

GENERATORS AND CLOSED CLASSES OF GROUPS

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Abstract: We show that, in the category of groups, every singly-generated class which is closed under isomorphisms, direct limits, and extensions is also singly-generated under isomorphisms and direct limits, and in particular is co-reflective. In this way, we extend to the group-theoretic framework the topological analogue proved by Chachólski, Parent, and Stanley in 2004. We also establish several new relations between singly-generated closed classes.

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1. Introduction

In a category of R -modules over a certain ring R , the study of the closure of different classes under different operations has been a hot research topic for decades (see [26] for a thorough approach). The operations include extensions, subgroups, quotients, etc. In the case of $R = \mathbb{Z}$, we are dealing, of course, with operations in the category of abelian groups.

There is another different but related problem. Given a collection \mathbf{D} of objects in a class \mathcal{C} , the smallest class that contains \mathbf{D} and is closed under certain operations is called the class generated by \mathbf{D} under these operations. When this class is equal to \mathcal{C} , it is said that \mathcal{C} is *generated* by \mathbf{D} under the operations, and the elements of \mathbf{D} are called *generators* of \mathcal{C} . In this situation, if \mathbf{D} consists of only one object, the class \mathcal{C} is said to be *singly-generated* under the operations. Now the question is clear: given a class \mathcal{C} defined by a certain property, is it possible to construct a generator for it? A standard reference for this topic is [1], and related problems have been recently treated in [10].

In this paper we will deal with different versions of these two problems in the category of groups. Rather than from the vast literature in Module and Category Theory related to these topics, our motivation and strategy

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come from the field of Homotopy Theory. Since the nineties, Bousfield, Chachólski, and Farjoun ([6], [13], [18]), among others, developed an ambitious program whose main goal was to classify spaces and spectra in terms of classes defined by certain operations, particularly homotopy direct limits (cellular classes) and fibrations (periodicity classes). Taking account of this framework, the fundamental group functor gives a natural (but nontrivial) way to understand closed classes of groups from the same viewpoint.

We are mainly interested in three classes of operations in the class of groups, namely taking direct limits, constructing extensions from a given kernel and cokernel, and taking quotients. The relation between the closed classes under these operations has been already studied in [14] from a different point of view. We focus on singly-generated classes, and in fact our first main result is a generation result. Given a group A , we denote by $\mathcal{C}(A)$ the class generated by A under direct limits and isomorphisms, and by $\overline{\mathcal{C}(A)}$ the class generated by A under direct limits, extensions, and isomorphisms. The elements of $\mathcal{C}(A)$ are usually called A -cellular groups, and the elements of $\overline{\mathcal{C}(A)}$ are called A -acyclic groups. Then:

Theorem 3.12. *For any group A , there exists a group B such that $\overline{\mathcal{C}(A)} = \mathcal{C}(B)$.*

Moreover, we are able to describe B in terms of groups of the class $\overline{\mathcal{C}(A)}$. This statement allows us to generalize results of [34], where the case in which the Schur multiplier of A is trivial was studied, and also of [3] for the case of groups with trivial ordinary homology. It can also be seen as the group-theoretic version of the main theorem of [16], and it is in the spirit of the generators of radicals described in [11]; see Section 5. We remark that the proof of [16] cannot be translated to the category of groups, because the simplicial structure of the category of spaces is crucial in it.

Section 4 is devoted to establish relations between the closed classes under the operations mentioned above. The hierarchy is the following:

$$\begin{array}{ccc} \mathcal{C}(A) & \hookrightarrow & \overline{\mathcal{C}(A)} \\ \downarrow & & \downarrow \\ \mathcal{C}_q(A) & \hookrightarrow & \mathcal{C}_t(A) \end{array}$$

where $\mathcal{C}_q(A)$ is the closure of $\mathcal{C}(A)$ under quotients and $\mathcal{C}_t(A)$ is the closure of $\mathcal{C}(A)$ under quotients and extensions. All these classes turn out to be co-reflective. Furthermore, we find an example where the four classes are distinct (see Example 4.14).

The co-reflections associated to the classes $\mathcal{C}(A)$, $\mathcal{C}_q(A)$, and $\mathcal{C}_t(A)$ are given respectively by the A -cellular cover C_A , the A -socle S_A , and the A -radical T_A , which are well-known notions in Group/Module Theory (see definitions in Section 2). However, for the class $\overline{\mathcal{C}(A)}$, the existence of the corresponding co-reflection D_A had been previously established ([34]) only for groups for which there exists a two-dimensional Moore space, and this goal was achieved only by topological methods. This gap in the theory is filled in this paper making use of Theorem 3.12, that guarantees the existence of D_A for any group A . Roughly speaking, the philosophy is the following: given a group G , a singly-generated class \mathcal{C} , and the co-reflection F associated to a certain operation, the kernel and the cokernel of the homomorphism $F(G) \rightarrow G$ measure how far G is from belonging to the class generated from \mathcal{C} under the operation. We prove how the co-reflections define the corresponding closed classes, describe when possible the kernels, and discuss, for different instances of the generator, the (in)equality between the corresponding classes. In particular, we prove the following:

Proposition 4.17. *Suppose that for a group A the class $\mathcal{C}(A)$ is closed under extensions, i.e. $\mathcal{C}(A) = \overline{\mathcal{C}(A)}$. Then $\mathcal{C}_q(A) = \mathcal{C}_t(A)$.*

The converse of this result (stated as Question 4.19) is not true in general. Section 5 is devoted to give a counterexample that involves the Burnside idempotent radical and the Burnside group $B(2, q)$. It is also an interesting question to find groups for which this converse is true. In fact, we are not even aware of an example of an abelian group A such that $\mathcal{C}_q(A) = \mathcal{C}_t(A)$, but $\mathcal{C}(A) \neq \overline{\mathcal{C}(A)}$ (or equivalently $S_A = T_A$ and $C_A \neq D_A$); see Proposition 4.16 and the subsequent discussion.

All the classes that appear in this paper will be supposed to be closed under isomorphisms. Moreover, the axioms of ZFC will be assumed throughout the paper.

2. Co-reflections

2.1. Background on co-reflections. In this section we introduce the main notions that will be used throughout the rest of our paper. We will start with the classical categorical definition of *co-reflection*, that will be crucial for us.

Definition 2.1. Given an inclusion of categories $\mathbf{C}' \subseteq \mathbf{C}$, where \mathbf{C}' is full in \mathbf{C} , \mathbf{C}' is *co-reflective* in \mathbf{C} if the inclusion functor possesses a right adjoint (F, μ) , called *co-reflection*.

A good survey about the topic from the categorical point of view can be found in [1]. We recall here some basic facts and notations that will be used later.

- (1) Given a co-reflective subcategory $\mathbf{C}' \subseteq \mathbf{C}$, the associated co-reflection $F: \mathbf{C} \rightarrow \mathbf{C}'$ is unique up to natural isomorphism.
- (2) Objects X satisfying $FX \cong X$ are called *F-colocal*, and the morphisms $X \rightarrow Y$ inducing isomorphisms $FX \rightarrow FY$ are called *F-colocal equivalences*.
- (3) An object Z is *F-colocal* if and only if $\mathbf{C}(Z, X) \cong \mathbf{C}(Z, Y)$ for all *F-colocal* equivalences $X \rightarrow Y$, and also, a morphism $X \rightarrow Y$ is an *F-colocal* equivalence if and only if $\mathbf{C}(Z, X) \cong \mathbf{C}(Z, Y)$ of all *F-colocal* objects Z . In particular, $\eta: FX \rightarrow X$ is an *F-colocal* equivalence, and FX is *F-colocal*.
- (4) An inclusion $\mathbf{C}_1 \subseteq \mathbf{C}_2$ of two co-reflective subcategories induces a natural transformation $\tau: F_1 \rightarrow F_2$ between the corresponding co-reflections, and viceversa. Furthermore, τ is unique up to natural equivalence.

It is well-known that co-reflections preserve finite products in the categories of modules and spaces. Next we prove an analogous statement for arbitrary groups, as we are not aware of a proof of it in the literature (see [19, Lemma 2.3] for a proof in a concrete instance).

Proposition 2.2. *Let $F: \text{Groups} \rightarrow \mathcal{C}$ be a co-reflection in the category of groups, with $\mu_G: FG \rightarrow G$ its counit. Then, $F(G \times H) \cong FG \times FH$.*

Proof: By (3) above, we have that $\mu_G \times \mu_H: FG \times FH \rightarrow G \times H$ is an *F-colocal* equivalence, as $\text{Hom}(Z, FG \times FH) \simeq \text{Hom}(Z, FG) \times \text{Hom}(Z, FH) \simeq \text{Hom}(Z, G) \times \text{Hom}(Z, H) \simeq \text{Hom}(Z, G \times H)$, for all *F-colocal* groups Z , and the composition is induced by $\mu_G \times \mu_H$. To prove that $FG \times FH$ is *F-colocal*, set an ordering $FH = \{x_i : i < \alpha\}$ for some ordinal α , and consider the linear sequence

$$FG * FH \xrightarrow{f_0} FG * FH \xrightarrow{f_1} \dots \longrightarrow FG * FH \xrightarrow{f_i} FG * FH \xrightarrow{f_{i+1}} \dots,$$

where $f_i(g) = x_i g x_i^{-1}$ for $g \in FG$, and $f_i(h) = h$ for $h \in FH$. The colimit is isomorphic to the product $FG \times FH$. Thus $FG \times FH$ is a direct limit of *F-colocal* groups, and hence also *F-colocal*. Uniqueness, up to isomorphism, yields $F(G \times H) \cong FG \times FH$. □

This property will be useful in particular in Example 4.14.

2.2. Group-theoretic constructions. The following are well-known examples of group-theoretic constructions. As we shall see in Section 4, they give rise to different co-reflections over interesting (co-reflective) subcategories of the category of groups.

Definition 2.3. Let A and H be groups.

- (1) The A -socle $S_A H$ is the subgroup of H generated by the images of all the homomorphisms $A \rightarrow H$.
- (2) The A -radical $T_A H$ of H is the normal subgroup of H characterized by the following property: $H/T_A H$ is the largest quotient of H such that $\text{Hom}(A, H/T_A H) = 0$. Groups G that satisfy $\text{Hom}(A, G) = 0$ are called A -reduced (or A -null). The projection $\varphi: H \twoheadrightarrow H/T_A H$ is called the A -reduction (or A -nullification), where $H/T_A H$ is A -reduced and φ induces a bijection $\text{Hom}(H/T_A H, G) \simeq \text{Hom}(H, G)$, for all A -reduced groups G .
- (3) The A -cellular cover of H is the unique group $C_A H \in \mathcal{C}(A)$, up to isomorphism, that can be constructed from copies of A by iterated direct limits, and such that there exists a homomorphism $C_A H \rightarrow H$ which induces a bijection $\text{Hom}(A, C_A H) \simeq \text{Hom}(A, H)$ (an A -equivalence). Observe that this is equivalent to $\text{Hom}(G, C_A H) \simeq \text{Hom}(G, H)$ being a bijection for every $G \in \mathcal{C}(A)$.

Remark 2.4. Note that similar definitions are possible if we change A by a collection \mathcal{A} of groups indexed over a certain set I . In this case, the generator would be the free product of isomorphism classes of elements of \mathcal{A} . Moreover, it is possible to give versions of the socle and the radical relative to a class of epimorphisms of groups. See [1] for a general treatment of these topics. On the other hand, the A -reduction is a particular example of a localization functor L_f with respect to the constant homomorphism $A \rightarrow \{1\}$ (see for example [9] for background on localization of groups and spaces).

For the convenience of the reader, we recall more constructive versions of the radical and the cellular cover. The A -radical $T_A H$ is the union $T_A(H) = \bigcup_i T^i(H)$ of a chain starting at $T^0(H) = S_A(H)$ and defined inductively by $T^{i+1}(H)/T^i(H) = S_A(H/T^i(H))$, and $T^\lambda(H) = \bigcup_\alpha T^\alpha(H)$ if λ is a limit ordinal. This process yields an A -reduced group $H/T_A H$. Furthermore, if $\varphi: H \rightarrow G$ is any homomorphism with G A -reduced, then it factors through $H \rightarrow H/T_A H \rightarrow G$. To see this observe that, if $S_A H \rightarrow H \rightarrow G$ is trivial, φ factors through $H \rightarrow H/S_A(H) \rightarrow G$. Inductively, if $i + 1$ is a successor ordinal, since $S_A(H/T^i H) \rightarrow H \rightarrow G$ is trivial, then φ factors through $H/T^{i+1} H \rightarrow G$. If λ is an ordinal limit, $H/T^\alpha H \rightarrow G$ for $\alpha < \lambda$, and this yields $H/T^\lambda H \rightarrow G$.

In turn, the A -cellular cover $C_A H$ is constructed in Theorem 1 of [15] in the following explicit way. Let A and H be two groups, $C_0^H = \star_{A \rightarrow H} H$ the free product indexed over all homomorphisms $A \rightarrow H$, $\text{ev}: C_0^H \rightarrow H$ the evaluation homomorphism, and K_0^H the kernel of the evaluation. Then,

$$C_A^H(H) \cong (C_0^H/[C_0^H, K_0^H])/T_A(K_0^H/[C_0^H, K_0^H]).$$

An inductive description of $C_A H$ can be found in Section 3 of [34].

Next we will check that previous constructions are functorial, and then they provide an important source of co-reflections (see Section 4):

Proposition 2.5. *Given a group A , the A -socle, the A -radical, and the A -cellular cover define functors in the category of groups.*

Proof: Consider a group homomorphism $f: G \rightarrow H$. To check functoriality we should verify that f restricts to the corresponding A -socles, $f: S_A G \rightarrow S_A H$. Let x be an element of $S_A G$. Then there exist a finite set of homomorphisms $\phi_i: A \rightarrow G$, $i \leq n$, and elements $x_i \in A$ such that $x = \phi_1(x_1) \cdots \phi_n(x_n)$. Hence, $f(x) = (f \circ \phi_1(x_1)) \cdots (f \circ \phi_n(x_n))$, which implies $f(x) \in S_A H$.

By induction along the increasing chain $T_A H = \bigcup_i T^i H$, one can prove functoriality of all the T^i and hence of T_A . Let $f: G \rightarrow H$ be a group homomorphism. If $i + 1$ is a successor ordinal, and assuming that f restricts to $T^i H \rightarrow T^i G$, we obtain that f induces a homomorphism on the quotients $H/T^i H \rightarrow G/T^i G$. Now the functoriality of S_G yields the restriction $S_A(H/T^i H) \rightarrow S_A(G/T^i G)$, and therefore the restriction $T^{i+1} H \rightarrow T^{i+1} G$. For a limit ordinal λ , if f restricts to $T^\alpha H \rightarrow T^\alpha G$ for each $\alpha < \lambda$, then it can be extended to $T^\lambda H \rightarrow T^\lambda G$. Note that the previous argument also yields the functoriality of the quotients $H \rightarrow H/T^i H$ and hence that of the A -reduction homomorphism $H \rightarrow H/T_A H$.

To check functoriality of the A -cellular cover, consider a homomorphism $f: H \rightarrow H'$. Following the notation of the previous construction of $C_A H$, f induces a homomorphism $\tilde{f}: C_0^H \rightarrow C_0^{H'}$ and a commutative diagram

$$\begin{array}{ccccc} K_0^H & \longrightarrow & C_0^H & \longrightarrow & H \\ \downarrow & & \downarrow \tilde{f} & & \downarrow \\ K_0^{H'} & \longrightarrow & C_0^{H'} & \longrightarrow & H' \end{array}$$

whose rows are group extensions. Now, as $\tilde{f}(K_0^H) \subseteq K_0^{H'}$, it is clear that \tilde{f} (and then f) induces a homomorphism $C_0^H/[C_0^H, K_0^H] \rightarrow C_0^{H'}/[C_0^{H'}, K_0^{H'}]$. Now the result follows from the functoriality of the radical, taking account of the previous description of $C_A(H)$. □

Remark 2.6. A thorough analysis of cellular covers and their functorial behaviour is undertaken in Section 2 of [5].

Concerning cellular covers, a different characterization was obtained by J. Scherer and the second author:

Theorem 2.7 ([34, Theorem 3.7]). *Given groups A and H , there is a central extension*

$$(2.1) \quad K \hookrightarrow C_A(H) \twoheadrightarrow S_A(H)$$

such that $\text{Hom}(A_{\text{ab}}, K) = 0$ and the natural map $\text{Hom}(A, H) \rightarrow H^2(A; K)$ is trivial. Moreover, this extension is initial with respect to all the extensions with these two properties.

2.3. Covers of spaces. We will also need to review some constructions in the category of *spaces*. Analogously to groups, given a pointed space M , the *cellular class* $\mathcal{C}(M)$ generated by M is the smallest class that contains M and is closed under homotopy equivalences and pointed homotopy colimits. The corresponding augmentation $CW_M X \rightarrow X$ for a pointed space X , called the *M -cellular cover* of X , was explicitly described by Chachólski in his thesis [13] in the following way: consider the homotopy cofiber sequence

$$(2.2) \quad \bigvee_{[M, X]_*} M \xrightarrow{\text{ev}} X \longrightarrow \text{Cof}_M X,$$

where ev is the evaluation map. Denote by P_M the M -nullification functor, i.e. the homotopical localization with respect to the constant map $M \rightarrow *$ (see [18, Chapter 1]). Then, for any connected space X , $CW_M X$ fits into the following homotopy fiber sequence

$$(2.3) \quad CW_M X \longrightarrow X \longrightarrow P_{\Sigma M}(\text{Cof}_M X).$$

In [34, Theorem 3.3] the relation between cellular covers of groups and spaces was described: if M is a 2-dimensional complex with $\pi_1 M = A$ and X is an Eilenberg–MacLane space $K(H, 1)$, then $\pi_1 CW_M X = C_A H$ for every group H .

3. Generating classes of groups

Given a group A , the class $\overline{\mathcal{C}(A)}$ is the smallest class that contains A and is closed under direct limits and extensions. The elements of this class will be called *A -acyclic* groups. The main goal of this section will be to show that such a class is always singly-generated using only direct limits or, in other words, that this class possesses a *cellular generator*:

Definition 3.1. Given a class \mathcal{C} of groups closed under direct limits, a *cellular generator* for the class \mathcal{C} is a group B such that $\mathcal{C} = \mathcal{C}(B)$.

In our context, \mathcal{C} will usually be the class $\overline{\mathcal{C}(A)}$ for some group A . These cellular generators are not unique and, in fact, several models for particular instances of A have been proposed in the literature ([3], [34]). We now present a general construction, which is inspired in the methodology of Chachólski–Parent–Stanley [16], that solved successfully the same problem in the category of topological spaces. Concretely, we adopt to the group-theoretic context their idea of small space, and follow the general scheme of their proof; however, it should be remarked that the translation is highly nontrivial, as their methods are homotopical in nature and rely in the simplicial structure of space, and we deal with a completely different context.

We start with a set-theoretic definition:

Definition 3.2. For a given infinite regular uncountable cardinal λ a group H is called λ -small (or small with respect to λ) if $\#H < \lambda$.

In the sequel, unless stated otherwise, we will assume that λ is regular infinite.

Next we show the easiest example of λ -smallness.

Example 3.3. For $\lambda = \aleph_1$, λ -small means finite or countably infinite.

Using the previous definition, we can define the class $\mathcal{C}[\lambda]$.

Definition 3.4. Given a class \mathcal{C} of groups closed under direct limits, we denote by $\mathcal{C}[\lambda]$ the subclass of \mathcal{C} defined by the following property: H belongs to $\mathcal{C}[\lambda]$ if for each λ -small group K , every group homomorphism $K \rightarrow H$ factors as $K \rightarrow C \rightarrow H$ for some λ -small group C in \mathcal{C} .

Note that the definition of $\mathcal{C}[\lambda]$ only depends on \mathcal{C} and the cardinal λ . We remark that the group C depends on the homomorphism $K \rightarrow H$ (and in particular depends on K and H).

The following characterization says that we can restrict ourselves to subgroups of H .

Proposition 3.5. *Let λ be an infinite regular cardinal. Then, a group H is in $\mathcal{C}[\lambda]$ if and only if for each λ -small subgroup $S < H$, there exists a λ -small subgroup C in \mathcal{C} such that $S < C < H$.*

Proof: Let $K \rightarrow H$ be any group homomorphism with K λ -small. The image $\varphi(K)$ is a λ -small subgroup of H . Assuming the necessary condition, there exists a λ -small subgroup $C \in \mathcal{C}$ such that $\varphi(K) < C < H$. Hence, $K \rightarrow \varphi(K) \rightarrow C \rightarrow H$ is the desired factorization.

For the converse, suppose that S is any λ -small subgroup of H , and let $S \rightarrow C_0 \xrightarrow{\phi_0} H$ be a factorization of the inclusion $S \hookrightarrow H$, where C_0 is λ -small and belongs to \mathcal{C} . Then $\phi_0(C_0) < H$ is λ -small and $S \leq \phi_0(C_0)$. Denote $K_0 = \phi_0(C_0)$. If $S = K_0$, then S is a retract of C_0 , hence it belongs to \mathcal{C} . Therefore, we can take $C = S$ and we are done. If $S < K_0$, we can proceed in the same way with the inclusion $K_0 < H$, obtaining a factorization $K_0 \rightarrow C_1 \xrightarrow{\phi_1} H$. Iterating the process if necessary, we obtain a string of homomorphisms such that for every i there is a strict inclusion $K_i \lesssim K_{i+1}$:

$$S \hookrightarrow C_0 \twoheadrightarrow K_0 \hookrightarrow C_1 \twoheadrightarrow K_1 \hookrightarrow C_2 \twoheadrightarrow K_2 \hookrightarrow \dots,$$

where C_i is λ -small in \mathcal{C} for every i . Passing to the countable limit we get that the composition

$$\operatorname{colim}_{i < \omega} K_i \twoheadrightarrow \operatorname{colim}_{i < \omega} C_{i+1} \twoheadrightarrow \operatorname{colim}_{i < \omega} K_{i+1}$$

is the identity, so $\operatorname{colim} K_i$ is a retract of $\operatorname{colim} C_i$, hence it belongs to \mathcal{C} . Thus, $C = \operatorname{colim} K_i$ is the desired subgroup of H . □

The following proposition determines the scope of the class $\mathcal{C}[\lambda]$.

Proposition 3.6. *Let λ be an infinite regular cardinal. A group H is in $\mathcal{C}[\lambda]$ if and only if it is isomorphic to a direct limit of λ -small groups in \mathcal{C} .*

Proof: By the previous proposition, for each λ -small subgroup S of H there exists a λ -small subgroup in \mathcal{C} such that $S < C < H$. Hence, H is isomorphic to the direct system of all λ -small subgroups of H that belong to \mathcal{C} .

Conversely, let H be a direct limit of λ -small groups in \mathcal{C} and let us prove that H belongs to $\mathcal{C}[\lambda]$. Following [12, 4.3], and taking into account that π_1 commutes with colimits, every direct limit of groups can be obtained as an iteration of limits over linear categories (chains) and push-outs, so we need to check the statement in these two cases. Observe that H belongs to \mathcal{C} .

Let H be the direct limit of the λ -small groups $\{G_\alpha\}$ over a linear category \mathcal{I} , i.e. $G_1 \rightarrow G_2 \rightarrow \dots \rightarrow H$, κ -indexed for a certain ordinal κ . As H is a regular cardinal, it makes sense to distinguish between the following two cases. If \mathcal{I} is λ -filtered, a standard compactness argument states that every homomorphism $K \rightarrow H$ from a λ -small group factors through G_α for a certain α . On the other hand, if \mathcal{I} is not λ -filtered, this means that the cardinality of κ is smaller than λ , and then the diagram

is λ -small. As H is a quotient of the free product of the G_i indexed over a λ -small cardinal, the definition of regularity implies that H is λ -small (this is also a consequence of Proposition 1.16 in [1]). Hence, as H is also in \mathcal{C} , the factorization of every homomorphism $K \rightarrow H$ through the identity of H proves that H belongs to $\mathcal{C}[\lambda]$.

Now if H is a push-out of a diagram of groups $C \leftarrow A \rightarrow B$ with $A, B,$ and C λ -small groups in \mathcal{C} , H is in particular a quotient of the free product $B * C$, and thus $|H| \leq |B * C|$. If B and C are finite, then $|B * C| = \aleph_0 < \lambda$, and if not, $|B * C| = \max(|B|, |C|) < \lambda$, because $|B|$ and $|C|$ are λ -small. So we obtain the H is λ -small. As H also belongs to \mathcal{C} , we are done. \square

Now we can establish closure properties of $\mathcal{C}[\lambda]$. The first is a straightforward consequence of the previous result:

Corollary 3.7. *If \mathcal{C} is closed under direct limits, then so is $\mathcal{C}[\lambda]$ for every infinite regular cardinal λ .*

Proof: Let H be a direct limit of groups in $\mathcal{C}[\lambda]$. By the previous proposition, H is a direct limit of λ -small groups in \mathcal{C} , and every direct limit of λ -small groups in \mathcal{C} belongs to $\mathcal{C}[\lambda]$, so we are done. \square

Observe that the proof establishes that the λ -small groups in \mathcal{C} generate $\mathcal{C}[\lambda]$ under colimits.

The following result is the group-theoretical analogue of [16, Theorem 6.1], although the proof is very different and purely group-theoretical.

Proposition 3.8. *If \mathcal{C} is any class of groups closed under direct limits and extensions, then $\mathcal{C}[\lambda]$ is also closed under extensions.*

Proof: The idea throughout the proof is to find λ -small groups that are in \mathcal{C} from the starting short exact sequence. In the end, we will get into (non exact) sequences of small groups in \mathcal{C} , take normal closures to make them exact, and pass to the limit to get the desired factorization.

Let $1 \rightarrow X \rightarrow Y \xrightarrow{\pi} Z \rightarrow 1$ be a short exact sequence with X and Z in $\mathcal{C}[\lambda]$. We have to prove that $Y \in \mathcal{C}[\lambda]$. Let $j: K \rightarrow Y$ be any group homomorphism, with K λ -small. Since $Z \in \mathcal{C}[\lambda]$, πj factors through some λ -small group Z_0 in \mathcal{C} . Taking the pullback along $Z_0 \rightarrow Z$ we obtain a short exact sequence $1 \rightarrow X \cong \text{Ker } \pi_0 \rightarrow Y_0 \xrightarrow{\pi_0} Z_0 \rightarrow 1$. Let $j_0: K \rightarrow Y_0$ be the induced homomorphism and consider K_0 the subgroup of Y_0 generated by $j_0(K)$ and $s(Z_0)$, where s is a chosen section (non necessarily a homomorphism) of π_0 . Let K' be the kernel of $K_0 \rightarrow Z_0$. Clearly, $K' = K_0 \cap X$ and it is λ -small because K_0 is λ -small. Since

$X \in \mathcal{C}[\lambda]$, the inclusion $K' \hookrightarrow X$ factors as $K' \hookrightarrow X_0 \rightarrow X$ for some small group $X_0 \in \mathcal{C}$.

$$\begin{array}{ccccccc}
 & & K' & \longrightarrow & X_0 & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 K & \longrightarrow & K_0 & \longrightarrow & Y_0 & \longrightarrow & Y \\
 & & \downarrow & & \downarrow \scriptstyle \pi_0 & & \downarrow \\
 & & Z_0 & \xlongequal{\quad} & Z_0 & \longrightarrow & Z
 \end{array}$$

Let now P_0 be the push-out of $K_0 \leftarrow K' \rightarrow X_0$, and denote by $(X_0)^N$ the normal closure of X_0 in P_0 . The quotient $P_0/(X_0)^N$ is isomorphic to Z_0 , and P_0 is a λ -small group in \mathcal{C} , because both K' and X_0 are λ -small. Observe that $(X_0)^N$ need not be in \mathcal{C} . In any case, it is λ -small and therefore the inclusion $(X_0)^N \rightarrow X$ factors through some λ -small group $X_1 \in \mathcal{C}$. Iterating this process we get an increasing chain of short sequences over $X \rightarrow Y \rightarrow Z$ which are, alternatively, in \mathcal{C} but not exact $(X_i \rightarrow P_i \rightarrow Z_0)$, and exact but not in \mathcal{C} $(X_i^N \rightarrow P_i \rightarrow Z_0)$. Here the subscript “ ∞ ” denotes the (countable) colimit of each linear diagram of homomorphisms:

$$\begin{array}{cccccccccccc}
 X_0 & \rightarrow & (X_0)^N & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_i & \rightarrow & (X_i)^N & \rightarrow & \cdots & \rightarrow & X_\infty = (X_\infty)^N \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 P_0 & \longrightarrow & P_0 & \longrightarrow & P_1 & \longrightarrow & \cdots & \longrightarrow & P_i & \longrightarrow & P_i & \longrightarrow & \cdots & \longrightarrow & P_\infty \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 Z_0 & \longrightarrow & Z_0 & \longrightarrow & Z_0 & \longrightarrow & \cdots & \longrightarrow & Z_0 & \longrightarrow & Z_0 & \longrightarrow & \cdots & \longrightarrow & Z_0
 \end{array}$$

In the limit we get a short exact sequence $1 \rightarrow X_\infty = (X_\infty)^N \rightarrow P_\infty \rightarrow Z_0 \rightarrow 1$, where X_∞ is λ -small and is in \mathcal{C} . As the linear colimit is indexed over a countable ordinal and the cardinal of P_i is λ -small for every i , regularity of λ implies that P_∞ is λ -small (compare with the proof of the “if” implication in Proposition 3.6). Moreover, P_∞ is also in \mathcal{C} because \mathcal{C} is closed under extensions. Hence, $K \rightarrow Y$ factors through $K \rightarrow P_\infty \rightarrow Y$ and this shows that $\mathcal{C}[\lambda]$ is closed under extensions. \square

Remark 3.9. Observe that the hypothesis of \mathcal{C} being closed under direct limits can be weakened to being closed under linear direct limits, but we preferred the current statement for simplicity.

Definition 3.10. Given a class \mathcal{C} closed under direct limits and an infinite regular cardinal λ , let B_λ the free product of a set of representatives of isomorphism classes of λ -small groups in \mathcal{C} .

We will need the following result for the class $\mathcal{C} = \overline{\mathcal{C}(A)}$:

Lemma 3.11. *Let \mathcal{C} be a class of groups closed under direct limits. For every infinite regular cardinal, $\mathcal{C}[\lambda] = \mathcal{C}(B_\lambda)$.*

Proof: If K is an element of $\mathcal{C}[\lambda]$, it is by Proposition 3.6 a direct limit of λ -small groups in \mathcal{C} . But every such λ -small group is a retract of B_λ , so it belongs to $\mathcal{C}(B_\lambda)$, and hence so does K since $\mathcal{C}(B_\lambda)$ is closed under direct limits. Conversely, every group H in $\mathcal{C}(B_\lambda)$ is an (iterated) direct limit of copies of B_λ , and in particular a direct limit of λ -small groups, so $H \in \mathcal{C}[\lambda]$ and we are done. \square

Now we are ready to prove the main theorem in this section:

Theorem 3.12. *For any group A , there exists a group B such that $\overline{\mathcal{C}(A)} = \mathcal{C}(B)$.*

Proof: Let A be any group and λ an infinite regular limit cardinal whose cofinality is bigger than $\#A$. Then, by Lemma 3.11, we have $\overline{\mathcal{C}(A)}[\lambda] = \mathcal{C}(B)$, where $B = B_\lambda$. In particular, the class $\overline{\mathcal{C}(A)}[\lambda]$ is closed under direct limits. The conditions over λ imply that A is λ -small, and hence $\mathcal{C}(A) \subseteq \overline{\mathcal{C}(A)}[\lambda]$. But by Proposition 3.8, $\overline{\mathcal{C}(A)}[\lambda]$ is closed under extensions, so $\overline{\mathcal{C}(A)} \subseteq \overline{\mathcal{C}(A)}[\lambda]$. On the other hand, $\overline{\mathcal{C}(A)}[\lambda]$ is a subclass of $\overline{\mathcal{C}(A)}$ by definition, and the result follows. \square

It is likely that the assumptions over the cardinal may be relaxed, but this doesn't affect the existence of the cellular generator.

In the next section we will use different cellular generators for concrete computations and in particular, and perhaps more importantly, to define in general the co-reflection associated to the class $\overline{\mathcal{C}(A)}$ for every group A .

4. Closed classes

4.1. Classes of groups and associated co-reflections. Let A be a group. The goal of this section is to analyze the relations between four different classes that contain the group A , namely:

- The *cellular* class $\mathcal{C}(A)$, the closure of $\{A\}$ under direct limits. Their elements are called *A-cellular groups*.
- The *socular* class $\mathcal{C}_q(A)$, the closure of $\{A\}$ under direct limits and quotients. Their elements are called *A-generated* or *A-socular groups*.
- The *radical* class $\mathcal{C}_t(A)$, the closure of $\{A\}$ under direct limits, extensions, and quotients. Their elements are called *A-constructible* or *A-radical groups*.
- The *acyclic* class $\overline{\mathcal{C}(A)}$, the closure of $\{A\}$ under direct limits and extensions. Their elements are called *A-acyclic groups*.

The hierarchy between these classes can be depicted in the following way:

$$(4.1) \quad \begin{array}{ccc} \mathcal{C}(A) & \hookrightarrow & \overline{\mathcal{C}(A)} \\ \downarrow & & \downarrow \\ \mathcal{C}_q(A) & \hookrightarrow & \mathcal{C}_t(A) \end{array}$$

We use in particular the terminology of W. Chachólski, who studied the problem in the non singly-generated case in [14].

It is known that a group G is A -cellular if and only if $C_A G = G$ ([34, Section 3]). We will check now that similar statements hold for A -generated and A -constructible groups. Probably this result is known to experts, but we do not know of any explicit proof of it. We offer purely algebraic arguments here (see also [33, Theorem 6] for part (ii) below).

Proposition 4.1. *Let A and G be groups. Then:*

- (i) G is A -generated if and only if $S_A G = G$.
- (ii) G is A -constructible if and only if $T_A G = G$.

Proof: Let \mathcal{C} be the class of groups G such that $S_A G = G$. Clearly $A \in \mathcal{C}$. Suppose that $H \in \mathcal{C}$ and $\pi: H \rightarrow G$ is an epimorphism. If $\pi(x) \in G/N$, as $S_A G = G$, there exists $\varphi_i: A \rightarrow G$ and $a_i \in A$ such that $x = \varphi_1(a_1)\varphi_2(a_2) \cdots \varphi_n(a_n)$. Therefore $\pi(x) = \pi\varphi_1(a_1)\pi\varphi_2(a_2) \cdots \pi\varphi_n(a_n)$. Hence $\pi(x) \in S_A(G/N)$, and this ensures that \mathcal{C} is closed under quotients. A similar argument shows that \mathcal{C} is closed under direct limits, and hence $\mathcal{C}_q(A) \subset \mathcal{C}$. Conversely, if $S_A(G) = G$, then G is a quotient of a free product of copies of A , so $G \in \mathcal{C}_q(A)$. Thus $\mathcal{C} \subset \mathcal{C}_q(A)$, and we get $\mathcal{C}_q(A) = \mathcal{C}$.

Let us check the second statement. Consider now the class \mathcal{C} of groups such that $G = T_A G$. Clearly A belongs to \mathcal{C} , so let us see that \mathcal{C} is closed under quotients and extensions. First, for every homomorphism of groups $f: H \rightarrow G$ we have $f(T_A H) \subseteq T_A G$. Then, if $H = T_A H$ and G is a quotient of H , $G = T_A G$, and hence G belongs to \mathcal{C} . On the other hand, assume that G is an extension $G_1 \rightarrow G \rightarrow G_2$, with G_1 and G_2 in \mathcal{C} . As $G_1 = T_A G_1$, we have $G_1 \subseteq T_A G$. Hence, there is a projection $G_2 \rightarrow G/T_A G$. But $G_2 = T_A G_2$ and $T_A(G/T_A G) = 0$, and thus $G/T_A G = 0$ and $G = T_A G$. We obtain that \mathcal{C} is also closed under extensions, and then $\mathcal{C}_t(A) \subseteq \mathcal{C}$.

Now assume $G = T_A G$. Then G is constructed out of A by means of A -socles, extensions, and a direct union. Thus, by the previous case and the fact that a direct union is a direct limit, G belongs to the closure of A by direct limits, quotients, and extensions. So $\mathcal{C} \subseteq \mathcal{C}_t(A)$, and we are done. □

The previous result provides co-reflections for the respective classes.

Corollary 4.2. *In the previous notation, the categories $\mathcal{C}(A)$, $\mathcal{C}_q(A)$, and $\mathcal{C}_t(A)$ are co-reflective. The co-reflections are respectively given by the A -cellular cover, the A -socle, and the A -radical. \square*

Remark 4.3. Observe that the arguments of the proof of the previous proposition are algebraic and quite formal. Hence, they are supposed to work in more general categories, such as for example categories of modules.

The next question is immediate: given a group A , is the class $\overline{\mathcal{C}(A)}$ co-reflective? In order to undertake this problem, we introduce one of the main definitions in this context:

Definition 4.4. Given a group A and the acyclic class $\overline{\mathcal{C}(A)}$, the co-reflection associated to this class (if it exists) will always be denoted by D_A . For a group G , $D_A G$ will be called the A -acyclic cover of G .

We will see in this section that Theorem 3.12 gives a general answer to the previous question, and in particular assures the existence of D_A for every group A . Before establishing this result, it is worth to point out the partial solution to the question obtained in [34], where the authors assumed the existence of a 2-dimensional Moore space for A in order to construct the co-reflection. Recall that given a group A , a *Moore space* of type $M(A, 1)$ is a CW-complex X such that $\pi_1 X = A$ and $H_i X = 0$ for $i > 1$.

Theorem 4.5 ([34, Theorem 4.3]). *Let A be the fundamental group of a two-dimensional Moore space. Then the inclusion $\overline{\mathcal{C}(A)} \hookrightarrow \text{Groups}$ admits a right adjoint D_A . Furthermore, for each group H , there is a central extension*

$$L \hookrightarrow D_A(H) \twoheadrightarrow T_A(H)$$

such that $\text{Hom}(A_{\text{ab}}, L) = \text{Ext}(A_{\text{ab}}, L) = 0$, and this extension is initial with respect to this property.

This result has interesting features. First, for A in the conditions of the statement, it gives a manageable way to compute the value of the A -acyclic cover of any group H , as it is identified with the fundamental group of the homotopy fiber of the $M(A, 1)$ -nullification of $K(T_A H, 1)$, i.e. localization with respect to the constant map over $M(A, 1)$ (see [18, Chapter 1]). Moreover, the similarity with Theorem 2.7 should be re-

marked, and in fact for every group H there exists a natural homomorphism $C_A H \rightarrow D_A H$ which induces a commutative diagram of extensions:

$$(4.2) \quad \begin{array}{ccccc} K & \longrightarrow & C_A H & \longrightarrow & S_A H \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & D_A H & \longrightarrow & T_A H \end{array}$$

where K and L are the respective kernels and the homomorphism $S_A H \rightarrow T_A H$ is injective. Let $M = M(A, 1)$ be a two-dimensional Moore space for A and $Y = \text{Cof}_M K(S_A H, 1)$ be the corresponding Chachólski homotopy cofiber (2.2). Then, it is stated in the proof of Theorem 3.7 in [34] that $K = \pi_2 P_{\Sigma M} Y$. Analogously, let Y' be $\text{Cof}_M K(T_A H, 1)$. Then, the proof of Theorem 4.3 in the mentioned article proves that $L = \pi_2 P_M Y'$. This result will be useful in Proposition 5.3 below.

We remark that the co-reflection D_A is constructed in Theorem 4.5 in a homotopical (non-algebraic) way. This homotopical nature of the construction made it impossible to define D_A in this way for a general A , as the definition depended on the existence of a two-dimensional Moore space for the group. In the sequel we will avoid the difficulty by defining D_A as a cellular cover in a pure group-theoretical way.

We present now our general construction of the co-reflection D_A for every group A .

Proposition 4.6. *Let A be a group and B a cellular generator of the class $\overline{\mathcal{C}(A)}$. Then a group G is A -acyclic if and only if $C_B G = G$.*

Proof: This is straightforward from Theorem 3.12. □

Corollary 4.7. *For every group A , the class $\overline{\mathcal{C}(A)}$ is co-reflective and the co-reflection D_A is given by C_B .*

Remark 4.8. Observe in particular that, although Theorem 4.5 above proves the existence of D_A for some instances of A and provides a way to compute the acyclic cover, it does not describe a cellular generator of the class.

After establishing the existence of the co-reflection D_A , we conclude this subsection by stating some properties of the co-reflection D_A and the cellular generators of the acyclic class.

Proposition 4.9. *If B and B' are cellular generators of the class $\overline{\mathcal{C}(A)}$ and H is a group, the cellular covers of H with respect to B and B' are isomorphic.*

Proof: As B and B' are cellular generators of the acyclic class, we have $C_B = D_A$ and $C_{B'} = D_A$. □

Lemma 4.10. *Let A be a group and B a cellular generator of $\overline{\mathcal{C}(A)}$. Then $S_B = T_A$.*

Proof: Both S_B and T_A are co-reflections over the class $\mathcal{C}_t(A)$. The result follows from uniqueness of co-reflections. □

Corollary 4.11. *For every group A , we have $D_A = D_A T_A$.*

Proof: It is well-known that $C_A = C_A S_A$ for every group A . Now, if B is a cellular generator of $\mathcal{C}(A)$, $C_B = D_A$ and $S_B = T_A$. So we are done. □

In particular, in the computation of $D_A H$ for a group H it is enough to take as input the A -radical of H .

The following proposition provides concrete descriptions of D_A .

Proposition 4.12. *Let H and G be finite nontrivial groups, with H simple, $|H| \leq |G|$, and such that there is exactly one subgroup of G that is isomorphic to H . Then $D_H G = H$, and G/H does not belong to $\overline{\mathcal{C}(H)}$.*

Proof: As H is simple and normal in G , $S_H G = T_H G = H$. Thus,

$$D_H G = D_H T_H G = D_H H = H,$$

as H is clearly H -acyclic. Moreover, if G/H belonged to $\overline{\mathcal{C}(H)}$, then this would also be the case for G , and that is impossible because $D_H G \neq G$. □

Note that this result gives information for classes $\overline{\mathcal{C}(A)}$ with $H_2 A \neq 0$. For example, if $H = A_n$, $G = \Sigma_n$, and $n \geq 5$, the proposition implies that $D_{A_n} \Sigma_n = A_n$, and hence neither Σ_n nor $\mathbb{Z}/2$ can be constructed out of A_n by direct limits and extensions. The groups for which $H_2 A \neq 0$ are beyond the scope of Theorem 4.5, as the existence of the Moore space implies that $H_2 A$ should be trivial there.

Remark 4.13. We note that the use of the word “acyclic” in our context comes from Homotopy Theory, and especially from Bousfield, who thoroughly studied the classes of spaces for which a homology theory vanishes. We would like to emphasize that the acyclic cover owes its name to the fact that, in the cases studied by Rodríguez–Scherer in [34], it is the fundamental group of the homotopy fibre of a localization; in general this group may have nontrivial ordinary homology, so it is not acyclic in the usual sense of Group Homology.

4.2. Comparing the classes. From now on we will concentrate in measuring the difference between the classes above. Fix a group A , and let F be one of the four co-reflections. Then given a group G , FG measures in general how close the group G is from belonging to

the corresponding class. Moreover, the homomorphisms $S_A G \rightarrow G$, $T_A G \rightarrow G$, and $S_A G \rightarrow T_A G$ are always injective, and it is a consequence of [34, Proposition 3.1] and Corollary 4.7 above that $C_A G \rightarrow S_A G$ and $D_A G \rightarrow T_A G$ are always surjective. Hence, we have a commutative diagram

$$(4.3) \quad \begin{array}{ccc} C_A G & \twoheadrightarrow & S_A G \\ \downarrow & & \downarrow \\ D_A G & \twoheadrightarrow & T_A G \end{array}$$

which generalizes diagram (4.2) above to the case of a general A . Note that in general, $C_A G \rightarrow D_A G$ is neither surjective nor injective.

Using an example, we start by showing that the four inclusions of classes in diagram (4.1) can be strict. Of course, the previous results imply that two such classes are identical if and only if their associated co-reflections are the same.

Example 4.14. Consider G the direct product of the Mathieu group M_{12} with the cyclic group $\mathbb{Z}/9$. We use Proposition 2.2 to compute the value of the co-reflections over this group. As the socle and the radical are always normal subgroups and M_{12} is simple and contains elements of order 3, we have $S_{\mathbb{Z}/3} G = M_{12} \times \mathbb{Z}/3$ and $T_{\mathbb{Z}/3} G = G$.

Moreover, taking into account that $H_2(M_{12}) = \mathbb{Z}/2$ (see for example Section 11 in [5]) we have that $C_{\mathbb{Z}/3} M_{12} = \widetilde{M}_{12}$, the universal cover of M_{12} . Observe that the alternative $C_{\mathbb{Z}/3} M_{12} = \mathbb{Z}/2 \times M_{12}$ is not possible as $\mathbb{Z}/2 \times M_{12}$ is not $\mathbb{Z}/3$ -cellular. Hence, $C_{\mathbb{Z}/3} G = \widetilde{M}_{12} \times \mathbb{Z}/3$. On the other hand, $D_{\mathbb{Z}/3} G = \widetilde{M}_{12} \times \mathbb{Z}/9$, as G is $\mathbb{Z}/3$ -constructible.

Next we will discuss when the socular class $C_q(A)$ is closed under extensions, and hence equal to the radical class $C_t(A)$. For example, if $A = \mathbb{Z}/p$ for some prime p , it is known that there are many groups G such that $G/S_{\mathbb{Z}/p} G$ has again p -torsion, and the same happens if we change \mathbb{Z}/p by $\ast_{j \geq 1} \mathbb{Z}/p^j$. This contrasts with the case of abelian groups, in which for $A = \bigoplus_{j \geq 1} \mathbb{Z}/p^j$, $T_A N = S_A N$ for every abelian group.

When A is the additive group of the rational numbers, the situation is similar. If N is abelian, $S_{\mathbb{Q}} N = T_{\mathbb{Q}} N$, because the image of every homomorphism $\mathbb{Q} \rightarrow N$ is divisible and then splits out of N . However, if N is not abelian, the equality is not true in general. We have found no example of this situation in the literature, so we describe one in the sequel:

Example 4.15. Let G be the push-out of inclusions $\mathbb{Q} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$. Following [29, Chapter 5], a presentation of the group $(\mathbb{Q}, +)$ is given by:

$$\langle x_1, x_2, x_3, \dots \mid x_n^n = x_{n-1} \text{ for } n \geq 2 \rangle.$$

If we look at the rationals as a subgroup of the reals, the fraction $\frac{1}{m!}$ corresponds to the generator x_m for every $m \geq 1$.

Analogously, a presentation of $\mathbb{Z}[1/p]$ is:

$$\langle y_1, y_2, y_3, \dots \mid y_n^p = y_{n-1}, \text{ for } n \geq 2 \rangle.$$

Now identifying $\mathbb{Z}[1/p]$ with the irreducible fractions of \mathbb{R} with denominator a power of p , the fraction $\frac{1}{p^m}$ corresponds to y_m .

Using Seifert–van Kampen’s theorem, we obtain that a presentation for the group G is given by:

$$\langle x_1, y_1, x_2, y_2, \dots \mid x_n^n = x_{n-1}, y_n^p = y_{n-1}, x_1 = y_1, \text{ for } n \geq 2 \rangle.$$

Note that the amalgamation is reflected in the relation $x_1 = y_1$.

Consider now the normal subgroup N of G generated by the copy of \mathbb{Q} inside G given by the generators $\{x_i\}$. This group is generated by conjugates of this copy of \mathbb{Q} , and therefore $N < S_{\mathbb{Q}}G$. On the other hand, as G is torsion-free, every homomorphism $\phi: \mathbb{Q} \rightarrow G$ is injective, and then $\text{Im } \phi$ is isomorphic to \mathbb{Q} . Now invoking a result of Moldavanskii ([31]; see also [30, p. 229]), and taking into account that $\mathbb{Q} \not\leq \mathbb{Z}[1/p]$, we obtain that $\text{Im } \phi < N$ for every $\phi: \mathbb{Q} \rightarrow G$, and this already implies that $N = S_{\mathbb{Q}}G$.

Next we will describe the quotient G/N . A presentation of this group is obtained by trivializing the generators $\{x_i\}$ in the previous presentation of G (which in particular implies trivializing y_1); so we get (after a renumbering of generators):

$$\langle y_1, y_2, \dots \mid y_1^p = 1, y_n^p = y_{n-1}, \text{ for } n \geq 2 \rangle.$$

This is a classical presentation of the quasicyclic group \mathbb{Z}/p^∞ , being the generator of every \mathbb{Z}/p^n identified with y_n . As there exists an epimorphism $\mathbb{Q} \rightarrow \mathbb{Z}/p^\infty$, we conclude that $G = T_{\mathbb{Q}}G$, and then $S_{\mathbb{Q}}G$ is strictly contained in $T_{\mathbb{Q}}G$, as claimed. In fact, $G = T_{\mathbb{Q}}G$, as the only nontrivial quotients of the quasicyclic group are isomorphic to it.

Using amalgams $\mathbb{Z}[1/q] \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$, with $q \neq p$, and taking into account that there are always nontrivial homomorphisms $\mathbb{Z}[1/q] \rightarrow \mathbb{Z}/p^\infty$, it can be proved in an analogous way that $S_{\mathbb{Z}[1/q]} \neq T_{\mathbb{Z}[1/q]}$ for every prime q . Moreover, similar arguments can be used to prove that for a nontrivial set of primes J , the co-reflection $S_{\mathbb{Z}[J^{-1}]}$ is never equal to $T_{\mathbb{Z}[J^{-1}]}$. In particular we obtain:

Proposition 4.16. *If A is a nontrivial additive subgroup of \mathbb{Q} , $S_A = T_A$ if and only if $A = \mathbb{Z}$.*

Proof: By the work of Beaumont–Zuckerman [2], the additive subgroups of \mathbb{Q} are the rational subgroups of rank one. As stated in Example 6.2 of [34] (see also [21]), the only nontrivial subgroups A among them for which S_A can be equal to T_A are those of the shape $A = \mathbb{Z}[J^{-1}]$ for a set of primes J . But if J is non-empty and $p \in J$, the same argument of Example 4.15 above proves that the $\mathbb{Z}[J^{-1}]$ -socle of the push-out $\mathbb{Z}[J^{-1}] \leftarrow \mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$ is strictly contained in its $\mathbb{Z}[J^{-1}]$ -radical, which in turn equals the whole group. On the other hand, if $J = \emptyset$, every group coincide with its \mathbb{Z} -socle, and then $S_{\mathbb{Z}} = T_{\mathbb{Z}}$. So we are done. \square

The moral here is that it is not easy to find examples for which $S_A = T_A$. However, there are at least two ways of constructing such examples. The first family arises in the context of varieties of groups. Consider a set of equations and the verbal subgroup W defined by these equations in the free group F_{∞} . It is proved in [11] that for every W there exists a locally free group A such that the homomorphisms of W into any group G identify the W -perfect radical and, in particular, this fact guarantees that $S_A = T_A$. The inspiring (and first) “generator” of this kind was the group constructed by Berrick–Casacuberta in [3], and a closely related group will be the crucial ingredient of Proposition 5.3 below.

The second family of examples can be constructed as a byproduct of Theorem 3.12, taking account of the following two results:

Proposition 4.17. *Suppose that for a group A the class $\mathcal{C}(A)$ is closed under extensions, i.e. $\mathcal{C}(A) = \overline{\mathcal{C}(A)}$. Then $C_q(A) = C_t(A)$.*

Proof: In this case, $C_A G = D_A G$ for every group G , and then there is a diagram

$$\begin{array}{ccc} D_A G & \twoheadrightarrow & S_A G \\ \downarrow \text{Id} & & \downarrow j \\ D_A G & \twoheadrightarrow & T_A G \end{array}$$

As j should be an epimorphism, and since it is always a monomorphism, it is an isomorphism. \square

Corollary 4.18. *Let A be a group. Then for any cellular generator B of the class $\overline{\mathcal{C}(A)}$ we have $S_B = T_B$.*

Recall that by Theorem 3.12, it is always possible to construct such B out of any given A , and this provides many examples of (big) groups for which $S_A = T_A$. It is natural and interesting to ask if there is a converse of the previous proposition, so we consider the following problem:

Question 4.19. *If $T_A = S_A$ for a certain group A , is it true that $\mathcal{C}(A) = \overline{\mathcal{C}(A)}$?*

We will see in the next section that the general answer is negative. Before, we will discuss in which cases such an answer can be positive. The following result is easy to prove and gives a sufficient condition.

Proposition 4.20. *Let A be a group, and assume that $\mathcal{C}(A) = \mathcal{C}_q(A)$ and $S_A = T_A$. Then $\mathcal{C}(A) = \overline{\mathcal{C}(A)}$.*

Proof: As the members of the classes are defined precisely as the groups for which the corresponding co-reflection is the identity, $S_A = T_A$ implies $\mathcal{C}_q(A) = \mathcal{C}_t(A)$, and then by hypothesis $\mathcal{C}(A) = \mathcal{C}_t(A)$. But the inclusions $\mathcal{C}(A) \subseteq \overline{\mathcal{C}(A)} \subseteq \mathcal{C}_t(A)$ always hold, so the result follows. \square

It can be seen that the condition of the proposition holds for the universal group of Berrick–Casacuberta. However, the following example shows that the conditions of the previous proposition are not necessary in general for cellular generators of closures under extensions:

Example 4.21. Given a prime p , consider a cellular generator B of the class $\overline{\mathcal{C}(\mathbb{Z}/p)}$. If $\mathcal{C}(B) = \mathcal{C}_q(B)$, this would imply that $C_B = S_B$, and equivalently $D_{\mathbb{Z}/p} = T_{\mathbb{Z}/p}$. But this is not true in general, as can be deduced for instance from Example 4.14. Observe that we obtain in particular that $\mathcal{C}(B) = \overline{\mathcal{C}(B)} \neq \mathcal{C}_q(B) = \mathcal{C}_t(B)$ in this case.

In general, the relation between the group co-reflections C_A and C_B ($= D_A$) has been investigated when $H_2A = 0$, and it is quite well understood when A is a cyclic group or a subring of \mathbb{Q} (see for example [34]). As seen above, the latter is not useful in our context, as $S_A \neq T_A$ for these groups. However, in the next section, we will take profit of some features of the case $H_2A = 0$ to describe a counterexample for Question 4.19. We emphasize that in this proof we will not need the results of Section 3.

5. The Burnside radical and a free Burnside group

Recall from [11] that the *Burnside idempotent radical* for a prime p of a group G is the largest subgroup $H < G$ which is generated by p -powers of its elements. As is established in [11, Theorem 3.3], given any group G and a prime p , its corresponding Burnside radical can be always generated by the images of homomorphisms of a group F , called the *generator* of the radical. This group will be the key object in the proof of Proposition 5.3 below, so we will compile below some properties of it that will be useful in the sequel, in particular in the homology computations of Lemma 5.2.

Proposition 5.1. *The following statements hold for the group F :*

- (i) F generates the Burnside radical.
- (ii) F is an infinite free product of groups $F_{\mathbf{n}}$, where \mathbf{n} runs over all the non-decreasing sequences of integers.
- (iii) The groups $F_{\mathbf{n}}$ are countable.
- (iv) Every $F_{\mathbf{n}}$ is a directed system of finitely generated free groups $\{F_{\mathbf{n},r}\}$, and in particular is locally free and torsion-free.
- (v) For every \mathbf{n} and $r < s$, $\text{rank } F_{\mathbf{n},r} < \text{rank } F_{\mathbf{n},s}$.
- (vi) The image of every bonding homomorphism $\phi_r: F_{\mathbf{n},r} \rightarrow F_{\mathbf{n},r+1}$ is always contained in a product of p -powers of generators of $F_{\mathbf{n},r+1}$.
- (vii) For every \mathbf{n} and r , there exist distinguished systems $\{x_{\mathbf{n},r}\}$ of generators of $F_{\mathbf{n},r}$, such that for every $i \neq j$, $\phi_r(x_{i,r})$ and $\phi_r(x_{j,r})$ are spanned by disjoint subsets of the system $\{x_{\mathbf{n},r+1}\}$.
- (viii) F is locally free and torsion-free.

Proof: Statement (i) is exactly Theorem 3.3 in [11], and the remaining ones are established in the course of the proof of the mentioned result. We adopt the notation from there. The definition of F as a free product of (ii) appears in the second paragraph of that proof. Just after it, the construction of $F_{\mathbf{n}}$ as a directed system of free groups $F_{\mathbf{n},r}$ is undertaken, with a precise description of the generator of each group and the bonding homomorphisms ϕ . The fact (established in the proof of Theorem 3.3 in [11]) that the groups $F_{\mathbf{n},r}$ are finitely generated implies (iii) and the first part of (iv), and the construction as a directed system of free groups gives locally freeness and torsion-freeness of $F_{\mathbf{n}}$ (rest of statement (iv)). The labeling of the generators in the second paragraph of the proof of the mentioned result implies the inequalities of (v), and the fact that in this case the only nontrivial word that defines the variety is $w = x^p$ gives (vi), by the definition of the product in that paragraph. As all the symbols that appear in the definition of x_r appear again in the generators that span $\phi_r(x_r)$ we have (vii). Finally, (viii) is a straight consequence of (iv), as being locally free is closed under free products. □

Now we are ready to compute the ordinary homology of F .

Lemma 5.2. *The first homology group $H_1(F)$ is a free module over $\mathbb{Z}[1/p]$ of infinite rank, and $H_i F = 0$ if $i > 1$.*

Proof: We first compute the homology of every $F_{\mathbf{n}}$. As these groups are filtered colimits of free groups, $H_i F_{\mathbf{n}} = 0$ if $i > 1$. Moreover, the

construction of $F_{\mathbf{n}}$ in [11] as a telescope of homomorphisms that sends generators to products of p -powers (see statement (vi) of the previous proposition) implies that the first homology group is the direct limit of a system

$$\mathbb{Z} \longrightarrow \bigoplus_{I_1} \mathbb{Z} \longrightarrow \bigoplus_{I_2} \mathbb{Z} \longrightarrow \dots,$$

where:

- Every direct sum is taken over a finite set of indexes (statement (iv) above).
- If $j < k$, then $I_j < I_k$ (statement (v)).
- For every k , the homomorphism $\bigoplus_{I_k} \mathbb{Z} \rightarrow \bigoplus_{I_{k+1}} \mathbb{Z}$ takes the $|I_k|$ components of the leftmost group to the first $|I_k|$ of the rightmost group in their order, and the homomorphism is, up to a change of basis, multiplication by p (statements (vi) and (vii)).

Hence, the direct system that defines $H_1 F_{\mathbf{n}}$ is a countable direct sum of copies of the system $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \dots$ and then this homology group is a free module over $\mathbb{Z}[1/p]$. Finally, writing F as the colimit (under the inclusion) of all the subgroups of the shape $F_{\mathbf{i}_1} * \dots * F_{\mathbf{i}_n}$ and taking the corresponding limit in homology, the result follows. Note that the rank of $H_1(F)$ as a free module over $\mathbb{Z}[1/p]$ is uncountable. \square

Now given a prime $q > 10^{75}$, $q \neq p$, consider the *free Burnside group* $B(2, p)$ in two generators. It is a group of exponent q , and by Corollary 31.2 in [32] the bound $q > 10^{75}$ implies that its second homology group is free abelian in a countable number of generators. In the sequel we will denote $B(2, q)$ by G .

Proposition 5.3. *For the group F above defined, the equality $S_F = T_F$ holds, but $C_F \neq D_F$.*

Proof: As F generates the idempotent radical associated to a variety of groups, $S_F = T_F$. We will check now that $C_F \neq D_F$. As G has exponent q and q is coprime to p , multiplication by p is an isomorphism. Then G is equal to its idempotent Burnside radical, and then $G = S_F G = T_F G$. Hence, to prove that $C_F G \neq D_F G$ it is enough to check that the kernels of $D_F G \rightarrow G$ and $C_F G \rightarrow G$ are not isomorphic.

We first compute the kernel of the epimorphism $D_F G \rightarrow G$. Since F is locally free, there exists a 2-dimensional Moore space $M(F, 1)$ and therefore, we can use Theorem 4.5 directly to compute such kernel. Consider the Chachólski cofibration

$$(5.1) \quad \bigvee_{[M, K(G, 1)]_*} M \longrightarrow K(G, 1) \longrightarrow Y,$$

where Y denotes the homotopy cofiber. We proved just after Theorem 4.5 that the kernel of $D_F G \rightarrow G$ is isomorphic to $\pi_2(P_M Y)$, which in turn is the Ext- p -completion of the group $\pi_2 Y$ in the primes for which F_{ab} is uniquely p -divisible ([7, Theorem 7.5]). By Lemma 5.2, $F_{ab} = H_1 F$ is a $\mathbb{Z}[1/p]$ -free module, and hence we only need to complete in the prime p . Applying Mayer–Vietoris to (5.1) and the fact that $\pi(Y) = H_2(Y)$, as Y is simply-connected, we obtain an extension

$$(5.2) \quad A \hookrightarrow \pi_2 Y \xrightarrow{f} R.$$

Here $A = H_2(G)$ is free abelian in a countable number of generators and R is the kernel of the homomorphism $H_1(\bigvee_{[M, K(G,1)]_*} M) \rightarrow H_1 K(G, 1)$. In this case $R = \bigoplus_J \mathbb{Z}[1/p]$ is a free $\mathbb{Z}[1/p]$ -module for some set J of indexes, because $H_1(\bigvee_{[M, K(G,1)]_*} M)$ is and $\mathbb{Z}[1/p]$ is a principal ideal domain. Taking now account of the fact that $\text{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}) = 0$ and [8, VI.2.5], we obtain another extension:

$$\text{Ext}(\mathbb{Z}/p^\infty, A) \hookrightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_2 Y) \twoheadrightarrow \text{Ext}(\mathbb{Z}/p^\infty, R).$$

As the Ext- p -completion of a free $\mathbb{Z}[1/p]$ -module is trivial, then so is the term on the right-hand side, and the homomorphism on the left-hand side is an isomorphism. Now observe that again by [8, VI.2.5], the extension $\mathbb{Z} \hookrightarrow A \rightarrow R$ given by the inclusion of a factor induces another extension

$$\text{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}) \hookrightarrow \text{Ext}(\mathbb{Z}/p^\infty, A) \twoheadrightarrow \text{Ext}(\mathbb{Z}/p^\infty, R).$$

As $\text{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}) = \mathbb{Z}_p^\wedge$ we obtain that the kernel of $D_F G \rightarrow G$ contains at least a copy of the p -adic integers.

Next we will prove that $C_F G \neq D_F G$. In order to do so, we check that the kernel of $C_F G \rightarrow G$ cannot contain a subgroup isomorphic to \mathbb{Z}_p^\wedge . Again from the argument after Theorem 4.5 we have that this kernel is isomorphic to $\pi_2(P_{\Sigma M} Y)$, and this is the quotient of $\pi_2 Y$ by its F -radical T , which is in turn equal to the $\mathbb{Z}[1/p]$ -radical as $\pi_2 Y$ is abelian ([11, Section 3]).

Let us describe T and $(\pi_2 Y)/T$ more explicitly. For short, denote by $S = S_{\mathbb{Z}[1/p]} \pi_2 Y$, the $\mathbb{Z}[1/p]$ -socle of $\pi_2 Y$.

Consider all the $\mathbb{Z}[1/p]$ -submodules of R of rank one such that the inclusion $\mathbb{Z}[1/p] \hookrightarrow R$ lifts to $\pi_2 Y$. After a change of basis, we may assume that there is a set $I < J$ (that can be empty) such that (a) the module R decomposes as a direct sum $R = R_I \oplus R_{J \setminus I}$, being $R_I = \bigoplus_I \mathbb{Z}[1/p]$ and $R_{J \setminus I} = \bigoplus_{J \setminus I} \mathbb{Z}[1/p]$, and (b) every homomorphism $\mathbb{Z}[1/p] \rightarrow R_I$ lifts to $\pi_2 Y$.

Next we will prove $S \cong R_I$, via the restriction of f in (5.2). First, observe that the elements of S are precisely the elements of $\pi_2 Y$ that are uniquely p -divisible, and every such element is in the image of some homomorphism $\mathbb{Z}[1/p] \rightarrow \pi_2 Y$. Now for every homomorphism $g: \mathbb{Z}[1/p] \rightarrow \pi_2 Y$ the composition $f \circ g$ is again nontrivial; see 3F.11 in [27], for example, taking into account that A does not contain p -divisible elements. In particular, g is a lift of a homomorphism $\mathbb{Z}[1/p] \rightarrow R_I$. Hence, the restriction of f to the socle takes values in R_I . Moreover, if $x \in R_I$, then x is in the image of some homomorphism $\mathbb{Z}[1/p] \rightarrow R_I$, so by lifting this homomorphism we obtain a preimage of x in the socle, and $f|_S$ is then surjective. To check injectivity observe that, fixing a homomorphism $\mathbb{Z}[1/p] \rightarrow R_I$, the lifting to the socle is unique, because again by 3F.11 in [27] the induced homomorphism $\text{Hom}(\mathbb{Z}[1/p], \pi_2 Y) \rightarrow \text{Hom}(\mathbb{Z}[1/p], R_I)$ is injective. Then, the only homomorphism $\mathbb{Z}[1/p] \rightarrow S$ that lifts the trivial one is trivial, and then $f(x) = 0$ implies $x = 0$ for $x \in S$. So we have proved that f maps isomorphically the socle S onto R_I . Now we have a commutative diagram, where rows and columns are exact:

$$\begin{array}{ccccc}
 \{1\} & \longrightarrow & S & \xrightarrow{\cong} & R_I \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \longrightarrow & \pi_2 Y & \xrightarrow{f} & R \\
 \downarrow \cong & & \downarrow & & \downarrow \\
 A & \longrightarrow & (\pi_2 Y)/S & \longrightarrow & R_{J \setminus I}
 \end{array}$$

Observe that $(\pi_2 Y)/S$ has no uniquely p -divisible elements; otherwise, some homomorphism $\mathbb{Z}[1/p] \rightarrow R_{J \setminus I}$ could be lifted to that group and this is not possible by the definition of I . This implies that $S = T$. Now it is immediate that $(\pi_2 Y)/T$ cannot contain a copy of \mathbb{Z}_p^\wedge , taking into account for example that the p -adic integers admit nontrivial homomorphisms from $\mathbb{Z}[1/q]$ (q a prime different from p) and $(\pi_2 Y)/T$ does not. Hence $D_F G \neq C_F G$ and we are done. \square

Remark 5.4. In the previous proof we do not know if $(\pi_2 Y)/T$ is a free abelian group, because we are unable to show if the aforementioned extension that defines $\pi_2 Y$ splits or not. Recall that $\text{Ext}(\mathbb{Z}[1/p], \mathbb{Z}) = \mathbb{Z}_p^\wedge/\mathbb{Z}$ is uncountable, so there is an uncountable number of abelian extensions of $\mathbb{Z}[1/p]$ by \mathbb{Z} , but only the trivial extension of \mathbb{Z} by $\mathbb{Z}[1/p]$ splits.

We conclude by pointing out that the arguments of Proposition 5.3 are also valid when approaching C_F and D_F for the perfect groups defined by

Berrick–Miller in [4] for which the Schur multiplier is free abelian. Moreover, *mutatis mutandis*, they may also be used to describe the \mathbb{Z}/p -acyclic cover of these Burnside groups of high exponent. This is related with recent results about cellular covers of Burnside groups ([25], [28] [22], [23]), that in particular have been useful to solve some conjectures by Farjoun [17]. On the other hand, note that we have approached the description of the acyclic and cellular covers of a non-nilpotent group with respect to an uncountable non-nilpotent group. This kind of problems has been widely studied in the recent literature, but usually under restrictive hypotheses of finiteness or nilpotency of the groups involved. See for example [5], [19], [20], [24].

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