{D}^2_\alpha$ if and only if:

$$\sup_{\xi \in \mathbb{T}} \frac{1}{h^2} \int\limits_{W(\xi,h)} \frac{N_{\lambda_\theta,\alpha}(w)}{(1 - |w|^2)^\alpha}\, dA(w) \underset{h \to 0}{\longrightarrow} 0\,.$$

Since, for $w \in \Omega$:

$$1 - |\lambda_\theta^{-1}(w)| \approx (1 - |w|)^{1/\theta}\,, \tag{5.24}$$

we have:

$$\int\limits_{W(\xi,h)} \frac{N_{\lambda_\theta,\alpha}(w)}{(1 - |w|^2)^\alpha}\, dA(w) \approx h^{2 + \frac{1-\theta}{\theta}\alpha}\,;$$

so C_{λ_θ} is compact on $\mathcal{D}^2_\alpha$ for $\alpha > 0$, but is not compact on $\mathcal{D}^2_0$.

For the estimates on approximation numbers, the proof follows the line of that of Theorem 5.3; hence we only sketch it.

Lower estimate.

For $0 < \alpha \leq 1$, we can use Theorem 3.12, 3) a) and [23, Theorem 2.1]:

$$\mathrm{e}^{-c\sqrt{n}} \lesssim \big[a_{2n}\big(C_{\lambda_\theta}^{H^2}\big)\big]^2 \lesssim a_n\big(C_{\lambda_\theta}^{\mathcal{D}^2_\alpha}\big)\,,$$

since $H^2 = \mathcal{D}^2_1$ is dominated by $\mathcal{D}^2_\alpha$.

However, for all $\alpha > 0$, the proof given for the cusp map can be used also for the lens maps. The only difference is that, if $\lambda_\theta(u_j) = v_j$, we have $1 - u_j \approx (1 - v_j)^{1/\theta}$, via (5.24), and:

$$\lambda_\theta'(z) \approx (1 - z)^{\theta - 1} \qquad \text{for } z \in \mathbb{D} \text{ with } \mathfrak{Re}\, z > 0\,.$$

Hence we get:

$$|\lambda_\theta'(u_j)|\, \frac{\|K_{v_j}\|}{\|K_{u_j}\|} \gtrsim \frac{1}{(1 - v_j)^{\frac{1}{\theta} - 1}}\, \frac{(1 - v_j)^{\frac{1}{\theta}(1 + \frac{\alpha}{2})}}{(1 - v_j)^{1 + \frac{\alpha}{2}}} = (1 - v_j)^{\frac{\alpha}{2}(\frac{1}{\theta} - 1)}\,.$$

Choosing $v_j = 1 - \mathrm{e}^{-j\varepsilon}$, we get:

$$a_n^{\mathfrak{B}^2_\alpha}(M_{\lambda_\theta'} C_{\lambda_\theta}) \gtrsim \exp\bigg(- \Big[\frac{\alpha}{2}\Big(\frac{1}{\theta} - 1\Big)\, n\,\varepsilon + \frac{C}{\varepsilon}\Big]\bigg)\,.$$

Taking $\varepsilon = \sqrt{\frac{2C}{\alpha}\frac{\theta}{1-\theta}}\,\frac{1}{\sqrt{n}}$ gives now the result.

Upper estimate.

1) We have:

Lemma 5.11. *For the lens map λ_θ of parameter θ, $0 < \theta < 1$, we have, for $\alpha > 0$:*

$$\|\lambda_\theta^n\|_{\mathcal{D}^2_\alpha} \lesssim n^{-\alpha/2\theta} . \tag{5.25}$$

Proof. Let $\Omega = \{z \in \mathbb{D}\,;\ \mathfrak{Re}\, z > 0\}$. For $z \in \Omega$, we easily see that:

$$|\lambda_\theta(z)| \leq \exp(-c\,|1-z|^\theta) \quad \text{and} \quad |\lambda_\theta'(z)| \lesssim |1-z|^{\theta-1} . \tag{5.26}$$

We get, using $(1-|z|)^\alpha \leq |1-z|^\alpha$ and the symmetry of λ_θ:

$$\begin{aligned} \|\lambda_\theta^n\|^2_{\mathcal{D}^2_\alpha} &= 2(\alpha+1)\int_\Omega n^2|\lambda_\theta(z)|^{2n-2}|\lambda_\theta'(z)|^2(1-|z|^2)^\alpha \\ &\lesssim n^2\int_\Omega \exp\big(-2\,c(n-1)\,|1-z|^\theta\big)\,|1-z|^{2\theta-2}\,|1-z|^\alpha\,dA(z) . \end{aligned}$$

Using polar coordinates centered at 1 and then making the change of variable $x = cnr^\theta$, so $r = c^{-1/\theta}n^{-1/\theta}x^{1/\theta}$, we obtain:

$$\begin{aligned} \|\lambda_\theta^n\|^2_{\mathcal{D}^2_\alpha} &\lesssim n^2\int_0^{+\infty} \exp(-cnr^\theta)\,r^{2\theta-2}\,r^\alpha\,r\,dr \\ &\approx n^2\int_0^{+\infty} \mathrm{e}^{-x}n^{-\frac{2\theta-1+\alpha}{\theta}}x^{\frac{2\theta-1+\alpha}{\theta}}x^{\frac{1}{\theta}-1}n^{-1/\theta}\,dx \approx n^{-\alpha/\theta} . \qquad \square \end{aligned}$$

2) We have:

$$\|C_{\lambda_\theta} - C_{\lambda_\theta}S_N\|^2_{\mathcal{D}^2_\alpha} \lesssim \frac{1}{N^{\frac{1-\theta}{\theta}\,\alpha}} \cdot \tag{5.27}$$

In fact, using the Hilbert-Schmidt norm on $\mathcal{D}^2_\alpha$:

$$\begin{aligned} \|C_{\lambda_\theta} - C_{\lambda_\theta}S_N\|^2_{\mathcal{D}^2_\alpha} \leq \|C_{\lambda_\theta} - C_{\lambda_\theta}S_N\|^2_{HS} &\lesssim \sum_{n>N}\frac{1}{n^{\alpha/\theta}}\,\frac{1}{n^{1-\alpha}} \\ &= \sum_{n>N}\frac{1}{n^{\frac{1-\theta}{\theta}\,\alpha+1}} \approx \frac{1}{N^{\frac{1-\theta}{\theta}\,\alpha}} \cdot \end{aligned}$$

3) We have, exactly as for the cusp map:

$$a_n(C_{\lambda_\theta} S_N) \lesssim N^{\frac{\alpha+1}{2}}\, a_n(J)\,, \tag{5.28}$$

where $J\colon H^2 \to L^2(\mu)$ is the canonical injection.

4) We have:

$$a_n(J) \lesssim \mathrm{e}^{-c\,\sqrt{\alpha}\,\sqrt{n}}\,. \tag{5.29}$$

In fact, we take the Blaschke product B_0 as for the cusp map, except that here we take its length r as the largest integer $< \sqrt{n}$. We then take $B = B_0^{[\alpha\sqrt{n}]}$. With the notation used for the cusp map, when $l < r$ and $2^{-r} \leq |w-1| < 2^{-l}$, we have $|B(w)| \lesssim \kappa^{\alpha\sqrt{n}}$; and when $l \geq r$, we use (5.24).

5) Finally, we have:

$$a_n(C_{\lambda_\theta}) \lesssim \frac{1}{N^{\frac{1-\theta}{\theta}\,\alpha}} + N^{\frac{\alpha+1}{2}}\, \mathrm{e}^{-c\,\sqrt{\alpha n}}\,,$$

and the choice for N of the integer part of $\mathrm{e}^{c\,\frac{2\theta}{2\alpha+(1-\alpha)\theta}\,\sqrt{\alpha}\,\sqrt{n}}$ gives:

$$a_n(C_{\lambda_\theta}) \lesssim \exp\bigg(-c\,\frac{2(1-\theta)}{2\alpha+(1-\alpha)\theta}\,\alpha^{3/2}\,\sqrt{n}\bigg)\cdot \qquad \square$$

Remark. Since $a_n(C_{\lambda_\theta}) \gtrsim \mathrm{e}^{-\alpha b_\theta \sqrt{n}}$ for $\alpha > 0$, Theorem 3.12, 3) *a*), gives:

$$a_n^{\mathcal{D}_0^2}(C_{\lambda_\theta}) \gtrsim \Big(a_{2n}^{\mathcal{D}_\alpha^2}(C_{\lambda_\theta})\Big)^2 \gtrsim \mathrm{e}^{-2\alpha b_\theta \sqrt{2n}}\,;$$

letting α tend to 0, we get $a_n^{\mathcal{D}_0^2}(C_{\lambda_\theta}) \gtrsim 1$ and we recover that C_{λ_θ} is not compact on $\mathcal{D}_0^2$.

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