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The Extended Half-Skew Normal Distribution

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Abstract: A new class of densities for modelling non-negative data, which is based on the skew-symmetric family of distributions proposed by Azzalini is introduced.We focus on the model generated by the skew-normal distribution, called Extended Half Skew-Normal distribution. Its relevant properties are studied. These are pdf, cdf, moments, mgf, and stochastic representation. The parameters are estimated by moment and maximum likelihood methods. A simulation study to assess the performance of the maximum likelihood estimators in finite samples was carried out. Two real applications are included, in which the EHSN provides a better fit than other proposals in the literature.

Keywords: lifetime distributions; skew-symmetric distributions; maximum likelihood

MSC: 62E10; 62E15; 62F10

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1. Introduction

In the last few years, a considerable amount of research activity has been carried out in the field of skew-symmetric distributions theory and applications. Skew-symmetric distributions are of interest since they can be applied to datasets with an asymmetric structure without the need of applying arbitrary transformations to original data in order to reach symmetry. Quite often these transformations cause a loss of interpretability with respect to the original data. Nowadays, the focus of a number of statisticians is the development of non-symmetric parametric extensions of symmetric distributions. That is, to consider distributions with support in $\mathbb{R}$, to which parameters are incorporated to deal with skewness and kurtosis. In this sense, we can cite the pioneering paper by Azzalini [1] in which the skew-normal model was introduced. Other univariate parametric extensions of this model can be seen in the works by Azzalini [2], Azzalini and Capitano [3], Gupta et al. [4], Arellano-Valle et al. [5], DiCiccio and Monti [6], Gómez et al. [7], Adcock and Azzalini [8], and Gómez-Déniz et al. [9], among others. In all these papers, results and applications of great interest can be found.

The model studied in this paper is based on the univariate skew-normal distribution introduced by Azzalini [1]. There, a general method to build skew distributions on the basis of symmetric distributions was proposed. This key result is pointed out in next lemma, and will be one of the starting points to develop our proposal.

Lemma 1. *Let f be a symmetric at zero probability density function (pdf) and G an absolutely continuous cumulative distribution function (cdf), such that $G'(x) = G'(-x)$. Then*

$$h(x;\lambda) \quad = \quad 2f(x)G(\lambda x), \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{R}, \tag{1}$$

is the pdf of a random variable (rv) X. In (1), λ is a shape parameter related to the skewness of the distribution.

In literature, it is said that $h_\lambda = h(x; \lambda)$ introduced in (1) is the skew version of f with skewing function G.

Gómez et al. [7] showed that cases of great interest are obtained if the $N(0,1)$ cdf, Φ, is considered as skewing function. Specifically they proposed the following family of skew densities

$$g(x; \lambda) \quad = \quad 2 f_0(x) \Phi(\lambda x), \quad \lambda, x \in \mathbb{R}, \tag{2}$$

where f_0 is any symmetric around zero pdf and λ is a skewness parameter. Recall that, if $f_0 = \phi$ and $G = \Phi$ then g_λ reduces to the pdf of the skew normal $SN(\lambda)$ introduced by Azzalini [1]. As merit of (2), it can be cited that the models obtained are more flexible as for its skewness parameter than the distributions proposed by other authors, such as those proposed in Nadarajah and Kotz [10].

Flexible models can be of interest in real applications where is quite common to find non-negative data, which need for their modelling distributions with positive support. In this sense, we consider the model introduced in Elal-Olivero et al. [11], whose pdf is

$$f_Y(y; \alpha) \quad = \quad 2\left(\frac{\alpha + y^2}{\alpha + k}\right) f_0(y), \qquad y \geq 0, \;\; \alpha \geq 0, \tag{3}$$

where f_0 is the pdf of a symmetric around zero rv Y with $E[Y^2] = k < \infty$. If $f_0(\cdot) = \phi$, the pdf of the $N(0,1)$ distribution, then $k = 1$ and (3) is called the Extended Half-Normal (EHN) density, which was studied in [11]. Other choices of f_0 are possible, for instance, taken as $f_0(\cdot)$ the Power Exponential pdf, [12], the Extended Half-Power Exponential model was recently proposed in [13].

The aim of this paper is to obtain a new model of distributions, based on (2), which will be flexible enough to modelling non-negative datasets.

The outline of this paper is the following one. In Section 2, the new general family of distributions is introduced along with basic properties. In Section 3, the particular case based on the skew-normal pdf is considered. This model is called the Extended Half Skew-Normal (EHSN) distribution. Its cdf, moments and stochastic representation are studied in detail. Section 4 is devoted to inference in the EHSN model. The method of moments and maximum likelihood (ML) are discussed. In Section 5, a simulation study is carried out to assess the consistence of ML estimators in the EHSN model. Finally, two real applications are given in Section 6, which illustrate the usefulness of our proposal.

2. A New General Family of Distributions

In this section, based on (1), a new class of distributions for modelling non-negative data is introduced. Some basic properties of this family are also given.

Lemma 2. *Let $g_\lambda = g(x; \lambda)$ introduced in (2), and h a non-negative scalar function such that*

$$\int_0^\infty h(x) g_\lambda(x) dx < \infty.$$

Then,

$$f(x) = c g_\lambda(x) h(x), \qquad x > 0\,,$$

is a pdf in $\mathbb{R}^+$ with $c^{-1} = \int_0^\infty h(x) g_\lambda(x) dx < \infty$.

Note that, varying f_0 or h in Lemma 2, a diversity of distributions with non-negative support can be obtained. In Proposition 1, we apply the idea given in (3) with $k = 1$ to the skew density g_λ defined in (2).

Proposition 1. *Let $g_\lambda(\cdot)$ defined in (2) and $c_{\alpha,\lambda}^{-1} = \int_0^\infty \left(\frac{\alpha + x^2}{\alpha+1}\right) g_\lambda(x)dx$. Provided that $c_{\alpha,\lambda}^{-1} < \infty$, and by applying Lemma 2, the following family of densities is obtained*

$$f_X(x;\alpha,\lambda) = c_{\alpha,\lambda}\left(\frac{\alpha + x^2}{\alpha+1}\right) g_\lambda(x), \qquad x \geq 0, \tag{4}$$

where $\alpha > 0$ and $\lambda \in \mathbb{R}$.

Proof. Straightforward by using Lemma 2. □

Remark 1. *Particular cases of interest of (4) are:*

P1. Taking limit when $\lambda \to \infty$, then g_λ tends to the half (or folded at zero) density of f_0, $2f_0(x)$ with $x \geq 0$, and therefore $f_X(\cdot)$ tends to the pdf introduced in (3).
P2. If $\alpha \to \infty$, then f_X tends to the density g_λ truncated to $(0, +\infty)$, which can be denoted as $f_X(x;\alpha,\lambda) \to c g_\lambda(x)$, $x \geq 0$.

In a general setting, it is of interest to introduce a scale parameter $\delta > 0$. Then, the pdf of our family will be $\delta^{-1} f_X(x/\delta)$, that is

$$f_X(x;\alpha,\lambda,\delta) = \frac{c_{\alpha,\lambda}}{\delta}\left(\frac{\alpha + (x/\delta)^2}{\alpha+1}\right) g_\lambda\left(\frac{x}{\delta}\right), \tag{5}$$

with $c_{\alpha,\lambda}$ given in Proposition 1.

The notation $X \sim f_X(x;\alpha,\lambda,\delta)$ will be used to refer to the density defined in (5).

3. The Extended Half Skew-Normal

In this section, we focus on the particular case in which $g_\lambda(\cdot)$ given in (2), is the pdf of the skew normal model, that is , $f_0 = \phi$. The model, which results of applying (4), will be called the Extended Half Skew-Normal (EHSN). Results in this new model will be obtained by applying next lemmas, whose proofs can be seen in Nadarajah and Kotz [10] and Huang et al. [14], respectively.

Lemma 3 ((Nadarajah and Kotz [10])). *Let F_{t_r} be the cdf of a t-Student's distribution with $r > 0$ degrees of freedom. Then, for every positive integer r, F_{t_r} is given by*

$$F_{t_r}(t) = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{t}{\sqrt{r}}\right) + \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{(r-1)/2} \frac{\Gamma(i) r^{i-1/2}}{\Gamma(i+1/2)} \frac{t}{(r+t^2)^i}, & \text{if } r \text{ is odd} \\ \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{r/2} \frac{\Gamma(i-1/2) r^{i-1}}{\Gamma(i)} \frac{t}{(r+t^2)^{i-1/2}}, & \text{if } r \text{ is even} \end{cases} \tag{6}$$

where $t \in \mathbb{R}$ and $\sum_{i=1}^{0}$ is defined as 0.

Lemma 4 (Huang et al. [14]). *For $s \geq -1$ and $\lambda \in \mathbb{R}$, the following result holds*

$$\int_0^\infty v^s \phi(v)\Phi(\lambda v)dv = \frac{2^{s/2-1}\Gamma((s+1)/2)}{\sqrt{\pi}} F_{t_{s+1}}\left(\lambda\sqrt{s+1}\right). \tag{7}$$

Lemma 5. *Let X have pdf $f_X(x;\alpha,\lambda,\delta)$ as introduced in (5). Then, the normalizing constant, $c_{\alpha,\lambda}$, in (5) is*

$$c_{\alpha,\lambda} = \frac{2\pi}{\pi + 2\arctan(\lambda) + \frac{2\lambda}{(1+\lambda^2)(\alpha+1)}}. \tag{8}$$

Proof. Note that, making the change of variable $v = \frac{x}{\delta}$,

$$c_{\alpha,\lambda}^{-1} = \int_0^{\infty} \frac{2}{\delta^3}\left(\frac{\alpha\delta^2 + x^2}{\alpha+1}\right)\phi\left(\frac{x}{\delta}\right)\Phi\left(\lambda\frac{x}{\delta}\right)d = \int_0^{\infty} 2\left(\frac{\alpha + v^2}{\alpha+1}\right)\phi(v)\Phi(\lambda v)dv\,.$$

By applying the results given in Lemma 3 and Lemma 4 [10]

$$\int_0^{\infty}\phi(v)\Phi(\lambda v)dv = \frac{\Gamma(1/2)}{2\sqrt{\pi}}F_{t_1}(\lambda) = \frac{1}{4} + \frac{1}{2\pi}\arctan(\lambda)\,, \tag{9}$$

$$\int_0^{\infty} v^2\phi(v)\Phi(\lambda v)dv = \frac{\Gamma(3/2)}{\sqrt{\pi}}F_{t_3}\left(\lambda\sqrt{3}\right) = \frac{1}{4} + \frac{1}{2\pi}\left[\frac{\lambda}{1+\lambda^2} + \arctan(\lambda)\right]. \tag{10}$$

From (8)–(10) is obtained. □

Proposition 2. *Let $X \sim EHSN(\alpha,\delta,\lambda)$. Then, the pdf of X is given by*

$$f_X(x;\alpha,\delta,\lambda) = \frac{c_{\alpha,\lambda}}{\delta^3}\left(\frac{\alpha\delta^2 + x^2}{\alpha+1}\right)2\phi\left(\frac{x}{\delta}\right)\Phi\left(\lambda\frac{x}{\delta}\right),\ \ x \geq 0, \tag{11}$$

with $\alpha > 0$ shape parameter, $\delta > 0$ scale parameter, $\lambda \geq 0$ skewness parameter and $c_{\alpha,\lambda}$ given in (8).

Proof. It follows from Lemma 5. □

Remark 2. *In the model introduced in Proposition 2, we restrict the skewness parameter to $\lambda \geq 0$ since negative values of λ skew the distribution to negative values of x, and the resulting pdf's are not of interest for the purpose of modelling non-negative data.*

Corollary 1. *The following models are particular cases of the EHSN distribution:*

1. *If $\lambda \to 0$ or $\lambda \to +\infty$ then $EHSN(\alpha,\delta,\lambda)$ reduces to the Extended Half-Normal distribution, $EHN(\alpha,\delta)$, introduced in* [11].
2. *If $\lambda = 0$, $\alpha = 0$, and $\delta = 1$, then $EHSN(\alpha = 0, \delta = 1, \lambda = 0)$ reduces to the Right Half Bimodal Normal model proposed in* [15], *$RHBN(2)$.*
3. *If $\lambda = 0$ and $\alpha \to \infty$ then $EHPE(\alpha \to \infty, \delta, \lambda = 0)$ reduces to the Half-Normal distribution, HN(δ).*

Figure 1 summarizes the relationships among the EHSN and the particular cases previously cited.

In Figure 2, plots for the pdf of EHSN model are given. Without loss of generality, the scale parameter $\delta = 1$ is taken. Four values of λ are fixed ($\lambda = 0.1, 1, 2, 5$), and several values of α are considered.

Next, it is proven that the cdf of the EHSN model can be expressed in terms of the cdf of a skew-normal, cdf and pdf of a Generalized Gamma, and pdf of a $N(0,1)$ distribution. Details about the Generalized Gamma introduced by Stacy [16] are given in Appendix A.

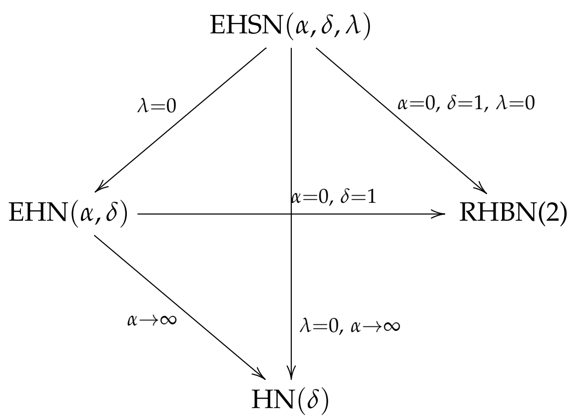


Figure 1. Particular cases for the EHSN distribution.

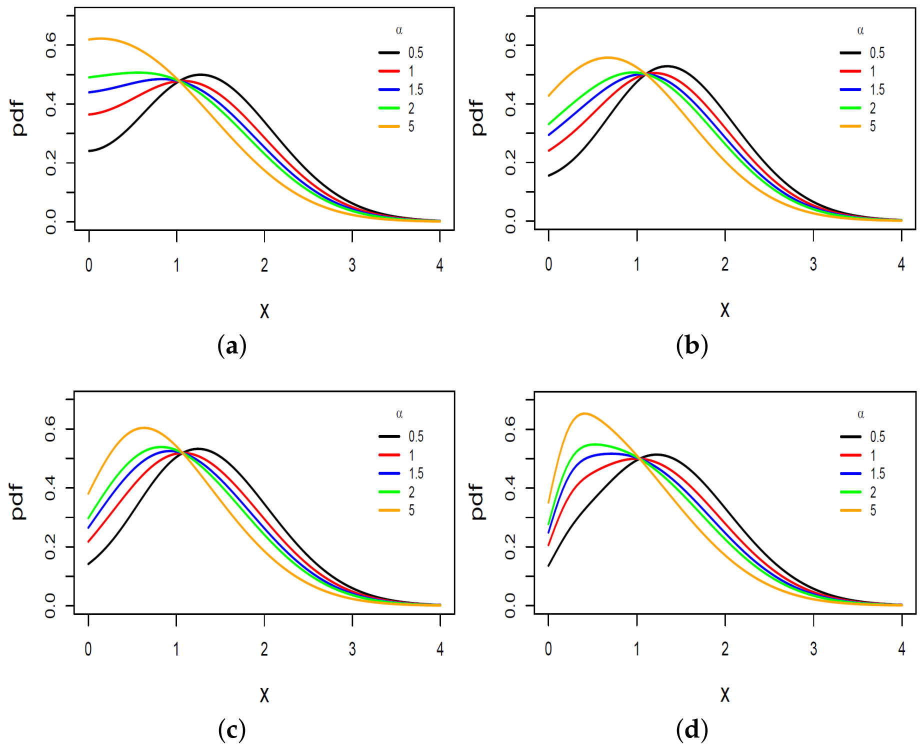


Figure 2. (**a**) Pdf of EHSN $(\alpha, \delta = 1, \lambda = 0.1)$ for different values of α. (**b**) Pdf of EHSN $(\alpha, \delta = 1, \lambda = 1)$ for different values of α. (**c**) Pdf of EHSN $(\alpha, \delta = 1, \lambda = 2)$ model for different values of α. (**d**) Pdf of EHSN $(\alpha, \delta = 1, \lambda = 5)$ model for different values of α.

Proposition 3. *Let $X \sim EHSN(\alpha, \delta, \lambda)$, $\lambda > 0$. Then, the cdf X, F_X, can be obtained as:*

$$F_X(x) \;=\; \frac{c_{\alpha,\lambda}}{(\alpha+1)}\left\{\alpha\left[\Phi\left(\frac{x}{\delta}\right) - 2O\left(\frac{x}{\delta},\lambda\right) - \frac{1}{2} + \frac{1}{\pi}\arctan(\lambda)\right] + \frac{1}{2}F_{GG}\left(\frac{x}{\delta}\right) + \int_0^{x/\delta} f_{GG}(t)\int_0^{\lambda t}\phi(u)dudt\right\}, \tag{12}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of a $N(0,1)$, F_{GG} and f_{GG} denotes the cdf and pdf of a Generalized Gamma distribution, $GG(\sqrt{2},3,2)$, and $O(\cdot,\lambda)$ is the Owen [17] *function.*

Proof. Given $X \sim EHSN(\alpha, \delta, \lambda)$, let us consider $Z = \frac{X}{\delta} \sim EHSN(\alpha, 1, \lambda)$. We have

$$F_X(x) = F_Z\left(\frac{x}{\delta}\right). \tag{13}$$

Next, we are going to obtain the cdf of Z for $z > 0$

$$\begin{aligned} F_Z(z) &= \int_0^z f_Z(t)dt \\ &= c_{\alpha,\lambda}\frac{\alpha}{(\alpha+1)}\int_0^z 2\phi(t)\Phi(\lambda t)dt + c_{\alpha,\lambda}\frac{1}{(\alpha+1)}\int_0^z 2t^2\phi(t)\Phi(\lambda t)dt\,. \end{aligned}$$

Note that $2\phi(t)\Phi(\lambda t)$ for $t \in \mathbb{R}$ is the pdf of a skew-normal distribution, $SN(\lambda)$, as it can be seen in [1]. Therefore, we can write

$$\int_0^z 2\phi(t)\Phi(\lambda t)dt = \Phi\left(\frac{x}{\delta}\right) - 2O\left(\frac{x}{\delta},\lambda\right) - \frac{1}{2} + \frac{1}{\pi}\arctan(\lambda)\,, \tag{14}$$

On the other hand, let us next consider

$$\int_0^z 2t^2\phi(t)\Phi(\lambda t)dt\,.$$

Note that $2t^2\phi(t)$ for $t > 0$ is the pdf of the Generalized Gamma distribution introduced by Stacy [16], $GG(a = \sqrt{2},\ d = 3,\ p = 2)$. By proceeding similarly to [18], we have that

$$\int_0^z 2t^2\phi(t)\Phi(\lambda t)dt = \int_0^z 2t^2\phi(t)\int_{-\infty}^{\lambda t}\phi(u)dudt\,,$$

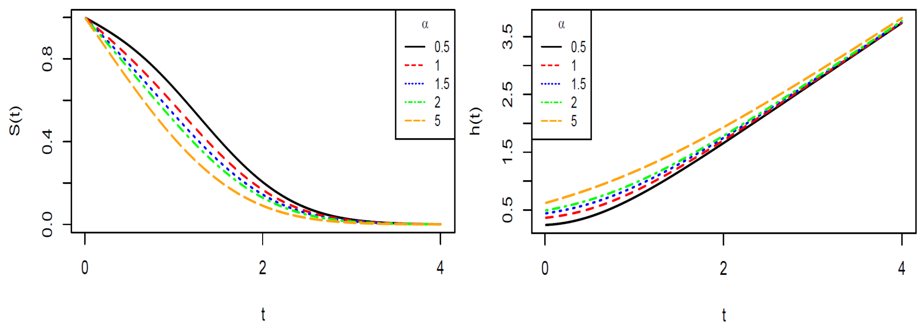
can be obtained in terms of the joint distribution of a random vector (T,U) with $T \sim GG(\sqrt{2},\ 3,\ 2)$, $f_{GG}(t) = 2t^2\phi(t)$ with $t > 0$, and $U \sim N(0,1)$ independent.

Specifically, for $\lambda > 0$

$$\int_0^z 2t^2\phi(t)\int_{-\infty}^{\lambda t}\phi(u)dudt = F_{GG}(z)\Phi(0) + \int_0^z 2t^2\phi(t)\int_0^{\lambda t}\phi(u)dudt\,.$$

Taking into account that $\Phi(0) = 1/2$, $f_{GG}(t) = 2t^2\phi(t)$ with $t > 0$, and (12) and (13) follows. □

Since the EHSN distribution can be used to model lifetime data is of interest to study its survival and hazard rate function, see [19,20]. These functions are next obtained. Some plots are given in Figure 3.



(a) **(b)**

Figure 3. (**a**) Plots of the survival function for the EHSN$(\alpha,\ 1,\ 0.1)$. (**b**) Plots of the hazard rate function for the EHSN$(\alpha,\ 1,\ 0.1)$.

Corollary 2. *Let $X \sim EHSN(\alpha,\delta,\lambda)$ with $\lambda > 0$. Then*

1. *The survival function, S_X, is $S_X(t) = 1 - F_X(t)$, $t > 0$, and F_X was given in (12).*
2. *The hazard rate function, h_X, is $h_X(t) = \frac{f_X(t)}{S_X(t)}$, $t > 0$, and f_X given in (11).*

Plots of the survival and hazard rate function of $X \sim EHSN(\alpha, \delta, \lambda)$ are given in Figure 3 for fixed $\lambda = 0.1, \delta = 1$, and $\alpha \in \{0.5, 1, 1.5, 2, 5\}$.

Remark 3. *Similar plots were obtained for the values of parameters considered in Figure 2. The plot obtained in Figure 3 for the hazard rate function suggests that for λ and δ fixed, the EHSN model is stochastically ordered with respect to α.*

3.1. Moments

From now on, the notation $Y \sim f_Y(y)$ will be used to denote that Y has pdf f_Y. In this section, first the moments of $EHSN$ model are obtained. Second it is proven that the moment generating function (mgf) of $EHSN$ model can be obtained from the mgf of $Y \sim c_1\phi(y)\Phi(\lambda y),\ y \geq 0$.

Proposition 4. *Let $X \sim EHSN(\alpha, \delta, \lambda)$. Then, the moment of order n can be obtained as*

$$E[X^n] = \frac{2c_{\alpha,\lambda}}{\alpha+1}\left\{\frac{\alpha 2^{n/2-1}\Gamma((n+1)/2)}{\sqrt{\pi}}F_{t_{n+1}}(\lambda\sqrt{n+1}) + \frac{2^{n/2}\Gamma((n+3)/2)}{\sqrt{\pi}}F_{t_{n+3}}(\lambda\sqrt{n+3})\right\},$$

where $F_{t_r}(\cdot)$ was given in (6).

Proof. From (6) and (7), the proposed result follows. □

Next, the moment generating function (mgf) is obtained.

Proposition 5. *Let $X \sim EHSN(\alpha, \delta, \lambda)$. Then the mgf of X, M_X, can be obtained as*

$$M_X(t) = M_Z(\delta t) \quad with \ \ Z \sim EHSN(\alpha, 1, \lambda)$$

and where the mgf of Z is given by

$$\begin{aligned} M_Z(t) &= \frac{c_{\alpha,\lambda}}{(\alpha+1)}(1+\alpha+t^2)\ M_Y(t) \\ &+ \frac{c_{\alpha,\lambda}}{(\alpha+1)}\left[\frac{t}{\sqrt{2\pi}} + t\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}M_{HN}(t) \ + \ \frac{\lambda}{\pi(1+\lambda^2)}\ M_{GG}(t)\right], \end{aligned}$$

where $M_Y(t)$ is the mgf of $Y \sim c_1 2\phi(y)\Phi(\lambda y)$, $y > 0$; $M_{HN}(t)$ is the mgf of a half normal distribution, $HN\left(\frac{1}{1+\lambda^2}\right)$; and M_{GG} is the mgf of a Generalized Gamma, $GG\left(\frac{\sqrt{2}}{\sqrt{1+\lambda^2}},\ 2,\ 2\right)$.

Proof.

$$M_Z(t) = E[\exp(tZ)] = \int_0^\infty e^{tz} f_z(z)dz,$$

with f_z the pdf of $Z \sim EHSN(\alpha, 1, \lambda)$.

Note that

$$\int_0^\infty e^{tz} 2\phi(z)\Phi(\lambda z)dz = M_Y(t) \tag{15}$$

with $Y \sim c_1 2\phi(y)\Phi(\lambda y)$, $y > 0$. That is, the mgf of a skew normal distribution, $SN(\lambda)$, truncated at zero.

Let us consider now

$$\int_0^\infty e^{tz} 2z^2\phi(z)\Phi(\lambda z)dz = I. \tag{16}$$

This integral is solved by applying integration by parts twice, taking in both cases $dv = z\phi(z)$, $(v = -\phi(z))$. First, we have

$$I = M_y(t) + 2t\int_0^\infty e^{tz} z\phi(z)\Phi(\lambda z)dz + 2\lambda\int_0^\infty e^{tz} z\phi(z)\phi(\lambda z)dz.$$

Second, we obtain

$$\begin{aligned} 2t\int_0^\infty e^{tz}z\phi(z)\Phi(\lambda z)dz &= \frac{t}{\sqrt{2\pi}} + t^2M_y(t) + \lambda t\int_0^\infty e^{tz}2\phi(z)\phi(\lambda z)dz \\ &= \frac{t}{\sqrt{2\pi}} + t^2M_y(t) + \lambda t\frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{1+\lambda^2}}M_{HN}(t), \end{aligned}$$

where $M_{HN}(t)$ is the mgf of a half normal distribution, $HN\left(\frac{1}{1+\lambda^2}\right)$.

Finally, we consider

$$\begin{aligned} 2\lambda\int_0^\infty e^{tz}z\phi(z)\phi(\lambda z)dz &= \frac{\lambda}{\pi}\int_0^\infty e^{tz}\, z\, exp\left\{-\frac{z^2(1+\lambda^2)}{2}\right\}dz \\ &= \frac{\lambda}{\pi(1+\lambda^2)}M_{GG}(t) \end{aligned}$$

where M_{GG} is the mgf of a Generalized Gamma, $GG\left(a = \frac{\sqrt{2}}{\sqrt{1+\lambda^2}},\ d = 2,\ p = 2\right)$. $\square$

As a direct consequence of Proposition 4, the first four moments of $EHSN(\alpha, \delta, \lambda)$ model are given, $\mu_k = E[X^k]$, with $k = 1, \ldots, 4$.

Corollary 3. *Let $X \sim EHSN(\alpha, \lambda, \delta)$. Then,*

1. $\mu_1 = \dfrac{2c_{\alpha,\lambda}\delta}{\sqrt{2\pi}(\alpha+1)}\left\{\alpha F_{t_2}\left(\lambda\sqrt{2}\right) + 2F_{t_4}\left(\lambda\sqrt{4}\right)\right\}.$
2. $\mu_2 = \dfrac{c_{\alpha,\lambda}\delta^2}{\alpha+1}\left\{\alpha F_{t_3}\left(\lambda\sqrt{3}\right) + 3F_{t_5}\left(\lambda\sqrt{5}\right)\right\}.$
3. $\mu_3 = \dfrac{2\sqrt{2}c_{\alpha,\lambda}\delta^3}{\sqrt{\pi}(\alpha+1)}\left\{\alpha F_{t_4}\left(\lambda\sqrt{4}\right) + 4F_{t_6}\left(\lambda\sqrt{6}\right)\right\}.$
4. $\mu_4 = \dfrac{c_{\alpha,\lambda}\delta^4}{\alpha+1}\left\{3\alpha F_{t_5}\left(\lambda\sqrt{5}\right) + 15F_{t_7}\left(\lambda\sqrt{7}\right)\right\}.$

Expressions of variance, skewness and kurtosis can be obtained from Corollary 3.

Corollary 4. *Let $X \sim EHSN(\alpha, \lambda, \delta)$. Then,*

1. *The variance of X, $Var[X] = E[X^2] - (E[X])^2$, is*

$$Var[X] = \frac{c_{\alpha,\lambda}\delta^2}{\alpha+1}\left\{\alpha F_{t_3}(\lambda\sqrt{3}) + 3F_{t_5}(\lambda\sqrt{5}) - \frac{4c_{\alpha,\lambda}}{2\pi(\alpha+1)}\left[\alpha F_{t_2}(\lambda\sqrt{2} + 2F_{t_4}(\lambda\sqrt{4}))\right]^2\right\}.$$

2. *The skewness, $\sqrt{\beta_1}$, and kurtosis, β_2, coefficients can be obtained by using*

$$\sqrt{\beta_1} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}}, \quad and \quad \beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}.$$

Without loss of generality, the scale parameter can be taken as one, $\delta = 1$. Plots for $\sqrt{\beta_1}$ and β_2, as functions of α and λ are given in Figure 4.

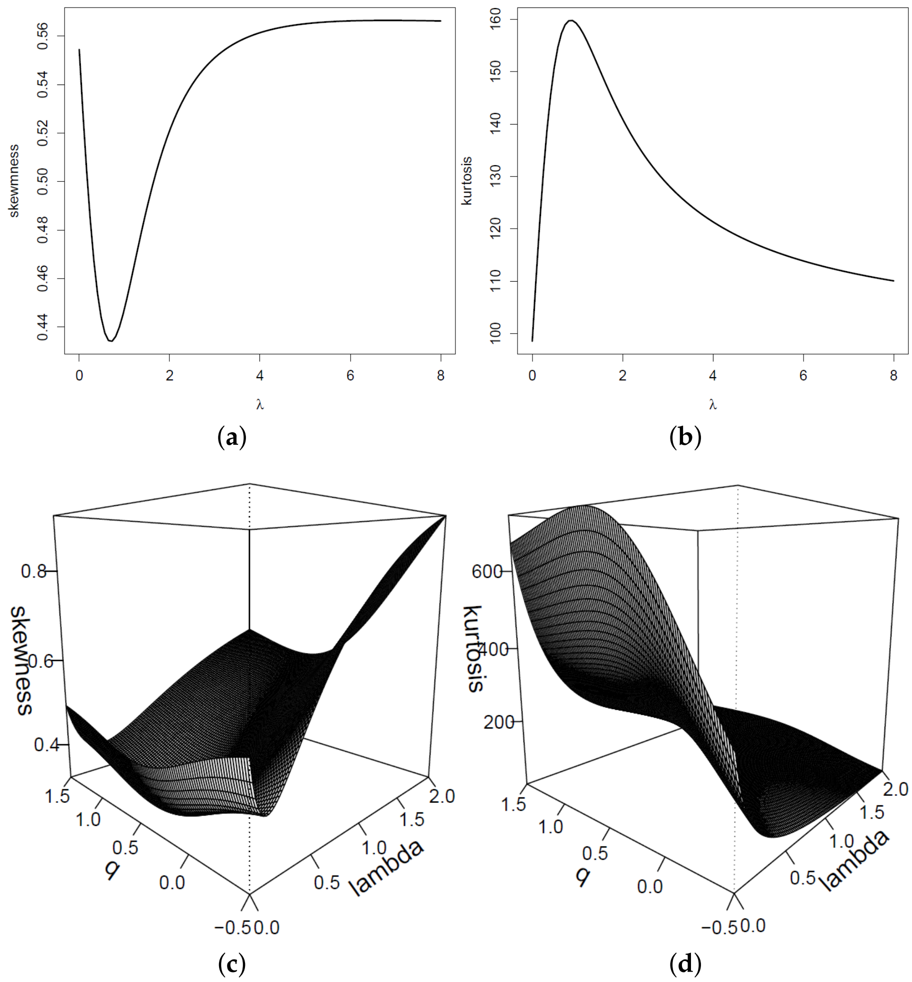


Figure 4. (**a**) $\sqrt{\beta_1}$ as function of λ ($\alpha = \delta = 1$); (**b**) β_2 as function of λ ($\alpha = \delta = 1$); (**c**) $\sqrt{\beta_1}$ as function of α and λ ($\delta = 1$); (**d**) β_2 as function of α and λ ($\delta = 1$).

3.2. Stochastic Representation

In this section, the stochastic representation of the EHSN model is given. This result will be used to generate random values of this model.

Next propositions allow us to obtain the stochastic representation of the EHSN model.

Proposition 6. *Let Y and U be independent rv's, with $Y^2 \sim \chi^2_3$ and $P(U = -1) = P(U = 1) = \frac{1}{2}$. Then, the rv $W = U|Y|$ is distributed as $W \sim w^2\phi(w)$ with $w \in \mathbb{R}$.*

Proof. It can be seen in Elal-Olivero et al. [11]. □

Proposition 7. *Let Z and W be independent rv's, with $Z \sim N(0,1)$ and $W \sim w^2\phi(w)$, $w \in \mathbb{R}$. Let us define $H = \sqrt{\frac{\alpha}{\alpha+1}}Z + \sqrt{\frac{1}{\alpha+1}}W$. Then $H \sim \dfrac{\alpha + h^2}{\alpha + 1}\phi(h)$, with $h \in \mathbb{R}$ and $\alpha > 0$.*

Proof. Let us denote $a = \sqrt{\frac{\alpha}{\alpha+1}}$ and $b = \sqrt{\frac{1}{\alpha+1}}$. We have that the cdf of H, F_H, is

$$F_H(h) \quad = \quad P(aZ + bW \leq h) = P\left(Z \leq \frac{h - bW}{a}\right) = \int_{-\infty}^{\infty} F_Z\left(\frac{h - bw}{a}\right) f_W(w)dw \, .$$

Taking derivative with respect to h, the pdf of H, f_H, is obtained

$$\begin{aligned} f_H(h) &= \frac{1}{a}\int_{-\infty}^{\infty} f_Z\left(\frac{h-bw}{a}\right) f_w(w)dw = \frac{1}{a}\phi(h)\int_{-\infty}^{\infty} w^2\phi\left(\frac{w-hb}{a}\right)dw \\ &= \left(\frac{\alpha+h^2}{\alpha+1}\right)\phi(h), \qquad h \in \mathbb{R}. \end{aligned}$$

□

Proposition 8. *Let $L = |H|$, where H was defined in Proposition 7. Then, $L \sim 2\left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l)$ with $l > 0$, that is, $L \sim EHN(\alpha)$.*

Proof. For $l > 0$, we have that

$$F_L(l) = F_{|H|}(l) = P(|H| < l) = P(H < l) - P(H < -l) = F_H(l) - F_H(-l).$$

Taking derivative with respect to l, and sice f_H is symmetrical about zero, we have that

$$f_L(l) = f_H(l) + f_H(-l) = 2f_H(l) = 2\left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l), \quad l > 0.$$

□

Proposition 9. *Let L and Z be independent rv's, such that $L \sim 2\left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l)$, with $l > 0$ and $Z \sim N(0,1)$. Then*

$$T \equiv L|Z < \lambda L \sim EHSN(\alpha,\lambda), \qquad \lambda \in \mathbb{R}\,.$$

Proof. The cdf of T, F_T, is

$$F_T(t) = P(L < t|Z < \lambda L) = \frac{P(L \le t, Z < \lambda L)}{P(Z < \lambda L)}, \qquad t > 0\,.$$

The numerator is

$$P(L \le t, Z < \lambda L) = \int_0^t P(Z < \lambda L|L = l) f_L(l)dl = 2\int_0^t \left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l)\Phi(\lambda l)dl\,.$$

The denominator is

$$P(Z < \lambda L) = \int_0^\infty P(Z < \lambda L|L = l) f_L(l)dl = 2\int_0^\infty \left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l)\Phi(\lambda l)dl\,.$$

From (8), $P(Z < \lambda L) = c_{\alpha,\lambda}^{-1}$, and therefore

$$F_T(t) = 2c_{\alpha,\lambda}\int_0^t \left(\frac{\alpha+l^2}{\alpha+1}\right)\phi(l)\Phi(\lambda l)dl\,.$$

Taking derivative with respect to t, the pdf of T is

$$f_T(t) = 2c_{\alpha,\lambda}\left(\frac{\alpha+t^2}{\alpha+1}\right)\phi(t)\Phi(\lambda t), \qquad t > 0.$$

□

Proposition 10. *Let $X = \delta T$ with $\delta > 0$ and T defined in Proposition 9. Then $X \sim EHSN(\alpha,\delta,\lambda)$.*

Proof. Immediate since the pdf of X is $f_X(x) = \frac{1}{\delta} f_T\left(\frac{x}{\delta}\right)$. $\square$

4. Inference

In this section, inference for the parameters in the $EHSN$ distribution is carried out from a classical point of view. Let $X \sim EHSN(\alpha, \delta, \lambda)$ and consider independent copies of X, that is $X_1, X_2, \ldots, X_n$ a random sample from X. The method of moments and maximum likelihood estimators are next discussed.

4.1. Method of Moment Estimators

The moments estimators result from the solution of the equations $E(X^j) = \overline{X^j}$, for $j = 1, 2, 3$, where $\overline{X^j} = n^{-1}\sum_{i=1}^{n} x_i^j$ denotes the j-th sample moment. Solving $E(X) = \overline{X}$, we have that

$$\delta = \frac{\overline{X}\sqrt{2\pi}(\alpha+1)}{2c_{\alpha,\lambda}\left\{\alpha F_{t_2}\left(\lambda\sqrt{2}\right) + 2F_{t_4}\left(\lambda\sqrt{4}\right)\right\}}. \tag{17}$$

Taking (17), and by substituting it into $E(X^j)$, $j = 2, 3$ given in Corollary 1, we get

$$\overline{X^2} = \frac{\overline{X}^2\pi(\alpha+1)\left\{\alpha F_{t_3}\left(\lambda\sqrt{3}\right) + 3F_{t_5}\left(\lambda\sqrt{5}\right)\right\}}{2c_{\alpha,\lambda}\left\{\alpha F_{t_2}\left(\lambda\sqrt{2}\right) + 2F_{t_4}\left(\lambda\sqrt{4}\right)\right\}^2}, \tag{18}$$

$$\overline{X^3} = \frac{\overline{X}^3\pi(\alpha+1)^2\left\{\alpha F_{t_4}\left(\lambda\sqrt{4}\right) + 4F_{t_6}\left(\lambda\sqrt{6}\right)\right\}}{c_{\alpha,\lambda}^2\left\{\alpha F_{t_2}\left(\lambda\sqrt{2}\right) + 2F_{t_4}\left(\lambda\sqrt{4}\right)\right\}^3}. \tag{19}$$

These equations must be solved by using mathematical software, such as the function `nleqslv` available in R software [21], to obtain the moment estimators $\widehat{\alpha}_{MM}$ and $\widehat{\lambda}_{MM}$. Finally, $\widehat{\delta}_{MM}$ is obtained from (17).

4.2. Maximum Likelihood

Given $X_1, X_2, \ldots, X_n$ a random sample of size n from $EHSN(\alpha, \delta, \lambda)$, then from (11), the log-likelihood function is given by

$$\ell(\boldsymbol{\theta}) \propto n\log(c_{\alpha,\lambda}) - 3n\log(\delta) - n\log(\alpha+1) + \sum_{i=1}^{n}\log(\alpha\delta^2 + x_i^2) - \frac{1}{2\delta^2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\log\left(\Phi\left(\lambda\frac{x_i}{\delta}\right)\right), \tag{20}$$

where $\propto$ means proportional to, and $\boldsymbol{\theta} = (\alpha, \delta, \lambda)$. Taking partial derivatives with respect to α, δ, and λ, the elements of the score vector are obtained, $S(\boldsymbol{\theta}) = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\lambda}\right)$, that is

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n}{\alpha+1} + \frac{n\lambda c_{\alpha,\lambda}}{\pi(\alpha+1)^2(1+\lambda^2)} + \sum_{i=1}^{n}\frac{\delta^2}{\alpha\delta^2 + x_i^2},$$

$$\frac{\partial\ell}{\partial\delta} = -\frac{3n}{\delta} + \frac{1}{\delta^3}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\frac{2\alpha\delta}{\alpha\delta^2 + x_i^2} - \lambda\sum_{i=1}^{n}\frac{x_i}{\delta^2}\xi\left(\lambda\frac{x_i}{\delta}\right),$$

$$\frac{\partial\ell}{\partial\lambda} = -\frac{c_{\alpha,\lambda}}{\pi(1+\lambda^2)}\left(1 + \frac{1-\lambda^2}{(1+\alpha)(1+\lambda^2)}\right) + \sum_{i=1}^{n}\frac{x_i}{\delta}\xi\left(\lambda\frac{x_i}{\delta}\right),$$

where $\xi(\cdot) = \phi(\cdot)/\Phi(\cdot)$. The MLEs of $\widehat{\boldsymbol{\theta}}$ are obtained as solution of $S(\boldsymbol{\theta}) = \mathbf{0}$. Numerical methods must be used to solve this system. For instance, we use the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm to obtain the estimators of $(\log\alpha, \log\delta, \lambda)$ and by invariance, we obtained the estimators of α and δ (See details in [22]).

4.3. Observed Fisher Information Matrix

The asymptotic variance of MLEs, $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\delta}, \widehat{\lambda})$, can be estimated from the Fisher information matrix, given by $\mathcal{I}(\boldsymbol{\theta}) = -E\left[\partial^2 \ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top\right]$ with $\ell(\boldsymbol{\theta})$ given in (20). Recall that, under regularity conditions,

$$\mathcal{I}(\boldsymbol{\theta})^{-1/2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{\mathcal{D}} N_3(\mathbf{0}_3, \boldsymbol{I}_3), \quad \text{as } n \to +\infty, \tag{21}$$

where $\mathcal{D}$ stands convergence in distribution and $N_3(\mathbf{0}_3, \boldsymbol{I}_3)$ denotes the standard trivariate normal distribution. Moreover, $\mathcal{I}(\boldsymbol{\theta})$ can be obtained from the matrix $-\partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top$, whose elements are given by $I_{\alpha\alpha} = -\partial^2\ell(\boldsymbol{\theta})/\partial\alpha^2$, $I_{\alpha\delta} = -\partial^2\ell(\boldsymbol{\theta})/\partial\alpha\partial\delta$, and so on. Explicitly, we have

$$\begin{aligned}
I_{\alpha\alpha} &= \sum_{i=1}^{n}\frac{\delta^4}{(\alpha\delta^2+x_i^2)^2} - \frac{n}{(\alpha+1)^2}\left[1+\frac{\lambda c_{\alpha,\lambda}}{\pi(\alpha+1)(1+\lambda^2)}\left(\frac{\lambda c_{\alpha,\lambda}}{\pi(\alpha+1)(1+\lambda^2)}-2\right)\right],\\
I_{\alpha\delta} &= \sum_{i=1}^{n}\frac{2\delta}{\alpha\delta^2+x_i^2}\left[\frac{\alpha\delta^2}{(\alpha\delta^2+x_i^2)}-1\right],\\
I_{\alpha\lambda} &= \frac{nc_{\alpha,\lambda}}{\pi(\alpha+1)^2(1+\lambda^2)}\left[\frac{\lambda c_{\alpha,\lambda}}{\pi(1+\lambda^2)}k_{\alpha,\lambda}+\frac{2\lambda^2}{1+\lambda^2}-1\right],\\
I_{\delta\delta} &= \frac{3}{\delta^2}\left[\sum_{i=1}^{n}\frac{x_i^2}{\delta^2}-n\right]+\sum_{i=1}^{n}\frac{2\alpha}{\alpha\delta^2+x_i^2}\left[\frac{2\alpha\delta^2}{\alpha\delta^2+x_i^2}-1\right]+\lambda\sum_{i=1}^{n}\frac{x_i}{\delta^3}\xi\left(\lambda\frac{x_i}{\delta}\right)\left[\lambda\frac{x_i}{\delta}\left(\lambda\frac{x_i}{\delta}+\xi\left(\lambda\frac{x_i}{\delta}\right)\right)-2\right],\\
I_{\delta\lambda} &= \sum_{i=1}^{n}\frac{x_i}{\delta^2}\xi\left(\lambda\frac{x_i}{\delta}\right)\left[1-\lambda\frac{x_i}{\delta}\left(\lambda\frac{x_i}{\delta}+\xi\left(\lambda\frac{x_i}{\delta}\right)\right)\right],\\
I_{\lambda\lambda} &= \frac{-c_{\alpha,\lambda}}{\pi(1+\lambda^2)^2}\left[k_{\alpha,\lambda}\left[\frac{c_{\alpha,\lambda}}{\pi}k_{\alpha,\lambda}+2\lambda\right]+\frac{4\lambda}{(1+\alpha)(1+\lambda^2)}\right]+\sum_{i=1}^{n}\frac{x_i^2}{\delta^2}\xi\left(\lambda\frac{x_i}{\delta}\right)\left[\lambda\frac{x_i}{\delta}+\xi\left(\lambda\frac{x_i}{\delta}\right)\right].
\end{aligned}$$

where $k_{\alpha,\lambda} = \left(1+\frac{1-\lambda^2}{(1+\alpha)(1+\lambda^2)}\right)$.

In practice, it is not possible to obtain a closed form to the expected value of previous expressions. So, the covariance matrix of MLEs, $\mathcal{I}(\boldsymbol{\theta})^{-1}$, can be estimated by $I(\widehat{\boldsymbol{\theta}})^{-1}$, where $I(\widehat{\boldsymbol{\theta}})$ denotes the observed information matrix, which is obtained by evaluating the previous derivatives at the MLE $\widehat{\boldsymbol{\theta}}$, i.e.

$$I(\widehat{\boldsymbol{\theta}}) = -\partial^2\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}. \tag{22}$$

The asymptotic variances of $\widehat{\alpha}$, $\widehat{\delta}$, and $\widehat{\lambda}$ are estimated by the diagonal elements of $I(\widehat{\boldsymbol{\theta}})^{-1}$, and their standard errors by the square root of asymptotic variances. Details about the theoretical results used in this subsection can be seen in [23].

5. Simulation Study

In this section, a simulation study is carried out to asses the performance of ML estimators. First an algorithm to generate samples from $EHSN(\alpha, \lambda, \delta)$ is given. The simulation algorithm is based on the stochastic representation introduced in Section 3.2.

Algorithm

(i) Simulate independently: $R \sim U(0,1)$, $Y \sim \chi_3^2$ and $Z, J \sim N(0,1)$.
(ii) If $R < \frac{1}{2}$, then $U = -1$. Otherwise $U = 1$.
(iii) Compute $W = U\sqrt{Y}$.
(iv) Compute $L = \left|\sqrt{\frac{\alpha}{\alpha+1}}Z + \frac{1}{\sqrt{\alpha+1}}W\right|$.
(v) Compute $L = |H|$.

(vi) If $J < \lambda L$, then $T = L$. Otherwise, repeat steps (i) to (v) until you get a new random value of T.
(vii) Take $X = \delta T$.

As values of parameters in our simulation, we consider for $\alpha \in \{0.25, 0.5, 1\}$; $\lambda \in \{0.5, 1, 2\}$ and $\delta \in \{1, 10\}$. As for the sample size, we consider $n \in \{100, 300, 500, 1000\}$. For each sample size, and every combination of α, λ, δ, we carry out 1000 replicates and the corresponding ML estimates are computed.

Results are given in Table 1. As summaries we provide the estimated bias (bias), the mean of the estimated standard errors (SE), and the root of the estimated mean squared error (RMSE).

Table 1. Estimated bias, SE and RMSE for ML estimators in finite samples from the EHSN model.

True Value				$n = 100$			$n = 300$			$n = 500$			$n = 1000$		
λ	α	δ	Estimator	Bias	SE	RMSE	Bias	SE	RMSE	Bias	SE	RMSE	Bias	SE	RMSE
0.5	0.25	1	$\widehat{\lambda}$	0.0887	4.1077	2.2876	0.0791	1.5435	1.5804	0.0763	0.8677	0.9295	0.0602	0.5491	0.6101
			$\widehat{\alpha}$	−0.0871	0.1766	0.1558	−0.0580	0.1117	0.1078	−0.0529	0.0906	0.0972	−0.0429	0.0702	0.0788
			$\widehat{\delta}$	−0.0001	2.9992	1.6374	−0.0061	0.6085	0.7113	−0.0073	0.2588	0.4069	−0.0091	0.0551	0.0886
		10	$\widehat{\lambda}$	0.0919	3.6774	2.0329	0.0881	1.3901	1.4581	0.0860	0.8025	0.8319	0.0753	0.5241	0.5291
			$\widehat{\alpha}$	−0.0875	0.1778	0.1570	−0.0594	0.1155	0.1060	−0.0524	0.0915	0.0965	−0.0419	0.0688	0.0793
			$\widehat{\delta}$	−0.0801	24.3377	12.8305	−0.0670	4.3303	5.0911	−0.0660	1.8821	2.6233	−0.0598	0.4906	0.6173
	0.5	1	$\widehat{\lambda}$	0.0774	3.9451	2.3954	0.0620	1.7896	1.9473	0.0557	1.2244	1.5575	0.0522	0.7375	0.9020
			$\widehat{\alpha}$	−0.1644	0.3192	0.2752	−0.1319	0.2071	0.2080	−0.1114	0.1689	0.1746	−0.1098	0.1404	0.2068
			$\widehat{\delta}$	−0.0238	2.3073	1.2299	−0.0230	0.6922	0.6892	−0.0229	0.3236	0.4405	−0.0236	0.2054	0.2725
		10	$\widehat{\lambda}$	0.0889	3.4827	2.2626	0.0703	1.6500	1.7088	0.0609	1.1596	1.4419	0.0576	0.6941	0.7828
			$\widehat{\alpha}$	−0.1657	0.3167	0.2721	−0.1314	0.2104	0.2031	−0.1122	0.1723	0.1786	−0.1070	0.1450	0.2083
			$\widehat{\delta}$	−0.2277	18.0318	9.9169	−0.2199	5.0375	4.5636	−0.2028	2.9008	3.5120	−0.1916	1.6328	1.9676
	1	1	$\widehat{\lambda}$	0.1357	4.1670	2.3474	0.1082	2.4756	1.9255	0.0942	1.3401	1.5183	0.0502	1.1993	1.2587
			$\widehat{\alpha}$	−0.3808	2.0775	10.3702	−0.3514	0.4394	0.5198	−0.2628	0.3321	0.3857	−0.3156	0.2642	0.4956
			$\widehat{\delta}$	−0.0504	2.3090	0.9321	−0.0362	1.3206	0.8033	−0.0446	0.3932	0.4115	−0.0329	0.5853	0.4356
		10	$\widehat{\lambda}$	0.1466	3.3898	2.1449	0.1280	2.3190	1.7919	0.1115	1.2832	1.4155	0.1089	1.1240	1.1980
			$\widehat{\alpha}$	−0.3788	2.0948	7.7521	−0.3559	0.4638	0.5231	−0.2558	0.3326	0.3834	−0.3078	0.2647	0.4881
			$\widehat{\delta}$	−0.5089	15.6118	7.6009	−0.3776	11.4282	6.3186	−0.4477	3.5177	3.3111	−0.3209	5.0068	3.5292
1	0.25	1	$\widehat{\lambda}$	0.2221	2.8231	2.1465	0.1679	1.7509	1.9720	0.1490	1.3535	1.7698	0.1119	0.9427	1.4323
			$\widehat{\alpha}$	−0.0774	0.1594	0.1417	−0.0688	0.0903	0.0983	−0.0611	0.0718	0.0849	−0.0548	0.0523	0.0696
			$\widehat{\delta}$	−0.0105	0.0758	0.1084	−0.0089	0.0361	0.0324	−0.0084	0.0288	0.0264	−0.0063	0.0216	0.0211
		10	$\widehat{\lambda}$	0.2056	2.8324	2.0857	0.1573	1.6851	1.8236	0.1186	1.3109	1.6966	0.0883	0.8754	1.3112
			$\widehat{\alpha}$	−0.0810	0.1605	0.1445	−0.0663	0.0911	0.0973	−0.0602	0.0718	0.0829	−0.0551	0.0527	0.0692
			$\widehat{\delta}$	−0.1147	0.9808	1.2949	−0.0940	0.3846	0.5770	−0.0700	0.2878	0.2646	−0.0708	0.2177	0.2161
	0.5	1	$\widehat{\lambda}$	0.1548	2.5072	2.0512	0.1337	1.6216	1.8140	0.1289	1.3093	1.5864	0.1132	0.8936	1.2625
			$\widehat{\alpha}$	−0.1769	0.4831	1.9959	−0.1482	0.1635	0.1823	−0.1362	0.1294	0.1634	−0.1207	0.0933	0.1414
			$\widehat{\delta}$	−0.0217	0.1388	0.1371	−0.0158	0.0418	0.0393	−0.0146	0.0325	0.0323	−0.0123	0.0237	0.0255
		10	$\widehat{\lambda}$	0.1530	2.5203	2.0469	0.1223	1.6141	1.7941	0.1119	1.3024	1.5971	0.1051	0.8922	1.2264
			$\widehat{\alpha}$	−0.1673	0.4137	3.3884	−0.1459	0.1637	0.1808	−0.1364	0.1293	0.1631	−0.1200	0.0932	0.1406
			$\widehat{\delta}$	−0.2053	1.2641	1.0703	−0.1623	0.4137	0.3855	−0.1484	0.3204	0.3188	−0.1157	0.2341	0.2521
	1	1	$\widehat{\lambda}$	0.3995	2.8717	2.0561	0.3470	1.6243	1.7523	0.3251	1.3054	1.6552	0.2763	0.8542	1.1454
			$\widehat{\alpha}$	−0.4039	1.8900	1.7652	−0.3382	0.3390	0.4111	−0.3009	0.2579	0.3659	−0.2563	0.1729	0.3070
			$\widehat{\delta}$	−0.0319	0.7001	0.4396	−0.0222	0.1432	0.1427	−0.0201	0.0806	0.1150	−0.0169	0.0348	0.0432
		10	$\widehat{\lambda}$	0.3930	2.7774	2.1070	0.3625	1.6437	1.7545	0.2641	1.2570	1.5088	0.2525	0.8522	1.0931
			$\widehat{\alpha}$	−0.3956	1.5302	1.4103	−0.3315	0.3347	0.4063	−0.2939	0.2572	0.3579	−0.2582	0.1758	0.3062
			$\widehat{\delta}$	−0.2763	5.6178	3.6497	−0.2195	1.1898	1.2777	−0.1972	0.7523	1.0015	−0.1619	0.3545	0.4032

Table 1. *Cont.*

True Value				$n = 100$			$n = 300$			$n = 500$			$n = 1000$		
λ	α	δ	**Estimator**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**
2	0.25	1	$\widehat{\lambda}$	0.1412	3.8490	2.3980	0.1226	2.5533	2.1963	0.1112	2.1624	2.1480	0.0968	1.5911	1.8478
			$\widehat{\alpha}$	−0.0698	0.1649	0.1486	−0.0639	0.0946	0.0984	−0.0601	0.0747	0.0855	−0.0546	0.0550	0.0724
			$\widehat{\delta}$	−0.0201	0.0660	0.0649	−0.0148	0.0325	0.0349	−0.0122	0.0257	0.0283	−0.0085	0.0180	0.0206
		10	$\widehat{\lambda}$	0.2016	3.7258	2.3354	0.1603	2.6031	2.2358	0.1375	2.0671	2.0419	0.1181	1.5362	1.7291
			$\widehat{\alpha}$	−0.0694	0.2949	10.3606	−0.0627	0.0941	0.0998	−0.0569	0.0752	0.0852	−0.0550	0.0553	0.0720
			$\widehat{\delta}$	−0.1857	0.5683	0.5554	−0.1456	0.3300	0.3430	−0.1145	0.2570	0.2810	−0.0900	0.1804	0.2075
	0.5	1	$\widehat{\lambda}$	0.1408	3.3738	2.4686	0.1244	2.2736	2.0844	0.1169	1.7768	1.7268	0.1078	1.2928	1.3723
			$\widehat{\alpha}$	−0.1466	1.1916	5.8004	−0.1248	0.1735	0.1786	−0.1130	0.1335	0.1509	−0.1044	0.0957	0.1301
			$\widehat{\delta}$	−0.0230	0.1689	0.1737	−0.0167	0.0403	0.0480	−0.0152	0.0295	0.0313	−0.0119	0.0202	0.0237
		10	$\widehat{\lambda}$	0.1595	3.3467	2.2779	0.1472	2.2529	2.1634	0.1353	1.8322	1.8594	0.1222	1.3190	1.3916
			$\widehat{\alpha}$	−0.1456	0.6574	6.9030	−0.1191	0.1703	0.1741	−0.1150	0.1322	0.1544	−0.1060	0.0967	0.1305
			$\widehat{\delta}$	−0.2085	1.2238	1.4289	−0.1672	0.3870	0.3939	−0.1490	0.3094	0.3834	−0.1166	0.2001	0.2341
	1	1	$\widehat{\lambda}$	0.2782	3.5365	2.5655	0.2777	2.0052	1.9471	0.2553	1.6272	1.6502	0.2483	1.1661	1.1607
			$\widehat{\alpha}$	−0.3187	2.1219	9.7211	−0.2470	0.3562	0.3716	−0.2172	0.2736	0.3068	−0.1987	0.1907	0.2553
			$\widehat{\delta}$	−0.0248	0.6483	0.4276	−0.0209	0.0957	0.1128	−0.0166	0.0634	0.1016	−0.0147	0.0258	0.0292
		10	$\widehat{\lambda}$	0.2533	3.3656	2.5756	0.2224	1.9569	1.8486	0.2087	1.5768	1.5812	0.1844	1.1453	1.1161
			$\widehat{\alpha}$	−0.3061	1.9547	4.2552	−0.2403	0.3623	0.3601	−0.2167	0.2723	0.3012	−0.2022	0.1897	0.2534
			$\widehat{\delta}$	−0.2574	4.2033	3.1281	−0.1998	0.8585	0.9176	−0.1704	0.4187	0.4742	−0.1452	0.2487	0.2929

For the ML estimators of λ, α and δ, note that when the sample size increases then the bias, SE and RMSE decrease. Additionally, note that, when the sample size increases then, the SE and RMSE are closer, which suggests that the standard errors of the MLE estimators are well estimated. However, we highlight that such convergence is slower for the ML of λ, suggesting that a big sample size is necessary in order to guarantee good statistical properties of this estimator.

6. Applications

In this section, two real applications are given. The aim is to compare the EHSN model to other models of interest. Specifically, the EHSN model is compared to its precedent, the Extended Half-Normal distribution (EHN) proposed in [11] and the log-skew-normal (LSN) introduced in [24] with location parameter α, scale parameter δ, and shape parameter λ, which includes as particular case the traditional log-normal (LN) distribution for $\lambda = 0$.

6.1. Application 1

We consider the dataset that corresponds to daily average wind speeds for 1961–1978 at 12 synoptic meteorological stations in the Republic of Ireland at the station number 7 (DUB) (see http://lib.stat.cmu.edu/datasets/wind.data, last accessed on 12 July 2022).

In Table 2, the descriptive summaries are provided: sample mean, sample variance, sample skewness ($\sqrt{b_1}$), and sample kurtosis coefficient (b_2). We highlight that we obtained a low value for the sample kurtosis coefficient, $b_2 = 4.0531$, which suggests that a distribution with flexible values for this coefficient, such as the EHSN can be used to model this dataset.

Table 2. Descriptive summaries for wind speeds dataset.

n	$\overline{x}$	s^2	$\sqrt{b_1}$	b_2
6574	6.3063	12.999	0.9031	4.0531

For this application, there is one observation as "zero". The EHN and EHSN do not have problem to accommodate this observation. However, the LSN (and LN) distribution cannot accommodate it. For this reason, we only consider as competitors the EHN and EHSN in this problem. The moments estimators for the EHSN model in this dataset are $\widehat{\alpha}_{MM} = 1.093$, $\widehat{\delta}_{MM} = 5.059$ and $\widehat{\lambda}_{MM} = 2.521$, which were used to initialize the maximization procedure to obtain the corresponding MLE's. In Table 3, EHN and EHSN models are compared. Akaike Information Criterion (AIC) [25] and Bayesian Information Criterion (BIC) [26] are used. Additionally, in Figure 5, the empirical cdf is plotted along with the cdf estimated for the EHSN model, whereas qqplots comparing both models are given in Figure 6. In Figure 7, the histogram and models fitted by maximum likelihood are given.

We highlight that following AIC and BIC criteria, the fitted EHSN model provides a better fit to this dataset. QQ-plots in Figure 6 and histogram with fitted densities in Figure 7 also support this statement.

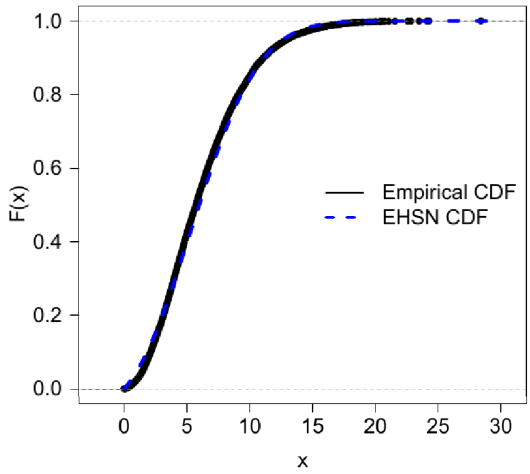


Figure 5. Empirical and fitted EHSN cdf for Speed of Wind in dataset 1.

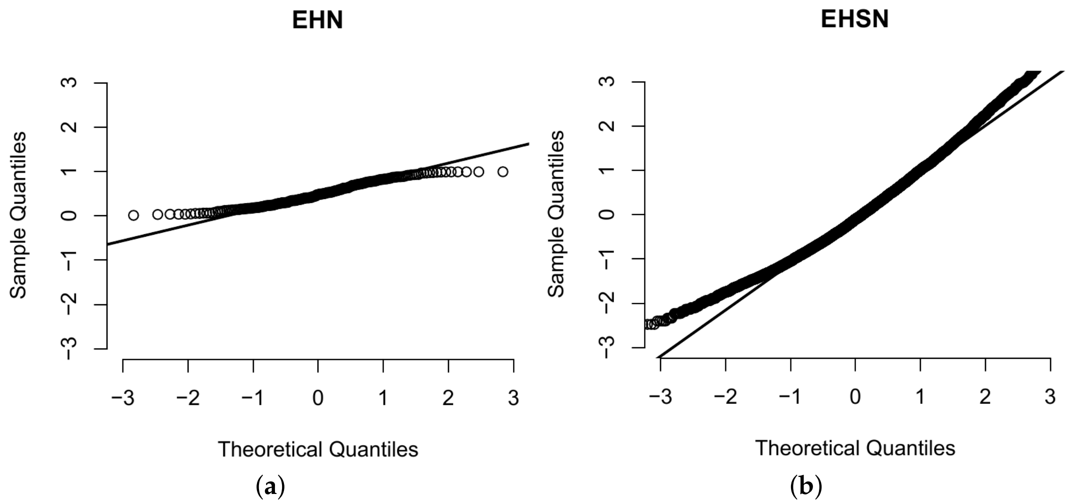


Figure 6. qqplots for Speed of Wind dataset: (**a**) EHN model, (**b**) EHSN model.

Table 3. Estimated parameters in the EHN and EHSN models.

Estimates of Parameters	EHN	EHSN
α	0.413 (0.029)	0.754 (0.065)
δ	4.674 (0.037)	4.851 (0.046)
λ		3.058 (0.293)
Log-likelihood	−17,404.03	−17,321.46
AIC	34,812.60	34,648.93
BIC	34,826.18	34,669.30

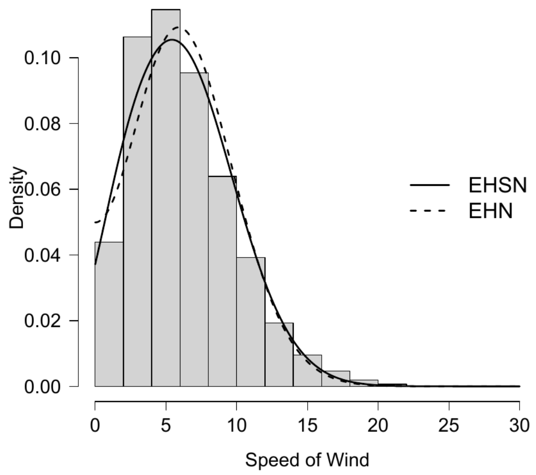


Figure 7. Fitted EHSN and EHN model by maximum likelihood for speed of wind dataset.

6.2. Application 2

This dataset corresponds to heights (100 × feet) of 219 volcanoes studied in [27]. In Table 4, the descriptive summaries are provided. These are: the sample mean ($\overline{x}$), the sample variance (s^2), the sample skewness coefficient ($\sqrt{b_1}$), and the sample kurtosis coefficient (b_2).

In this application, the EHSN model is compared to LSN and EHN. Moments estimates in the EHSN model are obtained by applying results in Section 4.1, these are $\widehat{\alpha}_{MM} = 1.579$, $\widehat{\delta}_{MM} = 59.668$, and $\widehat{\lambda}_{MM} = 2.788$. These estimates are used as initial values to obtain MLEs by using numerical methods. ML estimates for the LSN, EHN, and EHSN models, along with their standard errors are provided in Table 5. For purposes of comparison, log-likelihood, AIC and BIC are also included in this table. These summaries support the fact that the EHSN model provides a better fit to this dataset. As plots, the histogram, along with the estimated pdfs, are given in Figure 8. The QQ-plots comparing EHN and EHSN models are given in Figure 9. In Figure 10, the empirical cdf is plotted along with the cdf estimated for the EHSN model. All these plots support our conclusions.

Table 4. Descriptive summaries for volcano heights (in 100 × feet).

n	$\overline{x}$	s^2	$\sqrt{b_1}$	b_2
219	70.247	1850.548	0.8344	3.4439

Table 5. Estimates of parameters in EHN and EHSN models for volcanoes dataset.

Estimates of Parameters	LSN	EHN	EHSN
α	4.015 (1.051)	0.783 (0.286)	1.579 (0.923)
δ	0.770 (0.037)	56.514 (2.894)	59.668 (4.187)
λ	0.007 (1.709)		2.788 (1.616)
Log-likelihood	−1133.636	−1117.914	−1115.100
AIC	2273.272	2241.828	2236.200
BIC	2283.440	2251.996	2246.367

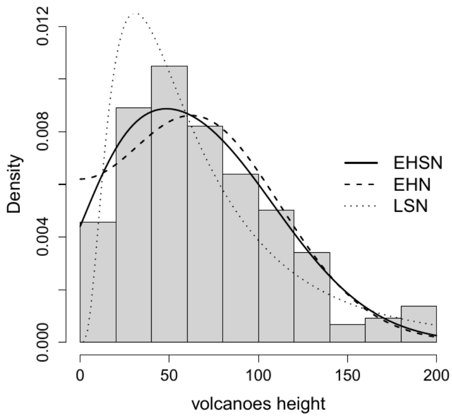


Figure 8. Histogram for volcano heights dataset and estimated models. EHSN (solid line), EHN (dashed line), LSN (dotted line).

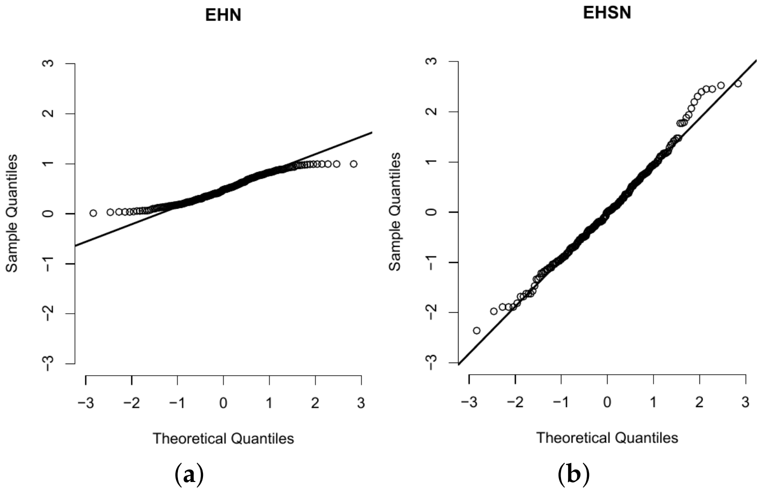


Figure 9. qqplots for volcano heights dataset: (**a**) EHN model, (**b**) EHSN model.

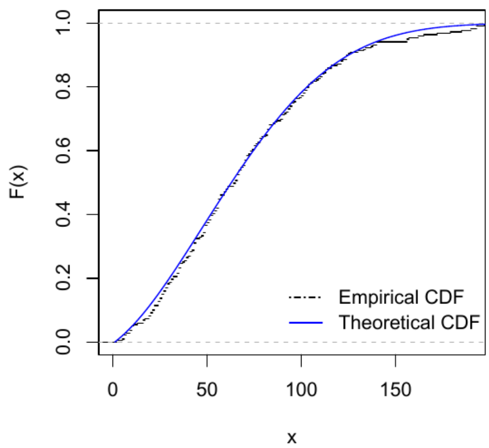


Figure 10. Empirical and fitted EHSN cdf for volcano heights in dataset 2.

7. Final Discussion and Conclusions

This study presents a new model with positive support based on the skew-normal distribution, which has been called the extended half skew-normal distribution. This distribution is useful as a more general model compared to the EHN model proposed by Elal-Olivero [11], pursuing to increase kurtosis and improve the modeling of positive datasets with high kurtosis. Relevant properties of the model are given. Closed expressions are given for the pdf, cdf, mgf, and moments. Additionally, a stochastic representation is proposed, which will be the basis to generate random values in this model. Estimation of parameters is carried out via maximum likelihood methods by using numerical techniques. The simulation study illustrates the good performance of estimators. Two applications with real datasets were also carried out, verifying that the new model performs better than the competing models.

One of the referees points out that the pdf's introduced throughout this paper, see, for instance, the general family defined in (4) or the pdf for the EHN introduced in (3), can be considered as weighted distributions, [28]. For the pdf introduced in (4), this would be to write

$$f_X(x) = \frac{w(x;\alpha)}{E_{g_\lambda I[x\geq 0]}[w(X;\alpha)]}\, g_\lambda(x)\; I[x \geq 0] \,,$$

with weight function $w(x;\alpha) = \alpha + x^2$. This idea may be studied in future works along with its implications.

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Data Availability Statement: Dataset studied in Section 6.1 can be found at http://lib.stat.cmu.edu/datasets/wind.data, (last accessed on 12 July 2022). The dataset in Section 6.2 has been taken from [27].

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Appendix A

In this Appendix details are given about the models under consideration in Section 3. These models are:

- Skew-Normal, $X \sim SN(\lambda)$, introduced in [1], whose pdf is

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x), \qquad x \in \mathbb{R}, \;\; \lambda \in \mathbb{R}. \tag{A1}$$

  The cdf is

$$F(x;\lambda) = \Phi(x) - 2\, O(x,\lambda), \qquad x \in \mathbb{R}, \;\; \lambda \in \mathbb{R}, \tag{A2}$$

  where $O(\cdot,\lambda)$ denotes the Owen function [17].

- Generalized Gamma, $GG(a,d,p)$, introduced in [16], whose pdf is

$$f(x;a,d,p) = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\, \frac{p}{a^d} x^{d-1} e^{-(x/a)^p}, \qquad x > 0, \;\; a,d,p > 0\,. \tag{A3}$$

  The cdf is

$$F(x;a,d,p) = \frac{\Gamma_z\left(\frac{d}{p}\right)}{\Gamma\left(\frac{d}{p}\right)} \tag{A4}$$

  where $z = (x/a)^p$ and $\Gamma_z\left(\frac{d}{p}\right) = \int_0^z v^{(d/p)-1} e^{-v} dv$, see [29].

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