Hasse–Schmidt derivations versus classical derivations

Luis Narváez Macarro

Dedicated to Lê Dũng Tráng

Abstract. In this paper we survey the notion and basic results on multivariate Hasse–Schmidt derivations over arbitrary commutative algebras and we associate to such an object a family of classical derivations. We study the behavior of these derivations under the action of substitution maps and we prove that, in characteristic 0, the original multivariate Hasse–Schmidt derivation can be recovered from the associated family of classical derivations. Our constructions generalize a previous one by M. Mirzavaziri in the case of a base field of characteristic 0.

Introduction

Let k be a commutative ring and A a commutative k-algebra. A Hasse–Schmidt derivation of A over k of length $m \geq 0$ (or $m = \infty$), is a sequence $D = (D_0, D_1, \ldots, D_m)$ (or $D = (D_0, D_1, \ldots))$ of k-linear endomorphisms of A such that D_0 is the identity map and

$$D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y), \quad \forall \alpha, \forall x, y \in A.$$

A such D can be seen as a power series $D = \sum_{\alpha=0}^m D_\alpha t^\alpha$ in the quotient ring $R[[t]]/\langle t^{m+1}\rangle$, with $R = \mathrm{End}_k(A)$ (the ring of endomorphisms of A as k-module). For $i \geq 1$, the ith component D_i turns out to be a k-linear differential operator of order $\leq i$ vanishing on 1, in particular D_1 is a k-derivation of A.

The notion of Hasse–Schmidt derivation was introduced in [**1**] in the case where k is a field of characteristic $p > 0$ and A a field of algebraic functions over k. This notion was used to understand, among others, Taylor expansions in this setting. But actually, Hasse–Schmidt derivations make sense in full generality.

If we are in characteristic 0 ($\mathbb{Q} \subset k$), then it is easy to produce examples of Hasse–Schmidt derivations: starting with a k-linear derivation $\delta : A \to A$ we

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consider its exponential:

$$e^{t\delta} = \sum_{\alpha=0}^{\infty} \frac{\delta^\alpha}{\alpha!} t^\alpha \in R[[t]], \quad R = \operatorname{End}_k(A).$$

It is clear that $e^{t\delta}$ is a Hasse–Schmidt derivation of A over k (of infinite length). This example also proves that, always under the characteristic 0 hypothesis, any k-linear derivation $\delta : A \to A$ appears as the 1-component of some Hasse–Schmidt derivation (of infinite length, and so, of any length $m \geq 1$) of A over k. This is what we call "to be ∞-integrable" (and so "m-integrable", for each $m \geq 1$) (see [**5**]). But if we are no more in characteristic 0, the situation becomes much more involved and integrable derivations deserve special attention (see [**8, 4, 13**] for several recent achievements in that direction).

As far as the author knows, two papers have been concerned with the description of Hasse–Schmidt derivations in terms of usual derivations, both in the case where k is a field of characteristic 0. In [**3**] it is proven that[1], if A is a (possibly non-commutative) k-algebra, then any Hasse–Schmidt derivation $D = (D_0 = \mathrm{Id}, D_1, D_2, \dots)$ of infinite length of A over k is determined by a unique sequence $\delta = (\delta_1, \delta_2, \dots)$ of classical derivations $\delta_i \in \operatorname{Der}_k(A)$. Namely, the expressions relating D and δ are:

$$\delta_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{\substack{n_1+\cdots+n_r=n\\ n_i>0}} D_{n_1} D_{n_2} \cdots D_{n_r}, \ D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{\substack{n_1+\cdots+n_r=n\\ n_i>0}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r},$$

or in other words:

$$\sum_{n=0}^{\infty} D_n t^n = \exp\left(\sum_{n=1}^{\infty} \delta_n t^n\right).$$

A similar result is proven in [**7**]: any Hasse–Schmidt derivation $D = (D_0 = \mathrm{Id}, D_1, D_2, \dots)$ of infinite length of A over k determines, and is determined by a sequence $\delta = (\delta_1, \delta_2, \dots)$ of classical derivations given by the following recursive formula:

$$(n+1) D_{n+1} = \sum_{r=0}^{n} \delta_{r+1} D_{n-r}, \quad n \geq 0.$$

An interesting reinterpretation of both results can be found in [**2**].

The goal of this paper is twofold: to give a survey of multivariate Hasse–Schmidt derivations over a general commutative base ring k and a general commutative k-algebra A, as defined in [**9**]; and to generalize the construction in [**7**] to this setting.

One of our motivations is to understand the relationship between HS-modules, as defined in [**10**], and classical integrable connections. The paper [**11**] is devoted to prove that both notions are equivalent in characteristic 0, and the proof strongly depends on the constructions and results of the present paper.

[1] This result has been "rediscovered" in [**12**] for A a commutative algebra over a field k of characteristic zero.

A (p,Δ)-variate Hasse–Schmidt derivation of A over k is a family $D=(D_\alpha)_{\alpha\in\Delta}$ of k-linear endomorphisms of A such that D_0 is the identity map and:

$$D_\alpha(xy)=\sum_{\beta+\gamma=\alpha} D_\beta(x)D_\gamma(y),\quad \forall\alpha\in\Delta,\forall x,y\in A,$$

where $\Delta\subset\mathbb{N}^p$ is a non-empty *co-ideal*, i.e. a subset of $\mathbb{N}^p$ such that everytime $\alpha\in\Delta$ and $\alpha'\leq\alpha$ (i.e. $\alpha-\alpha'\in\mathbb{N}^p$) we have $\alpha'\in\Delta$. A simple but important idea is to think on Hasse–Schmidt derivations as series $D=\sum_{\alpha\in\Delta}D_\alpha\mathbf{s}^\alpha$ in the quotient ring $R[[\mathbf{s}]]_\Delta$ of the power series ring $R[[\mathbf{s}]]=R[[s_1,\dots,s_p]]$, $R=\operatorname{End}_k(A)$, by the two-sided monomial ideal generated by all $\mathbf{s}^\alpha$ with $\alpha\in\mathbb{N}^p\setminus\Delta$.

The set $\operatorname{HS}^p_k(A;\Delta)$ of (p,Δ)-variate Hasse–Schmidt derivations is a subgroup of the group of units $(R[[\mathbf{s}]]_\Delta)^\times$, and it also carries the action of substitution maps: given a substitution map $\varphi: A[[s_1,\dots,s_p]]_\Delta\to A[[t_1,\dots,t_q]]_\nabla$ and a (p,Δ)-variate Hasse–Schmidt derivation $D=\sum_{\alpha\in\Delta}D_\alpha\mathbf{s}^\alpha$ we obtain a new (q,∇)-variate Hasse–Schmidt derivation given by:

$$\varphi\bullet D:=\sum_{\alpha\in\Delta}\varphi(\mathbf{s}^\alpha)D_\alpha.$$

This new structure is a key point in [**10**].

To generalize the construction in [**7**], we reinterpret the aforementioned recursive formula by means of the “logarithmic derivative type” maps:

$$\varepsilon^i: D\in\operatorname{HS}^p_k(A;\Delta)\longmapsto\varepsilon^i(D):=D^*\left(s_i\frac{\partial D}{\partial s_i}\right)\in R[[\mathbf{s}]]_\Delta,\quad i=1,\dots,p,$$

where D^* denotes the inverse of D. The starting point is to check that the coefficients of $\varepsilon(D)$ are always classical derivations, i.e. $\varepsilon(D)\in\operatorname{Der}_k(A)[[\mathbf{s}]]_\Delta$.

Let us comment on the content of the paper.

In section 1 we have gathered some notations and constructions on power series modules, powers series rings and substitution maps, most of them taken from [**9**], and we study the maps ε^i, and their conjugate $\overline{\varepsilon}^i$.

In section 2 we recall the notion and the basic properties of multivariate Hasse–Schmidt derivations and of the action of substitution maps on these objects.

Section 3 contains the main original results of this paper. First, we see how the ε^i or $\overline{\varepsilon}^i$ maps of section 1 allow us to associate to any multivariate Hasse–Schmidt derivation a power series whose coefficients are classical derivations, as explained before. When $\mathbb{Q}\subset k$ we obtain a characterization of multivariate Hasse–Schmidt derivations in terms of the ε^i (or $\overline{\varepsilon}^i$) maps, and we prove that any multivariate Hasse–Schmidt derivation can be constructed from a power series of classical derivations. To finish, we study the behavior of the ε^i maps of a multivariate Hasse–Schmidt derivation under the action of substitution maps.

1. Notations and preliminaries

1.1. Notations. Throughout the paper we will use the following notations:

-) k is a commutative ring and A a commutative k-algebra.

-) $\mathbf{s}=\{s_1,\dots,s_p\}$, $\mathbf{t}=\{t_1,\dots,t_q\}$, … are sets of variables.

-) $\mathscr{U}^p(R;\Delta)$: see Notation 1.2.4.

-) $\mathbf{C}_e(\varphi,\alpha)$: see (1.11).
-) $\varphi\bullet r, r\bullet\varphi$: see 1.3.5.
-) φ^D: see 2.2.3.

1.2. Some constructions on power series rings and modules. Throughout this section, k will be a commutative ring, A a commutative k-algebra and R a ring, not-necessarily commutative.

Let $p\geq 0$ be an integer and let us call $\mathbf{s}=\{s_1,\dots,s_p\}$ a set of p variables. The support of each $\alpha\in\mathbb{N}^p$ is defined as $\operatorname{supp}\alpha := \{i \mid \alpha_i\neq 0\}$. The monoid $\mathbb{N}^p$ is endowed with a natural partial ordering. Namely, for $\alpha,\beta\in\mathbb{N}^p$, we define:

$$\alpha\leq\beta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists\gamma\in\mathbb{N}^p \text{ such that } \beta=\alpha+\gamma \quad \Longleftrightarrow \quad \alpha_i\leq\beta_i \quad \forall i=1\dots,p.$$

We denote $|\alpha| := \alpha_1+\cdots+\alpha_p$.

Let $p\geq 1$ be an integer and $\mathbf{s}=\{s_1,\dots,s_p\}$ a set of variables. If M is an abelian group and $M[[\mathbf{s}]]$ is the abelian group of power series with coefficients in M, the *support* of a series $m=\sum_\alpha m_\alpha\mathbf{s}^\alpha\in M[[\mathbf{s}]]$ is $\operatorname{supp}(m) := \{\alpha\in\mathbb{N}^p \mid m_\alpha\neq 0\}\subset\mathbb{N}^p$. We have $m=0\Leftrightarrow\operatorname{supp}(m)=\emptyset$.

The abelian group $M[[\mathbf{s}]]$ is clearly a $\mathbb{Z}[[\mathbf{s}]]$-module, which will be always endowed with the $\langle\mathbf{s}\rangle$-adic topology.

Definition 1.2.1. We say that a subset $\Delta\subset\mathbb{N}^p$ is an *ideal* (resp. a *co-ideal*) of $\mathbb{N}^p$ if everytime $\alpha\in\Delta$ and $\alpha\leq\alpha'$ (resp. $\alpha'\leq\alpha$), then $\alpha'\in\Delta$.

It is clear that $\Delta\subset\mathbb{N}^p$ is an ideal if and only if its complement Δ^c is a co-ideal, and that the union and the intersection of any family of ideals (resp. of co-ideals) of $\mathbb{N}^p$ is again an ideal (resp. a co-ideal) of $\mathbb{N}^p$. Examples of ideals (resp. of co-ideals) of $\mathbb{N}^p$ are the $\beta+\mathbb{N}^p$ (resp. the $\{\alpha\in\mathbb{N}^p \mid \alpha\leq\beta\}$) with $\beta\in\mathbb{N}^p$. The $\{\alpha\in\mathbb{N}^p \mid |\alpha|\leq m\}$ with $m\geq 0$ are also co-ideals. Notice that a co-ideal $\Delta\subset\mathbb{N}^p$ is non-empty if and only if $\{0\}\subset\Delta$.

1.2.2 Let M be an abelian group. For each co-ideal $\Delta\subset\mathbb{N}^p$, we denote by Δ_M the closed sub-$\mathbb{Z}[[\mathbf{s}]]$-bimodule of $M[[\mathbf{s}]]$ whose elements are the formal power series $\sum_{\alpha\in\mathbb{N}^p} m_\alpha\mathbf{s}^\alpha$ such that $m_\alpha=0$ whenever $\alpha\in\Delta$, and $M[[\mathbf{s}]]_\Delta := M[[\mathbf{s}]]/\Delta_M$. The elements in $M[[\mathbf{s}]]_\Delta$ are power series of the form $\sum_{\alpha\in\Delta} m_\alpha\mathbf{s}^\alpha$, $m_\alpha\in M$. If $f:M\to M'$ is a homomorphism of abelian groups, we will denote by $\overline{f}: M[[\mathbf{s}]]_\Delta\to M'[[\mathbf{s}]]_\Delta$ the $\mathbb{Z}[[\mathbf{s}]]_\Delta$-linear map defined as $\overline{f}\left(\sum_{\alpha\in\Delta} m_\alpha\mathbf{s}^\alpha\right)=\sum_{\alpha\in\Delta} f(m_\alpha)\mathbf{s}^\alpha$.

If R is a ring, then Δ_R is a closed two-sided ideal of $R[[\mathbf{s}]]$ and so $R[[\mathbf{s}]]_\Delta$ is a topological ring, which we always consider endowed with the $\langle\mathbf{s}\rangle$-adic topology (= to the quotient topology). Similarly, if M is an $(A;A)$-bimodule (central over k), then $M[[\mathbf{s}]]_\Delta$ is an $(A[[\mathbf{s}]]_\Delta;A[[\mathbf{s}]]_\Delta)$-bimodule (central over $k[[\mathbf{s}]]_\Delta$).

For $\Delta'\subset\Delta$ non-empty co-ideals of $\mathbb{N}^p$, we have natural $\mathbb{Z}[[\mathbf{s}]]$-linear projections $\tau_{\Delta\Delta'}: M[[\mathbf{s}]]_\Delta\longrightarrow M[[\mathbf{s}]]_{\Delta'}$,that we call *truncations*:

$$\tau_{\Delta\Delta'}: \sum_{\alpha\in\Delta} m_\alpha\mathbf{s}^\alpha\in M[[\mathbf{s}]]_{\Delta'}\longmapsto\sum_{\alpha\in\Delta'} m_\alpha\mathbf{s}^\alpha\in M[[\mathbf{s}]]_\Delta.$$

If M is a ring (resp. an $(A;A)$-bimodule), then the truncations $\tau_{\Delta\Delta'}$ are ring homomorphisms (resp. $(A[[\mathbf{s}]]_\Delta;A[[\mathbf{s}]]_\Delta)$-linear maps). For $\Delta'=\{0\}$ we have

$M[[\mathbf{s}]]_{\Delta'} = M$ and the kernel of $\tau_{\Delta\{0\}}$ will be denoted by $M[[\mathbf{s}]]_{\Delta,+}$. We have a bicontinuous isomorphism:

$$M[[\mathbf{s}]]_{\Delta} = \varprojlim M[[\mathbf{s}]]_{\Delta'}$$

where Δ' runs over all finite co-ideals contained in Δ.

Definition 1.2.3. A k-algebra over A is a (not-necessarily commutative) k-algebra R endowed with a map of k-algebras $\iota : A \to R$. A map between two k-algebras $\iota : A \to R$ and $\iota' : A \to R'$ over A is a map $g : R \to R'$ of k-algebras such that $\iota' = g \circ \iota$.

It is clear that if R is a k-algebra over A, then $R[[\mathbf{s}]]_{\Delta}$ is a $k[[\mathbf{s}]]_{\Delta}$-algebra over $A[[\mathbf{s}]]_{\Delta}$.

Notation 1.2.4. Let R be a ring, $p \geq 1$ and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal. We denote by $\mathscr{U}^p(R;\Delta)$ the multiplicative sub-group of the units of $R[[\mathbf{s}]]_{\Delta}$ whose 0-degree coefficient is 1. The multiplicative inverse of a unit $r \in R[[\mathbf{s}]]_{\Delta}$ will be denoted by r^*. Clearly, $\mathscr{U}^p(R;\Delta)^{\mathrm{opp}} = \mathscr{U}^p(R^{\mathrm{opp}};\Delta)$. For $\Delta \subset \Delta'$ co-ideals we have $\tau_{\Delta'\Delta}(\mathscr{U}^p(R;\Delta')) \subset \mathscr{U}^p(R;\Delta)$ and the truncation map $\tau_{\Delta'\Delta} : \mathscr{U}^p(R;\Delta') \to \mathscr{U}^p(R;\Delta)$ is a group homomorphisms. Clearly, we have:

$$\mathscr{U}^p(R;\Delta) = \varprojlim_{\substack{\Delta' \subset \Delta \\ \sharp\Delta' < \infty}} \mathscr{U}^p(R;\Delta'). \tag{1.1}$$

If $p = 1$ and $\Delta = \{i \in \mathbb{N} \mid i \leq m\}$ we will simply denote $\mathscr{U}(R;m) := \mathscr{U}^1(R;\Delta)$.

For any ring homomorphism $f : R \to R'$, the induced ring homomorphism $\overline{f} : R[[\mathbf{s}]]_{\Delta} \to R'[[\mathbf{s}]]_{\Delta}$ sends $\mathscr{U}^p(R;\Delta)$ into $\mathscr{U}^p(R';\Delta)$ and so it induces natural group homomorphisms $\mathscr{U}^p(R;\Delta) \to \mathscr{U}^p(R';\Delta)$.

We recall the following easy result (cf. Lemma 2 in [**9**]).

Lemma 1.2.5. *Let R be a ring and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal. The units in $R[[\mathbf{s}]]_{\Delta}$ are those power series $r = \sum r_\alpha \mathbf{s}^\alpha$ such that r_0 is a unit in R. Moreover, in the special case where $r_0 = 1$, the inverse $r^* = \sum r^*_\alpha \mathbf{s}^\alpha$ of r is given by $r^*_0 = 1$ and*

$$r^*_\alpha = \sum_{d=1}^{|\alpha|} (-1)^d \sum_{\alpha^\bullet \in \mathscr{P}(\alpha,d)} r_{\alpha^1} \cdots r_{\alpha^d} \quad \textit{for} \ \ \alpha \neq 0,$$

where $\mathscr{P}(\alpha,d)$ is the set of d-uples $\alpha^\bullet = (\alpha^1,\ldots,\alpha^d)$ with $\alpha^i \in \mathbb{N}^{(\mathbf{s})}$, $\alpha^i \neq 0$, and $\alpha^1 + \cdots + \alpha^d = \alpha$.

1.2.6 Let E, F be A-modules. For each $r = \sum_\beta r_\beta \mathbf{s}^\beta \in \operatorname{Hom}_k(E,F)[[\mathbf{s}]]_{\Delta}$ we denote by $\widetilde{r} : E[[\mathbf{s}]]_{\Delta} \to F[[\mathbf{s}]]_{\Delta}$ the map defined by:

$$\widetilde{r}\left(\sum_{\alpha\in\Delta} e_\alpha \mathbf{s}^\alpha\right) := \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha} r_\beta(e_\gamma)\right)\mathbf{s}^\alpha,$$

which is obviously a $k[[\mathbf{s}]]_{\Delta}$-linear map. It is clear that the map:

$$r \in \operatorname{Hom}_k(E,F)[[\mathbf{s}]]_{\Delta} \longmapsto \widetilde{r} \in \operatorname{Hom}_{k[[\mathbf{s}]]_{\Delta}}(E[[\mathbf{s}]]_{\Delta}, F[[\mathbf{s}]]_{\Delta}) \tag{1.2}$$

is $(A[[\mathbf{s}]]_{\Delta}; A[[\mathbf{s}]]_{\Delta})$-linear.

If $f : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ is a $k[[\mathbf{s}]]_\Delta$-linear map, let us denote by $f_\alpha : E \to F$, $\alpha \in \Delta$, the k-linear maps defined by:

$$f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E.$$

If $g : E \to F[[\mathbf{s}]]_\Delta$ is a k-linear map, we denote by $g^e : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ the unique $k[[\mathbf{s}]]_\Delta$-linear map extending g to $E[[\mathbf{s}]]_\Delta = k[[\mathbf{s}]]_\Delta \widehat{\otimes}_k E$. It is given by:

$$g^e\left(\sum_\alpha e_\alpha \mathbf{s}^\alpha\right) := \sum_\alpha g(e_\alpha)\mathbf{s}^\alpha. \tag{1.3}$$

We have a $k[[\mathbf{s}]]_\Delta$-bilinear and $A[[\mathbf{s}]]_\Delta$-balanced map:

$$\langle -,-\rangle : (r,e) \in \operatorname{Hom}_k(E,F)[[\mathbf{s}]]_\Delta \times E[[\mathbf{s}]]_\Delta \longmapsto \langle r,e\rangle := \widetilde{r}(e) \in F[[\mathbf{s}]]_\Delta.$$

The following assertions are clear (see [**9**, Lemma 3]):

1) The map (1.2) is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodules. When $E = F$ it is an isomorphism of $k[[\mathbf{s}]]_\Delta$-algebras over $A[[\mathbf{s}]]_\Delta$.
2) The restriction map:
$$f \in \operatorname{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \mapsto f|_E \in \operatorname{Hom}_k(E, F[[\mathbf{s}]]_\Delta),$$
is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$-bimodules.

Let us call $R = \operatorname{End}_k(E)$. As a consequence of the above properties, the composition of the maps:

$$R[[\mathbf{s}]]_\Delta \xrightarrow{r\mapsto\widetilde{r}} \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \xrightarrow{f\mapsto f|_E} \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta) \tag{1.4}$$

is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$-bimodules, and so $\operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ inherits a natural structure of $k[[\mathbf{s}]]_\Delta$-algebra over $A[[\mathbf{s}]]_\Delta$. Namely, if $g, h : E \to E[[\mathbf{s}]]_\Delta$ are k-linear maps with:

$$g(e) = \sum_{\alpha\in\Delta} g_\alpha(e)\mathbf{s}^\alpha,\ h(e) = \sum_{\alpha\in\Delta} h_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E, \quad g_\alpha, h_\alpha \in \operatorname{Hom}_k(E,E),$$

then the product $hg \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ is given by:

$$(hg)(e) = \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha}(h_\beta\circ g_\gamma)(e)\right)\mathbf{s}^\alpha. \tag{1.5}$$

Notation 1.2.7. We denote:

$$\operatorname{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta) := \{f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta) \mid f(e) \equiv e \bmod \langle \mathbf{s}\rangle E[[\mathbf{s}]]_\Delta\ \forall e \in E\},$$

$$\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) :=$$
$$\left\{f \in \operatorname{Aut}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \mid f(e) \equiv e_0 \bmod \langle \mathbf{s}\rangle E[[\mathbf{s}]]_\Delta\ \forall e \in E[[\mathbf{s}]]_\Delta\right\}.$$

Let us notice that a $f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$, given by $f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha$, belongs to $\operatorname{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta)$ if and only if $f_0 = \operatorname{Id}_E$.

The isomorphism in (1.4) gives rise to a group isomorphism:

$$r \in \mathscr{U}^p(\operatorname{End}_k(E);\Delta) \overset{\sim}{\longmapsto} \widetilde{r} \in \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \tag{1.6}$$

and to a bijection:

$$f \in \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \overset{\sim}{\longmapsto} f|_E \in \operatorname{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta). \tag{1.7}$$

So, $\operatorname{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta)$ is naturally a group with the product described in (1.5).

If R is a (not necessarily commutative) k-algebra and $\Delta \subset \mathbb{N}^p$ is a co-ideal, any continuous k-linear map $h : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ induces a natural continuous left and right R-linear map:

$$h_R := \mathrm{Id}_R \widehat{\otimes}_k h : R[[\mathbf{s}]]_\Delta = R\widehat{\otimes}_k k[[\mathbf{s}]]_\Delta \longrightarrow R[[\mathbf{s}]]_\Delta = R\widehat{\otimes}_k k[[\mathbf{s}]]_\Delta$$

given by:

$$h_R\left(\sum_\alpha r_\alpha \mathbf{s}^\alpha\right) = \sum_\alpha r_\alpha h(\mathbf{s}^\alpha).$$

If $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ is k-derivation, it is continuous and $\mathfrak{d}_R : R[[\mathbf{s}]]_\Delta \to R[[\mathbf{s}]]_\Delta$ is a $(R;R)$-linear derivation, i.e. $\mathfrak{d}_R(sr) = s\mathfrak{d}_R(r)$, $\mathfrak{d}_R(rs) = \mathfrak{d}_R(r)s$, $\mathfrak{d}_R(rr') = \mathfrak{d}_R(r)r' + r\mathfrak{d}_R(r')$ for all $s \in R$ and for all $r, r' \in R[[\mathbf{s}]]_\Delta$.

The set of all $(R;R)$-linear derivations of $R[[\mathbf{s}]]_\Delta$ is a $k[[\mathbf{s}]]_\Delta$-Lie algebra and will be denoted by $\mathrm{Der}_R(R[[\mathbf{s}]]_\Delta)$. Moreover, the map:

$$\mathfrak{d} \in \mathrm{Der}_k(k[[\mathbf{s}]]_\Delta) \longmapsto \mathfrak{d}_R \in \mathrm{Der}_R(R[[\mathbf{s}]]_\Delta)$$

is clearly a map of $k[[\mathbf{s}]]_\Delta$-Lie algebras.

The following definition provides a particular family of k-derivations.

Definition 1.2.8. For each $i = 1, \ldots, p$, the *ith partial Euler k-derivation* is $\chi^i = s_i \frac{\partial}{\partial s_i} : k[[\mathbf{s}]] \to k[[\mathbf{s}]]$. It induces a k-derivation on each $k[[\mathbf{s}]]_\Delta$, which will be also denoted by χ^i.

The *Euler k-derivation* $\boldsymbol{\chi} : k[[\mathbf{s}]] \to k[[\mathbf{s}]]$ is defined as:

$$\boldsymbol{\chi} = \sum_{i=1}^p \chi^i, \quad \boldsymbol{\chi}\left(\sum_\alpha c_\alpha \mathbf{s}^\alpha\right) = \sum_\alpha |\alpha| c_\alpha \mathbf{s}^\alpha.$$

It induces a k-derivation on each $k[[\mathbf{s}]]_\Delta$, which will be also denoted by χ.

The proof of the following lemma is easy and it is left to the reader.

Lemma 1.2.9. *Let E be an A-module and $r = \sum_\beta r_\beta \mathbf{s}^\beta \in \mathrm{Hom}_k(A,E)[[\mathbf{s}]]_\Delta$ a formal power series with coefficients in $\mathrm{Hom}_k(A,E)$. The following properties are equivalent:*

1. $r \in \mathrm{Der}_k(A,E)[[\mathbf{s}]]_\Delta$.
2. *For any $a \in A[[\mathbf{s}]]_\Delta$ we have $[r,a] = \widetilde{r}(a)$.*
3. $\widetilde{r} \in \mathrm{Der}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta, E[[\mathbf{s}]]_\Delta)$.
4. $\widetilde{r}|_A \in \mathrm{Der}_k(A, E[[\mathbf{s}]]_\Delta)$.

In particular, for each $r \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$, we have that $\widetilde{r} \in \mathrm{Der}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta)$ (see 1.2.6) and that the $A[[\mathbf{s}]]_\Delta$-linear map

$$r \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta \longmapsto \widetilde{r} \in \mathrm{Der}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta) \tag{1.8}$$

is an isomorphism of $A[[\mathbf{s}]]_\Delta$-modules. Moreover, $\mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$ is a Lie algebra over $k[[\mathbf{s}]]_\Delta$, where the Lie bracket of $\delta = \sum_\alpha \delta_\alpha \mathbf{s}^\alpha, \varepsilon = \sum_\alpha \varepsilon_\alpha \mathbf{s}^\alpha \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$ is given by:

$$[\delta, \varepsilon] = \delta\varepsilon - \varepsilon\delta = \sum_\alpha \left(\sum_{\beta+\gamma=\alpha} [\delta_\beta, \varepsilon_\gamma]\right) \mathbf{s}^\alpha,$$

and the map (1.8) is also an isomorphism of $k[[\mathbf{s}]]_\Delta$-Lie algebras.

Lemma 1.2.10. *Let $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ be a k-derivation and $R = \operatorname{End}_k(A)$. Then, for each $r \in R[[\mathbf{s}]]_\Delta$ we have $\widetilde{\mathfrak{d}_R(r)} = [\mathfrak{d}_A, \widetilde{r}]$.*

Proof. We have to prove that $\mathfrak{d}_A(\langle r, a\rangle) = \langle \mathfrak{d}_R(r), a\rangle + \langle r, \mathfrak{d}_A(a)\rangle$ for all $a \in A[[\mathbf{s}]]_\Delta$. By continuity, it is enough to prove the identity for $r = r_\alpha \mathbf{s}^\alpha$, $a = a_\beta \mathbf{s}^\beta$ with $\alpha, \beta \in \Delta$, $r_\alpha \in R$, $a_\beta \in A$:

$$\mathfrak{d}_A(\langle r, a\rangle) = \mathfrak{d}_A(\widetilde{r}(a)) = \mathfrak{d}_A(r_\alpha(a_\beta)\mathbf{s}^\alpha \mathbf{s}^\beta) = r_\alpha(a_\beta)\mathfrak{d}(\mathbf{s}^\alpha)\mathbf{s}^\beta + r_\alpha(a_\beta)\mathbf{s}^\alpha \mathfrak{d}(\mathbf{s}^\beta) =$$
$$\widetilde{\mathfrak{d}_R(r)}(a) + \widetilde{r}(\mathfrak{d}_A(a)) = \langle \mathfrak{d}_R(r), a\rangle + \langle r, \mathfrak{d}_A(a)\rangle.$$

$\square$

Definition 1.2.11. For any k-derivation $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ and any $r \in \mathscr{U}^p(R;\Delta)$ we define:

$$\varepsilon^{\mathfrak{d}}(r) := r^* \mathfrak{d}_R(r), \quad \overline{\varepsilon}^{\mathfrak{d}}(r) := \mathfrak{d}_R(r) r^*,$$

and we will write:

$$\varepsilon^{\mathfrak{d}}(r) = \sum_\alpha \varepsilon^{\mathfrak{d}}_\alpha(r)\mathbf{s}^\alpha, \quad \overline{\varepsilon}^{\mathfrak{d}}(r) = \sum_\alpha \overline{\varepsilon}^{\mathfrak{d}}_\alpha(r)\mathbf{s}^\alpha.$$

We will simply denote:

-) $\varepsilon^i(r) := \varepsilon^{\mathfrak{d}}(r)$, $\overline{\varepsilon}^i(r) := \overline{\varepsilon}^{\mathfrak{d}}(r)$ if $\mathfrak{d} = \chi^i$ (the ith partial Euler derivation), $i = 1, \dots, p$.
-) $\varepsilon(r) := \varepsilon^{\mathfrak{d}}(r)$, $\overline{\varepsilon}(r) := \overline{\varepsilon}^{\mathfrak{d}}(r)$ if $\mathfrak{d} = \boldsymbol{\chi}$ is the Euler derivation.

Observe that $\overline{\varepsilon}^{\mathfrak{d}}(r) = r\,\varepsilon^{\mathfrak{d}}(r)\,r^*$ and, for any co-ideal $\Delta' \subset \Delta$, we have $\tau_{\Delta\Delta'}(\varepsilon^{\mathfrak{d}}(r)) = \varepsilon^{\mathfrak{d}}(\tau_{\Delta\Delta'}(r))$, $\tau_{\Delta\Delta'}(\overline{\varepsilon}^{\mathfrak{d}}(r)) = \overline{\varepsilon}^{\mathfrak{d}}(\tau_{\Delta\Delta'}(r))$. Moreover, if E is an A-module and $R = \operatorname{End}_k(A)$, then

$$\widetilde{\varepsilon^{\mathfrak{d}}(r)} = \widetilde{r}^{-1}[\mathfrak{d}_A, \widetilde{r}] = \widetilde{r}^{-1}\mathfrak{d}_A\widetilde{r} - \mathfrak{d}_A, \quad \widetilde{\overline{\varepsilon}^{\mathfrak{d}}(r)} = [\mathfrak{d}_A, \widetilde{r}]\widetilde{r}^{-1} = \mathfrak{d}_A - \widetilde{r}\mathfrak{d}_A\widetilde{r}^{-1}.$$

The proof of the following lemma is straightforward.

Lemma 1.2.12. *For each $r \in \mathscr{U}^p_k(R;\Delta)$, the maps:*

$$\mathfrak{d} \in \operatorname{Der}_k(k[[\mathbf{s}]]_\Delta) \longmapsto \varepsilon^{\mathfrak{d}}(r) \in R[[\mathbf{s}]]_\Delta, \quad \mathfrak{d} \in \operatorname{Der}_k(k[[\mathbf{s}]]_\Delta) \longmapsto \overline{\varepsilon}^{\mathfrak{d}}(r) \in R[[\mathbf{s}]]_\Delta$$

are $k[[\mathbf{s}]]_\Delta$-linear.

In particular:

$$\varepsilon(r) = \sum_{i=1}^p \varepsilon^i(r), \quad \overline{\varepsilon}(r) = \sum_{i=1}^p \overline{\varepsilon}^i(r).$$

Lemma 1.2.13. *Let $\mathfrak{d}, \mathfrak{d}' : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ be k-derivations and $r, r' \in \mathscr{U}^p_k(R;\Delta)$. Then, the following identities hold:*

(i) $\overline{\varepsilon}^{\mathfrak{d}}(1) = \varepsilon^{\mathfrak{d}}(1) = 0$, $\varepsilon^{\mathfrak{d}}(r'\,r) = \varepsilon^{\mathfrak{d}}(r) + r^*\,\varepsilon^{\mathfrak{d}}(r')\,r$, $\overline{\varepsilon}^{\mathfrak{d}}(r\,r') = \overline{\varepsilon}^{\mathfrak{d}}(r) + r\,\overline{\varepsilon}^{\mathfrak{d}}(r')\,r^*$.
(ii) $\varepsilon^{\mathfrak{d}}(r^*) = -r\,\varepsilon^{\mathfrak{d}}(r)\,r^* = -\overline{\varepsilon}^{\mathfrak{d}}(r)$.
(iii) $\varepsilon^{[\mathfrak{d},\mathfrak{d}']}(r) = \left[\varepsilon^{\mathfrak{d}}(r), \varepsilon^{\mathfrak{d}'}(r)\right] + \mathfrak{d}_R\left(\varepsilon^{\mathfrak{d}'}(r)\right) - \mathfrak{d}'_R\left(\varepsilon^{\mathfrak{d}}(r)\right)$.

Proof. The proof of (i) is straightforward. For (ii) and (iii) one uses that $\mathfrak{d}_R(r^*) = -r^*\,\mathfrak{d}_R(r)\,r^*$. $\square$

1.2.14 For each $r \in \mathscr{U}^p(R;\Delta)$ and each $i = 1,\dots,p$ we have:

$$\varepsilon^i(r) = r^* \chi^i_R(r) = \left(\sum_\alpha r^*_\alpha \mathbf{s}^\alpha\right)\left(\sum_\alpha \alpha_i r_\alpha \mathbf{s}^\alpha\right) = \sum_\alpha \left(\sum_{\beta+\gamma=\alpha} \gamma_i r^*_\beta\, r_\gamma\right)\mathbf{s}^\alpha,$$

$$\varepsilon(r) = r^* \boldsymbol{\chi}_R(r) = \left(\sum_\alpha r^*_\alpha \mathbf{s}^\alpha\right)\left(\sum_\alpha |\alpha| r_\alpha \mathbf{s}^\alpha\right) = \sum_\alpha \left(\sum_{\beta+\gamma=\alpha} |\gamma| r^*_\beta\, r_\gamma\right)\mathbf{s}^\alpha,$$

and so, by using Lemma 1.2.5, we obtain:

$$\varepsilon^i(r) = \sum_{\substack{\alpha\in\Delta\\ \alpha_i>0}} \left(\sum_{d=1}^{|\alpha|} (-1)^{d-1}\left(\sum_{\alpha^\bullet\in\mathscr{P}(\alpha,d)} \alpha^d_i r_{\alpha^1}\cdots r_{\alpha^d}\right)\right)\mathbf{s}^\alpha,$$

$$\varepsilon(r) = \sum_{\substack{\alpha\in\Delta\\ |\alpha|>0}} \left(\sum_{d=1}^{|\alpha|} (-1)^{d-1}\left(\sum_{\alpha^\bullet\in\mathscr{P}(\alpha,d)} |\alpha^d| r_{\alpha^1}\cdots r_{\alpha^d}\right)\right)\mathbf{s}^\alpha.$$

In a similar way we obtain:

$$\overline{\varepsilon}^i(r) = \sum_{\substack{\alpha\in\Delta\\ \alpha_i>0}} \left(\sum_{d=1}^{|\alpha|} (-1)^{d-1}\left(\sum_{\alpha^\bullet\in\mathscr{P}(\alpha,d)} \alpha^1_i r_{\alpha^1}\cdots r_{\alpha^d}\right)\right)\mathbf{s}^\alpha,$$

$$\overline{\varepsilon}(r) = \sum_{\substack{\alpha\in\Delta\\ |\alpha|>0}} \left(\sum_{d=1}^{|\alpha|} (-1)^{d-1}\left(\sum_{\alpha^\bullet\in\mathscr{P}(\alpha,d)} |\alpha^1| r_{\alpha^1}\cdots r_{\alpha^d}\right)\right)\mathbf{s}^\alpha.$$

In particular, we have $\varepsilon^i_\alpha(r) = \overline{\varepsilon}^i_\alpha(r) = 0$ whenever $\alpha_i = 0$, i.e. whenever $i \notin \operatorname{supp}\alpha$, and $\varepsilon_0(r) = \overline{\varepsilon}_0(r) = 0$:

$$\varepsilon^i(r) = \sum_{i\in\operatorname{supp}\alpha} \varepsilon^i_\alpha(r)\mathbf{s}^\alpha,\quad \overline{\varepsilon}^i(r) = \sum_{i\in\operatorname{supp}\alpha} \overline{\varepsilon}^i_\alpha(r)\mathbf{s}^\alpha,$$

$$\varepsilon(r) = \sum_{|\alpha|>0} \varepsilon_\alpha(r)\mathbf{s}^\alpha,\quad \overline{\varepsilon}(r) = \sum_{|\alpha|>0} \overline{\varepsilon}_\alpha(r)\mathbf{s}^\alpha,$$

and $\varepsilon^i(r), \overline{\varepsilon}^i(r), \varepsilon(r), \overline{\varepsilon}(r) \in R[[\mathbf{s}]]_{\Delta,+}$ (see (1.2.2)). The following recursive identities hold:

$$\alpha_i r_\alpha = \sum_{\substack{\beta+\gamma=\alpha\\ \gamma_i>0}} r_\beta\, \varepsilon^i_\gamma(r) = \sum_{\substack{\beta+\gamma=\alpha\\ \gamma_i>0}} \overline{\varepsilon}^i_\gamma(r)\, r_\beta,$$

$$\varepsilon^i_\alpha(r) = \alpha_i r_\alpha - \sum_{\substack{\beta+\gamma=\alpha\\ |\beta|,\gamma_i>0}} r_\beta\, \varepsilon^t_\gamma(r),\quad \overline{\varepsilon}^t_\alpha(r) = \alpha_t r_\alpha - \sum_{\substack{\beta+\gamma=\alpha\\ |\beta|,\gamma_i>0}} \overline{\varepsilon}^i_\gamma(r)\, r_\beta,$$

for all $\alpha \in \Delta$ with $\alpha_i > 0$, and:

$$|\alpha| r_\alpha = \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|>0}} r_\beta\, \varepsilon_\gamma(r) = \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|>0}} \overline{\varepsilon}_\gamma(r)\, r_\beta, \tag{1.9}$$

$$\varepsilon_\alpha(r) = |\alpha| r_\alpha - \sum_{\substack{\beta+\gamma=\alpha\\ |\beta|,|\gamma|>0}} r_\beta\, \varepsilon_\gamma(r),\quad \overline{\varepsilon}_\alpha(r) = |\alpha| r_\alpha - \sum_{\substack{\beta+\gamma=\alpha\\ |\beta|,|\gamma|>0}} \overline{\varepsilon}_\gamma(r)\, r_\beta,$$

for all $\alpha \in \Delta$.

Remark 1.2.15. After (1.9), our definition of $\overline{\varepsilon}$ generalizes the construction in [**7**].

LEMMA 1.2.16. *For any $r \in \mathscr{U}^p(R;\Delta)$ and any $i,j = 1,\dots,p$ the following identity holds:*

$$\chi_R^j\left(\varepsilon^i(r)\right) - \chi_R^i\left(\varepsilon^j(r)\right) = [\varepsilon^i(r), \varepsilon^j(r)].$$

PROOF. Since $[\chi^i,\chi^j]=0$, it is a consequence of Lemma 1.2.13, (iii). $\square$

NOTATION 1.2.17. Under the above conditions, we will denote by $\Lambda^p(R;\Delta)$ the subset of $(R[[\mathbf{s}]]_{\Delta,+})^p$ whose elements are the families $\{\delta^i\}_{1\leq i\leq p}$ satisfying the following properties:

(a) If $\delta^i = \sum_{|\alpha|>0} \delta^i_\alpha \mathbf{s}^\alpha$, we have $\delta^i_\alpha = 0$ whenever $\alpha_i = 0$.

(b) For all $i,j=1,\dots,p$ we have $\chi_R^j\left(\delta^i\right) - \chi_R^i\left(\delta^j\right) = [\delta^i,\delta^j]$.

Let us notice that property (b) may be explicitly written as:

$$\alpha_j \delta^i_\alpha - \alpha_i \delta^j_\alpha = \sum_{\substack{\beta+\gamma=\alpha\\ \beta_i,\gamma_j>0}} [\delta^i_\beta, \delta^j_\gamma] \tag{1.10}$$

for all $i,j=1,\dots,p$ and for all $\alpha\in\Delta$ with $\alpha_i,\alpha_u>0$. Let us also consider the map:

$$\mathbf{\Sigma} : \{\delta^i\} \in \Lambda^p(R;\Delta) \longmapsto \sum_{i=1}^p \delta^i \in R[[\mathbf{s}]]_{\Delta,+}.$$

After Lemma 1.2.16, we can consider the map:

$$\boldsymbol{\varepsilon} : D \in \mathscr{U}^p(R;\Delta) \longmapsto \{\varepsilon^i(r)\}_{1\leq i\leq p} \in \Lambda^p(R;\Delta)$$

and we obviously have $\varepsilon = \mathbf{\Sigma}\circ\boldsymbol{\varepsilon}$.

PROPOSITION 1.2.18. *Assume that $\mathbb{Q}\subset k$. Then, the three maps in the following commutative diagram:*

$$\begin{tikzcd} \mathscr{U}^p(R;\Delta) \arrow[r, "\boldsymbol{\varepsilon}"] \arrow[dr, "\varepsilon"'] & \Lambda^p(R;\Delta) \arrow[d, "\mathbf{\Sigma}"] \\ & R[[\mathbf{s}]]_{\Delta,+} \end{tikzcd}$$

are bijective.

PROOF. The injectivity of ε is a straightforward consequence of (1.9). Let us prove the surjectivity of ε. Let $\overline{r} = \sum_\alpha \overline{r}_\alpha \mathbf{s}^\alpha$ be any element in $R[[\mathbf{s}]]_{\Delta,+}$. Since $\mathbb{Q}\subset k$, the differential equation

$$\boldsymbol{\chi}(Y) = Y\overline{r}, \quad Y\in R[[\mathbf{s}]]_\Delta,$$

has a unique solution $r\in R[[\mathbf{s}]]_\Delta$ with initial condition $r_0=1$, i.e. $r\in\mathscr{U}^p(R;\Delta)$. It is given recursively by:

$$|\alpha| r_\alpha = \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|>0}} r_\beta\,\overline{r}_\gamma, \quad \alpha\in\Delta, |\alpha|>0,$$

and so $\varepsilon(r)=\overline{r}$. To finish, the only missing point is the injectivity of $\mathbf{\Sigma}$. Let $\{\delta^i\},\{\eta^i\}\in\Lambda^p(R;\Delta)$ be with $\sum_i\delta^i = \sum_i \eta^i$. It is clear that $\delta^i_\alpha = \eta^i_\alpha$ whenever $|\alpha|=1$. Assume that $\delta^i_\beta = \eta^i_\beta$ for all $i=1,\dots,p$ whenever $|\beta|<m$ and consider $\alpha\in\Delta$ with $|\alpha|=m$. By using (1.10) and the induction hypothesis we obtain:

$$\alpha_j\delta^i_\alpha - \alpha_i\delta^j_\alpha = \sum_{\substack{\beta+\gamma=\alpha\\ \beta_i,\gamma_j>0}} [\delta^i_\beta,\delta^j_\gamma] = \sum_{\substack{\beta+\gamma=\alpha\\ \beta_i,\gamma_j>0}} [\eta^i_\beta,\eta^j_\gamma] = \alpha_j\eta^i_\alpha - \alpha_i\eta^j_\alpha \quad \forall i,j\in\operatorname{supp}\alpha.$$

The above system of linear equations with rational coefficients joint with the linear equation:

$$\sum_{i\in \operatorname{supp}\alpha} \delta^i_\alpha = \sum_{i\in \operatorname{supp}\alpha} \eta^i_\alpha,$$

gives rise to a non singular system and we deduce that $\delta^i_\alpha = \eta^i_\alpha$ for all $i \in \operatorname{supp}\alpha$, and so $\delta^i_\alpha = \eta^i_\alpha$ for all $i = 1,\dots,p$. $\square$

Notice that Lemma 1.2.16 and Proposition 1.2.18 can also be stated with the $\overline{\varepsilon}^i$ and $\overline{\varepsilon}$ instead of the ε^i and ε.

1.3. Substitution maps. In this section we give a summary of sections 2 and 3 of [**9**]. Let k be a commutative ring, A a commutative k-algebra, $\mathbf{s} = \{s_1,\dots,s_p\}, \mathbf{t} = \{t_1,\dots,t_q\}$ two sets of variables and $\Delta \subset \mathbb{N}^p, \nabla \subset \mathbb{N}^q$ non-empty co-ideals.

Definition 1.3.1. An A-algebra map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ will be called a *substitution map* whenever $\operatorname{ord}(\varphi(s_i)) \geq 1$ for all $i = 1,\dots,p$. A such map is continuous and uniquely determined by the family $c = \{\varphi(s_i), i = 1,\dots,p\}$.

The *trivial* substitution map $A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is the one sending any s_i to 0. It will be denoted by $\mathbf{0}$.

The composition of substitution maps is obviously a substitution map. Any substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ determines and is determined by a family:

$$\{\mathbf{C}_e(\varphi,\alpha), e\in\nabla, \alpha\in\Delta, |\alpha|\leq|e|\} \subset A, \quad \text{with} \quad \mathbf{C}_0(\varphi,0) = 1,$$

such that:

$$\varphi\left(\sum_{\alpha\in\Delta} a_\alpha \mathbf{s}^\alpha\right) = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) a_\alpha\right)\mathbf{t}^e. \tag{1.11}$$

In section 3, 2., of [**9**] the reader can find the explicit expression of the $\mathbf{C}_e(\varphi,\alpha)$ in terms of the $\varphi(s_i)$. The following lemma is clear.

Lemma 1.3.2. *If $\Delta' \subset \Delta \subset \mathbb{N}^p$ are non-empty co-ideals, the truncation $\tau_{\Delta\Delta'} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_{\Delta'}$ is clearly a substitution map and $\mathbf{C}_\beta(\tau_{\Delta\Delta'},\alpha) = \delta_{\alpha\beta}$ for all $\alpha\in\Delta$ and for all $\beta\in\Delta'$ with $|\alpha|\leq|\beta|$.*

Definition 1.3.3. We say that a substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ has *constant coefficients* if $\varphi(s_i) \in k[[\mathbf{t}]]_\nabla$ for all $i = 1,\dots,p$. This is equivalent to saying that $\mathbf{C}_e(\varphi,\alpha) \in k$ for all $e\in\nabla$ and for all $\alpha\in\Delta$ with $|\alpha|\leq|e|$. Substitution maps which constant coefficients are induced by substitution maps $k[[\mathbf{s}]]_\Delta \to k[[\mathbf{t}]]_\nabla$.

1.3.4 Let M be an $(A;A)$-bimodule.

Any substitution map map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ induces $(A;A)$-linear maps:

$$\varphi_M := \varphi\widehat{\otimes}\mathrm{Id}_M : M[[\mathbf{s}]]_\Delta \equiv A[[\mathbf{s}]]_\Delta\widehat{\otimes}_A M \longrightarrow M[[\mathbf{t}]]_\nabla \equiv A[[\mathbf{t}]]_\nabla\widehat{\otimes}_A M$$

and

$$_M\varphi := \mathrm{Id}_M\widehat{\otimes}\varphi : M[[\mathbf{s}]]_\Delta \equiv M\widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \longrightarrow M[[\mathbf{t}]]_\nabla \equiv M\widehat{\otimes}_A A[[\mathbf{t}]]_\nabla.$$

We have:

$$\varphi_M\left(\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta}\varphi(\mathbf{s}^\alpha)m_\alpha = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}}\mathbf{C}_e(\varphi,\alpha)m_\alpha\right)\mathbf{t}^e,$$

$$_M\varphi\left(\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta}m_\alpha\varphi(\mathbf{s}^\alpha) = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}}m_\alpha\mathbf{C}_e(\varphi,\alpha)\right)\mathbf{t}^e$$

for all $m\in M[[\mathbf{s}]]_\Delta$. If M is a trivial bimodule, then $\varphi_M = {}_M\varphi$. If $\varphi' : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Omega$ is another substitution map and $\varphi'' = \varphi\circ\varphi'$, we have $\varphi''_M = \varphi_M\circ\varphi'_M$, ${}_M\varphi'' = {}_M\varphi\circ {}_M\varphi'$.

For all $m\in M[[\mathbf{s}]]_\Delta$ and all $a\in A[[\mathbf{s}]]_\nabla$, we have:

$$\varphi_M(am) = \varphi(a)\varphi_M(m),\ \ {}_M\varphi(ma) = {}_M\varphi(m)\varphi(a),$$

i.e. φ_M is $(\varphi;A)$-linear and ${}_M\varphi$ is $(A;\varphi)$-linear. Moreover, φ_M and ${}_M\varphi$ are compatible with the augmentations, i.e.:

(1.12)

$$\varphi_M(m)\equiv m_0 \bmod \langle\mathbf{t}\rangle M[[\mathbf{t}]]_\nabla,\ \ {}_M\varphi(m)\equiv m_0 \bmod \langle\mathbf{t}\rangle M[[\mathbf{t}]]_\nabla,\ \ m\in M[[\mathbf{s}]]_\Delta.$$

If φ is the trivial substitution map (i.e. $\varphi(s_i)=0$ for all $s_i\in\mathbf{s}$), then $\varphi_M : M[[\mathbf{s}]]_\Delta\to M[[\mathbf{t}]]_\nabla$ and ${}_M\varphi : M[[\mathbf{s}]]_\Delta\to M[[\mathbf{t}]]_\nabla$ are also trivial, i.e. $\varphi_M(m) = {}_M\varphi(m) = m_0$, for all $m\in M[[\mathbf{s}]]_\nabla$.

1.3.5 The above constructions apply in particular to the case of any k-algebra R over A, for which we have two induced continuous maps: $\varphi_R = \varphi\widehat{\otimes}\mathrm{Id}_R : R[[\mathbf{s}]]_\Delta\to R[[\mathbf{t}]]_\nabla$, which is $(A;R)$-linear, and ${}_R\varphi = \mathrm{Id}_R\widehat{\otimes}\varphi : R[[\mathbf{s}]]_\Delta\to R[[\mathbf{t}]]_\nabla$, which is $(R;A)$-linear. For $r\in R[[\mathbf{s}]]_\Delta$ we will denote $\varphi\bullet r := \varphi_R(r)$, $r\bullet\varphi := {}_R\varphi(r)$. Explicitly, if $r=\sum_\alpha r_\alpha\mathbf{s}^\alpha$ with $\alpha\in\Delta$, then:

$$\varphi\bullet r = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}}\mathbf{C}_e(\varphi,\alpha)r_\alpha\right)\mathbf{t}^e,\quad r\bullet\varphi = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}}r_\alpha\mathbf{C}_e(\varphi,\alpha)\right)\mathbf{t}^e. \tag{1.13}$$

From (1.12), we deduce that:

$$\varphi\bullet\,\mathscr{U}^p(R;\Delta)\subset\mathscr{U}^q(R;\nabla),\quad \mathscr{U}^p(R;\Delta)\bullet\varphi\subset\mathscr{U}^q(R;\nabla),$$

and $\varphi\bullet 1 = 1\bullet\varphi = 1$.

If φ is a substitution map with constant coefficients, then $\varphi_R = {}_R\varphi$ is a ring homomorphism over φ. In particular, $\varphi\bullet r = r\bullet\varphi$ and $\varphi\bullet(rr') = (\varphi\bullet r)(\varphi\bullet r')$.

If $\varphi = \mathbf{0} : A[[\mathbf{s}]]_\Delta\to A[[\mathbf{t}]]_\nabla$ is the trivial substitution map, then $\mathbf{0}\bullet r = r\bullet\mathbf{0} = r_0$ for all $r\in R[[\mathbf{s}]]_\Delta$. In particular, $\mathbf{0}\bullet r = r\bullet\mathbf{0} = 1$ for all $r\in\mathscr{U}^p(R;\Delta)$.

If $\mathbf{u}=\{u_1,\dots,u_r\}$ is another set of variables, $\Omega\subset\mathbb{N}^r$ is a non-empty co-ideal and $\psi : R[[\mathbf{t}]]_\nabla\to R[[\mathbf{u}]]_\Omega$ is another substitution map, one has:

$$\psi\bullet(\varphi\bullet r) = (\psi\circ\varphi)\bullet r,\quad (r\bullet\varphi)\bullet\psi = r\bullet(\psi\circ\varphi).$$

Since $(R[[\mathbf{s}]]_\Delta)^{\mathrm{opp}} = R^{\mathrm{opp}}[[\mathbf{s}]]_\Delta$, for any substitution map $\varphi : A[[\mathbf{s}]]_\Delta\to A[[\mathbf{t}]]_\nabla$ we have $(\varphi_R)^{\mathrm{opp}} = {}_{R^{\mathrm{opp}}}\varphi$ and $({}_R\varphi)^{\mathrm{opp}} = \varphi_{R^{\mathrm{opp}}}$.

For each substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ we define the $(A;A)$-linear map:

$$\varphi_* : f \in \operatorname{Hom}_k(A, A[[\mathbf{s}]]_\Delta) \longmapsto \varphi_*(f) = \varphi \circ f \in \operatorname{Hom}_k(A, A[[\mathbf{t}]]_\nabla)$$

which induces another one $\overline{\varphi_*} : \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta) \longrightarrow \operatorname{End}_{k[[\mathbf{t}]]_\nabla}(A[[\mathbf{t}]]_\nabla)$ given by:

$$\overline{\varphi_*}(f) := \left(\varphi_*\left(f|_A\right)\right)^e = \left(\varphi \circ f|_A\right)^e \quad \forall f \in \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta).$$

More generally, for any left A-modules E, F we have $(A;A)$-linear maps:

$$(\varphi_F)_* : f \in \operatorname{Hom}_k(E, F[[\mathbf{s}]]_\Delta) \longmapsto (\varphi_F)_*(f) = \varphi_F \circ f \in \operatorname{Hom}_k(E, F[[\mathbf{t}]]_\nabla),$$
$$\overline{(\varphi_F)_*} : \operatorname{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \longrightarrow \operatorname{Hom}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla, F[[\mathbf{t}]]_\nabla),$$
$$\overline{(\varphi_F)_*}(f) := \left(\varphi_F \circ f|_E\right)^e.$$

Let us consider the $(A;A)$-bimodule $M = \operatorname{Hom}_k(E,F)$. For each $m \in M[[\mathbf{s}]]_\Delta$ and for each $e \in E$ we have $\widetilde{\varphi_M(m)}(e) = \varphi_F\left(\widetilde{m}(e)\right)$, i.e.:

$$\widetilde{\varphi_M(m)}|_E = \varphi_F \circ \left(\widetilde{m}|_E\right), \tag{1.14}$$

or more graphically, the following diagram is commutative (see (1.4)):

$$\begin{CD}
M[[\mathbf{s}]]_\Delta @>{\sim}>{m\mapsto\widetilde{m}}> \operatorname{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) @>{\sim}>{\text{restr.}}> \operatorname{Hom}_k(E, F[[\mathbf{s}]]_\Delta)\\
@V{\varphi_M}VV @VV{\overline{(\varphi_F)_*}}V @V{(\varphi_F)_*}VV\\
M[[\mathbf{t}]]_\nabla @>{\sim}>{m\mapsto\widetilde{m}}> \operatorname{Hom}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla, F[[\mathbf{t}]]_\nabla) @>{\sim}>{\text{restr.}}> \operatorname{Hom}_k(E, F[[\mathbf{t}]]_\nabla).
\end{CD} \tag{1.15}$$

In order to simplify notations, we will also write:

$$\varphi \bullet f := \overline{(\varphi_F)_*}(f) \quad \forall f \in \operatorname{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta),$$

and so we have $\widetilde{\varphi \bullet m} = \varphi \bullet \widetilde{m}$ for all $m \in M[[\mathbf{s}]]_\Delta$. Let us notice that $(\varphi \bullet f)(e) = (\varphi_F \circ f)(e)$ for all $e \in E$, i.e.:

$$\boxed{(\varphi \bullet f)|_E = (\varphi_F \circ f)|_E = \varphi_F \circ (f|_E), \text{ but in general } \varphi \bullet f \neq \varphi_F \circ f.} \tag{1.16}$$

If $\varphi = \mathbf{0}$ is the trivial substitution map, then for each k-linear map $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha : E \to E[[\mathbf{s}]]_\Delta$ (resp. $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha \in \operatorname{End}_k(E)[[\mathbf{s}]]_\Delta \equiv \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$), we have $\mathbf{0} \bullet f = f \bullet \mathbf{0} = f_0 \in \operatorname{End}_k(E) \subset \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ (resp. $\mathbf{0} \bullet f = f \bullet \mathbf{0} = f_0^e = \overline{f_0} \in \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$).

If $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is a substitution map, we have:

$$\varphi \bullet (af) = \varphi(a)\,(\varphi \bullet f)\,,\ (fa) \bullet \varphi = (f \bullet \varphi)\,\varphi(a)$$

for all $a \in A[[\mathbf{s}]]_\Delta$ and for all $f \in \operatorname{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ (or $f \in \operatorname{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$). Moreover:

$$(\varphi_E)_*(\operatorname{Hom}_k^\circ(E, M[[\mathbf{s}]]_\Delta)) \subset \operatorname{Hom}_k^\circ(E, E[[\mathbf{t}]]_\nabla),$$
$$\varphi \bullet \left(\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)\right) \subset \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla),$$

and so we have a commutative diagram:

$$\begin{CD}
\mathscr{U}^p(R;\Delta) @>{\sim}>{r\mapsto\widetilde{r}}> \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) @>{\sim}>{\text{restr.}}> \operatorname{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta)\\
@V{\varphi\bullet(-)}VV @VV{\varphi\bullet(-)}V @VV{(\varphi_E)_*}V\\
\mathscr{U}^q(R;\nabla) @>{\sim}>{r\mapsto\widetilde{r}}> \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla) @>{\sim}>{\text{restr.}}> \operatorname{Hom}_k(E, F[[\mathbf{t}]]_\nabla).
\end{CD} \tag{1.17}$$

Now we are going to see how the $\varepsilon^i(r), \overline{\varepsilon}^i(r), \varepsilon(r), \overline{\varepsilon}(r)$ (see 1.2.14) can be expressed in terms of the action of substitution maps.

Let us consider the power series ring $A[[\mathbf{s},\tau]] = A[[\mathbf{s}]]\widehat{\otimes}_A A[[\tau]]$, and for each $i = 1,\ldots,p$ we denote $\sigma^i : A[[\mathbf{s}]] \to A[[\mathbf{s},\tau]]$ the substitution map (with constant coefficients) defined by:

$$\sigma^i(s_j) = \begin{cases} s_i + s_i\tau & \text{if} \quad j = i \\ s_j & \text{if} \quad j \neq i. \end{cases}$$

Let us also denote $\sigma : A[[\mathbf{s}]] \to A[[\mathbf{s},\tau]]$ the substitution map (with constant coefficients) defined by:

$$\sigma(s_i) = s_i + s_i\tau \quad \forall i = 1,\ldots,p,$$

and $\iota : A[[\mathbf{s}]] \to A[[\mathbf{s},\tau]]$ the substitution map induced by the inclusion $\mathbf{s} \hookrightarrow \mathbf{s}\sqcup\{\tau\}$. We often consider ι as an inclusion $A[[\mathbf{s}]] \hookrightarrow A[[\mathbf{s},\tau]]$.

It is clear that for each non-empty co-ideal $\Delta \subset \mathbb{N}^p$, the substitution maps $\sigma^i, \sigma, \iota : A[[\mathbf{s}]] \to A[[\mathbf{s},\tau]]$ induce new substitution maps $A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s},\tau]]_{\Delta\times\{0,1\}}$, which will be also denoted by the same letters. Moreover, as a consequence of Taylor's expansion we have:

$$\sigma^i(a) = a + \chi^i_A(a)\tau, \quad \sigma(a) = a + \chi_A(a)\tau$$

where $\chi^i = s_i\frac{\partial}{\partial s_i}$ and $\chi = \sum_i \chi^i$ (see Definition1.2.8).

The proof of the following lemma is clear.

Lemma 1.3.6. *The map $\xi : R[[\mathbf{s}]]_{\Delta,+} \to \mathscr{U}^{p+1}(R;\Delta\times\{0,1\})$ defined as:*

$$\xi\left(\sum_{\alpha\in\Delta,|\alpha|>0} r_\alpha \mathbf{s}^\alpha\right) = 1 + \sum_{\alpha\in\Delta,|\alpha|>0} r_\alpha \mathbf{s}^\alpha\tau$$

is a group homomorphism.

Let us notice that the map ξ above is injective and its image is the set of $r \in \mathscr{U}^{p+1}(R;\Delta\times\{0,1\})$ such that $\operatorname{supp} r \subset \{(0,0)\}\cup((\Delta\setminus\{0\})\times\{1\})$.

Proposition 1.3.7. *For each $r \in \mathscr{U}^p(R;\Delta)$, the following properties hold:*

(1) $r^*(\sigma^i \bullet r) = \xi(\varepsilon^i(r))$, $(\sigma^i \bullet r)r^* = \xi(\overline{\varepsilon}^i(r))$.
(2) $r^*(\sigma \bullet r) = \xi(\varepsilon(r))$, $(\sigma \bullet r)r^* = \xi(\overline{\varepsilon}(r))$.

Proof. It is a straightforward consequence of Taylor's expansion formula:

$$\sigma^i \bullet r = r + \chi^i_R(r)\tau, \quad \sigma \bullet r = r + \chi_R(r)\tau.$$

$\square$

Let us notice that in the above proposition, the action $\iota\bullet(-) : R[[\mathbf{s}]]_\Delta \to R[[\mathbf{s},\tau]]_{\Delta\times\{0,1\}}$ is simply considered as an inclusion.

2. Multivariate Hasse–Schmidt derivations

2.1. Basic definitions. In this section we recall some notions and results of the theory of Hasse–Schmidt derivations [**1, 6**] as developed in [**9**].

From now on k will be a commutative ring, A a commutative k-algebra, $\mathbf{s} = \{s_1, \dots, s_p\}$ a set of variables and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal.

Definition 2.1.1. A (p, Δ)*-variate Hasse–Schmidt derivation*, or a (p, Δ)*-variate HS-derivation* for short, of A over k is a family $D = (D_\alpha)_{\alpha\in\Delta}$ of k-linear maps $D_\alpha : A \longrightarrow A$, satisfying the following Leibniz type identities:

$$D_0 = \mathrm{Id}_A, \quad D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y)$$

for all $x, y \in A$ and for all $\alpha \in \Delta$. We denote by $\mathrm{HS}^p_k(A; \Delta)$ the set of all (p, Δ)-variate HS-derivations of A over k. For $p = 1$, a 1-variate HS-derivation will be simply called a *Hasse–Schmidt derivation* (a HS-derivation for short), or a *higher derivation*[2], and we will simply write $\mathrm{HS}_k(A; m) := \mathrm{HS}^1_k(A; \Delta)$ for $\Delta = \{q \in \mathbb{N} \mid q \leq m\}$.[3]

Any (p, Δ)-variate HS-derivation D of A over k can be understood as a power series:

$$\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]]_\Delta, \quad R = \mathrm{End}_k(A),$$

and so we consider $\mathrm{HS}^p_k(A; \Delta) \subset R[[\mathbf{s}]]_\Delta$. Actually $\mathrm{HS}^p_k(A; \Delta)$ is a (multiplicative) sub-group of $\mathscr{U}^p(R; \Delta)$. The group operation in $\mathrm{HS}^p_k(A; \Delta)$ is explicitly given by:

$$(D, E) \in \mathrm{HS}^p_k(A; \Delta) \times \mathrm{HS}^p_k(A; \Delta) \longmapsto D \circ E \in \mathrm{HS}^p_k(A; \Delta)$$

with:

$$(D \circ E)_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta \circ E_\gamma,$$

and the identity element of $\mathrm{HS}^p_k(A; \Delta)$ is $\mathbb{I}$ with $\mathbb{I}_0 = \mathrm{Id}$ and $\mathbb{I}_\alpha = 0$ for all $\alpha \neq 0$. The inverse of a $D \in \mathrm{HS}^p_k(A; \Delta)$ will be denoted by D^*.

For $\Delta' \subset \Delta \subset \mathbb{N}^p$ non-empty co-ideals, we have truncations:

$$\tau_{\Delta\Delta'} : \mathrm{HS}^p_k(A; \Delta) \longrightarrow \mathrm{HS}^p_k(A; \Delta'),$$

which obviously are group homomorphisms. Since any $D \in \mathrm{HS}^p_k(A; \Delta)$ is determined by its finite truncations, we have a natural group isomorphism

$$\mathrm{HS}^p_k(A) = \lim_{\substack{\longleftarrow \\ \Delta'\subset\Delta \\ \sharp\Delta'<\infty}} \mathrm{HS}^p_k(A; \Delta'). \tag{2.1}$$

The proof of the following proposition is straightforward and it is left to the reader (see Notation 1.2.4 and 1.2.6).

Proposition 2.1.2. *Let us denote* $R = \mathrm{End}_k(A)$ *and let* $D = \sum_\alpha D_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]]_\Delta$ *be a power series. The following properties are equivalent:*

(a) *D is a $(\mathbf{s}, \Delta)$-variate HS-derivation of A over k.*
(b) *The map $\widetilde{D} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_\Delta$ is a (continuous) $k[[\mathbf{s}]]_\Delta$-algebra homomorphism compatible with the natural augmentation $A[[\mathbf{s}]]_\Delta \to A$.*

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[2] This terminology is used for instance in [**6**].
[3] These HS-derivations are called of length m in [**8**].

(c) $D \in \mathscr{U}^p(R;\Delta)$ *and for all* $a \in A[[\mathbf{s}]]_\Delta$ *we have* $Da = \widetilde{D}(a)D$.
(d) $D \in \mathscr{U}^p(R;\Delta)$ *and for all* $a \in A$ *we have* $Da = \widetilde{D}(a)D$.

Moreover, in such a case $\widetilde{D}$ *is a* $k[[\mathbf{s}]]_\Delta$*-algebra automorphism of* $A[[\mathbf{s}]]_\Delta$*.*

Notation 2.1.3. Let us denote:

$$\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) := \{f \in \operatorname{Hom}_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta),\ f(a) \equiv a \bmod \langle \mathbf{s}\rangle\ \forall a \in A\},$$
$$\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) :=$$
$$\left\{f \in \operatorname{Aut}_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \mid f(a) \equiv a_0 \bmod \langle \mathbf{s}\rangle\ \forall a \in A[[\mathbf{s}]]_\Delta\right\}.$$

It is clear that $\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) \subset \operatorname{Hom}^\circ_k(A, A[[\mathbf{s}]]_\Delta)$ and

$$\operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \subset \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta)$$

(see Notation 1.2.7) are subgroups, and we have group isomorphisms (see (1.7) and (1.6)):

$$\operatorname{HS}^p_k(A;\Delta) \xrightarrow[\simeq]{D\mapsto \widetilde{D}} \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \xrightarrow[\simeq]{\text{restr.}} \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta). \tag{2.2}$$

The composition of the above isomorphisms is given by:
(2.3)

$$D \in \operatorname{HS}^p_k(A;\Delta) \overset{\sim}{\longmapsto} \Phi_D := \left[a \in A \mapsto \sum_{\alpha\in\Delta} D_\alpha(a)\mathbf{s}^\alpha\right] \in \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta).$$

For each HS-derivation $D \in \operatorname{HS}^p_k(A;\Delta)$ we have $\widetilde{D} = (\Phi_D)^e$, i.e.:

$$\widetilde{D}\left(\sum_{\alpha\in\Delta} a_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta} \Phi_D(a_\alpha)\mathbf{s}^\alpha$$

for all $\sum_\alpha a_\alpha \mathbf{s}^\alpha \in A[[\mathbf{s}]]_\Delta$, and for any $E \in \operatorname{HS}^p_k(A;\Delta)$ we have $\Phi_{D\circ E} = \widetilde{D} \circ \Phi_E$. If $\Delta' \subset \Delta$ is another non-empty co-ideal and we denote by $\pi_{\Delta\Delta'} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_{\Delta'}$ the projection (or truncation), one has $\Phi_{\tau_{\Delta\Delta'}(D)} = \pi_{\Delta\Delta'} \circ \Phi_D$.

2.2. The action of substitution maps on HS-derivations. Now, we recall the action of substitution maps on HS-derivations [**9**, §6]. Let $\mathbf{s} = \{s_1,\dots,s_p\}$, $\mathbf{t} = \{t_1,\dots,t_p\}$ be sets of variables, $\Delta \subset \mathbb{N}^p$, $\nabla \subset \mathbb{N}^q$ non-empty co-ideals and let us write $R = \operatorname{End}_k(A)$.

Let us recall Proposition 10 in [**9**].

Proposition 2.2.1. *For any substitution map* $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$*, we have:*

1) $\varphi_*\left(\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta)\right) \subset \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{t}]]_\nabla)$,
2) $\varphi \bullet \operatorname{HS}^p_k(A;\Delta) \subset \operatorname{HS}^q_k(A;\nabla)$,
3) $\varphi \bullet \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \subset \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla-\mathrm{alg}}(A[[\mathbf{t}]]_\nabla)$.

Then we have a commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) & \xleftarrow[\Phi_D \leftarrow\!\shortmid D]{\sim} & \operatorname{HS}^p_k(A;\Delta) & \xrightarrow{\sim} & \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta)n \\
{\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi\bullet(-)} & & \downarrow{\scriptstyle\varphi\bullet(-)} \\
\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{t}]]_\nabla) & \xleftarrow[\Phi_D \leftarrow\!\shortmid D]{\sim} & \operatorname{HS}^q_k(A;\nabla) & \xrightarrow{\sim} & \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla-\mathrm{alg}}(A[[\mathbf{t}]]_\nabla).
\end{array} \tag{2.4}$$

In particular, for any HS-derivation $D \in \mathrm{HS}_k^p(A;\Delta)$ we have $\varphi\bullet D \in \mathrm{HS}_k^q(A;\nabla)$ (see 1.3.5). Moreover $\Phi_{\varphi\bullet D} = \varphi\circ\Phi_D$.

It is clear that for any co-ideals $\Delta' \subset \Delta$ and $\nabla' \subset \nabla$ with $\varphi(\Delta'_A/\Delta_A) \subset \nabla'_A/\nabla_A$ we have:

$$\tau_{\nabla\nabla'}(\varphi\bullet D) = \varphi'\bullet\tau_{\Delta\Delta'}(D), \tag{2.5}$$

where $\varphi' : A[[\mathbf{s}]]_{\Delta'} \to A[[\mathbf{t}]]_{\nabla'}$ is the substitution map induced by φ.

2.2.2 Let $\mathbf{u} = \{u_1,\dots,u_r\}$ be another set of variables, $\Omega \subset \mathbb{N}^r$ a non-empty co-ideal, $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$, $\psi : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Omega$ substitution maps and $D, D' \in \mathrm{HS}_k^p(A;\Delta)$ HS-derivations. From 1.3.5 we deduce the following properties:

-) If we denote $E := \varphi\bullet D \in \mathrm{HS}_k^q(A;\nabla)$, we have

$$E_0 = \mathrm{Id},\quad E_e = \sum_{\substack{\alpha\in\Delta\\ |\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha)D_\alpha,\quad \forall e\in\nabla. \tag{2.6}$$

-) If $\varphi = \mathbf{0}$ is the trivial substitution map or if $D = \mathbb{I}$, then $\varphi\bullet D = \mathbb{I}$.

-) If φ has constant coefficients, then $\varphi\bullet(D\circ D') = (\varphi\bullet D)\circ(\varphi\bullet D')$ and $(\varphi\bullet D)^* = \varphi\bullet D^*$ (the general case is treated in Proposition 2.2.3).

-) $\psi\bullet(\varphi\bullet D) = (\psi\circ\varphi)\bullet D$.

The following result is proven in Propositions 11 and 12 of [**9**].

Proposition 2.2.3. *Let $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ be a substitution map. Then, the following assertions hold:*

(i) *For each $D \in \mathrm{HS}_k^p(A;\Delta)$ there is a unique substitution map $\varphi^D : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ such that $\left(\widetilde{\varphi\bullet D}\right)\circ\varphi^D = \varphi\circ\widetilde{D}$. Moreover, $(\varphi\bullet D)^* = \varphi^D\bullet D^*$, $\varphi^{\mathbb{I}} = \varphi$ and:*

$$\mathbf{C}_e(\varphi, f+\nu) = \sum_{\substack{\beta+\gamma=e\\ |f+g|\leq|\beta|,|\nu|\leq|\gamma|}} \mathbf{C}_\beta(\varphi,f+g)D_g(\mathbf{C}_\gamma(\varphi^D,\nu))$$

for all $e\in\Delta$ and for all $f,\nu\in\nabla$ with $|f+\nu|\leq|e|$.

(ii) *For each $D, E \in \mathrm{HS}_k^p(A;\nabla)$, we have $\varphi\bullet(D\circ E) = (\varphi\bullet D)\circ(\varphi^D\bullet E)$ and $\left(\varphi^D\right)^E = \varphi^{D\circ E}$. In particular, $\left(\varphi^D\right)^{D^*} = \varphi$.*

(iii) *If ψ is another composable substitution map, then $(\varphi\circ\psi)^D = \varphi^{\psi\bullet D}\circ\psi^D$.*

(iv) *If φ has constant coefficients then $\varphi^D = \varphi$.*

3. Main results

3.1. The derivations associated with a Hasse–Schmidt derivation. In this section k will be a commutative ring, A a commutative k-algebra, $R = \mathrm{End}_k(A)$, $\mathbf{s} = \{s_1,\dots,s_p\}$ a set of variables and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal.

Lemma 3.1.1. *Let $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ be a k-derivation and $D \in \mathrm{HS}_k^p(A;\Delta)$ a HS-derivation. Then, for each $a \in A[[\mathbf{s}]]_\Delta$ we have $\mathfrak{d}_R(D)a = \widetilde{\mathfrak{d}_R(D)}(a)D + \widetilde{D}(a)\mathfrak{d}_R(D)$.*

PROOF. By using Lemma 1.2.10 we have:

$$\mathfrak{d}_R(D)a = \mathfrak{d}_R(Da) - D\mathfrak{d}_A(a) = \mathfrak{d}_R(\widetilde{D}(a)D) - \widetilde{D}(\mathfrak{d}_A(a))D =$$
$$\mathfrak{d}_A(\widetilde{D}(a))D + \widetilde{D}(a)\mathfrak{d}_R(D) - \widetilde{D}(\mathfrak{d}_A(a))D = \widetilde{\mathfrak{d}_R(D)}(a)D + \widetilde{D}(a)\mathfrak{d}_R(D).$$

□

From now on, we will denote to simplify $\mathfrak{d}_R = \mathfrak{d}$ and $\mathfrak{d}_A = \mathfrak{d}$.

PROPOSITION 3.1.2. *Let $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ be a k-derivation. Then, for any $D \in \mathrm{HS}^p_k(A;\Delta)$ we have $\varepsilon^{\mathfrak{d}}(D), \overline{\varepsilon}^{\mathfrak{d}}(D) \in \mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+} = \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta \cap R[[\mathbf{s}]]_{\Delta,+}$.*

PROOF. Remember that $\overline{\varepsilon}^{\mathfrak{d}}(D) = \mathfrak{d}(D)D^*$ and $\varepsilon^{\mathfrak{d}}(D) = D^*\mathfrak{d}(D)$ (Definition 1.2.11). We will use Lemma 1.2.9 and Lemma 3.1.1. For any $a \in A[[\mathbf{s}]]_\Delta$ we have:

$$(\mathfrak{d}(D)D^*)\, a = \mathfrak{d}(D)\widetilde{D^*}(a)D^* = \left[\widetilde{\mathfrak{d}(D)}(\widetilde{D^*}(a))D + \widetilde{D}(\widetilde{D^*}(a))\mathfrak{d}(D)\right] D^* =$$
$$\widetilde{\mathfrak{d}(D)}(\widetilde{D^*}(a)) + a\mathfrak{d}(D)D^* = (\widetilde{\mathfrak{d}(D)D^*})(a) + a\,(\mathfrak{d}(D)D^*)\,,$$

and so $[\mathfrak{d}(D)D^*, a] = (\widetilde{\mathfrak{d}(D)D^*})(a)$ and $\mathfrak{d}(D)D^* \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$. The proof for $\varepsilon^{\mathfrak{d}}(D)$ is completely similar. □

EXAMPLE 3.1.3. If $D \in \mathrm{HS}_k(A;m)$ is a 1-variate HS-derivation of length m, then:

$$\varepsilon_1(D) = D_1,\ \varepsilon_2(D) = 2D_2 - D_1^2,\ \varepsilon_3(D) = 3D_3 - 2D_1D_2 - D_2D_1 + D_1^3, \dots$$

Let us recall that the map $\xi : R[[\mathbf{s}]]_{\Delta,+} \to \mathscr{U}^{p+1}(R;\Delta\times\{0,1\})$ has been defined in Lemma 1.3.6. The proof of the following lemma is clear.

LEMMA 3.1.4. *For each $\delta \in \mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+}$ we have $\xi(\delta) \in \mathrm{HS}^{p+1}_k(A;\Delta\times\{0,1\})$.*

So we have a group homomorphism $\xi : \mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+} \to \mathrm{HS}^{p+1}_k(A;\Delta\times\{0,1\})$ whose image is the set of $D \in \mathrm{HS}^{p+1}_k(A;\Delta\times\{0,1\})$ such that $\operatorname{supp} D \subset \{(0,0)\} \cup ((\Delta\setminus\{0\})\times\{1\})$.

The following proposition provides a characterization of HS-derivations in characteristic 0.

PROPOSITION 3.1.5. *Assume that $\mathbb{Q} \subset k$, $R = \mathrm{End}_k(A)$ and $D \in \mathscr{U}^p(R;\Delta)$. The following properties are equivalent:*

(a) $D \in \mathrm{HS}^p_k(A;\Delta)$.
(b) $\varepsilon^{\mathfrak{d}}(D) \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$ *for all k-derivations* $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$.
(c) $\varepsilon(D) \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$.

PROOF. The implication (a) ⇒ (b) comes from Proposition 3.1.2 and (b) ⇒ (c) is obvious. Let us prove (c) ⇒ (a). Write $\delta = \varepsilon(D)$, i.e. $\boldsymbol{\chi}_R(D) = D\,\delta$. After Proposition 2.1.2, we need to prove that $D\,a = \widetilde{D}(a)D$ for all $a \in A$, and since $D\,a - \widetilde{D}(a)D$ belongs to the augmentation ideal of $R[[\mathbf{s}]]_\Delta$ and $\mathbb{Q} \subset k$, it is enough to prove that $\boldsymbol{\chi}_R\left(D\,a - \widetilde{D}(a)D\right) = 0$. By using that $\boldsymbol{\chi}_A(a) = 0$ and Lemma 1.2.10

we have:

$$\begin{gathered}\boldsymbol{\chi}_R\left(D\,a-\widetilde{D}(a)D\right)=\boldsymbol{\chi}_R(D)\,a+D\,\boldsymbol{\chi}_A(a)-\boldsymbol{\chi}_A(\widetilde{D}(a))D-\widetilde{D}(a)\boldsymbol{\chi}_R(D)=\\ D\,\delta\,a-\widetilde{\boldsymbol{\chi}_R(D)}(a)D-\widetilde{D}(\boldsymbol{\chi}_A(a))D-\widetilde{D}(a)D\,\delta=\\ D\,a\,\delta+D\,\widetilde{\delta}(a)-\widetilde{\boldsymbol{\chi}_R(D)}(a)D-\widetilde{D}(a)D\,\delta=\\ \widetilde{D}(a)D\,\delta+\widetilde{D}(\widetilde{\delta}(a))D-\widetilde{(D\,\delta)}(a)D-\widetilde{D}(a)D\,\delta=0.\end{gathered}$$

□

THEOREM 3.1.6. *Assume that $\mathbb{Q}\subset k$. Then, all the maps in the following commutative diagram:*

$$\begin{array}{ccc}\mathrm{HS}^p_k(A;\Delta) & \xrightarrow{\boldsymbol{\varepsilon}} & \Lambda^p(R;\Delta)\bigcap(\mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+})^p\\ & \searrow^{\varepsilon} & \downarrow{\scriptstyle\boldsymbol{\Sigma}}\\ & & \mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+}\end{array}$$

are bijective.

PROOF. It is a consequence of Proposition 1.2.18 and Proposition 3.1.5. □

A similar result holds for $\overline{\varepsilon}$ instead of ε.

REMARK 3.1.7. Let us notice that, in Theorem 3.1.6, $\mathrm{HS}^p_k(A;\Delta)$ carries a group structure (non-commutative in general) and an action of substitution maps, and on the other hand $\mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+}$ carries an $A[[\mathbf{s}]]_\Delta$-module structure and a $k[[\mathbf{s}]]_\Delta$-Lie algebra structure, but the bijection $\varepsilon:\mathrm{HS}^p_k(A;\Delta)\xrightarrow{\sim}\mathrm{Der}_k(A)[[\mathbf{s}]]_{\Delta,+}$ is not compatible with these structures. The formulas expressing the behavior of ε with respect to the group operation on HS-derivations or the behavior of ε^{-1} with respect to the addition of power series with coefficients in $\mathrm{Der}_k(A)$, turn out to be complicated and have a similar flavor to Baker-Campbell-Hausdorff formula.

3.2. The behavior under the action of substitution maps.

DEFINITION 3.2.1. Let S be a k-algebra over A, $r\in\mathscr{U}^p(S;\Delta)$, $D\in\mathrm{HS}^p_k(A;\Delta)$, $r'\in S[[\mathbf{s}]]_\Delta$ and $\delta\in\mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$. We say that

-) r is a *D-element* if $r\,a=\widetilde{D}(a)\,r$ for all $a\in A[[\mathbf{s}]]_\Delta$.
-) r' is a *δ-element* if $r'\,a=ar'+\widetilde{\delta}(a)\,1_S$ for all $a\in A[[\mathbf{s}]]_\Delta$.

It is clear that $D\in\mathrm{HS}^p_k(A;\Delta)\subset\mathscr{U}^p(\mathrm{End}_k(A);\Delta)$ is a D-element. For $D=\mathbb{I}$ the identity HS-derivation, a $r\in\mathscr{U}^p(S;\Delta)$ is an *$\mathbb{I}$-element* if and only if r commutes with all $a\in A[[\mathbf{s}]]_\Delta$. If $E\in\mathrm{HS}^p_k(A;\Delta)$ is another HS-derivation, $r\in\mathscr{U}^p(S;\Delta)$ is a D-element and $s\in\mathscr{U}^p(S;\Delta)$ is an E-element, then rs is a $(D\circ E)$-element.

The following lemma provides a characterization of D-elements. Its proof is easy and it is left to the reader.

LEMMA 3.2.2. *With the above notations, for each $r=\sum_\alpha r_\alpha\mathbf{s}^\alpha\in\mathscr{U}^p(S;\Delta)$ the following properties are equivalent:*

-) *r is a D-element.*
-) *$br=r\,\widetilde{D^*}(b)$ for all $b\in A[[\mathbf{s}]]_\Delta$.*
-) *r^* is a D^*-element.*

-) *If* $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha$, *we have* $r_\alpha\, a = \sum_{\beta+\gamma=\alpha} D_\beta(a) r_\gamma$ *for all* $a \in A$ *and for all* $\alpha \in \Delta$.
-) $r\,a = \widetilde{D}(a) r$ *for all* $a \in A$.

The following proposition reproduces Proposition 2.2.6 of [**10**].

PROPOSITION 3.2.3. *Let* S *be a* k*-algebra over* A, $D \in \mathrm{HS}^p_k(A;\Delta)$, $\varphi : A[[\mathbf{s}]]_\Delta \to A[\mathbf{u}]]_\nabla$ *a substitution map and* $r \in \mathscr{U}^p(S;\Delta)$ *a* D*-element. Then the following identities hold:*

(a) $\varphi_S(r)$ *is a* $(\varphi\bullet D)$*-element.*
(b) $\varphi_S(rr') = \varphi_S(r)\varphi^D_S(r')$ *for all* $r' \in R[[\mathbf{s}]]_\Delta$. *In particular,* $\varphi_S(r)^* = \varphi^D_S(r^*)$.

Moreover, if E *is an* A*-module and* $S = \mathrm{End}_k(E)$*, then the following identity holds:*

(c) $\langle \varphi\bullet r, \varphi^D_E(e)\rangle = \varphi_E(\langle r, e\rangle)$ *for all* $e \in E[[\mathbf{s}]]_\Delta$. *In other words:* $\varphi_E \circ \widetilde{r} = (\varphi\bullet\widetilde{r}) \circ \varphi^D_E$.

Lemma 1.2.10 and Proposition 3.1.2 can be generalized in the following way.

PROPOSITION 3.2.4. *Let* S *be a* k*-algebra over* A, $D \in \mathrm{HS}^p_k(A;\Delta)$, $r \in \mathscr{U}^p(S;\Delta)$ *a* D*-element and* $\mathfrak{d} : k[[\mathbf{s}]]_\Delta \to k[[\mathbf{s}]]_\Delta$ *a* k*-derivation. Then, the following properties hold:*

(1) $\mathfrak{d}(r)\, a = \langle \mathfrak{d}(D), a\rangle\, r + \langle D, a\rangle\, \mathfrak{d}(r)$ *for all* $a \in A$
(2) $\varepsilon^{\mathfrak{d}}(r)$ *is a* $\varepsilon^{\mathfrak{d}}(D)$*-element and* $\overline{\varepsilon}^{\mathfrak{d}}(r)$ *is a* $\overline{\varepsilon}^{\mathfrak{d}}(D)$*-element.*

PROOF. (1) Since $\delta(a) = 0$ we have:

$$\mathfrak{d}(r)\, a = \delta(r\, a) = \mathfrak{d}\left(\widetilde{D}(a) r\right) = \mathfrak{d}\left(\langle D, a\rangle\, r\right) = \langle \mathfrak{d}(D), a\rangle\, r + \langle D, a\rangle\, \mathfrak{d}(r).$$

(2) For all $a \in A$ we have:

$$\varepsilon^{\mathfrak{d}}(r)\, a = r^*\, \mathfrak{d}(r)\, a \stackrel{(1)}{=} r^* \left(\widetilde{\mathfrak{d}(D)}(a)\, r + \widetilde{D}(a)\, \mathfrak{d}(r)\right) =$$

$$\widetilde{D^*}(\widetilde{\mathfrak{d}(D)}(a)) r^*\, r + \widetilde{D^*}(\widetilde{D}(a)) r^* \mathfrak{d}(r) = \widetilde{\varepsilon^{\mathfrak{d}}(D)}(a)\, 1_S + a\, \varepsilon^{\mathfrak{d}}(r).$$

The proof for $\overline{\varepsilon}^{\mathfrak{d}}(r)$ is similar. □

Let us consider two sets of variables $\mathbf{s} = \{s_1, \dots, s_p\}$ and $\mathbf{u} = \{u_1, \dots, u_q\}$, and let us denote by $\{v^1, \dots, v^p\}$ the canonical basis of $\mathbb{N}^p$: $v^i_l = \delta_{il}$.

THEOREM 3.2.5. *For each non-empty co-ideals* $\Delta \subset \mathbb{N}^p, \Omega \subset \mathbb{N}^q$, *each substitution map* $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{u}]]_\Omega$ *and each HS-derivation* $D \in \mathrm{HS}^p_k(A;\Delta)$, *there exists a family*

$$\left\{ \mathbf{N}^{j,i}_{e,h} \mid 1 \le j \le q, 1 \le i \le p, e \in \Omega, h \in \Delta, |h| \le |e| \right\} \subset A$$

such that for any k*-algebra* S *over* A *and any* D*-element* $r \in \mathscr{U}^p(S;\Delta)$*, we have:*

$$\varepsilon^j_e(\varphi\bullet r) = \sum_{\substack{0<|h|\le|e| \\ i\in\operatorname{supp} h}} \mathbf{N}^{j,i}_{e,h} \varepsilon^i_h(r) \quad \forall e \in \Omega, \forall j = 1, \dots, q. \tag{3.1}$$

Moreover, $\mathbf{N}^{j,i}_{e,h} = \sum_{f,g,\beta} g_j \mathbf{C}_f(\varphi^D, \beta + h - v^i) D^*_\beta(\mathbf{C}_g(\varphi, v^i))$, *where* $f, g \in \Omega$, $\beta \in \Delta$, $f+g = e$, $|\beta+h|-1 \le |f|$ *and* $g_j > 0$, *whenever* $e_j, h_i > 0$, *and* $\mathbf{N}^{j,i}_{e,h} = 0$ *otherwise.*

PROOF. Let us write $r = \sum_{\alpha\in\Delta} r_\alpha \mathbf{s}^\alpha$. For each $\alpha \in \Delta$ and each $j = 1, \ldots, q$ we have:

$$\chi^j(\varphi(\mathbf{s}^\alpha)) = \chi^j\left(\prod_{i=1}^p \varphi(s_i)^{\alpha_i}\right) = \sum_{\alpha_i\neq 0} \alpha_i \varphi(s_i)^{\alpha_i-1}\left(\prod_{\substack{1\leq l\leq p\\ l\neq i}} \varphi(s_l)^{\alpha_l}\right)\chi^j(\varphi(s_i)) =$$

$$\sum_{\alpha_i\neq 0}\alpha_i\varphi\left(\mathbf{s}^{\alpha-v^i}\right)\chi^j(\varphi(s_i)) = \sum_{\alpha_i\neq 0}\alpha_i\left(\sum_{|e|\geq|\alpha|-1}\mathbf{C}_e(\varphi,\alpha-v^i)\mathbf{u}^e\right)\left(\sum_{e_j>0} e_j\mathbf{C}_e(\varphi,v^i)\mathbf{u}^e\right) =$$

$$\sum_{\alpha_i\neq 0}\alpha_i\left(\sum_{\substack{|e|\geq|\alpha|\\ e_j>0}}\left(\sum_{\substack{e'+e''=e\\ |e'|\geq|\alpha|-1\\ e''_j>0}} e''_j\mathbf{C}_{e'}(\varphi,\alpha-v^i)\mathbf{C}_{e''}(\varphi,v^i)\right)\mathbf{u}^e\right) = \sum_{\alpha_i\neq 0}\alpha_i\left(\sum_{\substack{|e|\geq|\alpha|\\ e_j>0}}\mathbf{M}^{j,i}_{\alpha,e}\mathbf{u}^e\right)$$

with:

$$\mathbf{M}^{j,i}_{\alpha,e} := \sum_{\substack{e'+e''=e\\ |e'|\geq|\alpha|-1\\ e''_j>0}} e''_j\mathbf{C}_{e'}(\varphi,\alpha-v^i)\mathbf{C}_{e''}(\varphi,v^i)$$

for $i\in\operatorname{supp}\alpha$ and $e\in\Omega$ with $e_j>0$ and $|e|\geq|\alpha|$. If either $\alpha_i=0$ or $e_j=0$, we set $\mathbf{M}^{j,i}_{\alpha,e}=0$. So, for each $j=1,\ldots,q$ we have:

$$\varepsilon^j(\varphi\bullet r) = (\varphi\bullet r)^*\chi^j(\varphi\bullet r) = \cdots = (\varphi^D\bullet r^*)\left(\sum_{\alpha\in\Delta}\chi^j(\varphi(\mathbf{s}^\alpha))r_\alpha\right) =$$

$$\left(\sum_{e\in\Omega}\left(\sum_{|\alpha|\leq|e|}\mathbf{C}_e(\varphi^D,\alpha)r^*_\alpha\right)\mathbf{u}^e\right)\left(\sum_{e_j>0}\left(\sum_{\substack{|\alpha|\leq|e|\\ \alpha_i\neq 0}}\alpha_i\mathbf{M}^{j,i}_{\alpha,e}r_\alpha\right)\mathbf{u}^e\right) =$$

$$\sum_{e_j>0}\left(\sum_{\substack{f+g=e\\ |\mu|\leq|f|,|\nu|\leq|g|\\ g_j>0, i\in\operatorname{supp}\nu}}\nu_t\mathbf{C}_f(\varphi^D,\mu)r^*_\mu\mathbf{M}^{j,i}_{\nu,g}r_\nu\right)\mathbf{u}^e \overset{(\star)}{=}$$

$$\sum_{e_j>0}\left(\sum_{\substack{f+g=e\\ |\beta+\gamma|\leq|f|,0<|\nu|\leq|g|\\ g_j>0, i\in\operatorname{supp}\nu}}\nu_i\mathbf{C}_f(\varphi^D,\beta+\gamma)D^*_\beta(\mathbf{M}^{j,i}_{\nu,g})r^*_\gamma r_\nu\right)\mathbf{u}^e =$$

$$\sum_{e_j>0}\left(\sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp}h}}\sum_{\substack{\gamma+\nu=h\\ i\in\operatorname{supp}\nu}}\left(\sum_{\substack{f+g=e\\ |\beta+\gamma|\leq|f|\\ |\nu|\leq|g|,g_j>0}}\mathbf{C}_f(\varphi^D,\beta+\gamma)D^*_\beta(\mathbf{M}^{j,i}_{\nu,g})\right)\nu_i r^*_\gamma r_\nu\right)\mathbf{u}^e =$$

$$\sum_{e_j>0}\left(\sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp}h}}\sum_{\substack{\gamma+\nu=h\\ i\in\operatorname{supp}\nu}} N^{j,i}_{e,\nu,\gamma}\nu_i r^*_\gamma r_\nu\right)\mathbf{u}^e,$$

where equality $(\star)$ comes from the fact that r^* is a D^*-element (see Lemma 3.2.2) and

$$N^{j,i}_{e,\nu,\gamma} = \begin{cases} \displaystyle\sum_{\substack{f+g=e\\ |\beta+\gamma|\leq|f|\\ |\nu|\leq|g|, g_j>0}} \mathbf{C}_f(\varphi^D,\beta+\gamma) D^*_\beta(\mathbf{M}^{j,i}_{\nu,g}) & \text{if } \nu_i>0, e_j>0, |e|\geq|\nu+\gamma|,\\ 0 & \text{otherwise.}\end{cases}$$

But, for $h=\nu+\gamma$, we have:

$$N^{j,i}_{e,\nu,\gamma} = \sum_{\substack{f+g=e\\ |\beta+\gamma|\leq|f|\\ |\nu|\leq|g|, g_j>0}} \mathbf{C}_f(\varphi^D,\beta+\gamma) D^*_\beta(\mathbf{M}^{j,i}_{\nu,g}) =$$

$$\sum_{\substack{f+g=e\\ |\beta+\gamma|\leq|f|\\ |\nu|\leq|g|, g_j>0}} \mathbf{C}_f(\varphi^D,\beta+\gamma)\, D^*_\beta\left(\sum_{\substack{g'+g''=g\\ |g'|\geq|\nu|-1\\ g''_j>0}} g''_j \mathbf{C}_{g'}(\varphi,\nu-v^i)\mathbf{C}_{g''}(\varphi,v^i)\right) =$$

$$\sum_{\substack{f+g'+g''=e\\ |\beta'+\beta''+\gamma|\leq|f|\\ |\nu|-1\leq|g'|, g''_j>0}} g''_u \mathbf{C}_f(\varphi^D,\beta'+\beta''+\gamma)\, D^*_{\beta'}\left(\mathbf{C}_{g'}(\varphi,\nu-v^i)\right)\, D^*_{\beta''}\left(\mathbf{C}_{g''}(\varphi,v^i)\right) =$$

$$\sum_{\substack{\rho+g''=e\\ |\beta''+h|-1\leq|\rho|\\ g''_j>0}} g''_j \left(\sum_{\substack{f+g'=\rho\\ |\beta''+\gamma+\beta'|\leq|f|\\ |\nu|-1\leq|g'|}} \mathbf{C}_f(\varphi^D,\beta''+\gamma+\beta')\, D^*_{\beta'}\left(\mathbf{C}_{g'}(\varphi,\nu-v^i)\right)\right) D^*_{\beta''}\left(\mathbf{C}_{g''}(\varphi,v^i)\right) \stackrel{(\star)}{=}$$

$$\sum_{\substack{\rho+g''=e\\ |\beta''+h|-1\leq|\rho|\\ g''_j>0}} g''_j\, \mathbf{C}_\rho(\varphi^D,\beta''+\gamma+\nu-v^i)\, D^*_{\beta''}\left(\mathbf{C}_{g''}(\varphi,v^i)\right),$$

where equality $(\star)$ comes from Proposition 2.2.3, and so $N^{j,i}_{e,\nu,\gamma}$ only depends on $h=\gamma+\nu$. By setting:

$$\mathbf{N}^{j,i}_{e,h} := \sum_{\substack{f+g=e\\ |\beta+h|-1\leq|f|\\ g_j>0}} g_j \mathbf{C}_f(\varphi^D,\beta+h-v^i) D^*_\beta(\mathbf{C}_g(\varphi,v^i))$$

for $e_j, h_i>0, |e|\geq|h|$ and $\mathbf{N}^{j,i}_{e,h} := 0$ otherwise, we have $N^{j,i}_{e,\nu,\gamma} = \mathbf{N}^{j,i}_{e,\gamma+\nu}$, and so:

$$\varepsilon^j(\varphi\bullet r) = \cdots = \sum_{e_j>0} \left(\sum_{\substack{0<|h|\leq|e|\\ o\in\operatorname{supp} h}} \sum_{\substack{\gamma+\nu=h\\ i\in\operatorname{supp}\nu}} N^{j,i}_{e,\nu,\gamma} \nu_i r^*_\gamma r_\nu \right) \mathbf{u}^e =$$

$$\sum_{e_j>0} \left(\sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp} h}} \sum_{\substack{\gamma+\nu=h\\ i\in\operatorname{supp}\nu}} \mathbf{N}^{j,i}_{e,\gamma+\nu} \nu_i r^*_\gamma r_\nu \right) \mathbf{u}^e = \sum_{e_j>0} \left(\sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp} h}} \mathbf{N}^{j,i}_{e,h} \sum_{\substack{\gamma+\nu=h\\ i\in\operatorname{supp}\nu}} \nu_i r^*_\gamma r_\nu \right) \mathbf{u}^e =$$

$$\sum_{e_j>0} \left(\sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp} h}} \mathbf{N}^{j,i}_{e,h} \varepsilon^i_h(r) \right) \mathbf{u}^e.$$

□

Corollary 3.2.6. *Under the hypotheses of Theorem 3.2.5, we have:*

$$\varepsilon^j_e(\varphi\bullet D) = \sum_{\substack{0<|h|\leq|e|\\ i\in\operatorname{supp} h}} \mathbf{N}^{j,i}_{e,h} \varepsilon^i_h(D) \quad \forall e\in\Omega, \forall j=1,\dots,q.$$

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Luis Narváez Macarro, Departamento de Álgebra & Instituto de Matemáticas (IMUS), Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia s/n, 41012 Sevilla, Spain.

E-mail address: `narvaez@us.es`