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Research article

Fundamental theorems of Morse theory on posets

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Abstract: We prove a version of the fundamental theorems of Morse theory in the setting of finite partially ordered sets. By using these results we extend Forman's discrete Morse theory to more general cell complexes and derive the Morse-Pitcher inequalities in that context.

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1. Introduction

Morse theory was originally introduced as a tool for the study of variational problems on manifolds by relating the homology of a space with some critical objects arising after defining a map and considering its induced dynamics ([25]). Since its introduction, Morse theory has been an active field of research and connections with many different areas of mathematics have been found. That interaction has led to several adaptations of Morse theory in different contexts: PL versions by Banchoff ([2, 3]), by Bestvina and Brady ([8]), and a purely combinatorial approach by Forman ([15, 17]). Nowadays, not only pure mathematics benefit from that interaction but also applied mathematics ([18]) due to the importance of discrete settings.

Roughly speaking, Morse theory addresses the study of the topology (homotopy or homology, originally) of a space by breaking it into "elementary" pieces. That is achieved by the so-called Fundamental or Structural Theorems of Morse theory, which assert that the object of study (for example a smooth manifold or a simplicial complex) has the homotopy type of a CW-complex with a given cell structure determined by the criticality of a Morse function defined on it ([15, 23]).

Finite spaces and finite partially ordered sets (posets) are essentially the same objects considered from different points of view, as observed by Alexandroff ([1]). In this way, Stong ([29]) proved how the combinatorics of posets classify the (strong) homotopy types of their associated finite spaces. These ideas were continued in the work of Raptis ([28]) on the homotopy theory of posets.

More or less at the same time as Stong's work, McCord ([22]) discovered the relationship between finite spaces and finite simplicial complexes. Given a finite space X, its associated "order complex" $\mathcal{K}(X)$ is a simplicial complex having the same weak homotopy type as X. Conversely, for each finite simplicial complex K, its "face poset" $\mathcal{X}(K)$ is a finite space which is weak homotopy equivalent to K. In this way, weak homotopy types of finite spaces coincide with homotopy types of finite simplicial or regular CW-complexes. Notice that by Whitehead's Theorem, the distinction between weak homotopy types and homotopy types is lost when we only consider the associated complexes, while for finite spaces there is a distinction between weak homotopy types and homotopy types.

The Morse theory developed in this work sees at most the simple homotopy type of the posets but does not recover the (strong) homotopy type (see Example 4.6). This phenomenon may suggest that it is possible to study the Morse theory of posets by studying a Morse theory in their associated order complexes. However, this is not possible since the order complex functor does not carry Morse functions nor matchings.

Finite homology manifolds (see Subsection 2.2) are of special relevance among finite spaces. This is due to several reasons. First, these spaces play the role of simplicial homology manifolds in the setting of posets. Second, and related to the first one, studying finite homology manifolds provides information about the topology of simplicial homology manifolds. This is the reason why we develop our results for down-wide and two-wide posets, which are families of finite spaces that include the finite homology manifolds.

The idea of approaching problems regarding posets by using topological methods dates back at least to the works of Brown ([10]), Quillen ([27]), and Stong ([30]) dealing with the way the homotopy type of the poset of non-trivial p-subgroups of a finite group G ordered by inclusion determines algebraic properties of the group. Recently, Barmak and Minian ([6]), and later Chocano, Morón, and Ruiz del Portal ([12]) have made use of the topology of finite spaces in order to study groups as automorphism groups of certain posets. A general reference for different problems dealt with through the study of poset topology is the survey of Wachs ([31]). Therefore, it makes sense to study the (strong and weak) homotopy type and homology of finite spaces by means of some adapted version of Morse theory.

The aim of this work is to develop an extension of Morse theory to finite posets introduced by Minian ([24]) in order to prove the Fundamental Theorems of Morse theory in this setting. More precisely, we study the evolution of level subposets with or without critical values. Moreover, some of the consequences of the Fundamental Theorems are exploited. For instance, we provide an alternative proof of Forman's decomposition theorem [15, Corollary 3.5], an extension of discrete Morse theory to more general cell complexes, and the Morse-Pitcher inequalities.

In order to achieve our goal, we work with finite topological spaces (that is, posets endowed with the Alexandroff topology) with their intrinsic topology instead of using the topology of the associated (order) simplicial complex. This is for several reasons. First, it allows us to establish more general hypotheses and to take advantage of more efficient algorithms and computations. For the second and more crucial reason, we need to elaborate a bit. Given a simplicial complex K and a Morse function $f\colon K \to \mathbb{R}$, the function induces a combinatorial Morse function on its face poset $\mathcal{X}(f)\colon \mathcal{X}(K) = X \to \mathbb{R}$ such that $\mathcal{X}(f)$ retains the topological information regarding the level subcomplexes of K (see Section 5). However, this technique does not work well the other way around. That is, given a poset X and a discrete Morse function $f\colon X \to \mathbb{R}$, there is no canonical way of constructing a discrete Morse function on its order complex $\mathcal{K}(X)$ such that it retains the topological information of the level subposets of X. Therefore, it is more general to study Morse theory in the setting of posets than in the context of simplicial or regular cell complexes, and the first setting can not be reduced to the second one.

The organization of this paper is as follows. In Section 2 we recall some definitions and standard results about posets and finite topological spaces originally stated by Alexandroff ([1]) and Stong ([29]). Section 3 is devoted to the study of discrete Morse theory in the context of posets, proving the correspondence between Morse matchings and order preserving Morse functions on two-wide graded posets. In Section 4 we prove the Fundamental Theorems of Morse theory in this setting. Among these results are the so-called Structural Theorems, which show that the topology of the level subsets does not change in absence of critical values and how it changes when some critical point arises. We then show an example that illustrates the results of this section. Finally in Section 5, we show some consequences of these results, such as extending Forman's original result ([15, Corollary 3.5]) regarding the homotopy type of a regular CW-complex with a Morse function defined on it and obtaining the Morse inequalities. Moreover, we study a method to reduce the number of critical points of a Morse function defined on a poset (Theorem 5.7).

2. Finite spaces, posets, and simplicial complexes

This section is devoted to introducing the objects we will work with. In particular, we are interested in two kinds of posets: two-wide (introduced by Bloch ([9])), and down-wide. It is in this setting where we shall establish our main results. Most of the material presented in this section is well established in the literature. For further details, the reader is referred to [4, 7, 9, 14, 24, 31]. All posets will be assumed to be finite. By a finite space, we mean a finite T_0-space.

2.1. Preliminaries on finite posets

It is well known that finite posets and finite T_0-spaces are in bijective correspondence ([1]). If $(X, \leq)$ is a poset, a topology on X is defined by taking as basis the open sets

$$U_x := \{y \in X : y \leq x\}$$

for each $x \in X$. On the other hand, if X is a finite T_0-space, for any $x \in X$, define the *minimal open set* U_x as the intersection of all open sets containing x. Then X may be given a poset structure by defining $y \leq x$ if and only if $U_y \subseteq U_x$. It is easy to see that these correspondences are mutually inverses of each other. Moreover, a map between posets $f\colon X \to Y$ is order preserving if and only if it is continuous when considered as a map between the associated finite spaces. As a consequence of the correspondence between posets and finite T_0-spaces, we will use both notions interchangeably.

We need to introduce some basic notions and results.

Definition 2.1. *A chain in the poset X is a subset $C \subseteq X$ such that if $x, y \in C$, then either $x \leq y$ or $y \leq x$. The chain $x_0 < x_1 < \ldots < x_n$ has length n. The height of X is the maximum length of the chains in X. The height of an element $x \in X$ is the height of U_x with the induced order. A poset X is said to be homogeneous of degree n if all maximal chains in X have length n. A poset is graded if U_x is* *homogeneous for every $x \in X$. In that case, the degree of x is its height. If $x \in X$ has height p, then we denote $x = x^{(p)}$. We denote by* $\deg(X)$ *the degree of any maximal element.*

Let X be a finite poset, $x, y \in X$. If $x < y$ and there is no $z \in X$ such that $x < z < y$, we write $x \prec y$. For $x \in X$ we also define $\widehat{U}_x := \{w \in X \colon w < x\}$ as well as $F_x := \{y \in X : y \geq x\}$ and $\widehat{F}_x := \{y \in X : y > x\}$. We denote $\widehat{C}_x = (U_x \cup F_x) - \{x\}$.

Due to the correspondence between posets and finite topological spaces we can also study the homotopy type of a poset by means of this intrinsic topology. The homotopy type of finite spaces can be studied combinatorially by using the notion of beat points. This was first developed by Stong ([29]).

Definition 2.2. *([29]) A point $x \in X$ is a beat point if either $\widehat{U}_x$ has a maximum (down beat point) or $\widehat{F}_x$ has a minimum (up beat point).*

Notice that, by Corollary 3 in [29], a poset with a maximum or a minimum is contractible. Consequently, if $x \in X$ is a down beat point (respectively up beat point) then $\widehat{U}_x$ is contractible (respectively $\widehat{F}_x$ is contractible).

The next proposition states that removing beat points from a poset does not change its homotopy type as a T_0-space.

Proposition 2.3. *[4, Proposition 1.3.4] Let $x \in X$ be a beat point. Then $X - \{x\}$ is a strong deformation retract of X.*

We now recall McCord functors between posets and simplicial complexes. The reader less familiarized with this topic may consult [4]. Given a poset X, we define its *order complex* $\mathcal{K}(X)$ as the simplicial complex whose k-simplices are the non-empty chains of X of length k. Furthermore, given an order preserving map $f \colon X \to Y$ between posets, we define a simplicial map $\mathcal{K}(f) \colon \mathcal{K}(X) \to \mathcal{K}(Y)$ by $\mathcal{K}(f)(x) = f(x)$.

Conversely, if K is a simplicial complex, we define the *face poset* of K, denoted by $\mathcal{X}(K)$, as the poset of simplices of K ordered by inclusion. Given a simplicial map $\phi \colon K \to L$ we define the order preserving map $\mathcal{X}(\phi) \colon \mathcal{X}(K) \to \mathcal{X}(L)$ by $\mathcal{X}(\phi)(\sigma) = \phi(\sigma)$ for each simplex σ of K.

The face poset functor is defined analogously for CW-complexes. That is, given a CW-complex K, $\mathcal{X}(K)$ is the poset of cells of K ordered by inclusion. Given a cellular map $\phi \colon K \to L$ we define the order preserving map $\mathcal{X}(\phi) \colon \mathcal{X}(K) \to \mathcal{X}(L)$ by $\mathcal{X}(\phi)(\sigma) = \phi(\sigma)$ for each cell σ of K. Note that for a simplicial complex K, $\mathcal{K}\mathcal{X}(K)$ is $\mathrm{sd}(K)$, the first barycentric subdivision of K. Recall that $|K|$ denotes the geometric realization of a complex K.

We will now recall two results which relate the topology of posets and that of their order complexes and the topology of complexes and that of their face posets. These results will be very useful in order to obtain information about the homotopy type of complexes equipped with a discrete Morse function without using simple homotopy type. For details and a proof of the following result, see [4, p.12–15]:

Theorem 2.4. *The following statements hold:*

1. *Let X be a finite T_0-space. Then there is a map $\mu_X \colon |\mathcal{K}(X)| \to X$ which is a weak homotopy equivalence.*
2. *Let K be a simplicial complex. Then there is a map $\mu_K \colon |K| \to \mathcal{X}(K)$ which is a weak homotopy equivalence.*

We recall now McCord's Theorem, which plays a central role in the proof of Theorem 2.4.

Let X be a topological space. An open cover $\mathcal{U}$ of X is called a *basis like open cover* for X if $\mathcal{U}$ is a basis for a topology in the underlying set of X. Note that given a finite space X, the cover by minimal open sets $\{U_x\}_{x\in X}$ is a basis like open cover.

We reproduce the statement of McCord's Theorem:

Theorem 2.5. *[22, Theorem 6] Let $f\colon X \to Y$ be a continuous map between topological spaces. Suppose there is a basis like open cover $\mathcal{U}$ of Y such that for every $U \in \mathcal{U}$, the restriction*

$$f_{|f^{-1}(U)}\colon f^{-1}(U) \to U$$

is a weak homotopy equivalence. Then $f\colon X \to Y$ is a weak homotopy equivalence.

2.2. Two-wide and down-wide posets

In this subsection, we work with two classes of posets which will play a key role in the later development of Morse theory in the context of finite posets.

Definition 2.6 ([9]). *A poset X is two-wide if for any $x, z, y \in X$ such that $x \prec z \prec y$, there is some $z' \in X$ such that $z' \neq z$ and $x \prec z' \prec y$.*

Remark 2.7. *Let X be a finite poset. If X is two-wide, then for any pair of elements $x, y \in X$ such that $x < y$ and $x \nprec y$, $\#\{z\colon x \le z \le y\} \ge 4$.*

It is easy to show that face posets of regular CW-complexes are two-wide. We recall this result below.

Lemma 2.8. *[21, Lemma 4.1, p. 168] Given a regular CW-complex K, its face poset $\mathcal{X}(K)$ is two-wide.*

We now introduce the notion of *down-wide poset*, which is one of the properties that satisfy the family of posets we are interested in.

Definition 2.9. *Given a poset X and $x \in X$, we define the* down-incidence number *of x as the cardinality of the set $\partial x = \{y \in X\colon y \prec x\}$. The poset X is* down-wide *if $\#\partial x \neq 1$ for every x in X.*

Observe that a down-wide poset is exactly a poset without down beat points, and ∂x is the set of maximal elements of $\hat{U}_x$.

It is easy to check the following result:

Lemma 2.10. *For any regular CW-complex K, its face poset $\mathcal{X}(K)$ is down-wide.*

Therefore all posets coming from simplicial complexes by the McCord functor $\mathcal{X}$ are down-wide. However, not every down-wide poset is the face poset of some simplicial complex.

Example 2.11. *The poset X pictured in Figure 1 is down-wide. However, it is not the face poset of any simplicial complex K. Otherwise K would have two 0-simplices and two 1-simplices.*

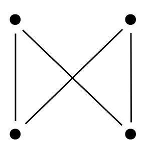


Figure 1. A down-wide poset which is not the face poset of any simplicial complex.

Definition 2.12 ([24]). *A poset X is a model of a CW-complex K if the geometric realization of $\mathcal{K}(X)$ is homotopy equivalent to K.*

Observe that a subposet Y of a poset X is an open subspace with the finite topology if whenever $x \in Y$ and $y \leq x$, then $y \in Y$. Therefore, an open subspace of a down-wide poset is down-wide, and an open subspace of a two-wide poset is two-wide.

Note that the properties of being two-wide and down-wide do not imply each other. For example, the poset depicted in Figure 3 is down-wide but it is not two-wide. Conversely, the poset depicted in Figure 2 is two-wide but it is not down-wide.

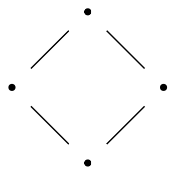


Figure 2. A two-wide poset which is not down-wide.

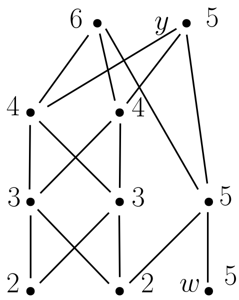


Figure 3. Lemma 3.6 may not hold for posets which are not two-wide.

We recall the notion of homologically admissible edges introduced by Minian ([24]). We denote by $\mathcal{H}(X)$ the Hasse diagram associated to the poset X; that is, the directed graph with edges (x, y) for each $x \prec y$ in the poset X.

Definition 2.13 ([24]). *Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is homologically admissible if $\widehat{U}_x - \{w\}$ is acyclic. A poset is homologically admissible if all its edges are homologically admissible.*

Definition 2.14. *Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is* 1*-admissible if $\widehat{U}_x - \{w\}$ is simply connected. A poset is* 1*-admissible if all its edges are* 1*-admissible.*

Remark 2.15. *The face poset of a regular CW-complex is homologically admissible [24, Remark 2.13].*

A simplicial complex K is a *closed homology manifold* of dimension n if the link of every k-simplex has the homology of a sphere $\mathbb{S}^{n-k-1}$. A poset X is a *finite closed homology manifold* if its order complex $\mathcal{K}(X)$ is a closed homology manifold.

By [24, Theorem 4.6], if the poset X is a finite closed homology manifold, then it is homologically admissible.

As a consequence, the theory that we will develop for posets which are homologically admissible can be applied to study the topology of closed triangulable homology manifolds by means of their triangulations.

We end the subsection by relating the properties of being homologically admissible with those of being two-wide and down-wide. This is important since it guarantees that all results in this work hold for homologically admissible posets.

In order to prove that homologically admissible posets are down-wide and two-wide, we recall a particular case of a construction of cellular homology for posets due to Farmer ([14]) and Minian ([24]).

Given a homologically admissible poset X, we define its cellular homology with $\mathbb{Z}_2$ coefficients. First, we define its *cellular chain complex* (C_*, d) as follows: for each p, $C_p(X)$ is a $\mathbb{Z}_2$-vector space (where $\mathbb{Z}_2$ stands for the field with two elements) with one generator for each element of X of degree p. The differential $d\colon C_p(X) \to C_{p-1}(X)$ is defined as $d(x) = \sum_{w \prec x} w$. It follows that $H_*(C_*(X)) \cong H_*(X; \mathbb{Z}_2)$ ([24, Theorem 3.7]) where $H_*(X; \mathbb{Z}_2)$ stands for the singular homology with $\mathbb{Z}_2$ coefficients.

Lemma 2.16. *Let X be a poset. If X is homologically admissible, then it is down-wide.*

Proof. For any $x^{(1)}$ and $w \prec x$, since $\widehat{U}_x - \{w\}$ is acyclic, then $\widehat{U}_x - \{w\}$ is non-empty, so there exists $w' \neq w$ such that $w' \prec x$. For any $x^{(p)}$ with $p > 1$, suppose that there is a unique $w \in X$ such that $w \prec x$. We will arrive to a contradiction. Using cellular homology,

$$d^2(x) = d(w) = \sum_{q \prec w} q \neq 0$$

which is a contradiction. □

Proposition 2.17. *Let X be a poset. If X is homologically admissible, then it is two-wide.*

Proof. Suppose there are elements $x \prec z \prec y$. We have to show that there is some $z' \neq z$ such that $x \prec z' \prec y$. Using cellular homology,

$$dy = \pm z + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} \pm \tilde{z}$$

and

$$dz = \pm x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z} \pm \tilde{x}.$$

Since $d^2 = 0$,

$$0 = d^2 y = \pm dz + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} \pm d\tilde{z} = \pm x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z} \pm \tilde{x} + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} \pm d\tilde{z}.$$

Since this equation holds, there must be some $z' \neq z$ such that

$$dz' = \pm x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z'} \pm \tilde{x},$$

that is, there is some $z' \neq z$ such that $x \prec z' \prec y$. □

3. Morse Theory on posets

3.1. Definition of Morse functions

We recall the definition of Morse functions for posets introduced by Minian ([24]). It is an adaptation of Forman's theory ([15, 17]) to the context of posets.

Definition 3.1. *Let X be a finite poset. A Morse function on X is a function $f\colon X \to \mathbb{R}$ such that, for every $x \in X$, we have*

$$\#\{y \in X\colon\ x \prec y \text{ and } f(x) \geq f(y)\} \leq 1$$

and

$$\#\{w \in X\colon\ w \prec x \text{ and } f(w) \geq f(x)\} \leq 1.$$

Definition 3.2. *If f is a Morse function on a finite poset X, the point $x \in X$ is said to be critical if*

$$\#\{y \in X\colon\ x \prec y \text{ and } f(x) \geq f(y)\} = 0$$

and

$$\#\{w \in X\colon\ w \prec x \text{ and } f(w) \geq f(x)\} = 0.$$

The set of critical points is denoted by crit$f$. The images of the critical points are called *critical values*. The points (respectively values) which are not critical are said to be *regular points* (respectively *regular values*).

3.2. Technical Lemmas for Morse functions

We begin by stating a result that plays the role of two important theorems developed by Forman in the simplicial setting ([15, Theorems 1.2 and 1.3]). In fact, the proof of the following Key Lemma is much simpler in this context than in the simplicial setting.

Lemma 3.3 (Key Lemma)**.** *Suppose that X is a finite two-wide poset and there are two elements $w < y$ such that $w \nprec y$. Then there are elements $x \neq \tilde{x}$ such that $w \prec x < y$ and $w \prec \tilde{x} < y$.*

Proof. Since $w < y$ and $w \nprec y$, there is a chain $w \prec x_0 \prec \cdots \prec x_n \prec y$ with $n \geq 0$. If $n = 0$, the result just follows from the definition. Otherwise, apply the definition to the chain $w \prec x_0 \prec x_1$. Therefore, there is an $\tilde{x}_0 \neq x_0$ verifying that $w \prec x_0 < x_1$ and $w \prec \tilde{x}_0 < x_1$. Take $x = x_0$ and $\tilde{x} = \tilde{x}_0$. It holds that $w \prec x < y$ and $w \prec \tilde{x} < y$ as we wanted to prove. □

Remark 3.4. *Observe that Lemma 3.3 does not hold in general for finite posets. As an example, consider the poset of Figure 3, and take the points labelled as w and y.*

Definition 3.5. *Given a poset X, a Morse function $f\colon X \to \mathbb{R}$ is said to satisfy the Exclusion condition if for every regular point $x \in X$, exactly one of the following conditions holds:*

1. *There exists exactly one $y \in X$, $x \prec y$, such that $f(x) \geq f(y)$.*
2. *There exists exactly one $w \in X$, $w \prec x$, such that $f(w) \geq f(x)$.*

The following result plays the role of the Exclusion Lemma ([15, Lemma 2.5]) in the context of finite spaces.

Lemma 3.6 (Exclusion Lemma)**.** *Let X be a finite two-wide poset and $f\colon X \to \mathbb{R}$ a Morse function on X. Then $f\colon X \to \mathbb{R}$ satisfies the Exclusion condition.*

Proof. Let x be a regular point. Since x is not critical, then at least one of the conditions in Definition 3.5 holds. We will see that these conditions are mutually exclusive. By way of contradiction, suppose that both conditions hold, and take $x \prec y$ and $w \prec x$ as in conditions 1 and 2, respectively. Then $w \prec x \prec y$. Since X is two-wide, by the Key Lemma (Lemma 3.3) there exists $x' \neq x$ such that $w \prec x' \prec y$. By the definition of Morse function applied to w with $w \prec x'$, we get $f(w) < f(x')$ since we already have $f(w) \geq f(x)$ and $w \prec x$. Similarly with y and $x' \prec y$, we obtain $f(x') < f(y) \leq f(x)$.

As a consequence we obtain the following chain of inequalities:

$$f(x) \leq f(w) < f(x') < f(y) \leq f(x).$$

Therefore $f(x) < f(x)$, which is a contradiction. □

It is interesting to point out that the Exclusion Lemma does not necessarily hold in general for posets which are not two-wide, as the following example shows.

Example 3.7. *Lemma 3.6 may not hold for arbitrary posets as we show below. Consider the down-wide model of $\mathbb{S}^3$ (taken from [24, Fig.2]) with the Morse function f represented in Figure 3 by the labelling of the points.*

3.3. Matchings

We recall the definition of a (Morse) matching introduced by Chari ([11]) and further developed by Minian ([24]).

Definition 3.8. *A matching $\mathcal{M}$ in a poset X is a subset $\mathcal{M} \subseteq X \times X$ such that*

- *$(x, y) \in \mathcal{M}$ implies $x \prec y$;*
- *each $x \in X$ is in at most one element in $\mathcal{M}$.*

We introduce some terminology which will be used later. Given a matching $\mathcal{M}$ on the poset X, we will decompose X as the disjoint union of three subsets $X = \mathrm{crit}(\mathcal{M}) \sqcup s(\mathcal{M}) \sqcup t(\mathcal{M})$. For each edge $(x, y) \in \mathcal{M}$, we say that x is the source of the edge and y is the target. We define the *source of the matching* $s(\mathcal{M})$ as the set whose elements are the sources of the edges in the matching. Analogously, we define the *target of the matching* $t(\mathcal{M})$ as the set whose elements are the targets of the edges in the matching. For convenience, we define the *source* and *target maps* (only defined for elements in the matching $\mathcal{M}$) as follows: given $(x, y) \in \mathcal{M}$, $s(y) = x$ and $t(x) = y$.

Definition 3.9. *A matching is homologically admissible if each element of the matching is homologically admissible. The notion of a* 1*-admissible matching is defined analogously.*

Let $\mathcal{H}(X)$ be the Hasse diagram of a poset X. If $\mathcal{M}$ is a matching in X, write $\mathcal{H}_{\mathcal{M}}(X)$ for the directed graph obtained from $\mathcal{H}(X)$ by reversing the orientations of the edges which are not in $\mathcal{M}$.

Definition 3.10. *The matching $\mathcal{M}$ is a Morse matching on X if $\mathcal{H}_{\mathcal{M}}(X)$ is acyclic as a directed graph. Any point of $\mathcal{H}(X)$ not incident with an edge of $\mathcal{M}$ is called critical. The set of all critical points of $\mathcal{M}$ is denoted by* $\mathrm{crit}(\mathcal{M})$.

Inspired by the notions of h-regular and cellular posets introduced in [24], we present the following definition.

Definition 3.11. *A matching $\mathcal{M}$ in X is homology-regular if for every $x^{(p)} \in \mathrm{crit}(\mathcal{M})$, the subspace $\widehat{U}_x$ has the homology of a sphere $\mathbb{S}^{p-1}$ where p is the height of x. A matching $\mathcal{M}$ in X is homotopy-regular if for every $x^{(p)} \in \mathrm{crit}(\mathcal{M})$, the subspace $\widehat{U}_x$ is a finite model of $\mathbb{S}^{p-1}$ where p is the height of x.*

Example 3.12. *Any matching defined in the poset pictured in Figure 3 is homotopy-regular, and therefore homology-regular, since for every $x^{(p)} \in \mathrm{crit}(\mathcal{M})$, the subspace $\widehat{U}_x$ has the homotopy type of a finite model of $\mathbb{S}^{p-1}$.*

3.4. Relation between matchings and Morse functions

In order to study the relation between Morse functions and matchings on posets, we introduce the notions of $\mathcal{M}$-path and generalized $\mathcal{M}$-path.

Definition 3.13. *Let X be a graded poset and let $\mathcal{M}$ be a matching on X. An $\mathcal{M}$-path of index p from $x^{(p)}$ to $\tilde{x}^{(p)}$ is a sequence*

$$\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}$$

such that for each $i \in \{0, \dots, r-1\}$:

1. $(x_i, y_i) \in \mathcal{M}$,
2. $x_i \neq x_{i+1}$.

For a general poset X, not necessarily graded, a generalized $\mathcal{M}$-path from x to z is a sequence of one of the following two forms:

1. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} = z$
2. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} \succ x_r = z$

such that for each $i \in \{0, \dots, r-1\}$, the sequence satisfies the same conditions as before.

Minian ([24, Lemma 3.12]) proved an integration result for matchings which can be slightly improved as follows.

Theorem 3.14. *Let X be a finite graded poset and let $\mathcal{M}$ be a a Morse matching on X. Then, there is a Morse function $f\colon X \to \mathbb{R}$ such that $\mathrm{crit}(f) = \mathrm{crit}(\mathcal{M})$. Moreover, the function $f\colon X \to \mathbb{R}$:*

1. *satisfies the Exclusion condition.*
2. *It is self-indexing, that is, for every critical element $x^{(p)}$, $f(x) = p$.*
3. *It is order preserving, that is, if $x \leq y$, then $f(x) \leq f(y)$.*
4. *If $(x, y) \in \mathcal{M}$, then $f(x) = f(y)$.*

Proof. First, we define an auxiliary map $l\colon X \to \mathbb{N}$ given by

$$l(x) = \max\{r\colon \exists\, \mathcal{M}\text{-path}$$
$$\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)}\}.$$

Second, we define $L = \max_{x \in X} l(x)$. Now, we define the function $f\colon X \to \mathbb{R}$ inductively on the degree of the poset. Given $x^{(p)} \in X$, we define $f(x)$ as follows:

1. If $x^{(p)}$ is a critical point of $\mathcal{M}$, then $f(x) = p$.
2. If $x \in s(\mathcal{M})$, then
$$f(x) = p + \frac{l(x)}{2 \cdot L}.$$
Note that this guarantees that
$$p < f(x) \leq p + \frac{1}{2}$$
due to $l(x) \geq 1$ in this case.
3. If $x \in t(\mathcal{M})$, then there exists $w^{(p-1)}$ such that $t(w) = x$ and $f(w)$ was defined in (2). We set $f(x) = f(w)$ and it follows that
$$p - 1 < f(x) \leq p - \frac{1}{2}.$$

By construction, the function $f: X \to \mathbb{R}$ satisfies the Exclusion condition, is self-indexing, and order preserving. It remains to check that f is Morse. We split the verification into cases.

First, suppose $x^{(p)}$ is critical. Then by construction of f, for any $w^{(p-1)}$, it follows that $f(w) \leq p - 1 + \frac{1}{2} < p$, and for any $y^{(p+1)}$, $f(y) > p$. Second, assume that $x^{(p)}$ is not critical and $y^{(p+1)} > x$.

1. If $t(x) = y$, then $f(y) = f(x)$, so
$$f(x) \geq f(y).$$
2. If $t(x) \neq y$, we consider several cases again:
   (a) If y is a critical point, then
   $$f(y) = p + 1 > p + 1/2 \geq f(x).$$
   (b) If $y \in s(\mathcal{M})$, then
   $$f(y) > p + 1 > p + 1/2 \geq f(x).$$
   (c) If $y \in t(\mathcal{M})$. Then there exists an unique $\tilde{x}^{(p)} \neq x$ such that $t(\tilde{x}) = y$. Since x is not critical, there are two cases:
      i. If $x \in t(\mathcal{M})$, then
      $$f(y) = f(\tilde{x}) \geq p > p - 1/2 \geq f(x).$$
      ii. If $x \in s(\mathcal{M})$ and $\gamma: x \prec \cdots$ is any $\mathcal{M}$-path beginning at x, then
      $$\tilde{\gamma}: \tilde{x} \prec y \succ x \prec \cdots$$
      is a $\mathcal{M}$-path beginning at $\tilde{x}$. Therefore
      $$l(\tilde{x}) \geq l(x) + 1,$$
      hence
      $$f(y) = f(\tilde{x}) > f(x).$$

Third, assume that $x^{(p)}$ is not critical and $w^{(p-1)} < x$. This case is analogous to the second one. □

As a consequence of our Exclusion Lemma for Morse functions on two-wide posets (Lemma 3.6), we obtain a converse result.

Theorem 3.15. *Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then there exists an associated Morse matching $\mathcal{M}_f$ satisfying the condition that for $(x, y) \in X \times X$ such that $x \prec y$, $(x, y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$. As a consequence,* $\mathrm{crit}(f) = \mathrm{crit}(\mathcal{M}_f)$*. In particular, given a finite two-wide poset X and a Morse function $f\colon X \to \mathbb{R}$, there exists an associated Morse matching $\mathcal{M}_f$ with* $\mathrm{crit}(f) = \mathrm{crit}(\mathcal{M}_f)$*.*

Proof. Define the matching $\mathcal{M}_f$ as follows. Let $(x, y) \in X \times X$ such that $x \prec y$. Then $(x, y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$ (so $\mathrm{crit}(f) = \mathrm{crit}(\mathcal{M}_f)$). It remains to check that the matching $\mathcal{M}_f$ is a Morse matching on X; that is, $\mathcal{H}_{\mathcal{M}_f}(X)$ is acyclic as a directed graph. First, observe that the paths in $\mathcal{H}_{\mathcal{M}_f}(X)$ are just generalized $\mathcal{M}_f$-paths. Now observe that for any generalized $\mathcal{M}_f$-path of any of the forms

1. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} = z$
2. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} \succ x_r = z$

it holds that

1. $f(x) = f(x_0) \geq f(y_0) > f(x_1) \geq f(y_1) > \cdots \geq f(y_{r-1}) = f(z)$
2. $f(x) = f(x_0) \geq f(y_0) > f(x_1) \geq f(y_1) > \cdots \geq f(y_{r-1}) > f(x_r) = f(z)$

respectively. Hence there cannot exist loops, and $\mathcal{M}_f$ is a Morse matching on X. □

Corollary 3.16. *Let X be a finite graded poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then there exists an order preserving and self-indexing Morse function $f'\colon X \to \mathbb{R}$ satisfying the Exclusion condition with the same associated Morse matching as that of f (hence,* $\mathrm{crit}(f') = \mathrm{crit}(f)$*).*

Proof. First apply Theorem 3.15 to $f\colon X \to \mathbb{R}$ to obtain a Morse matching $\mathcal{M}_f$ with the same critical set. Then apply Theorem 3.14 to $\mathcal{M}_f$. □

Therefore we can establish a correspondence between Morse matchings and order preserving Morse functions satisfying the Exclusion condition on graded posets. However, the correspondence is not bijective since given a Morse function $f\colon X \to \mathbb{R}$, a function $f'\colon X \to \mathbb{R}$ given by $f'(x) = 2f(x)$ is again Morse and both functions share the same associated matching.

4. Fundamental Theorems

4.1. First observations

We introduce the following notation: given a finite poset X and a discrete Morse function $f\colon X \to \mathbb{R}$, for each $a \in \mathbb{R}$ we denote

$$X_a^f = \bigcup_{f(x) \leq a} U_x.$$

Observe that for each $a \in \mathbb{R}$, the subposet X_a^f is an open subset of X. When the Morse function f is clear from the context, we simply write X_a for X_a^f.

We denote by $b_0(X)$ the number of connected components of X, which coincides with the number of path-connected components. Given a discrete Morse function on a simplicial complex $f\colon K \to \mathbb{R}$, it holds that new connected components of K_a arise as critical vertices as $a \in \mathbb{R}$ increases. The following result asserts this phenomenon for down-wide posets.

Proposition 4.1. *Let X be a path-connected finite down-wide poset and let $f\colon X \to \mathbb{R}$ be an injective discrete Morse function. Given $a, b \in \mathbb{R}$, $a < b$, if $b_0(X_a) < b_0(X_b)$, then there exists a critical value $c \in (a, b]$ such that $b_0(X_c) = b_0(X_a) + 1$. Moreover, the critical value c corresponds to a minimal element of X.*

Proof. Consider the values $c_1, \ldots, c_n$ attained by f in the interval $(a, b]$, such that $c_i < c_j$ if $i < j$. Since f is injective, we have a unique x_i such that $f(x_i) = c_i$. Set $c_0 = a$. For each i, we have that $X_{c_i} = X_{c_{i-1}} \cup U_{x_i}$, and either $b_0(X_{c_i}) \le b_0(X_{c_{i-1}})$ or $b_0(X_{c_i}) = b_0(X_{c_{i-1}})+1$. The latter condition holds if and only if $X_{c_{i-1}} \cap U_{x_i} = \emptyset$ (this can be seen by inspecting the long exact sequence of the reduced homology groups at degrees 0 and -1). Set $c = c_i$ and $x = x_i$, where i is an index such that $X_{c_{i-1}} \cap U_{x_i} = \emptyset$ and $b_0(X_{c_{i-1}}) = b_0(X_a)$. These conditions can be guaranteed since $b_0(X_a) < b_0(X_b)$.

If x is not minimal, then for some $y \prec x$ we have $f(y) < f(x)$ by the down-wide hypothesis and since f is a Morse function. Hence $f(y) \le c_{i-1}$ and $y \in X_{c_{i-1}} \cap U_x$, a contradiction. This shows that x is a minimal element.

If c is regular value, there exists y such that $x \prec y$ and $f(y) < f(x)$ (since x is minimal and f is injective). Then $x \in X_{f(y)} \cap U_x \subseteq X_{c_{i-1}} \cap U_x = \emptyset$, a contradiction. Therefore c is a critical value. □

Example 4.2. *Consider the Morse function represented in Figure 4. The value 3 must correspond to a critical point since we are adding a new path-component ($b_0(X_3) = b_0(X_2) + 1$). Moreover, the point corresponding to the value 3 is of zero height.*

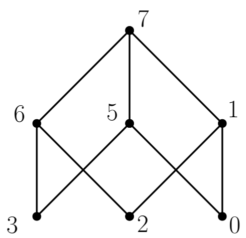


Figure 4. Regular values and path-components in general posets.

With the following result we prove that the addition of regular elements does not create new connected components.

Proposition 4.3. *Let X be a path-connected down-wide finite poset and let $f\colon X \to \mathbb{R}$ be an injective discrete Morse function satisfying the Exclusion condition. Consider $a, b \in \mathbb{R}$, $a < b$. If the interval $(a, b]$ does not contain critical values and only contains one regular value $f(y) = c$, then there exists $z \in X_a$ such that $z \prec y$ or $y \prec z$.*

Proof. Take z as in the Exclusion condition for y. If $y \prec z$ and $f(z) < f(y)$, then $f(z) \le a$ and $z \in X_a$. If $z \prec y$ and $f(y) < f(z)$, take $z' \neq z$ with $z' \prec y$ by the down-wide condition. Then $f(z') < f(y)$ and $z' \in X_a$.

In any case, we obtain an element $z' \in X_a$ with $y \prec z'$ or $z' \prec y$. □

Remark 4.4. *Observe that the Exclusion Lemma (Lemma 3.6) does not guarantee the conclusion of Proposition 4.3. Moreover, it is necessary the hypothesis of down-wide. See Figure 5 where y is the element with the value 4.*

Propositions 4.1 and 4.3 may not hold for arbitrary posets, as the following example shows.

Example 4.5. *Consider the Morse function represented in Figure 5. The value* 4 *is regular. However,* $b_0(X_4) \neq b_0(X_3)$ *while there are no critical values in* $(3, 4]$.

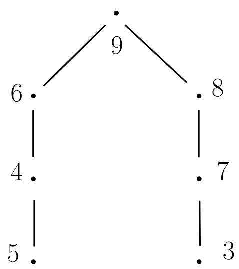


Figure 5. Regular values and path-components in general posets.

4.2. Structural Theorems

Both in smooth and discrete Morse Theory, manifolds and cell complexes can be recovered up to homotopy equivalence from Morse functions defined on them by means of the so called fundamental theorems of Morse Theory. The next example shows that this is not possible in Morse Theory defined on posets.

Example 4.6. *Consider the face poset of the simplicial complex depicted in Figure 6. It does not have the homotopy type of a point [4, Example 5.1.12].*

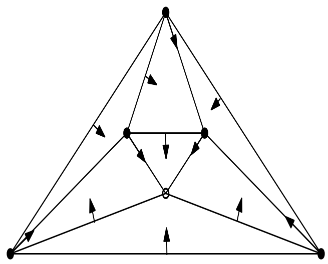


Figure 6. The Triangle.

However, there is a Morse function with only one critical point, namely, the Morse function associated to the matching drawn in the figure.

Example 4.6 shows that this Morse theory sees at most the simple homotopy type of the posets but does not recover by any means their (strong) homotopy type. This phenomenon may suggest that it is then possible to study the Morse theory of posets by studying a Morse theory in their associated order complexes. However, this is not possible since the order complex functor carries neither Morse functions nor matchings.

This subsection is devoted to proving the substitutes of the fundamental theorems of Morse Theory in this context, that is, two collapsing theorems and an adjunction theorem. The first collapsing theorem is a homological collapsing theorem, which asserts that in the absence of critical values, the homology remains unchanged provided the matching is homologically admissible. This result, combined with the adjunction theorem, is enough to prove the Morse inequalities. The second collapsing theorem guarantees that, in the absence of critical values, the weak homotopy type remains unchanged, provided that the matching is 1-admissible and homologically admissible. This result is analogous to [15, Theorem 3.3] in discrete Morse theory and plays the role of [23, Theorem 3.1] in smooth Morse Theory. Note that this result highlights the need of using the topology of posets and not just their combinatorial properties.

Proposition 4.7. *Let X be a finite path-connected down-wide and two-wide poset. Let $f\colon X \to \mathbb{R}$ be a discrete Morse function. Suppose that $(a, b]$ for $a < b$ contains no critical values of f and contains at most one regular value c. Then either $X_b = X_a$ or $X_b - X_a = \{v_i, w_i\}_{i=1}^r$, where*

1. *$f(v_i) = c$ for all i.*
2. *$w_i \prec v_i$ with $f(v_i) \leq f(w_i)$ for all i.*
3. *$\{v_i, w_i\} \cap \{v_j, w_j\} = \emptyset$ for all $i \neq j$.*
4. *For all i, w_i is an up beat point of X_b.*

Proof. Let $V = \{v \in X\colon f(v) = c\}$. Since c is a regular value, for each $v \in V$ there exists

1. a unique $w \in X$, $w \prec v$ and $f(v) \leq f(w)$,
2. or else, a unique $w \in X$, $v \prec w$ and $f(w) < f(v)$ (in case $v \prec w$ and $f(w) = f(v)$ we rename v and w to be in the first case).

Observe that by the Exclusion Lemma (Lemma 3.6), exactly one of these two possibilities can hold. During the proof, we will refer to an arbitrary v_i as v and to its correspondent w_i as w.

Suppose that we are in the second case. Then $f(w) \leq a$ since $f(v)$ is the unique regular value in $(a, b]$ and $f(w) < f(v) \leq b$. Therefore $X_b = X_a$. So, let us assume that we are in the first situation. We have to check that $w \notin X_a$, that is, there is no $u \in X$, $w < u$ such that $f(u) \leq a$. Suppose that there exists such an $u \in X$. We will reach a contradiction. First, observe that $w \nprec u$ because of the definition of Morse function ($f(u) \leq f(w)$ and $f(v) \leq f(w)$ cannot hold simultaneously). So there exists v' such that $w \prec v' < u$. Since X is two-wide, there exists $v'' \neq v'$ such that $w \prec v'' < u$. By the definition of Morse function, since $f(v) \leq f(w)$, it follows that $f(v') > f(w)$ and $f(v'') > f(w)$. By repeating this argument (taking v' instead of w in the first iteration) a finite number of times, we arrive to a contradiction with the definition of Morse function. Therefore, we have proved (1) and (2).

Condition (3) follows as a straightforward consequence of the definition of Morse function. It remains to check assertion (4). That is, we have to see that w is an up beat point in X_b. So, suppose on the contrary that there exists $u \neq v$, such that $w \prec u$ and $u \in X_b$. Then $f(w) < f(u)$ by the definition of Morse function ($w \prec u$ and $f(w) > f(v)$) and therefore $f(u) > a$. By the claim $u \in X_a$, then there exists $z > u$ such that $f(z) \leq a$, but we get $w \prec u < z$ and so $w \in X_a$, which is a contradiction. Then w is an up beat point. □

Proposition 4.8. *Under the conditions of Proposition 4.7, the inclusion $i\colon X_a \hookrightarrow X_b$ induces an isomorphism in all homology groups if and only if (w_i, v_i) is a homologically admissible edge for the Morse matching associated to the function f for all i. In case $X_a = X_b$, this holds trivially.*

Proof. First, perturb the Morse function f to obtain a Morse function f' such that there exist real numbers $\{a_i, b_i\} \subseteq (a, b]$ satisfying

$$a < a_0 < b_0 < a_1 < b_1 \cdots < a_i < b_i < \cdots a_r < b_r = b,$$

$f^{-1}((a_i, b_i]) \cap \{v_j\}_j = \{v_i\}$ and $X_a^{f'} = X_a^f$ and $X_b^{f'} = X_b^f$. So, we can assume that there is only one homologically admissible edge (w, v) for the Morse matching associated to the function f in $(a, b]$. Second, since w is an up beat point in X_b, $i\colon X_b - \{w\} \hookrightarrow X_b$ is a homotopy equivalence, so it is enough to prove that $i\colon X_a = X_b - \{w, v\} \hookrightarrow X_b - \{w\}$ induces an isomorphism in homology. By applying the Long Exact Sequence of homology to the pair $(X_b - \{w\}, X_a)$, it follows that $i\colon X_a \hookrightarrow X_b - \{w\}$ induces an isomorphism in all homology groups if and only if $H_*(X_b - \{w\}, X_a) \cong 0$. As a consequence of the Excision Theorem [20, Theorem 2.20], given two open sets A and B which cover $X_b - \{w\}$, then there is an isomorphism $H_*(B, A \cap B) \cong H_*(X_b - \{w\}, A)$. Considering $A = X_a$ and $B = U_v$, it follows that

$$H_*(U_v, \widehat{U}_v - \{w\}) \cong H_*(X_b - \{w\}, X_a).$$

Since $w \prec v$ is an element in the matching and the matching is homologically admissible, then $\widehat{U}_v - \{w\}$ is acyclic. By applying the Long Exact Sequence of homology to the pair $(U_v, \widehat{U}_v - \{w\})$ and using the fact that U_v is contractible, it follows that $H_*(U_v, \widehat{U}_v - \{w\}) \cong H_*(\widehat{U}_v - \{w\})$, so $H_*(U_v, \widehat{U}_v - \{w\}) \cong 0$ if and only if the element of the matching $w \prec v$ is homologically admissible. □

Remark 4.9. *In the above proof, $\widehat{U}_v - \{w\}$ is always nonempty due to the fact that X is a down-wide poset.*

Theorem 4.10. *Let X be a finite, path-connected, down-wide, two-wide poset. Let $f\colon X \to \mathbb{R}$ be a discrete Morse function. Suppose that $(a, b]$ for $a < b$ contains no critical values of f. If the Morse matching associated to the function f is homologically admissible, then the inclusion $i\colon X_a \hookrightarrow X_b$ induces an isomorphism in homology.*

Proof. This follows by combining Propositions 2.3, 4.7, and 4.8. □

Our next goal is to state the weak homotopical collapsing theorem, where again it is necessary to use the topology of the posets rather than their combinatorial properties. We need to add the extra hypothesis that the Morse matching associated to the function f is 1-admissible.

There is a weaker notion of beat point which we recall now.

Definition 4.11. *([5],[4, Definition 6.2.1]) The point $x \in X$ is a γ−point if $\widehat{C}_x = (U_x \cup F_x) - \{x\}$ is homotopically trivial.*

Proposition 4.12. *Under the conditions of Proposition 4.7, the inclusion $i\colon (X_b - \{w\}) - \{v\} \hookrightarrow X_b - \{w\}$ is a weak homotopy equivalence if the element $w \prec v$ of the Morse matching associated to the function f is 1-admissible and homologically admissible. Moreover, v is a γ−point in $X_b - \{w\}$.*

Proof. We will apply McCord's Theorem (Theorem 2.5) to the base $\{U_x\colon x \in X_b - \{w\}\}$. There are two cases to consider:

1. If $x \neq v$, then $i^{-1}(U_x)$ has a maximum and therefore is contractible, so $i_{|i^{-1}(U_x)}\colon i^{-1}(U_x) \to U_x$ is a weak homotopy equivalence.
2. If $x = v$, then $i_{|i^{-1}(U_x)}\colon i^{-1}(U_x) \to U_x$ is the map $i\colon \widehat{U}_v - \{w\} \hookrightarrow U_v$. The subspace U_v is contractible so it is homotopically trivial. Therefore $i\colon \widehat{U}_v - \{w\} \hookrightarrow U_v$ is a weak homotopy equivalence if and only if $\widehat{U}_v - \{w\}$ is homotopically trivial. Now, since $\widehat{U}_v - \{w\}$ is simply connected and acyclic, by Hurewicz Theorem it is homotopically trivial. □

Theorem 4.13. *Let X be a finite path-connected down-wide and two-wide poset. Let $f\colon X \to \mathbb{R}$ be a discrete Morse function. Assume that $(a, b]$ for $a < b$ contains no critical values of f.*

1. *If the Morse matching associated to the function f is 1-admissible and homologically admissible, then the inclusion $i\colon X_a \hookrightarrow X_b$ is a weak homotopy equivalence.*
2. *Moreover, in case $(a, b]$ contains no critical values of f and contains at most one regular value $f(v)$. Then $X_b = X_a$ or $X_b - X_a = \{v, w\}$ where $w \prec v$ and w is an up beat point in X_b and v is a γ−point in $X_b - \{w\}$.*

Proof. The result follows by combining Propositions 2.3, 4.7, and 4.12. □

Remark 4.14. *Theorem 4.13 does not necessarily hold for arbitrary posets, as Example 4.5 shows.*

The following result explains what happens with the homotopy type when we reach critical values. It plays the role of [23, Theorem 3.2] in the case of smooth Morse theory and [15, Theorem 3.4] in discrete Morse theory. Recall from Definition 2.9 that for a poset X and $x \in X$, $\partial x = \{y \in X\colon y \prec x\}$.

Theorem 4.15. *Let X be a path-connected finite poset and $f\colon X \to \mathbb{R}$ an order preserving Morse function. Suppose that $f^{-1}(a, b] = \{x\}$ is a critical element. Then $X_b = X_a \cup_{\partial x^{(p)}} x^{(p)}$, that is, X_b is obtained from the poset X_a by adding a new element x above the elements of the subposet $\partial x \subseteq X_a$.*

Proof. If $y > x$, then $f(x) \leq f(y)$ by the order preserving hypothesis. Moreover, since $f^{-1}(a, b] = \{x\}$ for $a < b$, $a < f(x) < f(y)$ and $f(y) > b$. Furthermore, if $y \in X_a$, then $y \leq z$ for some z with $f(z) \leq a$. Hence $b < f(y) \leq f(z) \leq a$, a contradiction. Therefore $X_a \cap F_x = \emptyset$.

If $y < x$, an analogous reasoning shows that $f(y) \leq a$. Therefore, $X_b = X_a \cup U_x$ and $X_a \cap U_x = \hat{U}_x$. In combination with the result of the previous paragraph, we arrive at the desired conclusion. □

Remark 4.16. *Theorem 4.15 is easily generalized to the case where $f^{-1}(a, b]$ is a collection of critical elements.*

The following example illustrates Theorems 4.10 and 4.15 by working out, step by step, the changes in homology of the level subposets of a poset endowed with a Morse function.

Example 4.17. *Let us denote by X the finite model of $\mathbb{R}P^2$ depicted in Figure 7 (see [4, Example 7.1.1], [19, Proposition 4.1] and [13, p. 138]). It can be checked that it is two-wide, down-wide, and homologically admissible. Consider the function $f\colon X \to \mathbb{R}$ given by the values depicted at the right side of the elements of X. It is clear that f is a Morse function. We will denote the level subposets by X_t.*

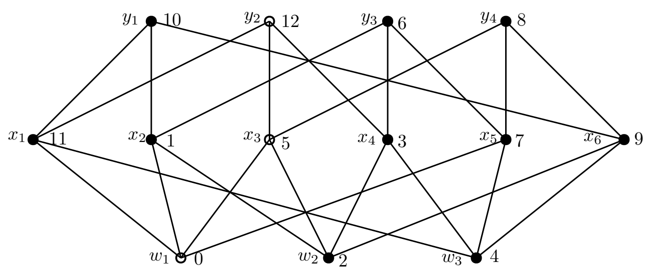


Figure 7. Morse function on a finite model of $\mathbb{R}P^2$.

We begin the analysis of the level subposets. First, as Proposition 4.1 claims, the minimum value of f, corresponding to the element w_1, is a critical value (we are adding a path-connected component) (see Figure 8):

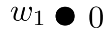
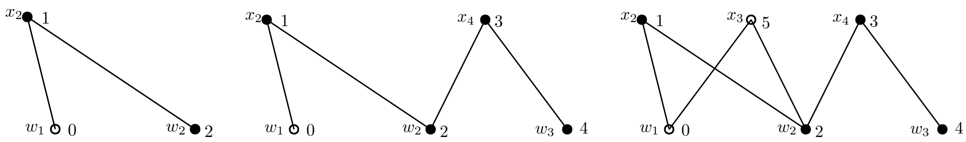


Figure 8. Level subposet $\mathbb{R}P^2_0$.

As we reach the value $t = 1$, the inclusion $i\colon X_0 \to X_1$ induces an isomorphism in homology. Observe that X_1 is contractible by removing beat points (see Figure 9 (a)).

The situation does not change when we reach the value $t = 3$ since X_3 is still contractible by removing beat points (see Figure 9 (b)).

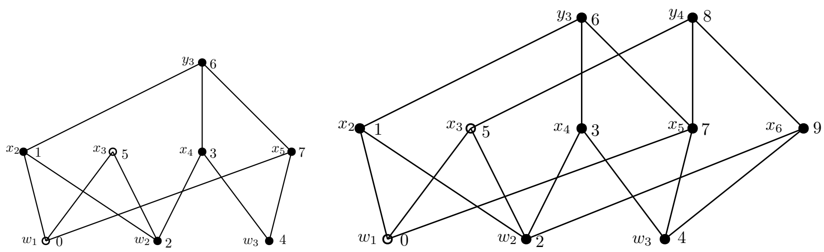
The value $t = 5$ is critical and the map $i\colon X_3 \to X_5$ does not longer induce an isomorphism in homology. Observe that X_5 has the homotopy type, by removing beat points, of a finite model of the circle $\mathbb{S}^1$ (see Figure 9 (c)).



(a) (b) (c)

Figure 9. Level subposets $\mathbb{R}P^2_1$, $\mathbb{R}P^2_3$ and $\mathbb{R}P^2_5$.

The value $t = 6$ is regular and it can be checked that the map $i\colon X_5 \to X_6$ induces an isomorphism in homology (see Figure 10 (a)).



(a) (b)

Figure 10. Level subposets $\mathbb{R}P^2_6$ and $\mathbb{R}P^2_8$.

The situation does not change when we reach the value $t = 8$ (see Figure 10 (b)) nor the value $t = 10$ (see Figure 11).

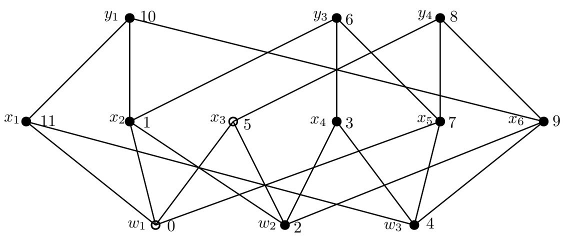


Figure 11. Level subposet $\mathbb{R}P^2_{10}$.

Finally, we reach the value $t = 12$, which is critical. The map $i\colon X_9 \to X_{12} = X$ induces an isomorphism in homology despite $t = 12$ is a critical value.

5. Consequences

5.1. Forman's Theory without simple homotopy types

It is clear that for any regular CW-complex K, given a Morse function $f\colon K \to \mathbb{R}$, it induces a combinatorial Morse function on its face poset $\mathcal{X}(f)\colon \mathcal{X}(K) = X \to \mathbb{R}$ such that the face poset functor satisfies $\mathcal{X}(K_a) = \mathcal{X}(K)_a$, where K_a denotes the level subcomplex corresponding to the value a, that is, the subcomplex of K defined as follows

$$K_a = \bigcup_{f(\tau)>a} \bigcup_{\sigma \leq \tau} \sigma.$$

It is worthwhile to mention that a function $f\colon K \to \mathbb{R}$ defined on the collection of simplices or regular cells of K is a (discrete) Morse function if and only if $\mathcal{X}(f)\colon \mathcal{X}(K) = X \to \mathbb{R}$ is a Morse function on the corresponding poset. Therefore, as a first consequence, by using Theorem 2.4 (2), we recover Forman's result [15, Corollary 3.5] without the use of simple homotopy types.

5.2. Morse-Pitcher inequalities

Another consequence of our structural theorems of Morse Theory for finite spaces is that we can reproduce step by step the classical proof (see [26, 23] for the standard argument) of Morse inequalities in this new context of posets. We consider coefficients in a principal ideal domain.

Definition 5.1. *Let $f\colon X \to \mathbb{R}$ be a Morse function. We denote by m_i the number of critical points of height i and by b_i the Betti number of dimension i.*

Corollary 5.2 (Strong Morse inequalities)**.** *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be an order preserving Morse function. Suppose that the Morse matching associated to f is homologically admissible and homology-regular. Then for every $i \geq 0$ and domain of coefficients:*

$$m_i - m_{i-1} + \cdots + (-1)^i m_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0.$$

Corollary 5.3 (Weak Morse inequalities)**.** *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be an order preserving Morse function. Suppose that the Morse matching associated to f is homologically admissible and homology-regular. Then*

1. *$m_i \geq b_i$ for every i.*
2. *The Euler-Poincaré Characteristic satisfies*

$$\chi(X) = \sum_{i=0}^{\deg(X)} (-1)^i b_i = \sum_{i=0}^{\deg(X)} (-1)^i m_i.$$

Remark 5.4. *The Morse inequalities for homologically admissible posets can also be derived by following a combinatorial Hodge-theoretic argument mimicking [16] since the arguments provided by Forman can be reproduced without changes in this context.*

Our next goal is to strengthen the Morse inequalities in the spirit of Pitcher's approach for smooth Morse theory ([26]). In order to proceed, we introduce some notation and an auxiliary result from homological algebra.

Let as denote by $\{c_0, \ldots, c_k, \ldots, c_n\}$ the image of the Morse function $f\colon X \to \mathbb{R}$. Then, there exist real numbers $\{a_k\}_k$ satisfying:

$$c_0 < a_0 < c_1 < a_1 \cdots < c_k < a_k < \cdots c_n < a_n.$$

To simplify our notation, we will denote X_{a_i} by X_i.

We denote the coefficient ring, which is assumed to be a principal ideal domain, by R. A a consequence of the Structure Theorem for finitely generated modules over a principal ideal domain, it follows that:

$$H_i(X) \cong R^{b_i} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_i})}.$$

For the case of relative homology, we use the notation:

$$H_i(X_k, X_{k-1}) \cong R^{b_i^k} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_i^k})}.$$

Set $M_i^k = b_i^k + \eta_i^k + \eta_i^{k-1}$ and $M_i = \sum_{k=0} M_i^k$.

Proposition 5.5 ([26, Theorem 14.1]). *Let $(C_*(X), \partial)$ be a free chain complex with singular homology groups $H_i(X)$, $i = 0, 1, \ldots$. Then there exists a free chain complex (L, ∂^L) such that:*

1. *For every $i \geq 0$, the group L_i has rank M_i.*
2. *There exists a monomorphism $i\colon L \hookrightarrow C$ which is a chain map an induces isomorphims in homology.*

Theorem 5.6 (Pitcher strengthening of Morse inequalities). *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be an order preserving Morse function whose associated Morse matching is homologically admissible and homology-regular. Then*

1. *For every $i \geq 0$:*
$$m_i \geq b_i + \eta_i + \eta_{i-1}.$$
2. *For every $i \geq 0$:*
$$m_i - m_{i-1} + \cdots + (-1)^i m_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0 + \eta_i.$$
Moreover, the equality is attained for $i = \deg(X)$.

Proof. First, apply Theorem 5.5 to the singular chain complex of X. Now, observe that:

$$\operatorname{rank} L_i = \operatorname{rank} H_i(X) + \operatorname{rank} \operatorname{Im} \partial_{i+1}^L + \operatorname{rank} \operatorname{Im} \partial_i^L.$$

The first set of inequalities follows from observing that $\operatorname{rank} \operatorname{Im} \partial_i^L \geq \eta_{i-1}$. The second set of inequalities follows from taking alternating sums in i. □

Observe that if $\deg(X) = n$, then $\mu_n = 0$ since $H_n(X)$ is a submodule of the free module $C_n(X)$. Moreover, $\mu_0 = 0$ and μ_{-1} is defined as 0.

5.3. Cancelling critical points

Both the Morse and Morse-Pitcher inequalities suggest the study of Morse functions with few critical points, the so-called *optimal* Morse functions. In order to obtain such functions we present an approach consisting of canceling pairs of critical elements, extending to the context of posets known results on smooth manifolds and simplicial complexes.

We present a result which can be seen as the adaptation of [15, Theorem 11.1] to our context.

Theorem 5.7 (Cancelling critical points). *Given a Morse matching $\mathcal{M}$ on a finite graded poset X, assume that $z^{(p+1)}$ and $x^{(p)}$ are critical points such that there is a unique $\mathcal{M}$-path*

$$\gamma\colon z \succ y = x_0 \prec z_0 \succ x_1 \prec z_1 \succ \cdots \prec z_r \succ x_r = x$$

with $y^{(p)} \prec z^{(p+1)}$ (there is no other $\mathcal{M}$-path from any p-face of $z^{(p+1)}$ to $x^{(p)}$). Then there is a matching $\mathcal{M}'$ such that:

- *The set of critical points of $\mathcal{M}'$ is*

$$\mathrm{crit}(\mathcal{M}') = \mathrm{crit}(\mathcal{M}) - \{x, z\}.$$

- *Moreover, $\mathcal{M}' = \mathcal{M}$ except along the unique $\mathcal{M}$-path from ∂z to x.*

Proof. We define $\mathcal{M}'$ as follows:

1. $t_{\mathcal{M}'}(w) = t_{\mathcal{M}}(w)$ if $w \notin \{y, z_0, x_1, z_1, \ldots z_r, x\}$ ($\mathcal{M}' = \mathcal{M}$ except along the unique gradient path from ∂z to x)
2. $t_{\mathcal{M}'}(x_i) = z_{i-1}$, $i = 1, \ldots, r$ (we reverse the gradient path from x to z_0 so x is no longer critical)
3. $t_{\mathcal{M}'}(y) = z$ (we reverse the arrow from y to z so z is no longer critical).

It remains to check that there are no closed $\mathcal{M}'$-paths. We argue by contradiction. Suppose there was a closed $\mathcal{M}'$-path δ.

Claim. *Under the above hypothesis, δ would contain at least one p-element from γ and one p-element not in γ.*

Proof of the Claim. The elements coming from γ can not give a closed $\mathcal{M}'$-path on their own since we have just reverted their arrows. The elements of X which are not in γ can not give a closed $\mathcal{M}'$-path since in that case we would also have a closed $\mathcal{M}$-path and $\mathcal{M}$ is a gradient vector field. Therefore in δ we must have at least one p-element in each of their sets. □

Hence, δ would contain a sequence of the form:

$$x_i \prec w_0 \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j$$

where $s \geq 0$, $w_l \neq x_k$, $w_l \neq z_k$, $s_l \neq x_k$, $s_l \neq z_k$, for all l and k. Since $t_{\mathcal{M}'}(w_l) = t_{\mathcal{M}}(w_l)$ and $t_{\mathcal{M}'}(s_l) = t_{\mathcal{M}}(s_l)$ for all l, we have a $\mathcal{M}$-path:

$$w_0 \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j.$$

Let us consider two cases:

1. If $i \neq 0$, then $s_1 \neq x_{i-1}, x_i$ and $s_1 \prec t_{\mathcal{M}'}(x_i) = z_{i-1}$. Therefore, we can define a second gradient $\mathcal{M}$-path $\gamma' \neq \gamma$ from ∂z to x:

$$\gamma' : y = x_0 \prec z_0 \succ x_1 \prec \cdots \succ x_{i-1} \prec z_{i-1} \succ s_1 \prec w_1 \succ \cdots$$
$$\succ x_j \prec z_j \succ \cdots \succ x_r = x.$$

   Which is a contradiction.
2. If $i = 0$, then $y = x_0 \neq s_1 \prec t_{\mathcal{M}'}(y) = z$. Therefore, we can define the following $\mathcal{M}$-path:

$$\gamma' : z \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j \prec z_j \succ \cdots \succ x_r = x$$

   which is different from γ and also goes from ∂z to x. Then we have a contradiction. □

Finally, there is a kind of dual result to Theorem 5.7 which allows us to create critical points. Both the statement and the proof are a straightforward translation of [15, Theorem 11.3].

6. Conclusions

We have studied Morse theory in the setting of finite partially ordered sets. Our approach covered Formans's discrete Morse theory for cell complexes as a particular case. Contrary to what happens for cell complexes, the Morse theory we studied in the setting of posets does not allow to control homotopy type. Still, we have proved two collapsing theorems and an adjunction theorem. The first collapsing result is a homological collapsing theorem, which asserts that in the absence of critical values, the homology remains unchanged provided the dynamics on the partially ordered set satisfies a certain homological condition. This result, combined with the adjunction theorem, sufficed to prove a strengthening of the Morse inequalities. The second collapsing theorem guaranteed that, in the absence of critical values, the weak homotopy type remains unchanged, provided the dynamics on the partially ordered set satisfies an extra condition. Finally, we extended the Morse inequalities by taking torsion into account.

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Conflict of interest

We declare no conflict of interest.

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